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A (ONE-DIMENSIONAL) FREE BRUNN-MINKOWSKI INEQUALITY

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Abstract. – *We present a one-dimensional version of the functional form of the geometric Brunn-Minkowski inequality in free (non-commutative) probability theory. The proof relies on matrix approximation as used recently by P. Biane and F. Hiai, D. Petz and Y. Ueda to establish free analogues of the logarithmic Sobolev and transportation cost inequalities for strictly convex potentials, that are recovered here from the Brunn-Minskowski inequality as in the classical case. The method is used to extend to the free setting the Otto-Villani theorem stating that the logarithmic Sobolev inequality implies the transportation cost inequality. It is used further to recover the free analogue of Shannon's entropy power inequality put forward by S. Szarek and D. Voiculescu.*

1. Classical Brunn-Minkowski and functional inequalities

In its multiplicative form, the classical geometric Brunn-Minkowski inequality indicates that for all bounded Borel measurable sets A, B in $\mathbb{R}^n$, and every $\theta \in (0,1)$,

$$\mathrm{vol}\big(\theta A + (1-\theta)B\big) \geq \mathrm{vol}\,(A)^\theta \mathrm{vol}\,(B)^{1-\theta}$$

where $\theta A + (1-\theta)B = \{\theta x + (1-\theta)y; x \in A, y \in B\}$ and where $\mathrm{vol}\,(\cdot)$ denotes the volume element in $\mathbb{R}^n$. Equivalently on functions (known as the Prékopa-Leindler theorem), whenever $\theta \in (0,1)$ and u_1, u_2, u_3 are non-negative measurable functions on $\mathbb{R}^n$ such that

$$u_3\big(\theta x + (1-\theta)y\big) \geq u_1(x)^\theta u_2(y)^{1-\theta} \quad \text{for all } x, y \in \mathbb{R}^n, \tag{1}$$

then

$$\int u_3 dx \geq \Big(\int u_1 dx\Big)^\theta \Big(\int u_2 dx\Big)^{1-\theta} \tag{2}$$

(cf. [Ga], [Ba2] for modern expositions).

The Brunn-Minkowski inequality has been used recently in the investigation of functional inequalities for strictly log-concave densities such as logarithmic Sobolev or transportation cost inequalities (cf. [B-G-L], [Le1], [Le2], [Vi]...). Let $d\mu = \mathrm{e}^{-Q}dx$ be a probability measure on $\mathbb{R}^n$ such that, for some $c > 0$, $Q(x) - \frac{c}{2}\,|x|^2$ is convex on $\mathbb{R}^n$. Therefore,

$$Q(\theta x + (1-\theta)y) - \theta Q(x) - (1-\theta)Q(y) \leq -\frac{c\theta(1-\theta)}{2}\,|x-y|^2$$

for all $x, y \in \mathbb{R}^n$. The typical example is the standard Gaussian measure $\mathrm{e}^{-|x|^2/2}\frac{dx}{(2\pi)^{n/2}}$ (with $c = 1$). Let then f and g be two (bounded continuous) functions on $\mathbb{R}^n$ such that $g(x) \leq f(y) + \frac{c}{2}|x-y|^2$, $x, y \in \mathbb{R}^n$. Choose $u_1 = \mathrm{e}^{(1-\theta)g - Q}$, $u_2 = \mathrm{e}^{-\theta f - Q}$ and $u_3 = \mathrm{e}^{-Q}$ satisfying thus (1). According to (2), for every $\theta \in (0,1)$,

$$\log\int \mathrm{e}^{(1-\theta)g}d\mu + \frac{1-\theta}{\theta}\,\log\int \mathrm{e}^{-\theta f}d\mu \leq 0.$$

