Hypercontractive Measures, Talagrand's Inequality, and Influences

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Abstract We survey several Talagrand type inequalities and their application to influences with the tool of hypercontractivity for both discrete and continuous, and product and non-product models. The approach covers similarly by a simple interpolation the framework of geometric influences recently developed by N. Keller, E. Mossel and A. Sen. Geometric Brascamp-Lieb decompositions are also considered in this context.

1 Introduction

In the famous paper [24], M. Talagrand showed that for every function f on the discrete cube $X = \{-1, +1\}^N$ equipped with the uniform probability measure μ,

$$\mathrm{Var}_\mu(f) = \int_X f^2 d\mu - \left(\int_X f d\mu\right)^2 \leq C \sum_{i=1}^{N} \frac{\|D_i f\|_2^2}{1 + \log\left(\|D_i f\|_2 / \|D_i f\|_1\right)} \quad (1)$$

for some numerical constant $C \geq 1$, where $\|\cdot\|_p$ denote the norms in $\mathrm{L}^p(\mu)$, $1 \leq p \leq \infty$, and for every $i = 1, \ldots, n$ and every $x = (x_1, \ldots, x_N) \in \{-1, +1\}^N$,

$$D_i f(x) = f(\tau_i x) - f(x) \quad (2)$$

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with $\tau_i x = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_N)$. Up to the numerical constant, this inequality improves upon the classical spectral gap inequality (see below)

$$\mathrm{Var}_\mu(f) \leq \frac{1}{4} \sum_{i=1}^N \|D_i f\|_2^2 \,. \tag{3}$$

The proof of (1) is based on an hypercontractivity estimate known as the Bonami-Beckner inequality [9], [7] (see below). Inequality (1) was actually deviced to recover (and extend) a famous result of J. Kahn, G. Kalai and N. Linial [12] about influences on the cube. Namely, applying (1) to the Boolean function $f = \mathbf{1}_A$ for some set $A \subset \{-1, +1\}^N$, it follows that

$$\mu(A)\big(1 - \mu(A)\big) \leq C \sum_{i=1}^N \frac{2I_i(A)}{1 + \log\big(1/\sqrt{2I_i(A)}\,\big)} \tag{4}$$

where, for each $i = 1, \ldots, N$,

$$I_i(A) = \mu\big(\{x \in A, \tau_i x \notin A\}\big)$$

is the so-called influence of the i-th coordinate on the set A (noticing that $\|D_i \mathbf{1}_A\|_p^p = 2I_i(A)$ for every $p \geq 1$). In particular, for a set A with $\mu(A) = a$, there is a coordinate i, $1 \leq i \leq N$, such that

$$I_i(A) \geq \frac{a(1-a)}{8CN} \, \log\Big(\frac{N}{a(1-a)}\Big) \geq \frac{a(1-a)\log N}{8CN} \tag{5}$$

which is the main result of [12]. (To deduce (5) from (4), assume for example that $I_i(A) \leq \big(\frac{a(1-a)}{N}\big)^{1/2}$ for every $i = 1, \ldots, N$, since if not the result holds. Then, from (4), there exists i, $1 \leq i \leq N$, such that

$$\frac{a(1-a)}{CN} \leq \frac{2I_i(A)}{1 + \log\big(1/\sqrt{2I_i(A)}\,\big)} \leq \frac{8I_i(A)}{4 + \log(N/4a(1-a))}$$

which yields (5)). Note that (5) remarkably improves by a (optimal) factor $\log N$ what would follow from the spectral gap inequality (3) applied to $f = \mathbf{1}_A$. The numerical constants like C throughout this text are not sharp.

The aim of this note is to amplify the hypercontractive proof of Talagrand's original inequality (1) to various settings, including non-product spaces and continuous variables, and in particular to address versions suitable to geometric influences. It is part of the folklore indeed (cf. e.g. [8]) that an inequality similar to (1), with the same hypercontractive proof, holds for the standard Gaussian measure μ on $\mathbb{R}^N$ (viewed as a product measure of one-dimensional factors), that is, for every smooth enough function f on $\mathbb{R}^N$ and some constant $C > 0$,

$$\mathrm{Var}_\mu(f) \leq C \sum_{i=1}^N \frac{\|\partial_i f\|_2^2}{1 + \log(\|\partial_i f\|_2 / \|\partial_i f\|_1)} \,. \tag{6}$$

(A proof will be given in Section 2 below.) However, the significance of the latter for influences is not clear, since its application to characteristic functions is not immediate (and requires notions of capacities). Recently, N. Keller, E. Mossel and A. Sen [13] introduced a notion of geometric influence of a Borel set A in $\mathbb{R}^N$ with respect to a measure μ (such as the Gaussian measure) simply as $\|\partial_i f\|_1$ for some smooth approximation f of $\mathbf{1}_A$, and proved for it the analogue of (5) (with $\sqrt{\log N}$ instead of $\log N$) for the standard Gaussian measure on $\mathbb{R}^N$. It is therefore of interest to seek for suitable versions of Talagrand's inequality involving only L^1-norms $\|\partial_i f\|_1$ of the partial derivatives. While the authors of [13] use isoperimetric properties, we show here how the common hypercontractive tool together with a simple interpolation argument may be developed similarly to reach the same conclusion. In particular, for the standard Gaussian measure μ on $\mathbb{R}^N$, we will see that for every smooth enough function f on $\mathbb{R}^N$ such that $|f| \leq 1$,

$$\mathrm{Var}_\mu(f) \leq C \sum_{i=1}^N \frac{\|\partial_i f\|_1 \big(1 + \|\partial_i f\|_1\big)}{\big[1 + \log^+ \big(1/\|\partial_i f\|_1\big)\big]^{1/2}} \,. \tag{7}$$

Applied to $f = \mathbf{1}_A$, this inequality indeed ensures the existence of a coordinate i, $1 \leq i \leq N$, such that the geometric influence of A along i is at least of the order of $\frac{\sqrt{\log N}}{N}$, that is one of the main conclusions of [13] (where it is shown moreover that the bound is sharp). In this continuous setting, the hypercontractive approach yields more general examples of measures with such an influence property in the range between exponential and Gaussian for which only a logarithmic Sobolev type inequality is needed while [13] required an isoperimetric inequality for the individual measures μ_i.

This note is divided into two main parts. In the first one, we present Talagrand type inequalities for various models, from the discrete cube to Gaussian and more general product measures, by the general principle of hypercontractivity of Markov semigroups. The method of proof, originating in Talagrand's work, has been used recently by R. O'Donnell and K. Wimmer [20], [21] to investigate non-product models such as random walks on some graphs which enter the general presentation below. Actually, most of the Talagrand inequalities we present in the discrete setting are already contained in the work by R. O'Donnell and K. Wimmer. It is worth mentioning that an approach to the Talagrand inequality (1) rather based on the logarithmic Sobolev inequality was deviced in [22] and [11] a few years ago. The abstract semigroup approach applies in the same way on the sphere along the decomposition of the Laplacian. Geometric Brascamp-Lieb decompositions within this setting are also discussed. In the second part, we address our new version (7) of Talagrand's inequality towards geometric influences and the recent results of [13] by a further interpolation step on the hypercontractive proof.

