REMARKS ON NOISE SENSITIVITY, BRASCAMP-LIEB AND SLEPIAN INEQUALITIES

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Abstract. – *In a recent work, E. Mossel and J. Neeman provided a heat flow monotonicity proof of Borell's noise sensitivity for the Ornstein-Uhlenbeck semigroup. The argument actually includes in a common framework noise sensitivity, Brascamp-Lieb inequalities (including hypercontractivity) and even a weak form of Slepian inequalities. The scheme applies furthermore to families of measures with are more log-concave than the Gaussian measure. The discrete cube raises some interesting issues on a class of concave functions on the plane.*

1. Hypercontractivity and Gaussian noise sensitivity

Borell's noise sensitivity theorem for the Ornstein-Uhlenbeck semigroup [Bor] expresses that if γ is the standard Gaussian measure $d\gamma(x) = d\gamma^n(x) = \mathrm{e}^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$ on $\mathbb{R}^n$, and if A, B are Borel measurable sets in $\mathbb{R}^n$ and H, K are parallel half-spaces with respectively the same Gaussian measures $\gamma(A) = \gamma(H)$, $\gamma(B) = \gamma(K)$, then, for every $t \geq 0$,

$$\int_A Q_t(1_B) d\gamma \leq \int_H Q_t(1_K) d\gamma. \tag{1}$$

Here $(Q_t)_{t \geq 0} = (Q_t^n)_{t \geq 0}$ is the Ornstein-Uhlenbeck semigroup defined, on suitable functions $f : \mathbb{R}^n \to \mathbb{R}$, by

$$Q_t f(x) = \int_{\mathbb{R}^n} f\big(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}}\, y\big) d\gamma(y), \quad t \geq 0, \ x \in \mathbb{R}^n. \tag{2}$$

Alternatively, $Q_t f(x)$ is given by the Mehler kernel

$$Q_t f(x) = \int_{\mathbb{R}^n} f(y) q_t(x, y) d\gamma(y) \tag{3}$$

where, for $t > 0$, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$q_t(x, y) = q_t^n(x, y) = \frac{1}{\sqrt{1 - \mathrm{e}^{-2t}}} \exp\bigg(- \frac{\mathrm{e}^{-2t}}{2(1 - \mathrm{e}^{-2t})} \big[|x|^2 + |y|^2 - 2\,\mathrm{e}^t x \cdot y \big] \bigg). \tag{4}$$

According to the representation (2), setting $\rho = \mathrm{e}^{-t}$, if $X = X^n$ and $Y = Y^n$ are independent with distribution $\gamma = \gamma^n$,

$$\int_A Q_t(1_B)d\gamma = \mathbb{P}\big(X \in A, \rho X + \sqrt{1-\rho^2}\, Y \in B\big)$$

so that the conclusion (1) reads equivalently as

$$\mathbb{P}\big(X \in A, \rho X + \sqrt{1-\rho^2}\, Y \in B\big) \leq \mathbb{P}\big(X \in H, \rho X + \sqrt{1-\rho^2}\, Y \in K\big).$$

In other words, if $Z = (X, \rho X + \sqrt{1-\rho^2}\, Y)$ is a (centered) Gaussian vector in $\mathbb{R}^n \times \mathbb{R}^n$ with covariance matrix

$$\begin{pmatrix} \mathrm{Id}_n & \rho\,\mathrm{Id}_n \\ \rho\,\mathrm{Id}_n & \mathrm{Id}_n \end{pmatrix}, \tag{5}$$

then

$$\mathbb{P}(Z \in A \times B) \leq \mathbb{P}(Z \in H \times K). \tag{6}$$

The result then extends to any $\rho \in [-1, +1]$ with however the inequality in (6) reversed when $\rho \in [-1, 0]$. For simplicity in the exposition, we mostly only consider $\rho \in [0, 1]$ below.

Note that since H and K are (parallel) half-spaces of the form (by rotational invariance)

$$H = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_1 \leq a\big\}, \quad K = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_1 \leq b\big\} \tag{7}$$

for some $a, b \in \mathbb{R}$,

$$\mathbb{P}\big(X^n \in H, \rho X^n + \sqrt{1-\rho^2}\, Y^n \in K\big) = \mathbb{P}\big(X^1 \leq a, \rho X^1 + \sqrt{1-\rho^2}\, Y^1 \leq b\big).$$

Then, by the kernel representation (3), if $\rho = \mathrm{e}^{-t}$,

$$\mathbb{P}\big(X^1 \leq a, \rho X^1 + \sqrt{1-\rho^2}\, Y^1 \leq b\big) = \int_{-\infty}^a \int_{-\infty}^b q_t^1(x,y)d\gamma^1(x)d\gamma^1(y). \tag{8}$$

Towards the proof of (1), C. Borell [Bor] developed symmetrization arguments with respect to the Gaussian measure introduced in [E]. Recently, E. Mossel and J. Neeman [M-N] proposed an alternate semigroup proof. This proof actually involves a specific function, called below Borell's noise sensitivity function, satisfying some particular concavity property. To describe this property, say that a C^2 function J on $\mathbb{R}^2$ or some open set $\mathcal{O}$ in $\mathbb{R}^2$ is ρ-convave for some $\rho \in \mathbb{R}$ if the matrix

$$\begin{pmatrix} \partial_{11}J & \rho\,\partial_{12}J \\ \rho\,\partial_{12}J & \partial_{22}J \end{pmatrix}$$

is (uniformly) semi-negative definite. $\rho = 1$ amounts to standard concavity while $\rho = 0$ amounts to concavity along each coordinate. Note that the preceding matrix is the point-wise (Hadamard) product of the Hessian of J with the matrix (5) (with $n = 1$).

In this class of ρ-concave functions, two examples are of most interest. Let first

$$J^{\mathrm{H}}(u, v) = u^\alpha v^\beta, \quad (u, v) \in [0, \infty)^2.$$

Since

$$\partial_{11} J^{\mathrm{H}} = \alpha(\alpha - 1)u^{\alpha-2}v^\beta, \quad \partial_{22} J^{\mathrm{H}} = \beta(\beta - 1)u^\alpha v^{\beta-2}, \quad \partial_{12} J^{\mathrm{H}} = \alpha\beta u^{\alpha-1}v^{\beta-1},$$

J^{H} is ρ-concave on $(0, \infty)^2$ as soon as $\alpha, \beta \in [0, 1]$ and

$$(\alpha - 1)(\beta - 1) \leq \rho^2 \alpha\beta. \tag{9}$$

The function J^{H} will be called the hypercontractive function in this context.

The second example is therefore Borell's noise sensitivity function considered in [M-N] defined for $(u, v) \in [0, 1]^2$ by

$$J^{\mathrm{B}}(u, v) = J^{\mathrm{B}}_\rho(u, v) = \mathbb{P}\big(X^1 \leq \Phi^{-1}(u), \rho X^1 + \sqrt{1 - \rho^2}\, Y^1 \leq \Phi^{-1}(v)\big)$$

where $\Phi(a) = \gamma^1((-\infty, a])$, $a \in \mathbb{R}$, is the distribution of the standard normal on $\mathbb{R}$ and $\rho = e^{-t}$. For the connection with Borell's theorem, observe that if H and K are half-spaces in $\mathbb{R}^n$ as in (7),

$$J^{\mathrm{B}}\big(\gamma(H), \gamma(K)\big) = \int_H Q_t(1_K) d\gamma. \tag{10}$$

In order to check the ρ-concavity of J^{B}, note for example that by (8),

$$J^{\mathrm{B}}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} q_t^1(x, y) d\gamma^1(x) d\gamma^1(y).$$

Elementary calculus therefore yields

$$\partial_1 J^{\mathrm{B}}(u, v) = \int_{-\infty}^{\Phi^{-1}(v)} q_t^1\big(\Phi^{-1}(u), y\big) d\gamma^1(y)$$

and

$$\partial_{12} J^{\mathrm{B}}(u, v) = q_t^1\big(\Phi^{-1}(u), \Phi^{-1}(v)\big).$$

On the other hand, by the integral representations (2) and (3), for h smooth enough,

$$\partial_x \int_{\mathbb{R}} h(y) q_t^1(x, y) d\gamma^1(y) = \partial_x Q_t^1 h(x) = \rho\, Q_t^1 h'(x) = \rho \int_{\mathbb{R}} h'(y) q_t^1(x, y) d\gamma^1(y).$$

With h a smooth approximation of $1_{(-\infty, b]}$,

$$\partial_x \int_\infty^b q_t^1(x, y) d\gamma^1(y) = -\rho\, q_t^1(x, b)\varphi(b)$$

where $\varphi = \Phi'$ is the density of γ^1. Therefore,

$$\partial_{11} J^{\mathrm{B}}(u,v) = -\rho\, q_t^1\big(\Phi^{-1}(u), \Phi^{-1}(v)\big)\, \frac{\varphi \circ \Phi^{-1}(v)}{\varphi \circ \Phi^{-1}(u)}\,.$$

Similarly,

$$\partial_{22} J^{\mathrm{B}}(u,v) = -\rho\, q_t^1\big(\Phi^{-1}(u), \Phi^{-1}(v)\big)\, \frac{\varphi \circ \Phi^{-1}(u)}{\varphi \circ \Phi^{-1}(v)}\,.$$

Hence, on $(0,1)^2$,

$$\partial_{11} J^{\mathrm{B}}\, \partial_{22} J^{\mathrm{B}} - \rho^2 (\partial_{12} J^{\mathrm{B}})^2 = 0$$

and $\partial_{11} J^{\mathrm{B}} \leq 0$, $\partial_{22} J^{\mathrm{B}} \leq 0$ so that J^{B} is indeed ρ-concave.