When $\theta \to 0$,

$$\log\int \mathrm{e}^{g}d\mu \leq \int f d\mu. \tag{3}$$

This inequality is actually the dual form of the quadratic transportation cost inequality

$$\mathrm{W}_2(\mu,\nu)^2 \leq \frac{1}{c}\,\mathrm{H}\,(\nu\,|\,\mu) \tag{4}$$

holding for all probability measures ν on $\mathbb{R}^n$, where W_2 is the Wasserstein distance between probability measures and $\mathrm{H}\,(\nu\,|\,\mu) = \int \log\frac{d\nu}{d\mu}\,d\nu$ is the relative entropy of $\nu \ll \mu$. The argument relies on the one side on the Monge-Kantorovitch-Rubinstein dual characterization of the Wasserstein metric as

$$\mathrm{W}_2(\mu,\nu)^2 = \sup\left[\int g d\nu - \int f d\mu\right]$$

where the supremum runs over all (bounded continuous) functions f and g such that $g(x) \leq f(y) + \frac{1}{2}|x-y|^2$ for all $x, y \in \mathbb{R}^n$ (cf. e.g. [Vi]), and on the other on the entropic inequality

$$\int g d\nu \leq \log\int \mathrm{e}^{g}d\mu + \mathrm{H}\,(\nu\,|\,\mu). \tag{5}$$

The Brunn-Minkowski theorem also covers the logarithmic Sobolev inequality for μ. Given $\nu \ll \mu$, set $f = \log\frac{d\nu}{d\mu}$ (assumed to be smooth enough), and let g_t, $t > 0$, be such that $g_t(x) \leq f(y) + \frac{1}{2t}|x-y|^2$, $x, y \in \mathbb{R}^n$. Apply Brunn-Minkowski to $u_1 = \mathrm{e}^{\frac{1}{\theta}g_t - Q}$ ($t = \frac{1-\theta}{c\theta}$), $u_2 = \mathrm{e}^{-Q}$, $u_3 = \mathrm{e}^{f-Q}$, to get that

$$\log\int \mathrm{e}^{(1+ct)g_t}d\mu \leq 0$$

for every $t > 0$. Now, the optimal choice for g_t is given by the infimum-convolution of f with the quadratic cost, solution of the classical Hamilton-Jacobi equation $\partial_t g_t + \frac{1}{2}|\nabla g_t|^2 = 0$ with initial condition f. Hence, $g_t \sim f - \frac{t}{2}\,|\nabla f|^2$ as $t \to 0$. In the limit, we thus get the logarithmic Sobolev inequality

$$\mathrm{H}\,(\nu\,|\,\mu) = \int f d\nu \leq \frac{1}{2c}\int |\nabla f|^2 d\nu = \frac{1}{2c}\,\mathrm{I}\,(\nu\,|\,\mu) \tag{6}$$

holding for every $\nu \ll \mu$, where $\mathrm{I}\,(\nu\,|\,\mu)$ is the Fisher information of ν with respect to μ.

It is worthwhile recalling that the Hamilton-Jacobi approach may also be used to produce a quick proof of the Otto-Villani theorem [O-V] stating that the logarithmic Sobolev inequality (6) implies the transportation cost inequality (4) for arbitrary probability measures $d\mu = \mathrm{e}^{-Q}dx$. Indeed, given $\nu \ll \mu$ and $f = \log \frac{d\nu}{d\mu}$ as before, let g_t be the infimum-convolution of f with the cost $\frac{1}{2t}\,|x|^2$, $t > 0$. Set, for any real number a, $f_t = (a+ct)g_t - j_t$ where $j_t = \log \int \mathrm{e}^{(a+ct)g_t} d\mu$. Define ν_t by $f_t = \log \frac{d\nu_t}{d\mu}$. The logarithmic Sobolev inequality (6) applied to ν_t yields

$$\int f_t d\nu_t \leq \frac{1}{2c}\int |\nabla f_t|^2 d\nu_t,$$

or, in other words,

$$c(a+ct)\int g_t d\nu_t - cj_t \leq -(a+ct)^2 \int \partial_t g_t d\nu_t.$$

Since $\int \partial_t f_t d\nu_t = 0$, it follows that $cj_t \geq (a+ct)\partial_t j_t$. Therefore, $(a+ct)^{-1} j_t$ is non-increasing in t. In particular, $\frac{1}{a+1}\,j_{1/c} \leq \frac{1}{a}\,j_0$, which for $a = 0$ amounts to (3).