In the last part of this introduction, we describe a convenient framework in order to develop hypercontractive proofs of Talagrand type inequalities. While of some abstract flavor, the setting easily covers two main concrete instances, probability measures on finite state spaces (as invariant measures of some Markov kernels) and continuous probability measures of the form $d\mu(x) = \mathrm{e}^{-V(x)}dx$ on the Borel sets of $\mathbb{R}^n$ where V is some (smooth) potential (as invariant measures of the associated diffusion operators $\Delta - \nabla V \cdot \nabla$). We refer for the material below to the general references [2], [10], [23], [1], [4]...

Let μ be a probability measure on a measurable space $(X, \mathcal{A})$. For a function $f : X \to \mathbb{R}$ in $\mathrm{L}^2(\mu)$, define its variance with respect to μ by

$$\mathrm{Var}_\mu(f) = \int_X f^2 d\mu - \bigg(\int_X f d\mu \bigg)^2.$$

Similarly, whenever $f > 0$, define its entropy by

$$\mathrm{Ent}_\mu(f) = \int_X f \log f d\mu - \int_X f d\mu \log \bigg(\int_X f d\mu \bigg)$$

provided it is well-defined. The $\mathrm{L}^p(\mu)$-norms, $1 \leq p \leq \infty$, will be denoted by $\| \cdot \|_p$.

Let then $(P_t)_{t \geq 0}$ be a Markov semigroup with generator L acting on a suitable class of functions on $(X, \mathcal{A})$. Assume that $(P_t)_{t \geq 0}$ and L have an invariant, reversible and ergodic probability measure μ. This ensures that the operators P_t are contractions in all $\mathrm{L}^p(\mu)$-spaces, $1 \leq p \leq \infty$. The Dirichlet form associated to the couple (L, μ) is then defined, on functions f, g of the Dirichlet domain, as

$$\mathcal{E}(f, g) = \int_X f(-\mathrm{L}g) d\mu.$$

Within this framework, the first example of interest is the case of a Markov kernel K on a finite state space X with invariant ($\sum_{x \in X} K(x, y)\mu(x) = \mu(y)$, $y \in X$) and reversible ($K(x, y)\mu(x) = K(y, x)\mu(y)$, $x, y \in X$) probability measure μ. The Markov operator $\mathrm{L} = K - \mathrm{Id}$ generates the semigroup of operators $P_t = \mathrm{e}^{t\mathrm{L}}$, $t \geq 0$, and defines the Dirichlet form

$$\mathcal{E}(f, g) = \int_X f(-\mathrm{L}g) d\mu = \frac{1}{2} \sum_{x, y \in X} \big[f(x) - f(y) \big] \big[g(x) - g(y) \big] K(x, y) \mu(x)$$

on functions $f, g : X \to \mathbb{R}$. The second class of examples is the case of $X = \mathbb{R}^n$ equipped with its Borel σ-field. Letting $V : \mathbb{R}^n \to \mathbb{R}$ be such that $\int_{\mathbb{R}^n} \mathrm{e}^{-V(x)} dx = 1$, under mild smoothness and growth conditions on the potential V, the second order operator $\mathrm{L} = \Delta - \nabla V \cdot \nabla$ admits $d\mu(x) = \mathrm{e}^{-V(x)}dx$ as symmetric and invariant probability measure. The operator L generates the Markov semigroup of operators $(P_t)_{t \geq 0}$ and defines by integration by parts the Dirichlet form

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^n} f(-\mathrm{L}g)d\mu = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, d\mu$$

for smooth functions f, g on $\mathbb{R}^n$.

Given such a couple (L, μ), it is said to satisfy a spectral gap, or Poincaré, inequality if there is a constant $\lambda > 0$ such that for all functions f of the Dirichlet domain,

$$\lambda \, \mathrm{Var}_\mu(f) \leq \mathcal{E}(f,f). \tag{8}$$

Similarly, it satisfies a logarithmic Sobolev inequality if there is a constant $\rho > 0$ such that for all functions f of the Dirichlet domain,

$$\rho \, \mathrm{Ent}_\mu(f^2) \leq 2\,\mathcal{E}(f,f). \tag{9}$$

One speaks of the spectral gap constant (of (L, μ)) as the best $\lambda > 0$ for which (8) holds, and of the logarithmic Sobolev constant (of (L, μ)) as the best $\rho > 0$ for which (9) holds. We still use λ and ρ for these constants. It is classical that $\rho \leq \lambda$.

Both the spectral gap and logarithmic Sobolev inequalities translate equivalently on the associated semigroup $(P_t)_{t \geq 0}$. Namely, the spectral gap inequality (8) is equivalent to saying that

$$\|P_t f\|_2 \leq \mathrm{e}^{-\lambda t} \, \|f\|_2$$

for every $t \geq 0$ and every mean zero function f in $\mathrm{L}^2(\mu)$. Equivalently for the further purposes, for every $f \in \mathrm{L}^2(\mu)$ and every $t > 0$,

$$\mathrm{Var}_\mu(f) \leq \frac{1}{1 - \mathrm{e}^{-\lambda t}} \big[\|f\|_2^2 - \|P_t f\|_2^2 \big]. \tag{10}$$

On the other hand, the logarithmic Sobolev inequality gives rise to hypercontractivity which is a smoothing property of the semigroup. Precisely, the logarithmic Sobolev inequality (9) is equivalent to saying that, whenever $p \geq 1 + \mathrm{e}^{-2\rho t}$, for all functions f in $\mathrm{L}^p(\mu)$,

$$\|P_t f\|_2 \leq \|f\|_p. \tag{11}$$

For simplicity, we say below that a probability measure μ in this context is hypercontractive with constant ρ.

A standard operation on Markov operators is the product operation. Let (L_1, μ_1) and (L_2, μ_2) be Markov operators on respective spaces X_1 and X_2. Then

$$\mathrm{L} = \mathrm{L}_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{L}_2$$

is a Markov operator on the product space $X_1 \times X_2$ equipped with the product probability measure $\mu_1 \otimes \mu_2$. The product semigroup $(P_t)_{t \geq 0}$ is similarly obtained as the tensor product $P_t = P_t^1 \otimes P_t^2$ of the semigroups on each factor. For the product Dirichlet form, the spectral gap and logarithmic Sobolev constants are stable in the sense that, with the obvious notation, $\lambda = \min(\lambda_1, \lambda_2)$ and $\rho = \min(\rho_1, \rho_2)$. This basic stability by products will allow for constants independent of the dimension in the Talagrand type inequalities under investigation. For the clarity of the exposition, we will not mix below products of continuous and discrete spaces, although this may easily be considered.