When $\rho \in [-1, 0]$, observe that

$$J_\rho^{\mathrm{B}}(u,v) = u - J_{-\rho}^{\mathrm{B}}(u, 1-v) \tag{11}$$

so that J^{B} is ρ-convex in this case.

The main result by E. Mossel and J. Neeman [M-N] expresses an integral concavity property along the Mehler kernel (from (4)) for ρ-concave functions.

Theorem 1. *Let $\rho \in [0,1]$ and let J be ρ-concave on $\mathcal{O} = I_1 \times I_2 \subset \mathbb{R}^2$ where I_1 and I_2 are open intervals. For every functions $f : \mathbb{R}^n \to I_1$, $g : \mathbb{R}^n \to I_2$ suitably integrable, and with $\rho = \mathrm{e}^{-t}$,*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J\big(f(x), g(y)\big) q_t(x,y) d\gamma(x) d\gamma(y) \leq J\bigg(\int_{\mathbb{R}^n} f\, d\gamma, \int_{\mathbb{R}^n} g\, d\gamma \bigg).$$

The proof by E. Mossel and J. Neeman [M-N] of Theorem 1 relies on heat flow monotonicity and will be emphasized in a more general context next. Before turning to the sketch of the argument, let us illustrate its application to the two previous examples of ρ-concave functions J^{H} and J^{B}, covering in this way hypercontractivity and noise sensitivity at the same time.

Concerning hypercontractivity, let $1 < p < q < \infty$ and let $\rho = \mathrm{e}^{-t} \in (0,1)$ be such that

$$\frac{1}{\rho^2} = \frac{q-1}{p-1}\,.$$

Denote by q' the conjugate of q, $\frac{1}{q} + \frac{1}{q'} = 1$. Then, according to (9), the function J^{H} with $\alpha = \frac{1}{q'}$ and $\beta = \frac{1}{p}$ is ρ-concave on $(0,\infty)^2$. For then strictly positive functions $f, g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^{1/q'}(x) g^{1/p}(y) q_t(x,y) d\gamma(x) d\gamma(y) \leq \bigg(\int_{\mathbb{R}^n} f\, d\gamma \bigg)^{1/q'} \bigg(\int_{\mathbb{R}^n} g\, d\gamma \bigg)^{1/p}.$$

In other words, changing f into $f^{q'}$ and g into g^p,

$$\int_{\mathbb{R}^n} f\, Q_t g\, d\gamma \leq \|f\|_{q'} \|g\|_p.$$

By duality

$$\|Q_t g\|_q \leq \|g\|_p$$

which amounts to hypercontractivity of the Ornstein-Uhlenbeck semigroup [Nel], [Gr] ([B-G-L]). Clearly, the conclusion of Theorem 1 for J^{H} is actually equivalent to hypercontractivity. Note that a prior to the proof of hypercontractivity along these lines may be found in [H].

Apply now on the other hand Theorem 1 to the function J^{B}. Since $J^{\mathrm{B}}(u,0) = J^{\mathrm{B}}(0,v) = 0$ and $J^{\mathrm{B}}(1,1) = 1$, for $f = 1_A$ and $g = 1_B$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J\big(f(x), g(y)\big) q_t(x,y) d\gamma(x) d\gamma(y) = \int_A\int_B q_t(x,y) d\gamma(x) d\gamma(y) = \int_A Q_t(1_B) d\gamma.$$

We then recover Borell's noise sensitivity theorem (1) since by (10)

$$J^{\mathrm{B}}\bigg(\int_{\mathbb{R}^n} f\, d\gamma, \int_{\mathbb{R}^n} g\, d\gamma\bigg) = J^{\mathrm{B}}\big(\gamma(A), \gamma(B)\big) = J^{\mathrm{B}}\big(\gamma(H), \gamma(K)\big) = \int_H Q_t(1_K) d\gamma$$

for parallel half-spaces H and K such that respectively $\gamma(A) = \gamma(H)$ and $\gamma(B) = \gamma(K)$. When $\rho \in [-1, 0]$, the conclusion of Theorem 1 for the function J^{B}_ρ is thus reversed by (11). As pointed out in [M-N], (1) on sets may actually be turned to Theorem 1 (for J^{B}) through epigraphs of functions on $\mathbb{R}^{n-1}$.

It is of interest to directly compare the conclusion of Theorem 1 for the hypercontractive function J^{H} and for Borell's noise sensitivity function J^{B}, and namely to show that noise sensivity is a stronger statement implying hypercontractivity. One way towards this end, however along a rather long detour, is to observe, as emphasized in [L], that Borell's noise sensitivity theorem may be used to reach the Gaussian isoperimetric inequality. Now, the latter implies in turn the standard logarithmic Sobolev inequality for the Gaussian measure, equivalent to hypercontractivity (cf. [L], [B-G-L]).

There is an alternate direct argument towards this relationship, applying the conclusion for J^{B} to εf and δg and letting $\varepsilon, \delta \to 0$. To this task, it is necessary to investigate the asymptotics of $J^{\mathrm{B}}(\varepsilon u, \delta v)$ as $\varepsilon, \delta \to 0$. Similar asymptotics are investigated in [DK-P-W].

Set $\rho = \mathrm{e}^{-t} > 0$ and fix $0 < u, v < 1$. Let furthermore $0 < \varepsilon < 1$, $\delta = \varepsilon^{\kappa^2}$ where $\rho < \kappa < \frac{1}{\rho}$, and

$$Z = \sqrt{2 \log \frac{1}{\varepsilon}}\,, \quad U = \log \frac{1}{u}\,, \quad V = \log \frac{1}{v}\,.$$

In this notation, after a change of variables,

$$J^{\mathrm{B}}(\varepsilon u, \delta v) = \frac{UV}{\kappa Z^2} \int_A^\infty \int_B^\infty \tilde{q}_t^1\Big(-Z - \frac{Ux}{Z}, -\kappa Z - \frac{Vy}{\kappa Z}\Big) dx dy$$

where

$$A = -\frac{Z}{U}\left[Z + \Phi^{-1}(\varepsilon u)\right] \quad \text{and} \quad B = -\frac{Z}{\kappa V}\left[\kappa Z + \Phi^{-1}(\delta v)\right]$$

and

$$\tilde{q}_t^1(x,y) = (2\pi)^{-1} q_t^1(x,y) \mathrm{e}^{-(x^2+y^2)/2}, \quad (x,y) \in \mathbb{R} \times \mathbb{R}.$$

After some algebra,

$$J^{\mathrm{B}}(\varepsilon u, \delta v) = \frac{UV\, \mathrm{e}^{\sigma Z^2}}{2\pi \sqrt{1-\rho^2}\kappa Z^2} \int_A^\infty \int_B^\infty \mathrm{e}^{-\alpha U x - \beta V y - R(x,y)} dx dy$$

where

$$\sigma = -\frac{1-2\kappa\rho+\kappa^2}{2(1-\rho^2)}, \quad \alpha = \frac{1-\kappa\rho}{1-\rho^2}, \quad \beta = \frac{1-\kappa^{-1}\rho}{1-\rho^2}$$

and

$$R(x,y) = -\frac{1}{2(1-\rho^2)} \bigg(\frac{U^2 x^2}{Z^2} + \frac{V^2 x^2}{\kappa^2 Z^2} - 2\rho \, \frac{UVxy}{\kappa Z^2} \bigg).$$

It is classical that

$$\Phi^{-1}(\varepsilon) = -\sqrt{2\log\frac{1}{\varepsilon}} + o\bigg(\sqrt{2\log\frac{1}{\varepsilon}}\bigg)$$

as $\varepsilon \to 0$, so that

$$\Phi^{-1}(\varepsilon u) = -Z - \frac{U}{Z} + o(Z)$$

as $Z \to \infty$. Moreover, $o(Z)$ can be made uniform over $\eta \le u \le 1-\eta$ for $\eta > 0$ fixed. As a consequence, as $\varepsilon \to 0$, $A, B \to 1$ and

$$2\pi \sqrt{1-\rho^2}\, \kappa Z^2 \mathrm{e}^{-\sigma Z^2} J^{\mathrm{B}}(\varepsilon u, \delta v) \to UV \int_1^\infty \int_1^\infty \mathrm{e}^{-\alpha U x - \beta V y} dx dy = \frac{1}{\alpha\beta}\, \mathrm{e}^{-\alpha U - \beta V}.$$

By definition of U and V, the right-hand side is $\frac{1}{\alpha\beta}\, u^\alpha v^\beta$.