For more details on this section, cf. [B-G-L], [Le2], [Vi].

2. Random matrix approximation

We now apply the preceding scheme to (one-dimensional) free probability theory by random matrix approximation following the recent investigations by P. Biane [Bi] and F. Hiai, D. Petz and Y. Ueda [H-P-U1], [H-P-U2]. This approach relies specifically on the large deviation asymptotics of spectral measures of unitary invariant Hermitian random matrices put forward by D. Voiculescu [Vo1] (as a microstate approximation) and G. Ben Arous and A. Guionnet [BA-G] (as a Sanov theorem), cf. e.g. [H-P] and the references therein. Given a continuous function $Q : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|x|\to\infty} |x|\,\mathrm{e}^{-\varepsilon Q(x)} = 0$ for every $\varepsilon > 0$, set

$$\widetilde{Z}_N(Q) = \int_A \Delta_N(x)^2\, \mathrm{e}^{-N\sum_{k=1}^N Q(x_k)} dx \tag{7}$$

where $A = \{x_1 < x_2 < \cdots < x_N\} \subset \mathbb{R}^N$ and $\Delta_N(x) = \prod_{1 \le k < \ell \le N}(x_\ell - x_k)$ is the Vandermonde determinant. In the random matrix context, $\widetilde{Z}_N(Q)$ is the partition function of the eigenvalue probability distribution. Namely, on the space $\mathcal{H}_N$ of Hermitian $N \times N$ matrices $X = X_N$, consider the probability density

$$f_N(X) = \frac{1}{Z_N(Q)} \, \mathrm{e}^{-N \mathrm{Tr}\,(Q(X))} \tag{8}$$

with respect to Lebesgue measure on $\mathcal{H}_N$. Then, as is classical (cf. [Me], [De], [H-P]...), the joint distribution of the (ordered list of the) eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ of $X = X_N \in \mathcal{H}_N$ under the law (8) is given by the probability distribution on $\mathbb{R}^N$

$$dP_N(x) = \Delta_N(x)^2 \, \mathrm{e}^{-N \sum_{k=1}^N Q(x_k)} \mathbf{1}_A(x) \, \frac{dx}{\widetilde{Z}_N(Q)} \, .$$

(Note that $\widetilde{Z}_N(Q) = \kappa_N Z_N(Q)$ where $\kappa_N > 0$ is the normalizing constant, only depending on N, from Lebesgue measure on $\mathcal{H}_N$ to the eigenvalue distribution.) The large deviation theorem of [Vo1] and [BA-G] (see also [Jo]) then indicates that

$$\lim_{N \to \infty} \frac{1}{N^2} \, \log \widetilde{Z}_N(Q) = \mathcal{E}_Q(\nu_Q) \tag{9}$$

where, for every probability measure ν on $\mathbb{R}$,

$$\mathcal{E}_Q(\nu) = \iint \log |x - y| d\nu(x) d\nu(y) - \int Q(x) d\nu(x)$$

is the weighted energy integral with extremal (compactly supported) measure ν_Q maximizing $\mathcal{E}_Q$ (cf. [S-T], [H-P]). (For the choice of $Q(x) = \frac{x^2}{2}$, ν_Q is the semicircle law.)

Let U_1, U_2, U_3 be real-valued continuous functions on $\mathbb{R}$ such that, for every $\varepsilon > 0$, $\lim_{|x| \to \infty} |x| \, \mathrm{e}^{-\varepsilon U_i(x)} = 0$, $i = 1, 2, 3$. The general idea would be to apply the Brunn-Minkowski theorem to the functions $\mathrm{e}^{-N \mathrm{Tr}\,(U_i(X))}$, $i = 1, 2, 3$, on $\mathcal{H}_N \simeq \mathbb{R}^{N^2}$, and to apply the large deviation asymptotics (9). However, while it is a classical fact that $\mathrm{Tr}\,(Q(X))$ is convex on $\mathcal{H}_N$ whenever $Q : \mathbb{R} \to \mathbb{R}$ is convex, it is not true that given U_1, U_2, U_3 such that $U_3(\theta x + (1-\theta) y) \le \theta U_1(x) + (1-\theta) U_2(y)$ for some $\theta \in (0,1)$ and all $x, y \in \mathbb{R}$, then