Let us illustrate the preceding definitions and properties on two basic examples. Consider first the two-point space $X = \{-1, +1\}$ with the measure $\mu = p\delta_{+1} + q\delta_{-1}$, $p \in [0,1]$, $p + q = 1$, and the Markov kernel $K(x,y) = \mu(y)$, $x, y \in X$. Then, for every function $f : X \to \mathbb{R}$,

$$\mathcal{E}(f,f) = \int_X f(-\mathrm{L}f)d\mu = \mathrm{Var}_\mu(f)$$

so that the spectral gap $\lambda = 1$. The logarithmic Sobolev constant is known to be

$$\rho = \frac{2(p-q)}{\log p - \log q} \quad (= 1 \text{ if } p = q). \tag{12}$$

The product chain on the discrete cube $X = \{-1, +1\}^N$ with the product probability measure $\mu = (p\delta_{+1} + q\delta_{-1})^{\otimes N}$ and generator $\mathrm{L} = \sum_{i=1}^N \mathrm{L}_i$ is associated to the Dirichlet form

$$\mathcal{E}(f,f) = \int_X \sum_{i=1}^N f(-\mathrm{L}_i f)d\mu = pq \int_X \sum_{i=1}^N |D_i f|^2 d\mu$$

where $D_i f$ is defined in (2). By the previous product property, it admits 1 as spectral gap and ρ given by (12) as logarithmic Sobolev constant. In its hypercontractive formulation, the case $p = q$ is the content of the Bonami-Beckner inequality [9], [7].

As mentioned before, M. Talagrand [24] used this hypercontractivity on the discrete cube $\{-1, +1\}^N$ equipped with the product measure $\mu = (p\delta_{+1} + q\delta_{-1})^{\otimes N}$ to prove that for any function $f : \{-1, +1\}^N \to \mathbb{R}$,

$$\mathrm{Var}_\mu(f) \le \frac{Cpq(\log p - \log q)}{p - q} \sum_{i=1}^N \frac{\|D_i f\|_2^2}{1 + \log\big(\|D_i f\|_2 / 2\sqrt{pq}\,\|D_i f\|_1\big)} \tag{13}$$

for some numerical constant $C > 0$ (this statement will be covered in Section 2 below). This in turn yields a version of the influence result of [12] on the biased cube.

In the continuous setting $X = \mathbb{R}^n$, the case of a quadratic potential V amounts to the Hermite or Ornstein-Uhlenbeck operator $\mathrm{L} = \Delta - x\cdot\nabla$ with invariant measure the standard Gaussian measure $d\mu(x) = (2\pi)^{-n/2}\, \mathrm{e}^{-|x|^2/2}dx$. It is known here that $\lambda = \rho = 1$ independently of the dimension. (More generally, if $V(x) - c\,\frac{|x|^2}{2}$ is convex for some $c > 0$, then $\lambda \ge \rho \ge c$.) Actually, L may also be viewed as the sum $\sum_{i=1}^n \mathrm{L}_i$ of one-dimensional Ornstein-Uhlenbeck operators along each coordinate, and μ as the product measure of standard normal distributions. Within this product structure, the analogue (6) of (13) has been known for some time, and will be recalled below.

2 Hypercontractivity and Talagrand's Inequality

This section presents the general hypercontractive approach to Talagrand type inequalities including the discrete cube, the Gaussian product measure and more general non-product models. The method of proof, directly inspired from [24], has been developed recently by R. O'Donnell and K. Wimmer [20], [21] towards non-product extensions on suitable graphs. Besides hypercontractivity, a key feature necessary to develop the argument is a suitable decomposition of the Dirichlet form along "directions" commuting with the Markov operator or its semigroup. These directions are immediate in a product space, but do require additional structure in more general contexts.

In the previous abstract setting of a Markov semigroup $(P_t)_{t\geq 0}$ with generator L, assume thus that the associated Dirichlet form $\mathcal{E}$ may be decomposed along directions Γ_i acting on functions on X as

$$\mathcal{E}(f,f) = \sum_{i=1}^{N} \int_X \Gamma_i(f)^2 d\mu \tag{14}$$

in such a way that, for each $i = 1, \ldots, N$, Γ_i commutes to $(P_t)_{t\geq 0}$ in the sense that, for some constant $\kappa \in \mathbb{R}$, every $t \geq 0$ and every f in a suitable family of functions,

$$\Gamma_i(P_t f) \leq \mathrm{e}^{\kappa t}\, P_t\big(\Gamma_i(f)\big). \tag{15}$$

These properties will be clearly illustrated on the main examples of interest below, with in particular explicit descriptions of the classes of functions for which (14) and (15) may hold.

We first present the Talagrand inequality in this context. The proof is the prototype of the hypercontractive argument used throughout this note and applied to various examples.

Theorem 1. *In the preceding setting, assume that* (L, μ) *is hypercontractive with constant* $\rho > 0$ *and that (14) and (15) hold. Then, for any function* f *in* $\mathrm{L}^2(\mu)$,

$$\mathrm{Var}_\mu(f) \leq C(\rho,\kappa) \sum_{i=1}^{N} \frac{\|\Gamma_i f\|_2^2}{1 + \log(\|\Gamma_i f\|_2 / \|\Gamma_i f\|_1)}$$

where $C(\rho,\kappa) = 4\,\mathrm{e}^{(1+(\kappa/\rho))^+}/\rho$.

Proof. The starting point is the variance representation along the semigroup $(P_t)_{t\geq 0}$ of a function f in the $\mathrm{L}^2(\mu)$-domain of the semigroup as

$$\mathrm{Var}_\mu(f) = -\int_0^\infty \left(\frac{d}{dt}\int_X (P_t f)^2 d\mu\right) dt = -2\int_0^\infty \left(\int_X P_t f \, \mathrm{L} P_t f d\mu\right) dt.$$

The time integral has to be handled both for the large and small values. For the large values of t, we make use of the exponential decay provided by the spectral gap in the form of (10) to get that, with $T = 1/2\rho$ for example since $\rho \leq \lambda$,

$$\mathrm{Var}_\mu(f) \leq 2\left[\|f\|_2^2 - \|P_T f\|_2^2\right].$$

We are thus left with the variance representation of

$$\|f\|_2^2 - \|P_T f\|_2^2 = -2\int_0^T \left(\int_X P_t f \, \mathrm{L} P_t f d\mu\right) dt = 2\int_0^T \mathcal{E}(P_t f, P_t f) dt.$$

Now by the decomposition (14),

$$\|f\|_2^2 - \|P_T f\|_2^2 = 2\sum_{i=1}^N \int_0^T \left(\int_X \big(\Gamma_i(P_t f)\big)^2 d\mu\right) dt.$$

Under the commutation assumption (15),

$$\int_X \big(\Gamma_i(P_t f)\big)^2 d\mu \leq \mathrm{e}^{2\kappa t}\int_X \big(P_t\big(\Gamma_i(f)\big)\big)^2 d\mu.$$