Let now f, g on $\mathbb{R}^n$ such that $\eta \le f, g \le 1-\eta$ for some fixed $\eta > 0$. Translating the preceding asymptotics in the inequality

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J^{\mathrm{B}}\big(\varepsilon f(x), \delta g(y)\big) q_t(x,y) d\gamma(x) d\gamma(y) \le J^{\mathrm{B}}\bigg(\varepsilon \int_{\mathbb{R}^n} f\, d\gamma, \delta \int_{\mathbb{R}^n} g\, d\gamma\bigg)$$

yields

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)^\alpha g(y)^\beta q_t(x,y) d\gamma(x) d\gamma(y) \le \bigg(\int_{\mathbb{R}^n} f\, d\gamma\bigg)^\alpha \bigg(\int_{\mathbb{R}^n} g\, d\gamma\bigg)^\beta.$$

This inequality extends to all positive measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ by homogeneity. Now, as is immediatley checked, for the values of α, β defined above,

$$(\alpha - 1)(\beta - 1) = \rho^2 \alpha\beta,$$

that is condition (9) of hypercontractivity holds. Given therefore any $\alpha, \beta \in (0,1)$ satisfying this relation, one may choose $\rho < \kappa < \frac{1}{\rho}$ such that $\alpha = \frac{1-\kappa\rho}{1-\rho^2}$ and $\beta = \frac{1-\kappa^{-1}\rho}{1-\rho^2}$ as above. The announced claim follows.

It is a further interesting observation due to R. O'Donnell, and communicated to us by J. Neeman, that the conclusion of Theorem 1 implies back the ρ-concavity property by by taking $f(x) = a + \varepsilon x$ and $g(y) = b + \varepsilon y$ and letting $\varepsilon \to 0$. It would also be of interest to find other relevant examples of function J.

As announced, let us briefly sketch at this stage the heat flow proof of Theorem 1 following [M-N], the detailed argument being developed in the more general context of Section 3. Consider, for t (> 0) fixed and (smooth) functions $f : \mathbb{R}^n \to I_1$, $g : \mathbb{R}^n \to I_2$,

$$\psi(s) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J\big(Q_s f(x), Q_s g(y)\big) q_t(x,y) d\gamma(x) d\gamma(y), \quad s \geq 0.$$

By ergodicity, $Q_s f \to \int_{\mathbb{R}^n} f d\gamma$ and $Q_s g \to \int_{\mathbb{R}^n} g d\gamma$ as $s \to \infty$ so that it is enough to show that ψ is non-decreasing in order that $\psi(0) \leq \psi(\infty)$ (which amounts to the conclusion of the theorem). Differentiating ψ and integrating by parts with respect to the infinitesimal generator $\mathrm{L} = \Delta - x \cdot \nabla$ of the Ornstein-Uhlenbeck semigroup $(Q_s)_{s \geq 0}$ yields (see the details in Section 3),

$$\psi'(s) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big[(-\partial_{11} J) |\nabla Q_s f|^2 + (-\partial_{22} J) |\nabla Q_s g|^2 - 2\,\rho\,\partial_{12} J \nabla Q_s f \cdot \nabla Q_s g \Big] q_t d\gamma d\gamma.$$

From the hypothesis of ρ-concavity on J, it follows that $\psi' \geq 0$ which is the result.

It may be mentioned that due to the product structure of both the Mehler kernel q^n and the Gaussian measure γ^n, the inequality of Theorem 1 immediately tensorizes so that it is actually enough to establish it in dimension one.

The purpose of this note is to somewhat broaden the scope of Theorem 1 by E. Mossel and J. Neeman [M-N] and of its proof to cover in the same mould various related inequalities such as Brascamp-Lieb or Slepian inequalities. Actually, heat flow arguments towards Brascamp-Lieb inequalities have been investigated in the recent years by E. Carlen, E. Lieb and M. Loss [C-L-L] and J. Bennett, A. Carbery, M. Christ and T. Tao [B-C-C-T] (see also [B-CE-L-M]) with a similar principle applied to multi-dimensional versions of the hypercontractive J^{H} function. In Section 2, we thus consider in this respect the multi-dimensional versions of Theorem 1 which were recently emphasized in [Nee], and discuss their applications to various families of concave functions towards applications to Brascamp-Lieb and Slepian-type inequalities. In the next section, we address extensions from the Gaussian model to families of measures $d\mu = \mathrm{e}^{-V} dx$ with a lower bound on the Hessian of V following the basic semigroup interpolation argument. The last part briefly comments on some analogous issues on the discrete cube following [D-M-N].

It would be worthwhile to examine similarly the noise sensitivity theorem for the Lebesgue measure λ with respect to the standard heat kernel expressing that for Borel sets A, B in $\mathbb{R}^n$ with finite volume,

$$\int_A H_t(1_B) dx \leq \int_C H_t(1_D) dx$$

where

$$H_t f(x) = \int_{\mathbb{R}^n} f(y)\, \mathrm{e}^{-|x-y|^2/4t} \frac{dy}{(4\pi t)^{n/2}}, \quad t > 0,\ x \in \mathbb{R}^n,$$

and C and D are centered balls in $\mathbb{R}^n$ such that $\lambda(A) = \lambda(C)$ and $\lambda(B) = \lambda(D)$. This classical result is going back to [B-L-L] and [B-T] by rearrangement tools and one might wonder for a heat flow proof. A similar question may be formulated on the sphere.

2. Multi-dimensional extensions

On the basis of the heat flow proof of Theorem 1, we address in this section multi-dimensional extensions and develop connections to Brascamp-Lieb and Slepian-type inequalities. The multi-dimensional versions of noise sensitivity were already put forward by J. Neeman in [Nee]. The Brascamp-Lieb applications are contained with the same approach in [C-L-L] and [B-C-C-T]. At the same time, the investigation provides a somewhat different analytical treatment of the conclusions of Section 1.

Let J be a (smooth) real-valued function on some open subset $\mathcal{O}$ of $\mathbb{R}^m$. It will be implicitly assumed below that a composition like $J \circ f$ is meant for functions f with values in $\mathcal{O}$.

Let $f_1, \ldots, f_m$ be (smooth) functions on $\mathbb{R}^n$ and consider, for $f = (f_1, \ldots, f_m)$,

$$\psi(s) = \int_{\mathbb{R}^n} J \circ Q_s f \, d\gamma, \quad s \geq 0,$$

where $(Q_s)_{s \geq 0}$ is the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^n$ (extended on functions with values in $\mathbb{R}^m$). Arguing as in Section 1, by integration by parts with respect to the Ornstein-Uhlenbeck generator,

$$\psi'(s) = -\sum_{k,\ell=1}^m \int_{\mathbb{R}^n} \partial_{k\ell} J \circ Q_s f\, \nabla Q_s f_k \cdot \nabla Q_s f_\ell \, d\gamma.$$

In the preceding, replace now n by qn, $q \geq 1$ integer, and assume that for every $k = 1, \ldots, m$,

$$f_k = g_k \circ A_k$$

where $g_k : \mathbb{R}^p \to \mathbb{R}$ and A_k is a (constant) $p \times qn$ matrix such that $A_k\, {}^t\!A_k$ is the identity matrix (of $\mathbb{R}^p$). By the integral representation (2) of Q_s,

$$\nabla Q_s f_k = \mathrm{e}^{-s}\ {}^t\!A_k \nabla Q_s g_k \circ A_k$$

where on the left-hand side the semigroup Q_s is acting on $\mathbb{R}^{qn}$ and on the right-hand side, it is acting on $\mathbb{R}^p$. Hence

$$\psi'(s) = -\,\mathrm{e}^{-2s} \sum_{k,\ell=1}^m \int_{\mathbb{R}^{qn}} \partial_{k\ell} J \circ Q_s f\, \Gamma_{k\ell} \nabla Q_s g_k \circ A_k \cdot \nabla Q_s g_\ell \circ A_\ell \, d\gamma$$

where $\Gamma_{k\ell} = A_\ell\, {}^t\!A_k$ (which is a $p \times p$ matrix). The following proposition summarizes the conclusion at this level of generality.