$$\mathrm{Tr}\,\big(U_3(\theta X + (1-\theta)Y)\big) \le \theta \mathrm{Tr}\,\big(U_1(X)\big) + (1-\theta) \mathrm{Tr}\,\big(U_2(Y)\big)$$

for matrices X, Y in $\mathcal{H}_N$. The following counter-example was kindly communicated to us by G. Aubrun, D. Cordero-Erausquin and M. Fradelizi. Denote by $\mathrm{sp}(X)$ and $\mathrm{sp}(Y)$ the spectrum of X and Y respectively, and choose for U_1 and U_2 (suitable approximations of) respectively $-\log \mathbf{1}_{\mathrm{sp}(X)}$ and $-\log \mathbf{1}_{\mathrm{sp}(Y)}$. The hypothesis, for

$\theta = \frac{1}{2}$, is then satisfied with $U_3 = -\log \mathbf{1}_{[\mathrm{sp}(X)+\mathrm{sp}(Y)]/2}$. The conclusion would however imply that

$$\mathrm{sp}(X+Y) \subset \mathrm{sp}(X) + \mathrm{sp}(Y),$$

which is not the case in general.

We work instead with the eigenvalue distribution. Given U_1, U_2, U_3 as above, set

$$u_i(x) = \Delta_N(x)^2 \, \mathrm{e}^{-N\sum_{k=1}^N U_i(x_k)} \mathbf{1}_A(x), \quad x \in \mathbb{R}^N, \ i = 1, 2, 3.$$

Since $-\log \Delta_N$ is convex on the convex set A, assuming that, for some $\theta \in (0,1)$ and all $x, y \in \mathbb{R}$,

$$U_3\big(\theta x + (1-\theta)y\big) \leq \theta U_1(x) + (1-\theta)U_2(y),$$

the Brunn-Minkowski theorem applies to u_1, u_2, u_3 on $\mathbb{R}^N$ to yield

$$\widetilde{Z}_N(U_3) \geq \widetilde{Z}_N(U_1)^\theta \widetilde{Z}_N(U_2)^{1-\theta}.$$

Taking the limit (9) immediately yields the following free analogue of the functional Brunn-Minkowski inequality on $\mathbb{R}$.

Theorem 1. *Let U_1, U_2, U_3 be real-valued continuous functions on $\mathbb{R}$ such that $\lim_{|x|\to\infty} |x| \, \mathrm{e}^{-\varepsilon U_i(x)} = 0$ for every $\varepsilon > 0$, $i = 1, 2, 3$. Assume that for some $\theta \in (0,1)$ and all $x, y \in \mathbb{R}$,*

$$U_3\big(\theta x + (1-\theta)y\big) \leq \theta U_1(x) + (1-\theta)U_2(y).$$

Then

$$\mathcal{E}_{U_3}(\nu_{U_3}) \geq \theta \mathcal{E}_{U_1}(\nu_{U_1}) + (1-\theta)\mathcal{E}_{U_2}(\nu_{U_2}).$$

3. Free logarithmic Sobolev and transportation cost inequalities

We next show how the preceding free (one-dimensional) Brunn-Minkowski inequality may be used, following the classical case, to recapture both the free logarithmic Sobolev inequality of D. Voiculescu [Vo3] (in the form put forward in [B-S] and extended in [Bi]) and the free quadratic transportation cost inequality of [B-V] and [H-P-U2] for quadratic and more general strictly convex potentials.