Since $(P_t)_{t\geq 0}$ is hypercontractive with constant $\rho > 0$, for every $i = 1, \ldots, N$ and $t \geq 0$,

$$\big\|P_t\big(\Gamma_i(f)\big)\big\|_2 \leq \big\|\Gamma_i(f)\big\|_p$$

where $p = p(t) = 1 + \mathrm{e}^{-2\rho t} \leq 2$. After the change of variables $p(t) = v$, we thus reached at this point the inequality

$$\mathrm{Var}_\mu(f) \leq \frac{2\,\mathrm{e}^{(1+(\kappa/\rho))^+}}{\rho}\sum_{i=1}^N \int_1^2 \big\|\Gamma_i(f)\big\|_v^2 dv. \tag{16}$$

This inequality actually basically amounts to Theorem 1. Indeed, by Hölder's inequality,

$$\big\|\Gamma_i(f)\big\|_v \leq \big\|\Gamma_i(f)\big\|_1^\theta \big\|\Gamma_i(f)\big\|_2^{1-\theta}$$

where $\theta = \theta(v) \in [0,1]$ is defined by $\frac{1}{v} = \frac{\theta}{1} + \frac{1-\theta}{2}$. Hence

$$\int_1^2 \big\|\Gamma_i(f)\big\|_v^2 dv \leq \big\|\Gamma_i(f)\big\|_2^2 \int_1^2 b^{2\theta(v)} dv$$

where $b = \|\Gamma_i(f)\|_1 / \|\Gamma_i(f)\|_2 \leq 1$. It remains to evaluate the latter integral with $2\theta(v) = s$,

$$\int_1^2 b^{2\theta(v)} dv \le \int_0^2 b^s ds \le \frac{2}{1+\log(1/b)}$$

from which the conclusion follows. □

Inequality (16) of the preceding proof may also be used towards a version of Theorem 1 with Orlicz norms as emphasized in [24]. As in [24], let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be convex such that $\varphi(x) = x^2/\log(\mathrm{e}+x)$ for $x \ge 1$, and $\varphi(0) = 0$, and denote by

$$\|g\|_\varphi = \inf\left\{c > 0\,;\int_X \varphi\big(|g|/c\big)d\mu \le 1\right\}$$

the associated Orlicz norm of a measurable function $g : X \to \mathbb{R}$. Then, for some numerical constant $C > 0$,

$$\int_1^2 \|g\|_v^2 \, dv \le C\,\|g\|_\varphi^2 \tag{17}$$

so that (16) yields

$$\mathrm{Var}_\mu(f) \le \frac{2C\,\mathrm{e}^{(1+(\kappa/\rho))^+}}{\rho} \sum_{i=1}^N \big\|\Gamma_i(f)\big\|_\varphi^2. \tag{18}$$

Since as pointed out in Lemma 2.5 of [24],

$$\|g\|_\varphi^2 \le \frac{C\,\|g\|_2^2}{1+\log(\|g\|_2/\|g\|_1)}\,,$$

we see that (18) improves upon Theorem 1. To briefly check (17), assume by homogeneity that $\int_X g^2/\log(e+g)d\mu \le 1$ for some non-negative function g. Then, setting $g_k = g\,1_{\{2^{k-1}<g\le 2^k\}}$, $k \ge 1$, and $g_0 = g\,1_{\{g\le 1\}}$,

$$\sum_{k\in\mathbb{N}} \frac{1}{k+1}\int_X g_k^2 d\mu \le C_1 \tag{19}$$

for some numerical constant $C_1 > 0$. Hence, since $g_k \le 2^k$ for every k,

$$\begin{aligned}
\int_1^2 \|g\|_v^2 \, dv &= \int_1^2 \bigg(\sum_{k\in\mathbb{N}} \int_X g_k^v d\mu\bigg)^{2/v} dv \\
&\le 4\int_1^2 \bigg(\sum_{k\in\mathbb{N}} 2^{-(2-v)k} \int_X g_k^2 d\mu\bigg)^{2/v} dv \\
&\le C_2 \sum_{k\in\mathbb{N}} \bigg(\int_1^2 (k+1)^{2/v} 2^{-2(2-v)k/v} dv\bigg) \frac{1}{k+1}\int_X g_k^2 d\mu
\end{aligned}$$

where we used (19) as convexity weights in the last step. Now, it is easy to check that

$$\int_1^2 (k+1)^{2/v} 2^{-2(2-v)k/v} dv \leq C_3$$

uniformly in k so that $\int_1^2 \|g\|_v^2 \, dv \leq C_1 C_2 C_3$ concluding thus the claim.

We next illustrate the general Theorem 1 on various examples of interest.

On a probability space $(X, \mathcal{A}, \mu)$, consider first the Markov operator $\mathrm{L}f = \int_X f d\mu - f$ acting on integrable functions (in other words $Kf = \int_X f d\mu$). This operator is symmetric with respect to μ with Dirichlet form

$$\mathcal{E}(f,f) = \int_X f(-\mathrm{L}f) d\mu = \mathrm{Var}_\mu(f).$$

In particular, it has spectral gap 1. Let now $X = X_1 \times \cdots \times X_N$ be a product space with product probability measure $\mu = \mu_1 \otimes \cdots \otimes \mu_N$. Consider the product operator $\mathrm{L} = \sum_{i=1}^N \mathrm{L}_i$ where L_i is acting on the i-th coordinate of a function f as $\mathrm{L}_i f = \int_{X_i} f d\mu_i - f$. The product operator L has still spectral gap 1. Its Dirichlet form is given by

$$\mathcal{E}(f,f) = \sum_{i=1}^N \int_X f(-\mathrm{L}_i f) d\mu = \sum_{i=1}^N \int_X (\mathrm{L}_i f)^2 d\mu.$$

We are therefore in the setting of a decomposition of the type (14). Moreover, it is immediately checked that $\mathrm{L}_i \, \mathrm{L} = \mathrm{L} \, \mathrm{L}_i$ for every $i = 1, \ldots, N$, and thus the commutation property (15) also holds (with $\kappa = 0$). Hence Theorem 1 applies for this model with hypercontractive constant $\rho = \min_{1 \leq i \leq N} \rho_i > 0$. In particular, Theorem 1 includes Talagrand's inequality (13) for the hypercube $X = \{-1, +1\}^N$ with the product measure $\mu = (p\delta_{+1} + q\delta_{-1})^{\otimes N}$ with hypercontractive constant given by (12), for which it is immediately checked that, for every $r \geq 1$ and every $i = 1, \ldots, N$,

$$\int_X |\mathrm{L}_i f|^r d\mu = (pq^r + p^r q) \int_X |D_i f|^r d\mu.$$

More generally, as pointed out to us by J. van den Berg and D. Kiss (private communication), we may consider similarly products of the complete graph $X_1 = \cdots = X_N = \{0, \ldots, k\}$, each factor being equipped with the probability measure $\mu_1 = \sum_{j=0}^k p_j \delta_j$. Talagrand's approach is known to extend to this case, as noted for instance in [14]. The hypercontractive constant of X_1 has been computed in [10] and is given by

$$\rho = \frac{2(1 - 2p^*)}{\log(1/p^* - 1)}$$

with $p^* = \min_{0 \le j \le k} p_j$, so that Theorem 1.3 from [14] follows from Theorem 1 above.