Proposition 2. *In the preceding notation, provided the Hessian of J is such that for all vectors v_k, $k = 1, \ldots, m$, in $\mathbb{R}^p$,*

$$\sum_{k,\ell=1}^{m} \partial_{k\ell} J\, \Gamma_{k\ell}\, v_k \cdot v_\ell \leq 0, \tag{12}$$

then $\int_{\mathbb{R}^{qn}} J \circ f\, d\gamma \leq J\big(\int_{\mathbb{R}^{qn}} f\, d\gamma\big)$, that is

$$\int_{\mathbb{R}^{qn}} J(g_1 \circ A_1, \ldots, g_m \circ A_m) d\gamma \leq J\bigg(\int_{\mathbb{R}^{qn}} g_1 \circ A_1 d\gamma, \ldots, \int_{\mathbb{R}^{qn}} g_m \circ A_m d\gamma \bigg).$$

To connect with Section 1, take for example $p = n$ and $q = m = 2$ and let A_1 and A_2 be the $n \times 2n$ matrices $A_1 = (\mathrm{Id}_n; 0_n)$ and $A_2 = (\rho\, \mathrm{Id}_n; \sqrt{1-\rho^2}\, \mathrm{Id}_n)$ so that

$$f_1(x,y) = g_1(x) \quad \text{and} \quad f_2(x,y) = g_2\big(\rho x + \sqrt{1-\rho^2}\, y\big), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Moreover, $\Gamma_{11} = \Gamma_{22} = \mathrm{Id}_n$ and $\Gamma_{12} = \Gamma_{21} = \rho\, \mathrm{Id}_n$. Therefore

$$\begin{aligned} \psi'(s) = -\,\mathrm{e}^{-2s} \Big[& \partial_{11} J \left| \nabla Q_s g_1 \circ A_1 \right|^2 + 2\rho\, \partial_{12} J\, \nabla Q_s g_1 \circ A_1 \cdot \nabla Q_s g_2 \circ A_2 \\ & + \partial_{22} J \left| \nabla Q_s g_2 \circ A_2 \right|^2 \Big] \end{aligned}$$

so that the monotonicity property similarly follows from the ρ-concavity of the Hessian of J expressed by (12).

We next systematically investigate illustrations of Proposition 2 for some main examples of interest. For simplicity, we consider the one-dimensional versions $p = q = 1$, the multi-dimensional cases being often obtained by tensor products with the identity matrix (as in the preceding example). In particular, the meaning of condition (12) is that the point-wise (Hadamard) multiplication of the Hessian of J and of Γ is (semi-) negative definite.

(i) The first illustration examines Brascamp-Lieb inequalities under geometric conditions. Consider unit vectors $A_1, \ldots, A_m$ which decompose the identity in $\mathbb{R}^n$ in the sense that for $0 \leq c_k \leq 1$, $k = 1, \ldots, m$,

$$\sum_{k=1}^{m} c_k A_k \otimes A_k = \mathrm{Id}_n. \tag{13}$$

Then, for

$$J(u_1, \ldots, u_m) = u_1^{c_1} \cdots u_m^{c_m}$$

on $(0, \infty)^m$ and $f_k(x) = g_k(A_k \cdot x)$, $g_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$, condition (12) of Proposition 2 amounts to

$$\sum_{k,\ell=1}^{m} c_k c_\ell A_k \cdot A_\ell\, v_k v_\ell \leq \sum_{k=1}^{m} c_k v_k^2 \tag{14}$$

for all $v_1, \ldots, v_m \in \mathbb{R}$. Now, if $x = \sum_{k=1}^m c_k A_k v_k$,

$$|x|^2 = \sum_{k=1}^m c_k A_k v_k \cdot x \leq \bigg(\sum_{k=1}^m c_k v_k^2 \bigg)^{1/2} \bigg(\sum_{k=1}^m c_k (A_k \cdot x)^2 \bigg)^{1/2}$$

Since by the decomposition (13) $|x|^2 = \sum_{k=1}^m c_k (A_k \cdot x)^2$, it follows that

$$|x|^2 = \bigg| \sum_{k=1}^m c_k A_k v_k \bigg|^2 \leq \sum_{k=1}^m c_k v_k^2$$

which is the precisely requested inequality (14). We therefore conclude to the following result.

Corollary 3. *Under the decomposition (13), for non-negative functions g_k on $\mathbb{R}$, $k = 1, \ldots, m$,*

$$\int_{\mathbb{R}^n} \prod_{k=1}^m g_k^{c_k}(A_k \cdot x) d\gamma \leq \prod_{k=1}^m \bigg(\int_{\mathbb{R}} g_k d\gamma \bigg)^{c_k}.$$

This inequality is part of the Brascamp-Lieb inequalities (under the geometric condition (13), cf. e.g. [B-CE-L-M]). It is more classically stated with respect to the Lebesgue measure as

$$\int_{\mathbb{R}^n} \prod_{k=1}^m f_k^{c_k}(A_k \cdot x) dx \leq \prod_{k=1}^m \bigg(\int_{\mathbb{R}} f_k dx \bigg)^{c_k}$$

which is immediately obtained after the change $f_k(x) = g_k(x) \mathrm{e}^{-x^2/2}$ (using that $\sum_{k=1}^m c_k = n$).

The heat flow proof of Corollary 3 is thus going back to [C-L-L] and [B-C-C-T] in which more general statements are considered and achieved in this way. One of the motivations of [C-L-L] was actually to investigate similar inequalities for coordinates on the sphere. Let $\mathbb{S}^{n-1}$ be the standard n-sphere in $\mathbb{R}^n$ and denote by σ the uniform (normalized) measure on it. In this framework, one result then reads as follows. If g_k, $k = 1, \ldots, n$, are, say bounded measurable, functions on $\mathbb{R}$, then

$$\int_{\mathbb{S}^{n-1}} J\big(g_1(x_1), \ldots, g_n(x_n)\big) d\sigma \leq J\bigg(\int_{\mathbb{S}^{n-1}} g_1(x_1) d\sigma, \ldots, \int_{\mathbb{S}^{n-1}} g_n(x_n) d\sigma \bigg)$$

as soon as $J : \mathbb{R}^n \to \mathbb{R}$ is separately concave in any two variables. The proof proceeds as the one of Proposition 2 along now the heat flow of the Laplace operator

$$\Delta = \frac{1}{2} \sum_{k,\ell=1}^n (x_k \partial_\ell - x_\ell \partial_k)^2$$

on $\mathbb{S}^{n-1}$. The condition (12) then takes the form

$$\sum_{k,\ell=1}^n \partial_{k\ell} J \, (\delta_{k\ell} - x_k x_\ell) v_k v_\ell \leq 0$$

which is easily seen to be satisfied under concavity of J in any two variables. The case considered in [C-L-L] simply corresponds to

$$J(u_1, \dots, u_n) = (u_1 \cdots u_n)^{1/2}$$

on $\mathbb{R}_+^n$. More general forms under decompositions (13) of the identity have been considered in [B-CE-M], [B-CE-L-M].

In the further illustrations, consider $X = (X_1, \dots, X_m)$ a centered Gaussian vector on $\mathbb{R}^m$ with covariance matrix $\Gamma = A\,{}^t\!A$ such that $\Gamma_{kk} = 1$ for every $k = 1, \dots, m$. The vector X has the distribution of Ax, $x \in \mathbb{R}^n$, under the standard normal distribution γ on $\mathbb{R}^n$. Applying the general Proposition 2 to the unit vectors ($1 \times n$ matrices) A_k, $k = 1, \dots, m$, which are the lines of the matrix A and to $f_k(x) = g_k(A_k \cdot x)$, $x \in \mathbb{R}^n$, where $g_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$, with respect to γ, yields that under condition (12), that is here

$$\sum_{k,\ell=1}^{m} \partial_{k\ell} J \, \Gamma_{k\ell} \, v_k v_\ell \leq 0 \tag{15}$$

for all $v_k \in \mathbb{R}$, $k = 1, \dots, m$ (and suitable integrability properties on the g_k's),

$$\mathbb{E}\Big(J\big(g_1(X_1), \dots, g_m(X_m)\big)\Big) \leq J\Big(\mathbb{E}\big(g_1(X_1)\big), \dots, \mathbb{E}\big(g_m(X_m)\big)\Big). \tag{16}$$

Note that, as in Section 1, the condition (15) is actually necessary and sufficient for (16) to hold.

(ii) This illustration deals with a correlation inequality for Gaussian vectors which covers in particular the classical hypercontractivity property. For a Gaussian vector X as above, let as in the first illustration,

$$J(u_1, \dots, u_m) = u_1^{c_1} \cdots u_m^{c_m}$$

on $(0, \infty)^m$, with $c_k \geq 0$, $k = 1, \dots, m$. This function J is the suitable multi-dimensional analogue of the hypercontractive function J^{H}. Applying the preceding conclusion (16) yields the following statement.