Let Q be a real-valued continuous function on $\mathbb{R}$ such that $\lim_{|x|\to\infty} |x| \, \mathrm{e}^{-\varepsilon Q(x)} = 0$ for every $\varepsilon > 0$. For ν, probability measure on $\mathbb{R}$, define the free entropy of ν (with respect to ν_Q) [Vo3], [B-S], [Bi], as

$$\widetilde{\Sigma}\big(\nu \,|\, \nu_Q\big) = \mathcal{E}_Q(\nu_Q) - \mathcal{E}_Q(\nu) \quad (\geq 0).$$

If $\varphi : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, it is convenient to set below

$$j_Q(\varphi) = \mathcal{E}_{Q-\varphi}(\nu_{Q-\varphi}) - \mathcal{E}_Q(\nu_Q).$$

For every probability measure ν on $\mathbb{R}$,

$$j_Q(\varphi) \geq \int \varphi d\nu + \mathcal{E}_Q(\nu) - \mathcal{E}_Q(\nu_Q) = \int \varphi d\nu - \widetilde{\Sigma}\big(\nu \,|\, \nu_Q\big) \tag{10}$$

with equality for $\nu = \nu_{Q-\varphi}$. In particular $j_Q(\varphi) \geq \int \varphi d\nu_Q$. (Note that in the case of the classical entropy from the Sanov theorem with respect to $d\mu = \mathrm{e}^{-Q}dx$, $j_Q(\varphi)$ would simply be $\log \int \mathrm{e}^{\varphi} d\mu$ and (10) then appears as the analogue of the entropic inequality (5).)

Assume now that (Q is C^1 and such that) $Q(x) - \frac{c}{2}\, x^2$ is convex for some $c > 0$. For bounded continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $g(x) \leq f(y) + \frac{c}{2}|x-y|^2$, we may apply the free Brunn-Minkowski theorem, as in the classical case, to $U_1 = Q - (1-\theta)g$, $U_2 = Q + \theta f$ and $U_3 = Q$. Thus, by Theorem 1,

$$j_Q\big((1-\theta)g\big) + \frac{1-\theta}{\theta}\, j_Q(-\theta f) \leq 0.$$

As $\theta \to 0$, it follows that for every probability measure ν,

$$\int g d\nu - \int f d\nu_Q \leq \widetilde{\Sigma}\big(\nu \,|\, \nu_Q\big).$$

In other words, $j_Q(g) \leq \int f d\nu_Q$ (cf. (3)). This is the dual form of the free quadratic transportation cost inequality

$$\mathrm{W}_2(\nu, \nu_Q)^2 \leq \frac{1}{c}\, \widetilde{\Sigma}\big(\nu \,|\, \nu_Q\big) \tag{11}$$

recently put forward in [B-V] for the semicircle law associated to the quadratic potential, and in [H-P-U2] for strictly convex potentials. (As discussed in [H-P-U2], (11) does not compare to (4).)

The free logarithmic Sobolev inequality of [Vo3], extended to strictly convex potentials in [Bi], follows in the same way from the Brunn-Minkowski theorem. We follow [Bi] where the matrix approximation is used similarly to this task. Fix a probability measure ν with compact support and smooth density p on $\mathbb{R}$. Define a C^1 function R on $\mathbb{R}$ such that $R(x) = 2\int \log|x-y| d\nu(y)$ on $\mathrm{supp}\,(\nu)$, $R(x) = Q(x)$ for $|x|$ large, and such that $R(x) \geq 2\int \log|x-y| d\nu(y)$ everywhere. By the uniqueness theorem of extremal measures of weighted potentials (cf. [S-T]), it is easily seen that the energy functional $\mathcal{E}_R$ is maximized at the unique point $\nu_R = \nu$. Define then f, with compact support, by $f = Q - R + C$ where the constant C $(= \mathcal{E}_Q(\nu_Q) - \mathcal{E}_R(\nu_R))$ is chosen so that $j_Q(f) = 0$. Let $g_t(x) = \inf_{y \in \mathbb{R}}[f(y) + \frac{1}{2t}(x-y)^2]$, $t > 0$, $x \in \mathbb{R}$, be the infimum-convolution of f with the quadratic cost, solution of the Hamilton-Jacobi equation $\partial_t g_t + \frac{1}{2} {g_t'}^2 = 0$ with initial condition f. As in the classical case, apply then Theorem 1 to $U_1 = Q - \frac{1}{\theta} g_t$ $(t = \frac{1-\theta}{c\theta})$, $U_2 = Q$, $U_3 = Q - f$, to get that

$$j_Q\big((1+ct)g_t\big) \leq j_Q(f) = 0$$

for every $t > 0$. In particular therefore,

$$\int (1+ct) g_t d\nu \le \widetilde{\Sigma}(\nu \,|\, \nu_Q)$$

and, since $\nu = \nu_R = \nu_{Q-f}$, as $t \to 0$,

$$\widetilde{\Sigma}\big(\nu \,|\, \nu_Q\big) = \int f d\nu \le \frac{1}{2c} \int {f'}^2 d\nu. \tag{12}$$