Non-product examples may be considered similarly as has been thus emphasized recently in [20] and [21] with similar arguments. Let for example G be a finite group, and let S be a symmetric set of generators of G. The Cayley graph associated to S is the graph with vertices the element of G and edges the couples (x, xs) where $x \in G$ and $s \in S$. The transition kernel associated to this graph is

$$K(x,y) = \frac{1}{|S|}\, \mathbf{1}_S(yx^{-1}), \quad x, y \in G,$$

where $|S|$ is the cardinal of S. The uniform probability measure μ on G is an invariant and reversible measure for K. This framework includes the example of $G = \mathcal{S}_n$ the symmetric group on n elements with the set of transpositions as generating set and the uniform measure as invariant and symmetric measure.

Given such a finite Cayley graph G with generator set S, kernel K and uniform measure μ as invariant measure, the associated Dirichlet form may be expressed on functions $f : G \to \mathbb{R}$ in the form (14)

$$\mathcal{E}(f,f) = \frac{1}{2|S|} \sum_{s\in S} \sum_{x\in G} \big[f(sx) - f(x)\big]^2 \mu(x) = \frac{1}{2|S|} \sum_{s\in S} \|D_s f\|_2^2$$

where for $s \in S$, $D_s f(x) = f(sx) - f(x)$, $x \in G$. In order that the operators D_s commute to K in the sense of (15) (with again $\kappa = 0$), it is necessary to assume that S is stable by conjugacy in the sense that

$$\text{for all } u \in S, \quad u\, S\, u^{-1} = S$$

as it is the case for the set of transpositions on the symmetric group $\mathcal{S}^n$. The following statement from [20] is thus an immediate consequence of the general Theorem 1.

Corollary 2. *Under the preceding notation and assumptions, denote by ρ the logarithmic Sobolev constant of the chain (K, μ). Then for every function f on G,*

$$\mathrm{Var}_\mu(f) \le \frac{2\mathrm{e}}{\rho|S|} \sum_{s\in S} \frac{\|D_s f\|_2^2}{1 + \log\big(\|D_s f\|_2 / \|D_s f\|_1\big)}\,.$$

One may wonder for the significance of this Talagrand type inequality for influences. For $A \subset G$ and $s \in S$, define the influence $I_s(A)$ of the direction s on the set A by

$$I_s(A) = \mu\big(\{x \in G; x \in A, sx \notin A\}\big).$$

As on the discrete cube, given $A \subset G$ with $\mu(A) = a$, Corollary 2 yields the existence of $s \in S$ such that

$$I_s(A) \geq \frac{1}{C}\, a(1-a)\rho \, \log\Big(1 + \frac{1}{C\rho\, a(1-a)}\Big) \geq \frac{1}{C}\, a(1-a)\,\rho \log\Big(1 + \frac{1}{C\rho}\Big) \quad (20)$$

(where $C \geq 1$ is numerical). However, with respect to the spectral gap inequality of the chain (K, μ)

$$\lambda \, \mathrm{Var}_\mu(f) \leq \frac{1}{2|S|} \sum_{s \in S} \|D_s f\|_2^2 \,,$$

we see that (20) is only of interest provided that $\rho \log(1 + (1/\rho)) \gg \lambda$. This is the case on the symmetric discrete cube $\{-1, +1\}^N$ for which, in the Cayley graph normalization of Dirichlet forms, $\lambda = \rho = 1/N$. On the symmetric group, it is known that the spectral gap λ is $\frac{2}{n-1}$ whereas its logarithmic Sobolev constant ρ is of the order of $1/n \log n$ ([10], [17]) so that $\rho \log(1 + (1/\rho))$ and λ are actually of the same order for large n, and hence yield the existence of a transposition τ with influence at least only of the order of $1/n$. It is pointed out in [21] that this result is however optimal. The paper [20] presents examples in the more general context of Schreier graphs for which (20) yields influences strictly better than the ones from the spectral gap inequality.

Theorem 1 may also be illustrated on continuous models such as Gaussian measures. While the next corollary is stated in some generality, it is already of interest for products of one-dimensional factors and covers in particular the example (6) of the standard Gaussian product measure.

Corollary 3. *Let $d\mu_i(x) = \mathrm{e}^{-V_i(x)}dx$, $i = 1, \ldots, N$, on $X_i = \mathbb{R}^{n_i}$ be hypercontractive with constant $\rho_i > 0$. Let $\mu = \mu_1 \otimes \cdots \otimes \mu_N$ on $X = X_1 \times \cdots \times X_N$. Assume in addition that $V_i'' \geq -\kappa$, $\kappa \in \mathbb{R}$, $i = 1, \ldots, N$. Then, for any smooth function f on X,*

$$\mathrm{Var}_\mu(f) \leq C(\rho, \kappa) \sum_{i=1}^N \frac{\|\nabla_i f\|_2^2}{1 + \log\big(\|\nabla_i f\|_2 / \|\nabla_i f\|_1\big)}$$

where $\rho = \min_{1 \leq i \leq N} \rho_i$, and where $\nabla_i f$ denotes the gradient of f in the direction X_i, $i = 1, \ldots, N$.

Corollary 3 again follows from Theorem 1. Indeed, the product structure immediately allows for the decomposition (14) of the Dirichlet form

$$\mathcal{E}(f, f) = \int_X |\nabla f|^2 d\mu = \sum_{i=1}^N \int_X |\nabla_i f|^2 d\mu$$

along smooth functions with thus $\varGamma_i(f) = |\nabla_i f|$. On the other hand, the basic commutation (15) between the semigroup and the gradients ∇_i is described here as a curvature condition. Namely, whenever the Hessian V'' of a smooth potential V on $\mathbb{R}^n$ is (uniformly) bounded below by $-\kappa$, $\kappa \in \mathbb{R}$, the semigroup

$(P_t)_{t\geq 0}$ generated by the operator $\mathrm{L} = \Delta - \nabla V \cdot \nabla$ commutes to the gradient in the sense that, for every smooth function f and every $t \geq 0$,

$$|\nabla P_t f| \leq \mathrm{e}^{\kappa t}\, P_t\big(|\nabla f|\big). \tag{21}$$

In the product setting of Corollary 3, the semigroup $(P_t)_{t\geq 0}$ is the tensor product of the semigroups along every coordinate so that (21) ensures that

$$|\nabla_i P_t f| \leq \mathrm{e}^{\kappa t}\, P_t\big(|\nabla_i f|\big) \tag{22}$$

along the partial gradients ∇_i, $i = 1, \ldots, N$ and hence (15) holds on smooth functions. This commutation property (with $\kappa = -1$) is for example explicit on the integral representation

$$P_t f(x) = \int_{\mathbb{R}^n} f\big(\mathrm{e}^{-t}x + (1 - \mathrm{e}^{-2t})^{1/2} y\big) d\mu(y), \quad x \in \mathbb{R}^n,\ t \geq 0, \tag{23}$$

of the Ornstein-Uhlenbeck semigroup with generator $\mathrm{L} = \Delta - x \cdot \nabla$ and invariant and symmetric measure the standard Gaussian distribution. The assumption $V'' \geq -\kappa$ describes a curvature property of the generator L and is linked to Ricci curvature on Riemannian manifolds. Since only $\kappa \in \mathbb{R}$ is required here, it appears as a mild property, shared by numerous potentials such as for example double-well potentials on the line of the form $V(x) = ax^4 - bx^2$, $a, b > 0$. Recall that the assumption $V'' \geq c > 0$ (for example the quadratic potential with the Gaussian measure as invariant measure) actually implies that μ satisfies a logarithmic Sobolev inequality, and thus hypercontractivity (with constant c). We refer for example to [2], [15], [4]... for an account on (21) and the preceding discussion.