Corollary 4. *Assume that $\Gamma_{kk} = 1$ for every $k = 1, \dots, m$, and that*

$$\sum_{k,\ell=1}^{m} c_k c_\ell \, \Gamma_{k\ell} \, v_k v_\ell \leq \sum_{k=1}^{m} c_k \, \Gamma_{kk} v_k^2 \tag{17}$$

for all $v_k \in \mathbb{R}$, $k = 1, \dots, m$. Then, for all non-negative functions $g_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$,

$$\mathbb{E}\bigg(\prod_{k=1}^{m} g_k^{c_k}(X_k)\bigg) \leq \prod_{k=1}^{m} \Big(\mathbb{E}\big(g_k(X_k)\big)\Big)^{c_k}. \tag{18}$$

Note that condition (17) amounts to the fact that $\Gamma \leq \Delta_c$ in the sense of symmetric matrices where Δ_c is the diagonal matrix $(\frac{1}{c_k})_{1\leq k\leq m}$. Observe also that if $\Gamma \geq \Delta_c$, the conclusion is reversed in (18). While Corollary 4 is somewhat part of the folklore (implicit for example in [B-CE-L-M]), it has been emphasized recently in [C-D-P] together with multi-dimensional versions.

One illustration concerns the Ornstein-Uhlenbeck process $Z = (Z_t)_{t\geq 0}$ (in dimension one) with stationary measure $\gamma = \gamma^1$ and associated Markov semigroup $(Q_t)_{t\geq 0} = (Q_t^1)_{t\geq 0}$. If X is the vector $(Z_{t_1}, \ldots, Z_{t_m})$ with $0 \leq t_1 \leq \cdots \leq t_m$, the covariance matrix Γ has entries $\Gamma_{k\ell} = \mathrm{e}^{-|t_k - t_\ell|}$, $k, \ell = 1, \ldots, m$. In particular, for $t_1 = 0$ and $t_2 = t > 0$, the condition (17) reads

$$2\,\mathrm{e}^{-t} c_1 c_2 v_1 v_2 \leq c_1(1-c_1)v_1^2 + c_2(1-c_2)v_2^2$$

for all $v_1, v_2 \in \mathbb{R}$ which amounts to (9)

$$(c_1 - 1)(c_2 - 1) \leq \mathrm{e}^{-2t} c_1 c_2$$

and the conclusion of Corollary 4 leads to hypercontractivity. The condition

$$\sum_{k,\ell=1}^{m} c_k c_\ell\, \mathrm{e}^{-|t_k - t_\ell|} v_k v_\ell \leq \sum_{k=1}^{m} c_k v_k^2$$

leads to a multi-dimensional form of hypercontractivity

$$\mathbb{E}\bigg(\prod_{k=1}^{m} g_k^{c_k}(Z_{s_k}) \bigg) \leq \prod_{k=1}^{m} \Big(\mathbb{E}\big(g_k(Z_{s_k}) \big) \Big)^{c_k}.$$

In terms of the Mehler kernel (4),

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=1}^{m} g_k^{c_k}(x_k)\, q_{t_2 - t_1}(x_1, x_2) \cdots q_{t_m - t_{m-1}}(x_{m-1}, x_m) d\gamma(x_1) \cdots d\gamma(x_m)$$
$$\leq \prod_{k=1}^{m} \bigg(\int_{\mathbb{R}} g_k d\gamma \bigg)^{c_k}.$$

(iii) We next turn to the multi-dimensional versions of noise sensitivity following [Nee]. Let $X = (X_1, \ldots, X_m)$ be a centered Gaussian vector on $\mathbb{R}^m$ with (non-degenerate) covariance matrix Γ. Define, for $u_1, \ldots, u_m$ in $(0,1)$,

$$J(u_1, \ldots, u_m) = \mathbb{P}\big(X_1 \leq \alpha_1(u_1), \ldots, X_m \leq \alpha_m(u_m) \big) \tag{19}$$

where $\alpha_1, \ldots, \alpha_m$ are smooth functions on $(0,1)$. For specific choices of α_k, this function will turn as the multi-dimensional analogue of the noise sensitivity function J^{B}. Denoting by p the density of the distribution of X with respect to Lebesgue measure, elementary (although a bit tedious, see [Nee]) differential calculus leads to

$$\partial_{k\ell} J = \alpha_k'(u_k) \alpha_\ell'(u_\ell) \int_{-\infty}^{\alpha_1(u_1)} \cdots \int_{-\infty}^{\alpha_m(u_m)} p_{k\ell}\, dx$$

for $k \neq \ell$ and

$$\partial_{kk} J = \left(\alpha_k''(u_k) - \frac{\alpha_k(u_k)\alpha_k'(u_k)^2}{\Gamma_{kk}}\right) \int_{-\infty}^{\alpha_1(u_1)} \cdots \int_{-\infty}^{\alpha_m(u_m)} p_k \, dx$$
$$- \alpha_k'(u_k)^2 \sum_{\ell \neq k} \frac{\Gamma_{k\ell}}{\Gamma_{kk}} \int_{-\infty}^{\alpha_1(u_1)} \cdots \int_{-\infty}^{\alpha_m(u_m)} p_{k\ell} \, dx$$

where

$$p_k = p\big(x_1, \ldots, \alpha_k(u_k), \ldots, x_m\big),$$

$$p_{k\ell} = p\big(x_1, \ldots, \alpha_k(u_k), \ldots, \alpha_\ell(u_\ell), \ldots, x_m\big).$$

Choose now $\alpha_k = \Phi^{-1}$, $k = 1, \ldots, m$, and where we recall the distribution function Φ of the standard normal. Since

$$\alpha_k' = \frac{1}{\varphi \circ \Phi^{-1}} \quad \text{and} \quad \alpha_k'' = \frac{\Phi^{-1}}{(\varphi \circ \Phi^{-1})^2} \,,$$

in order for the condition (15) to be satisfied, it is thus sufficient that $\Gamma_{kk} = 1$ for every $k = 1, \ldots, m$ and

$$\sum_{k=1}^m \sum_{\ell \neq k} \Gamma_{k\ell} \, p_{k\ell} \, v_k^2 - \sum_{k \neq \ell} p_{k\ell} \, \Gamma_{k\ell} \, v_k v_\ell \geq 0$$

for all $v_1, \ldots, v_m \in \mathbb{R}$. This holds as soon as $\Gamma_{k\ell} \geq 0$ for all k, ℓ.

For the application to the following corollary, recall that for the choice of $\alpha_k = \Phi^{-1}$, the function J of (19) is equal to 0 if one of the u_k's is (approaches) 0, and is equal to 1 if all the u_k's are equal to 1. The corollary then follows from the application of (16) to $g_k = 1_{B_k}$, $k = 1, \ldots, m$. The restriction $\Gamma_{kk} = 1$, $k = 1, \ldots, m$, is lifted after a simple scaling of the Gaussian vector and the Borel sets.

Corollary 5. *Let $X = (X_1, \ldots, X_m)$ be a centered Gaussian vector in $\mathbb{R}^m$ with (non-degenerate) covariance matrix Γ such that $\Gamma_{k\ell} \geq 0$ for all $k, \ell = 1, \ldots, m$. Then, for any Borel sets $B_1, \ldots, B_m$ in $\mathbb{R}$,*

$$\mathbb{P}(X_1 \in B_1, \ldots, X_m \in B_m) \leq \mathbb{P}(X_1 \leq b_1, \ldots, X_m \leq b_m)$$

where $\mathbb{P}(X_k \in B_k) = \Phi(b_k/\sigma_k)$, $k = 1, \ldots, m$, where $\sigma_k = \sqrt{\Gamma_{kk}}$.

When $\Gamma_{k\ell} \leq 0$ whenever $k \neq \ell$, the inequality in the conclusion of Corollary 5 is reversed. As developed in [Nee], the result applies similarly to Gaussian vectors $X_1, \ldots, X_m$ with covariance identity matrix. A related work by M. Isaksson and E. Mossel [I-M] establishes the conclusion of Corollary 5 under the hypothesis that the off-diagonal elements of the inverse of Γ are non-positive. Their approach relies on a rearrangement inequality for kernels on the sphere. Corollary 5 (as well as actually, after some work, the result of [I-M] – see [Nee]) covers the example of the Ornstein-Uhlenbeck process, and thus of C. Borell's result [Bor] in the form of the following corollary.

Corollary 6. *Let $(Z_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck process on the line, and let $0 \leq t_1 \leq \cdots \leq t_m$. For any Borel sets $B_1, \ldots, B_m$ in $\mathbb{R}$,*

$$\mathbb{P}(Z_{t_1} \in B_1, \ldots, Z_{t_m} \in B_m) \leq \mathbb{P}(Z_{t_1} \leq b_1, \ldots, Z_{t_m} \leq b_m)$$

where $\mathbb{P}(Z_{t_k} \in B_k) = \gamma(B_k) = \Phi(b_k)$, $k = 1, \ldots, m$.