Now, $f' = Q' - Hp$ where

$$Hp(x) = \text{p.v.} \int \frac{2p(y)}{x-y}\, dy$$

is the Hilbert transform (up to a multiplicative factor) of the (smooth) density p of ν. Hence (12) amounts to the free logarithmic Sobolev inequality

$$\widetilde{\Sigma}\big(\nu \,|\, \nu_Q\big) \le \frac{1}{2c} \int \big[Hp - Q'\big]^2 d\nu = \frac{1}{2c}\, \mathrm{I}\,\big(\nu \,|\, \nu_Q\big) \tag{13}$$

as established in [Bi], where $\mathrm{I}\,(\nu \,|\, \nu_Q)$ is known as the free Fisher information of ν with respect to ν_Q [Vo3], [B-S]. Careful approximation arguments to reach arbitrary probability measures ν (with density in $\mathrm{L}^3(\mathbb{R})$) are described in [H-P-U1].

As in the classical case, the Hamilton-Jacobi approach may be used to prove the free analogue of the Otto-Villani theorem [O-V]. To this task, given a compactly supported C^1 function f on $\mathbb{R}$, and $a \in \mathbb{R}$, set $j_t = j_Q((a+ct)g_t)$ and $f_t = (a+ct)g_t - j_t$ so that $j_Q(f_t) = 0$. Denote for simplicity by ν_t the extremal measure for the potential $Q - f_t$. Then the logarithmic Sobolev inequality (12) reads as

$$\int f_t d\nu_t \le \frac{1}{2c} \int {f'_t}^2 d\nu_t.$$

We may then repeat the classical case. By the Hamilton-Jacobi equation, the preceding logarithmic Sobolev inequality amounts to

$$c(a+ct) \int g_t d\nu_t - c j_t \le -(a+ct)^2 \int \partial_t g_t d\nu_t.$$

On the support of ν_t (cf. [S-T]),

$$2 \int \log|x-y| d\nu_t(y) = Q - f_t + C_t$$

where $C_t = \iint \log|x-y| d\nu_t d\nu_t + \mathcal{E}_{Q-f_t}(\nu_t)$. Since $j_Q(f_t) = \mathcal{E}_{Q-f_t}(\nu_t) - \mathcal{E}_Q(\nu_Q) = 0$, it follows that $\int \partial_t f_t d\nu_t = 0$. Therefore, $c j_t \ge (a+ct)\partial_t j_t$. Hence $(a+ct)^{-1} j_t$ is non-increasing in t, so that in particular, $\frac{1}{a+1}\, j_{1/c} \le \frac{1}{a}\, j_0$. Using that

$$\lim_{a \to 0} \frac{1}{a}\, j_Q(af) = \int f d\nu_Q, \tag{14}$$

the latter thus amounts to $j_Q(g) \le \int f d\nu_Q$, that is the dual form of the quadratic transportation cost inequality (11). To briefly check (14), recall that $j_Q(af) \ge a \int f d\nu_Q$. Conversely, we may write, for every $a > 0$,

$$\frac{1}{a}\, j_Q(af) = \sup \left[\int f d\rho - \frac{1}{a}\, \widetilde{\Sigma}\big(\rho \,|\, \nu_Q\big) \right]$$

where the supremum runs over the set $\mathcal{S}$ of all probability measures ρ supported on a given compact set, independent of a small enough (cf. [S-T]). Fix then $\delta > 0$, and set

$$A = \left\{ \rho \in \mathcal{S};\, \int f d\rho \le \int f d\nu_Q + \delta \right\}.$$