Corollary 3 admits generalizations in broader settings. Weighted measures on Riemannian manifolds with a lower bound on the Ricci curvature may be considered similarly with the same conclusions. In another direction, the hypercontractive approach may be developed in presence of suitable geometric decompositions. The next statements deal with the example of the sphere and with geometric decompositions of the identity in Euclidean space which are familiar in the context of Brascamp-Lieb inequalities (see [6] for further illustrations in a Markovian framework).

A non-product example in the continuous setting is the one of the standard sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ $(n \geq 2)$ equipped with its uniform normalized measure μ. Consider, for every $i, j = 1, \ldots, n$, $D_{ij} = x_i \partial_j - x_j \partial_i$. These will be the directions along which the Talagrand inequality may be considered since

$$\mathcal{E}(f,f) = \int_{\mathbb{S}^{n-1}} f(-\Delta f) d\mu = \frac{1}{2} \sum_{i,j=1}^{n} \int_{\mathbb{S}^{n-1}} (D_{ij} f)^2 d\mu.$$

The operators D_{ij} namely commute in an essential way to the spherical Laplacian $\Delta = \frac{1}{2}\sum_{i,j=1}^n D_{ij}^2$ so that (15) holds with $\kappa = 0$. Finally, the logarithmic Sobolev constant is known to be $n-1$ [2], [15], [4].... Corollary 4 thus again follows from the general Theorem 1.

Corollary 4. *For every smooth enough function* $f : \mathbb{S}^{n-1} \to \mathbb{R}$,

$$\mathrm{Var}_\mu(f) \leq \frac{4\mathrm{e}}{n} \sum_{i,j=1}^{n} \frac{\|D_{ij} f\|_2^2}{1 + \log\big(\|D_{ij} f\|_2 / \|D_{ij} f\|_1\big)} \,.$$

Up to the numerical constant, this inequality improves upon the Poincaré inequality for μ (with constant $\lambda = n-1$).

We turn to geometric Brascamp-Lieb decompositions. Consider thus E_i, $i = 1, \ldots, m$, subspaces in $\mathbb{R}^n$, and $c_i > 0$, $i = 1, \ldots, m$, such that

$$\mathrm{Id}_{\mathbb{R}^n} = \sum_{i=1}^{m} c_i \, Q_{E_i} \tag{24}$$

where Q_{E_i} is the projection onto E_i. In particular, for every $x \in \mathbb{R}^n$, $|x|^2 = \sum_{i=1}^m c_i |Q_{E_i}(x)|^2$ and thus, for every smooth function f on $\mathbb{R}^n$,

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu = \sum_{i=1}^{m} c_i \bigg(\int_{\mathbb{R}^n} \big| Q_{E_i}(\nabla P_t f) \big|^2 d\mu \bigg).$$

Furthermore, $Q_{E_i}(\nabla P_t f) = \mathrm{e}^{-t} P_t(Q_{E_i}(\nabla f))$ which may be examplified on the representation (23) of the Ornstein-Uhlenbeck semigroup with hypercontractive constant 1. Theorem 1 thus yields the following conclusion.

Corollary 5. *Under the decomposition (24), for* μ *the standard Gaussian measure on* $\mathbb{R}^n$*, and for every smooth function* f *on* $\mathbb{R}^n$,

$$\mathrm{Var}_\mu(f) \leq 4 \sum_{i=1}^{m} c_i \, \frac{\big\| Q_{E_i}(\nabla f) \big\|_2^2}{1 + \log\big(\|Q_{E_i}(\nabla f)\|_2 / \|Q_{E_i}(\nabla f)\|_1\big)} \,.$$

3 Hypercontractivity and Geometric Influences

In the continuous context of the preceding section, and as discussed in the introduction, the L^2-norms of gradients in Corollary 3 are not well-suited to the (geometric) influences of [13] which require L^1-norms. In order to reach L^1-norms through the hypercontractive argument, a further simple interpolation trick will be necessary.

To this task, we use an additional feature of the curvature condition $V'' \geq -\kappa$, $\kappa \geq 0$, namely that the action of the semigroup $(P_t)_{t \geq 0}$ with generator $\mathrm{L} = \Delta - \nabla V \cdot V$ on bounded functions yields functions with bounded gradients. More precisely (cf. [15], [4]...), for every smooth function f with

$|f| \leq 1$, and every $0 < t \leq 1/2\kappa$,

$$|\nabla P_t f| \leq \frac{1}{\sqrt{t}}\,. \tag{25}$$

This property may again be illustrated in case of the Ornstein-Uhlenbeck semigroup (22) for which, by integration by parts,

$$\nabla P_t f(x) = \frac{\mathrm{e}^{-t}}{(1-\mathrm{e}^{-2t})^{1/2}} \int_{\mathbb{R}^n} y\, f\big(\mathrm{e}^{-t}x + (1-\mathrm{e}^{-2t})^{1/2} y\big) d\mu(y).$$

With this additional tool, the following statement then presents the expected result. The setting is similar to the one of Corollary 3. Dependence on ρ and κ for the constant $C'(\rho,\kappa)$ below may be drawn from the proof. It will of course be independent of N.