(iv) This illustration is a variation on the previous multi-dimensional noise sensitivity result which actually leads to a weak form of the classical Slepian inequalities. Let as above $X = (X_1, \ldots, X_m)$ be a centered Gaussian vector on $\mathbb{R}^m$ with covariance matrix $\Gamma = \Gamma^X$ such that $\Gamma^X_{kk} = 1$ for every $k = 1, \ldots, m$. Consider furthermore $Y = (Y_1, \ldots, Y_m)$ a centered Gaussian vector on $\mathbb{R}^m$ with covariance matrix Γ^Y also such that $\Gamma^Y_{kk} = 1$ for every $k = 1, \ldots, m$, yielding a J function (19)

$$J(u_1, \ldots, u_m) = \mathbb{P}\big(Y_1 \leq \alpha_1(u_1), \ldots, Y_m \leq \alpha_m(u_m)\big), \quad u_1, \ldots, u_m \in (0,1).$$

Choose now again $\alpha_k = \Phi^{-1}$. Arguing as *(iii)* towards (15), the condition is now that

$$\sum_{k=1}^m \sum_{\ell \neq k} \Gamma^Y_{k\ell}\, p_{k\ell}\, v_k^2 - \sum_{k \neq \ell} p_{k\ell}\, \Gamma^X_{k\ell}\, v_k v_\ell \geq 0$$

for all $v_1, \ldots, v_m \in \mathbb{R}$ (where p is here the density of the law of Y). This holds as soon as $\Gamma^Y_{k\ell} \geq 0$ and

$$\big(\Gamma^X_{k\ell}\big)^2 \leq \big(\Gamma^Y_{k\ell}\big)^2$$

for all $k \neq \ell$. As a conclusion

Corollary 7. *Let $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ be centered Gaussian vectors on $\mathbb{R}^m$ with respective (non-degenerate) covariance matrices Γ^X and Γ^Y. Assume that $\Gamma^X_{kk} = \Gamma^Y_{kk} = 1$ and*

$$\big|\Gamma^X_{k\ell}\big| \leq \Gamma^Y_{k\ell}$$

for all for every $k, \ell = 1, \ldots, m$. Then, for any Borel sets $B_1, \ldots, B_m$ in $\mathbb{R}$,

$$\mathbb{P}(X_1 \in B_1, \ldots, X_m \in B_m) \leq \mathbb{P}(Y_1 \leq b_1, \ldots, Y_m \leq b_m)$$

where $\mathbb{P}(X_k \in B_k) = \Phi(b_k)$, $k = 1, \ldots, m$. In particular, for every $r_1, \ldots, r_m$ in $\mathbb{R}$,

$$\mathbb{P}(X_1 \leq r_1, \ldots, X_m \leq r_m) \leq \mathbb{P}(Y_1 \leq r_1, \ldots, Y_m \leq r_m).$$

This result is of course a (very) weak form (in particular through the constraint $\Gamma^Y_{k\ell} \geq 0$) of the classical Slepian lemma which indicates that for Gaussian vectors X and Y in $\mathbb{R}^m$, the conclusion of Corollary 7 holds whenever $\Gamma^X_{kk} = \Gamma^Y_{kk}$ and $\Gamma^X_{k\ell} \leq \Gamma^Y_{k\ell}$ for all $k, \ell = 1, \ldots, m$. Note that the traditional proof of Slepian's lemma ([S], [F], [Go]) is an interpolation between the covariances Γ^X and Γ^Y which is not the same as the one at the root of Corollary 7.

3. Log-concave measures

In this section, we develop the heat flow proof of Theorem 1 of E. Mossel and J. Neeman in the somewhat extended context of probability measures $d\mu = \mathrm{e}^{-V}dx$ on $\mathbb{R}^n$ such that V is smooth potential with a uniform lower bound on its Hessian. The typical application actually concerns potentials V which are more convex than the quadratic one, corresponding to Gaussian measures. The argument may be developed in the more general context of Markov diffusion semigroups and the Γ-calculus as exposed in [B-G-L] although for the simplicity of this note, we stay in the familiar Euclidean case.

Consider therefore a probability measure $d\mu = \mathrm{e}^{-V}dx$ on the Borel sets of $\mathbb{R}^n$, invariant and symmetric measure of the second order differential operator $\mathrm{L} = \Delta - \nabla V \cdot \nabla$ where V is a smooth potential on $\mathbb{R}^n$. The (symmetric) semigroup $(P_t)_{t\geq 0}$ with generator L may be represented by (smooth) probability kernels

$$P_t h(x) = \int_{\mathbb{R}^n} h(y)p_t(x,dy). \tag{20}$$

It will be assumed that $V - c\,\frac{|x|^2}{2}$ is convex for some $c \in \mathbb{R}$, in other words the Hessian of V is bounded from below by $c\,\mathrm{Id}_n$ as symmetric matrices. It is by now classical (cf. [B-G-L]) that this convexity assumption ensures that for all (smooth) $h : \mathbb{R}^n \to \mathbb{R}$,

$$|\nabla P_t h| \leq \mathrm{e}^{-ct} P_t\big(|\nabla h|\big). \tag{21}$$

The Gaussian example of the Ornstein-Uhlenbeck semigroup $(Q_t)_{t\geq 0}$ with invariant measure γ is included with $c = 1$. In this case, due to the representation (2), the gradient bound (20) actually turns into the identity

$$\nabla Q_t h = \mathrm{e}^{-t} Q_t(\nabla h).$$

We start with the analogue of Theorem 1 in this context following therefore the argument of [M-N].

Theorem 8. *Let J be ρ-concave, $\rho > 0$, on $\mathcal{O} = I_1 \times I_2 \subset \mathbb{R}^2$ where I_1 and I_2 are open intervals. Then, for every $f : \mathbb{R}^n \to I_1$, $g : \mathbb{R}^n \to I_2$ suitably integrable, and with $\rho = \mathrm{e}^{-ct}$, $t > 0$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J\big(f(x), g(y)\big)p_t(x,dy)d\mu(x) \leq J\bigg(\int_{\mathbb{R}^n} f\,d\mu, \int_{\mathbb{R}^n} g\,d\mu\bigg).$$

Proof. It is enough to assume that f and g are taking values in respective compact sub-intervals of I_1 and I_2. Set

$$\psi(s) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J\big(P_s f(x), P_s g(y)\big)p_t(x,dy)d\mu(x), \quad s \geq 0.$$

The task is to show that ψ is non-decreasing. Taking derivative in time s,

$$\begin{aligned}\psi'(s) = &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \mathrm{L} P_s f(x) p_t(x, dy) d\mu(x)\\ &+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_2 J\big(P_s f(x), P_s g(y)\big) \mathrm{L} P_s g(y) p_t(x, dy) d\mu(x).\end{aligned}$$

By integration by parts in space with respect to the operator L, expressed by (for smooth functions $\xi, \zeta : \mathbb{R}^n \to \mathbb{R}$),

$$\int_{\mathbb{R}^n} \xi(-\mathrm{L}\zeta) d\gamma = \int_{\mathbb{R}^n} \nabla \xi \cdot \nabla \zeta \, d\gamma,$$

it holds

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \mathrm{L} P_s f(x) p_t(x, dy) d\mu(x)\\ &\qquad = -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla_x \Big[\partial_1 J\big(P_s f(x), P_s g(y)\big) p_t(x, dy)\Big] \cdot \nabla P_s f(x) d\mu(x)\\ &\qquad = -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_{11} J\big(P_s f(x), P_s g(y)\big) \big|\nabla P_s f(x)\big|^2 p_t(x, dy) d\mu(x)\\ &\qquad\quad - \int_{\mathbb{R}^n\times\mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \nabla P_s f(x) \cdot \nabla_x p_t(x, dy) d\mu(x).\end{aligned}$$

For $x \in \mathbb{R}^n$ fixed, consider $h(y) = \partial_1 J(P_s f(x), P_s g(y))$, $y \in \mathbb{R}^n$. Since

$$\nabla P_t h(z) = \int_{\mathbb{R}^n} h(y) \nabla_z p_t(z, dy), \quad z \in \mathbb{R}^n,$$

at $z = x$,

$$\int_{\mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \nabla P_s f(x) \cdot \nabla_x p_t(x, dy) = \nabla P_t h(x) \cdot \nabla P_s f(x).$$

Now, by (21),

$$\big|\nabla P_t h(x)\big| \le \mathrm{e}^{-ct} P_t\big(|\nabla h|\big)(x) = \mathrm{e}^{-ct} \int_{\mathbb{R}^n} \big|\nabla h(y)\big| p_t(x, dy).$$

Since

$$\nabla h(y) = \partial_{12} J\big(P_s f(x), P_s g(y)\big) \nabla P_s g(y),$$

it follows that

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \nabla_x p_t(x, dy) \cdot \nabla P_s f(x) d\mu(x)\\ &\qquad \le \mathrm{e}^{-ct} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\partial_{12} J|\big(P_s f(x), P_s g(y)\big) \big|\nabla P_s g(y)\big| \big|\nabla P_s f(x)\big| p_t(x, dy) d\mu(x).\end{aligned}$$

Summarizing, and by the symmetric conclusion in the y variable, $\psi'(s)$ is bounded from below by

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big[(-\partial_{11}J)|\nabla P_s f|^2+(-\partial_{22}J)|\nabla P_s g|^2-2\,\mathrm{e}^{-cs}|\partial_{12}J||\nabla P_s f||\nabla P_s g|\Big]p_t(x,dy)d\mu(x).$$

From the hypothesis on the Hessian of J, it follows that $\psi'\geq 0$ which is the result. □

As in the Gaussian case, the examples of illustration of Theorem 8 cover both hypercontractivity and noise sensitivity for the choices of $J=J^{\mathrm{H}}$ or $J=J^{\mathrm{B}}$. The noise sensitivity part, with $c>0$, actually turns into a comparison theorem.