Theorem 6. *Let $d\mu_i(x) = \mathrm{e}^{-V_i(x)}dx$, $i = 1, \ldots, N$, on $X_i = \mathbb{R}^{n_i}$ be hypercontractive with constant $\rho_i > 0$. Let $\mu = \mu_1 \otimes \cdots \otimes \mu_N$ on $X = X_1 \times \cdots \times X_N$, and set as before $\rho = \min_{1 \leq i \leq N} \rho_i$. Assume in addition that $V_i'' \geq -\kappa$, $\kappa \geq 0$, $i = 1, \ldots, N$. Then, for some constant $C'(\rho,\kappa) \geq 1$ and for any smooth function f on X such that $|f| \leq 1$,*

$$\mathrm{Var}_\mu(f) \leq C'(\rho,\kappa) \sum_{i=1}^{N} \frac{\|\nabla_i f\|_1 \big(1 + \|\nabla_i f\|_1\big)}{\big[1 + \log^+\big(1/\|\nabla_i f\|_1\big)\big]^{1/2}}\,.$$

Proof. We follow the same line of reasoning as in the proof of Theorem 1, starting on the basis of (10) from

$$\begin{aligned}
\|f\|_2^2 - \|P_T f\|_2^2 &= 2 \sum_{i=1}^{N} \int_0^T \bigg(\int_X |\nabla_i P_t f|^2 d\mu \bigg) dt \\
&\leq 4 \sum_{i=1}^{N} \int_0^T \bigg(\int_X |\nabla_i P_{2t} f|^2 d\mu \bigg) dt
\end{aligned}$$

for some $T > 0$. By (22) along each coordinate, for each $t \geq 0$,

$$|\nabla_i P_{2t} f| \leq \mathrm{e}^{\kappa t}\, P_t\big(|\nabla_i P_t f|\big).$$

Hence, by the hypercontractivity property as in Theorem 1,

$$\|\nabla_i P_{2t} f\|_2 \leq \mathrm{e}^{\kappa t}\, \|\nabla_i P_t f\|_p$$

where $p = p(t) = 1 + \mathrm{e}^{-2\rho t} \leq 2$. We then proceed to the interpolation trick. Namely, by (25) and the tensor product form of the semigroup, $|\nabla_i P_t f| \leq t^{-1/2}$ for $0 < t \leq 1/2\kappa$, so that in this range,

$$\|\nabla_i P_{2t} f\|_2 \leq \mathrm{e}^{\kappa(1+1/p)t}\, t^{-(1-1/p)/2}\, \|\nabla_i f\|_1^{1/p}$$

(where we used again (22)). As a consequence, provided $T \le 1/2\kappa$,

$$\|f\|_2^2 - \|P_T f\|_2^2 \le 4\,\mathrm{e}^{4\kappa T} \sum_{i=1}^N \|\nabla_i f\|_1 \int_0^T t^{-(1-1/p(t))} \|\nabla_i f\|_1^{(2/p(t))-1} dt.$$

We are then left with the estimate of the latter integral that only requires elementary calculus. Set $b = \|\nabla_i f\|_1$ and $\theta(t) = \frac{2}{p(t)} - 1 \le 1$. Assuming $T \le 1$,

$$\int_0^T t^{-(1-1/p(t))}\, b^{\theta(t)} dt \le \int_0^T t^{-1/2}\, b^{\theta(t)} dt.$$

Distinguish between two cases. When $b \ge 1$,

$$\int_0^T t^{-1/2}\, b^{\theta(t)} dt \le b \int_0^T t^{-1/2} dt \le 2b\sqrt{T}.$$

When $b \le 1$, use that $\theta(t) \ge \rho t/2$ for every $0 \le t \le 1/2\rho$. Hence, provided $T \le 1/2\rho$,

$$\int_0^T t^{-1/2}\, b^{\theta(t)} dt \le \int_0^T t^{-1/2}\, b^{\rho t/2} dt \le \frac{C}{\sqrt{\rho}} \cdot \frac{1}{\big[1 + \log(1/b)\big]^{1/2}}$$

where $C \ge 1$ is numerical. Summarizing, in all cases, provided T is chosen smaller than $\min\big(1, \frac{1}{2\rho}\big)$, we have

$$\int_0^T t^{-(1-1/p(t))} b^{\theta(t)} dt \le \frac{2C}{\sqrt{\rho}} \cdot \frac{1+b}{\big[1 + \log^+(1/b)\big]^{1/2}}\,.$$

Choosing for example $T = \min\big(1, \frac{1}{2\rho}, \frac{1}{2\kappa}\big)$ and using (10), Theorem 6 follows with $C'(\rho,\kappa) = C'/\rho^{3/2}T$ for some further numerical constant C'. If $\kappa \le c\rho$, then this constant is of order $\rho^{-1/2}$. $\square$

The preceding proof may actually be adapted to interpolate between Corollary 3 and Theorem 6 as

$$\mathrm{Var}_\mu(f) \le C \sum_{i=1}^N \frac{\|\nabla_i f\|_q^q \big(1 + \|\nabla_i f\|_1^2 / \|\nabla_i f\|_q^q\big)}{\big[1 + \log^+\big(\|\nabla_i f\|_q^q / \|\nabla_i f\|_1^2\big)\big]^{q/2}}$$

for any smooth function f on X such that $|f| \le 1$, and any $1 \le q \le 2$ (where C depends on ρ, κ and q).

As announced in the introduction, the conclusion of Theorem 6 may be interpreted in terms of influences. Namely, for $f = \mathbf{1}_A$ (or some smooth approximation), define $\|\nabla_i f\|_1$ as the geometric influence $I_i(A)$ of the i-th coordinate on the set A. In other words, $I_i(A)$ is the surface measure of the section of A along the fiber of $x \in X = X_1 \times \cdots \times X_N$ in the i-th direction, $1 \leq i \leq N$, averaged over the remaining coordinates (see [13]). Then Theorem 6 yields that

$$\mu(A)\big(1-\mu(A)\big) \leq C(\rho,\kappa) \sum_{i=1}^{N} \frac{I_i(A)\big(1+I_i(A)\big)}{\big[1+\log^+\big(1/I_i(A)\big)\big]^{1/2}} \,.$$

Proceeding as in the introduction for influences on the cube, the following consequence holds.

Corollary 7. *In the setting of Theorem 6, for any Borel set A in X with $\mu(A) = a$, there is a coordinate i, $1 \leq i \leq N$, such that*

$$I_i(A) \geq \frac{a(1-a)}{CN} \left(\log \frac{N}{a(1-a)} \right)^{1/2} \geq \frac{a(1-a)(\log N)^{1/2}}{CN}$$

where C only depends on ρ and κ.

It is worthwhile mentioning that when $N = 1$, $I_1(A)$ corresponds to the surface measure (Minkowski content)

$$\mu^+(A) = \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \big[\mu(A_\varepsilon) - \mu(A)\big]$$

of $A \subset \mathbb{R}^{n_1}$, so that Corollary 7 contains the quantitative form of the isoperimetric inequality for Gaussian measures

$$\mu^+(A) \geq \frac{1}{C}\, a(1-a) \left(\log \frac{1}{a(1-a)} \right)^{1/2}.$$

Recall indeed (cf. e.g. [15,16]) that the Gaussian isoperimetric inequality indicates that $\mu^+(A) \geq \varphi \circ \Phi^{-1}(a)$ ($a = \mu(A)$) where $\varphi(x) = (2\pi)^{-1/2}\, \mathrm{e}^{-x^2/2}$, $x \in \mathbb{R}$, $\Phi(t) = \int_{-\infty}^{t} \varphi(x)dx$, $t \in \mathbb{R}$, and that $\varphi \circ \Phi^{-1}(u) \sim u(2\log \frac{1}{u})^{1/2}$ as $u \to 0$. This conclusion, for hypercontractive log-concave measures, was established previously in [3]. See [18,19] for recent improvements in this regard.