Corollary 9. *Let $(P_t)_{t\geq 0}$ be the Markov semigroup with invariant reversible measure $d\mu=\mathrm{e}^{-V}dx$ where V is a smooth potential on $\mathbb{R}^n$ such that* $\mathrm{Hess}(V)\geq c\,\mathrm{Id}_n$ *with $c>0$. Then, whenever A,B are Borel sets in $\mathbb{R}^n$ and H,K are respective parallel half-spaces such that $\mu(A)=\gamma(H)$, $\mu(B)=\gamma(K)$, then*

$$\int_A P_t(1_B)d\mu\leq\int_H Q_{ct}(1_K)d\gamma.$$

As in the Gaussian (cf. [L]), this property may be shown to imply the isoperimetric comparison theorem of [B-L] (see [B-G-L]) comparing the isoperimetric profile of measures $d\mu=\mathrm{e}^{-V}dx$ with $c>0$ to the Gaussian one. The choice of J^{H} yields hypercontractivity of the semigroup associated to this family of invariant measures, and thus the equivalent logarithmic Sobolev inequality for μ (cf. [B-G-L]).

We next turn to the multi-dimensional version of the preceding result, with therefore in the following $c>0$. Let $X=(X_t)_{t\geq 0}$ be the Markov process with generator $\mathrm{L}=\Delta-\nabla V\cdot\nabla$ and initial invariant distribution $d\mu=\mathrm{e}^{-V}dx$. We are interested in the distribution of $(X_{t_1},\ldots,X_{t_m})$ where $0\leq t_1\leq\cdots\leq t_m$. Consider the covariance matrix Γ the Ornstein-Uhlenbeck process at speed ct, that is $\Gamma_{k\ell}=\mathrm{e}^{-c|t_k-t_\ell|}$, $k,\ell=1,\ldots,m$. In the Gaussian case, this extension (for thus the Ornstein-Uhlenbeck process) was achieved by the study of general Gaussian vectors. In the present case, we deal with the kernels as given by (20), for simplicity one-dimensional.

Theorem 10. *In the preceding notation, assume that the point-wise product of $(|\partial_{k\ell}J|)_{1\leq k,\ell\leq m}$ and Γ is (semi-) negative-definite. Then, for every $f_i:\mathbb{R}\to I_i$, $i=1,\ldots,m$, suitably integrable,*

$$\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}J\big(f_1(x_1),\ldots,f_m(x_m)\big)p_{t_m-t_{m-1}}(x_{m-1},dx_m)\cdots p_{t_2-t_1}(x_1,dx_2)d\mu(x_1)$$
$$\leq J\bigg(\int_{\mathbb{R}}f_1\,d\mu,\ldots,\int_{\mathbb{R}}f_m\,d\mu\bigg).$$

We outline the argument when $m=3$. Consider

$$\psi(s)=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}J\big(P_sf(x),P_sg(y),P_sh(z)\big)p_{t-u}(y,dz)p_t(x,dy)d\mu(x),\quad s\geq 0,$$

for $t > u > 0$ and three functions f, g, h. Differentiating ψ and integrating by parts in space leads to consider expressions such as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_1 J \, p_{t-u}(y, dz) \partial_x p_u(x, dy) \partial_x P_s f \, d\mu(x).$$

Arguing as in the proof of Theorem 8, this expression is equal to

$$\int_{\mathbb{R}} \partial_x P_s k \, \partial_x P_s f \, d\mu(x)$$

where $k = k(y) = \int_{\mathbb{R}} \partial_1 J \, p_{t-u}(y, dz)$. Now by (21)

$$|\partial_x P_s k| \leq \mathrm{e}^{-cs} P_s\big(|\partial_y k|\big).$$

Since

$$\partial_y k = \int_{\mathbb{R}} \partial_{12} J \, p_{t-u}(y, dz) + \int_{\mathbb{R}} \partial_1 J \, \partial_y p_{t-u}(y, dz),$$

similarly

$$|\partial_y k| \leq \int_{\mathbb{R}} |\partial_{12} J| p_{t-u}(y, dz) + \mathrm{e}^{-c(t-u)} \int_{\mathbb{R}} |\partial_{13} J| \partial_y p_{t-u}(y, dz).$$

The proof is then completed in the same way.

With the J function (19) associated to a finite-dimensional distribution of the Ornstein-Uhlenbeck process, the following consequence holds true.

Corollary 11. *Let $c > 0$ and $0 \leq t_1 \leq \cdots \leq t_m$. For any Borel sets $B_1, \ldots, B_m$ in $\mathbb{R}$,*

$$\mathbb{P}(X_{t_1} \in B_1, \ldots, X_{t_m} \in B_m) \leq \mathbb{P}(Z_{ct_1} \leq b_1, \ldots, Z_{ct_m} \leq b_m)$$

where $\mathbb{P}(X_{t_k} \in B_k) = \mu(B_k) = \Phi(b_k)$, $k = 1, \ldots, m$ and where $(Z_{ct})_{t \geq 0}$ is the Ornstein-Uhlenbeck with speed ct.

As suggested by J. Neeman following his arguments developed in [Nee], Corollary 11 may be used towards a comparison property between hitting times. For a Borel set B in $\mathbb{R}$, let $e_B^X = \inf\{t \geq 0 \,; X_t \notin B\}$ be the exit time of the Markov process $X = (X_t)_{t \geq 0}$ from the set B.

Corollary 12. *Under the preceding notation, for any $s \geq 0$,*

$$\mathbb{P}\big(e_B^X \geq s\big) \leq \mathbb{P}\big(e_H^Z \geq s\big)$$

where H is a half-line in $\mathbb{R}$ such that $\gamma(H) = \mu(B)$ and $Z = (Z_{ct})_{t \geq 0}$ the Ornstein-Uhlenbeck process at speed ct.

4. The discrete cube

To conclude this note, we briefly address in this last section the corresponding noise sensitivity issue on the discrete cube and collect a few remarks and questions.

The discrete framework actually leads to a 4-point definition of ρ-concavity similar to the standard characterization of concavity. Say namely that a function J on some open convex set $\mathcal{O}$ of $\mathbb{R}^2$ is strongly ρ-concave for some $\rho \in \mathbb{R}$ if for all $(u,v) \in \mathcal{O}$, $(u',v') \in \mathcal{O}$,

$$\begin{aligned}\frac{1+\rho}{4}\,J(u,v) + \frac{1-\rho}{4}\,J(u',v) + \frac{1-\rho}{4}\,J(u,v') + \frac{1+\rho}{4}\,J(u',v') \\ \leq J\Big(\frac{u+u'}{2}, \frac{v+v'}{2}\Big),\end{aligned} \tag{22}$$

Lemma 13. *Strong ρ-concavity implies ρ-concavity (for smooth functions).*

Proof. By a Taylor expansion, at any $(a,b) \in \mathcal{O}$, $(h,k) \in \mathbb{R}^2$, such that $(a \pm h, b \pm k) \in \mathcal{O}$,

$$\begin{aligned}&(1+\rho)\big[J(a+h,b+k) + J(a-h,b-k) - 2J(a,b)\big] \\ &+(1-\rho)\big[J(a+h,b-k) + J(a-h,b+k) - 2J(a,b)\big] \\ &\qquad = 2h^2\partial_{11}J(a,b) + 4\rho hk\partial_{12}J(a,b) + 2k^2\partial_{22}J(a,b) + o(h^2+k^2)\end{aligned}$$

With $u = a+h$, $v = b+k$, $u' = a-h$, $v' = b-k$, (22) implies the ρ-concavity of J as $h,k \to 0$. $\square$

The definition of strong concavity actually amounts to Theorem 1 on the two-point space $\Sigma = \{-1,+1\}$. Namely, for the kernel $K_\rho(x,y) = 1 + \rho xy$, $(x,y) \in \Sigma \times \Sigma$, (22) is equivalent to saying that for every functions $f, g : \Sigma \to \mathbb{R}$,

$$\int_\Sigma \int_\Sigma J\big(f(x), g(y)\big) K_\rho(x,y) d\mu(x) d\mu(y) \leq J\Big(\int_\Sigma f\, d\mu, \int_\Sigma g\, d\mu\Big)$$

where μ is the uniform probability measure on $\Sigma = \{-1,+1\}$.