Theorem 6 admits also generalizations in broader settings such as weighted measures on Riemannian manifolds with a lower bound on the Ricci curvature (this ensures that both (21) and (25) hold).

Besides the Gaussian measure, N. Keller, E. Mossel and A. Sen [13] also investigate with isoperimetric tools products of one-dimensional distributions of the type $c_\alpha \mathrm{e}^{-|x|^\alpha} dx$, $1 < \alpha < \infty$, for which they produce influences at least of the order of $\frac{(\log N)^{\beta/2}}{N}$ where $\beta = 2(1 - \frac{1}{\alpha})$ ($\alpha = 2$ corresponding to the Gaussian case). The proof of Theorem 6 may be adapted to cover this result but only seemingly for $1 < \alpha < 2$. Convexity of the potentials $|x|^\alpha$ ensures (21) and (25). When $1 < \alpha < 2$, measures $c_\alpha \mathrm{e}^{-|x|^\alpha} dx$ are not hypercontractive. Nevertheless, the hypercontractive theorems in Orlicz norms of [5] still indicate that the semigroup $(P_t)_{t \geq 0}$ generated by the potential $|x|^\alpha$ is such that, for every bounded function g with $\|g\|_\infty = 1$ and every $0 \le t \le 1$,

$$\|P_t g\|_2^2 \le C\, \|g\|_1 \exp\big(-c\, t \log^\beta(1 + (1/\|g\|_1))\big) \tag{26}$$

for $\beta > 0$ and some constants $C, c > 0$, and similarly for the product semigroup with constants independent of N. The hypercontractive step in the proof of Theorem 6 is then modified into

$$\big\||\nabla_i P_{2t} f|\big\|_2^2 \le C\|\nabla_i f\|_1 \int_0^1 t^{-1/2} \exp\big(-ct\log^\beta(1 + (1/\|\nabla_i f\|_1))\big)dt.$$

As a consequence, for any smooth f with $|f| \le 1$,

$$\mathrm{Var}_\mu(f) \le C \sum_{i=1}^N \frac{\|\nabla_i f\|_1 \big(1 + \|\nabla_i f\|_1\big)}{\big[1 + \log^+\big(1/\|\nabla_i f\|_1\big)\big]^{\beta/2}}\,. \tag{27}$$

We thus conclude to the influence result of [13] in this range. When $\alpha > 2$ ($\beta \in (1,2)$), the potentials are hypercontractive in the usual sense so that the preceding proofs yield (27) but only for $\beta = 1$. We do not know how to reach the exponent $\beta/2$ in this case by the hypercontractive argument.

We conclude this note by the L^1 versions of Corollaries 4 and 5. In the case of the sphere, the proof is identical to the one of Theorem 6 provided one uses that $|D_{ij} f| \le |\nabla f|$ which ensures that $|D_{ij} P_t f| \le 1/\sqrt{t}$. The behavior of the constant is drawn from the proof of Theorem 6.

Theorem 8. *For every smooth enough function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ such that $|f| \le 1$,*

$$\mathrm{Var}_\mu(f) \le \frac{C}{\sqrt{n}} \sum_{i,j=1}^n \frac{\|D_{ij} f\|_1 \big(1 + \|D_{ij} f\|_1\big)}{\big[1 + \log^+\big(1/\|D_{ij} f\|_1\big)\big]^{1/2}}\,.$$

Application to geometric influences $I_{ij}(A)$ as the limit of $\|D_{ij} f\|_1$ as f approaches the characteristic function of the set A may be drawn as in the previous corresponding statements. From a geometric perspective, $I_{ij}(A)$ can be viewed as the average over x of the boundary of the section of A in the 2-plane $x + \mathrm{span}(e_i, e_j)$. We do not know if the order $n^{-1/2}$ of the constant in Theorem 8 is optimal.

As announced, the last statement is the L^1-version of the geometric decompositions of Corollary 5 which seems again of interest for influences. Under the corresponding commutation properties, the proof is developed similarly.

Proposition 9. *Under the decomposition* (24)*, for μ the standard Gaussian measure on $\mathbb{R}^n$ and for every smooth function f on $\mathbb{R}^n$ such that $|f| \le 1$,*

$$\mathrm{Var}_\mu(f) \le C \sum_{i=1}^m c_i \, \frac{\|Q_{E_i}(\nabla f)\|_1 \big(1 + \|Q_{E_i}(\nabla f)\|_1\big)}{\big[1 + \log^+\big(1/\|Q_{E_i}(\nabla f)\|_1\big)\big]^{1/2}}$$

where $C > 0$ is numerical.

Let us illustrate the last statement on a simple decomposition. As in the Loomis-Whitney inequality, consider the decomposition

$$\mathrm{Id}_{\mathbb{R}^n} = \sum_{i=1}^{n} \frac{1}{n-1}\, Q_{E_i}$$

with $E_i = {e_i}^{\perp}$, $i = 1, \ldots, n$, $(e_1, \ldots, e_n)$ orthonormal basis. Proposition 9 applied to $f = \mathbf{1}_A$ for a Borel set A in $\mathbb{R}^n$ with $\mu(A) = a$ then shows that there is a coordinate i, $1 \le i \le n$, such that

$$\big\| Q_{E_i}(\nabla f) \big\|_1 \ge \frac{1}{C}\, a(1-a) \bigg(\log \frac{1}{a(1-a)} \bigg)^{1/2}$$

for some constant $C > 0$. Now, $\| Q_{E_i}(\nabla f) \|_1$ may be interpreted as the boundary measure of the hyperplane section

$$A^{x\cdot e_i} = \big\{ (x\cdot e_1, \ldots, x\cdot e_{i-1}, x\cdot e_{i+1}, \ldots, x\cdot e_n);\, (x\cdot e_1, \ldots, x\cdot e_i, \ldots, x\cdot e_n) \in A \big\}$$

along the coordinate $x\cdot e_i \in \mathbb{R}$ averaged over the standard Gaussian measure. By Fubini's theorem, there is $x \cdot e_i \in \mathbb{R}$ (or even a set with measure as close to 1 as possible) such that

$$\mu^+(A^{x\cdot e_i}) \ge \frac{1}{C}\; a(1-a) \bigg(\log \frac{1}{a(1-a)} \bigg)^{1/2}. \tag{28}$$

The interesting point here is that a is the full measure of A. Indeed, recall that the isoperimetric inequality for μ indicates that $\mu^+(A) \ge \varphi \circ \Phi^{-1}(a)$, hence a quantitative lower bound for $\mu^+(A)$ of the same form as (28). When A is a half-space in $\mathbb{R}^n$, thus extremal set for the isoperimetric problem and satisfying $\mu^+(A) = \varphi \circ \Phi^{-1}(a)$, it is easy to see that there is indeed a coordinate $x\cdot e_i$ such that $A^{x\cdot e_i}$ is again a half-space in the lower-dimensional space. The preceding (28) therefore extends this property to all sets.

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