It is immediately seen that strong ρ-concavity is stable by product on the discrete cube. On $\Sigma^n = \{-1,+1\}^n$ equipped with the uniform product measure μ, let for $\rho \in \mathbb{R}$ and $x = (x_1,\ldots,x_n) \in \Sigma^n$, $y = (y_1,\ldots,y_n) \in \Sigma^n$,

$$K_\rho(x,y) = \prod_{i=1}^n (1 + \rho\, x_i y_i).$$

Proposition 14. *Let J be strongly ρ-concave on $\mathcal{O} = I_1 \times I_2$ where I_1 and I_2 are open intervals. Then for every $f : \Sigma^n \to I_1$, $g : \Sigma^n \to I_2$*

$$\int_{\Sigma^n} \int_{\Sigma^n} J\big(f(x), g(y)\big) K_\rho(x,y) d\mu(x) d\mu(y) \leq J\Big(\int_{\Sigma^n} f d\mu, \int_{\Sigma^n} g d\mu\Big).$$

It is a main result, namely the Bonami-Beckner hypercontrativity theorem [Bon], [Be], that the hypercontractive function J^{H} is strongly ρ-concave under (9) (along the equivalence between hypercontractivity and Theorem 1 described in Section 1 for the Ornstein-Uhlenbeck semigroup). However, we could not establish directly the strong ρ-concavity of J^{H} in this case. Such a proof could give a better understanding of the strong ρ-concavity property.

On the other hand, it is not true in general that ρ-concavity implies back strong ρ-concavity and one example, taken from [D-M-N], is simply Borell's noise sensitivity function J^{B} (with parameter $\rho \in (0,1)$). Indeed, for $u = v = 1$ and $u' = v' = 0$, (22) would imply that

$$1 + \rho \leq 4\, J^{\mathrm{B}}\Big(\frac{1}{2}, \frac{1}{2}\Big) \tag{23}$$

since $J^{\mathrm{B}}(1,1) = 1$ and $J^{\mathrm{B}}(1,0) = J^{\mathrm{B}}(0,1) = J^{\mathrm{B}}(0,0) = 0$. But

$$J^{\mathrm{B}}\Big(\frac{1}{2}, \frac{1}{2}\Big) = \int_{-\infty}^{0} \int_{-\infty}^{0} q_t^1(x,y) d\gamma^1(x) d\gamma^1(y) = \int_0^\infty \Phi(\alpha x) d\gamma^1(x)$$

where $\alpha = \frac{\rho}{\sqrt{1-\rho^2}}$ and $\rho = \mathrm{e}^{-t}$. Taking the derivative in α easily shows that

$$4 \int_0^\infty \Phi(\alpha x) d\gamma^1(x) = 1 + \frac{2}{\pi} \arctan(\alpha)$$

so that (23) indeed fails as $\rho \to 0$.

It would be of interest to understand which additional property to ρ-concavity ensures strong ρ-concavity. In this direction, A. De, E. Mossel and J. Neeman [D-M-N] recently observed by a suitable Taylor expansion that there exists, for any $\rho \in (-1,+1)$, $C(\rho) > 0$ such that

$$\bigg| \frac{\partial^3 J_\rho^{\mathrm{B}}(u,v)}{\partial^i u\, \partial^j v} \bigg| \leq C(\rho) \big[uv(1-u)(1-v) \big]^{-C(\rho)}$$

for all $i, j \geq 0$ with $i + j = 3$. This property then implies that for every $u, u', v, v' \in [\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$,

$$\begin{aligned} \frac{1+\rho}{4}\, J_\rho^{\mathrm{B}}(u,v) + & \frac{1-\rho}{4}\, J_\rho^{\mathrm{B}}(u',v) + \frac{1-\rho}{4}\, J_\rho^{\mathrm{B}}(u,v') + \frac{1+\rho}{4}\, J_\rho^{\mathrm{B}}(u',v') \\ & \leq J_\rho^{\mathrm{B}}\Big(\frac{u+u'}{2}, \frac{v+v'}{2}\Big) + C(\rho)\, \varepsilon^{-C(\rho)} \big(|u-u'|^3 + |v-v'|^3 \big). \end{aligned}$$

As a main achievement, the authors of [D-M-N] recover from this conclusion the majority is stablest result of [M-OD-O]. One further observation of [D-M-N] is that the preceding two-point inequality is still good enough to reach, after tensorization and the central limit theorem, Borell's noise sensitivity theorem for the Ornstein-Uhlenbeck semigroup.

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References

[B-T] A. Baernstein II, B. A. Taylor. Spherical rearrangements, subharmonic functions and *-functions in n-space. Duke Math. J. 43, 245–268 (1976).

[B-G-L] D. Bakry, I. Gentil, M. Ledoux. Analysis and geometry of Markov diffusion operators. Grundlehren der Mathematischen Wissenschaften 348. Springer (2013).

[B-L] D. Bakry, M. Ledoux. Lévy-Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator. Invent. Math. 123, 259–281 (1996).

[B-CE-M] F. Barthe, D. Cordero-Erausquin, B. Maurey. Entropy of spherical marginals and related inequalities. J. Math. Pures Appl. 86, 89–99 (2006).

[B-CE-L-M] F. Barthe, D. Cordero-Erausquin, M. Ledoux, B. Maurey. Correlation and Brascamp-Lieb inequalities for Markov semigroups. Int. Math. Res. Not. IMRN 10, 2177–2216 (2011).

[B-C-C-T] J. Bennett, A. Carbery, M. Christ, T. Tao. The Brascamp-Lieb inequalities: finiteness, structure and extremals. Geom. Funct. Anal. 17, 1343–1415 (2008).

[Be] W. Beckner. Inequalities in Fourier analysis. Ann. of Math. 102, 159–182 (1975).

[Bon] A. Bonami. Étude des coefficients de Fourier des fonctions de $L^p(G)$. Ann. Inst. Fourier 20, 335–402 (1970).

[Bor] C. Borell. Geometric bounds on the Ornstein-Uhlenbeck velocity process. Z. Wahrsch. Verw. Gebiete 70, 1–13 (1985).

[B-L-L] H. Brascamp, E. Lieb, J. M. Luttinger. A general rearrangement inequality for multiple integrals. J. Funct. Anal. 17, 227–237 (1974).

[C-L-L] E. Carlen, E. Lieb, M. Loss. A sharp analog of Young's inequality on S^N and related entropy inequalities. J. Geom. Anal. 14, 487–520 (2004).

[C-D-P] W.-K. Chen, N. Dafnis, G. Paouris. Improved Hölder inequalities for correlated Gaussian random vectors (2013).

[D-M-N] A. De, E. Mossel, J. Neeman. Majority is Stablest: Discrete and SoS (2013).

[DK-P-W] E. De Klerk, D. Pasechnik, J. Warners. On approximate graph colouring and MAX-k-CUT algorithms based on the #-function. J. Comb. Optim. 8, 267–294 (2004).

[E] A. Ehrhard. Symétrisation dans l'espace de Gauss. Math. Scand. 53, 281–301 (1983).

[F] X. Fernique. Régularité des trajectoires des fonctions aléatoires gaussiennes. École d'Été de Probabilités de St-Flour 1974. Lecture Notes in Math. 480, 1–96 (1975). Springer.

[Go] Y. Gordon. Some inequalities for Gaussian processes and applications. Israel J. Math. 50, 265–289.

[Gr] L. Gross. Logarithmic Sobolev inequalities. Amer. J. Math. 97, 1061–1083 (1975).

[H] Y. Hu. A unified approach to several inequalities for Gaussian and diffusion measures. Séminaire de Probabilités XXXIV. Lecture Notes in Math. 1729, 329–335 (2000). Springer.

[I-M] M. Isaksson, E. Mossel. New maximally stable Gaussian partitions with discrete applications. Israel J. Math. 189, 347–396 (2012).

[L] M. Ledoux. Isoperimetry and Gaussian Analysis. Ecole d'Été de Probabilités de St-Flour 1994. Lecture Notes in Math. 1648, 165–294 (1996). Springer.

[M-N] E. Mossel, J. Neeman. Robust optimality of Gaussian noise stability (2012).

[M-OD-O] E. Mossel, R. O'Donnell, K. Oleszkiewicz. Noise stability of functions with low influences: invariance and optimality. Ann. of Math. 171, 295–341 (2010).

[Nee] J. Neeman. A multidimensional version of noise stability (2013).

[Nel] E. Nelson. The free Markoff field. J. Funct. Anal. 12, 211–227 (1973).

[S] D. Slepian. The one-sided barrier problem for Gaussian noise. Bell. System Tech. J. 41, 463–501 (1962).

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