Then

$$\frac{1}{a}\, j_Q(af) \le \max\left(\int f d\nu_Q + \delta,\, \|f\|_\infty - \frac{1}{a} \inf_{\rho \notin A} \widetilde{\Sigma}\big(\rho \,|\, \nu_Q\big) \right).$$

By the lower semicontinuity of $\widetilde{\Sigma}$ and uniqueness of the extremal measure ν_Q,

$$\inf_{\rho \notin A} \widetilde{\Sigma}\big(\rho \,|\, \nu_Q\big) > 0.$$

Therefore

$$\limsup_{a \to 0} \frac{1}{a}\, j_Q(af) \le \int f d\nu_Q + \delta.$$

Since $\delta > 0$ is arbitrary, the claim (14) is proved. We may thus conclude to the following statement. The preceding approach through Hamilton-Jacobi equations has some similarities with the use of the (complex) Burgers equation in [B-V].

Theorem 2. *Let Q be a real-valued C^1 function on $\mathbb{R}$ such that, for every $\varepsilon > 0$, $\lim_{|x| \to \infty} |x|\, \mathrm{e}^{-\varepsilon Q(x)} = 0$. Then the free logarithmic Sobolev inequality (13) (holding for some constant $c > 0$ and every probability measure ν on $\mathbb{R}$) implies the free quadratic transportation inequality (11) (holding, with the same constant $c > 0$, for every probability measure ν on $\mathbb{R}$).*

4. Shannon's entropy power inequality

In this last part, we briefly indicate how the matrix approximation approach may be used similarly to yield a direct proof of the free analogue of Shannon's entropy power inequality due to S. Szarek and D. Voiculescu [S-V1], [S-V2]. Their proof is based on the microstate approximation of entropy together with an improved Brunn-Minkowski inequality for restricted sums (see also [Ba1]).

Recall first that whenever f and g are probability densities on $\mathbb{R}^n$, Shannon's entropy power inequality expresses that

$$\mathrm{e}^{\frac{2}{n} S(f*g)} \ge \mathrm{e}^{\frac{2}{n} S(f)} + \mathrm{e}^{\frac{2}{n} S(g)} \tag{15}$$

where $S(f) = -\int f \log f dx$ is the (classical) entropy of the density f (cf. e.g. [Ba2]).

Let ν be a compactly supported probability measure (with smooth density) on $\mathbb{R}$. As for the free logarithmic Sobolev inequality in Section 3, define a C^1 potential R such that $\nu = \nu_R$ (with for example $Q(x) = \frac{x^2}{2}$). Let, as in (8),

$$f_N(X) = \frac{1}{Z_N(R)} \,\mathrm{e}^{-N\mathrm{Tr}\,(R(X))}$$

be the probability density on $\mathcal{H}_N$ (with respect to Lebesgue measure) induced by the potential R. Then

$$S(f_N) = \log Z_N(R) + \int N\mathrm{Tr}\,\big(R(X)\big) f_N dX.$$

Under f_N, the eigenvalue distribution $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ converges almost surely to $\nu_R = \nu$, and furthermore

$$\lim_{N\to\infty} \frac{1}{N} \int \mathrm{Tr}\,\big(R(X)\big) f_N(X) dX = \int R d\nu.$$

On the other hand, recall the normalizing constant κ_N, only depending on N, from Lebesgue measure on $\mathcal{H}_N$ to the eigenvalue distribution. That is, $\widetilde{Z}_N(R) = \kappa_N Z_N(R)$ (where $\widetilde{Z}_N(R)$ is defined in (7)). It follows from the large deviation property (9) that

$$\lim_{N\to\infty} \frac{1}{N^2} \big[\log Z_N(R) + \log \kappa_N\big] = \mathcal{E}_R(\nu).$$

Hence

$$\lim_{N\to\infty} \frac{1}{N^2} \big[S(f_N) + \log \kappa_N\big] = \Sigma(\nu) \tag{16}$$

where we set

$$\Sigma(\nu) = \iint \log|x-y| d\nu(x) d\nu(y).$$

(Actually, to overcome several integrability questions, the argument, here and below, should rely on the large deviations for restricted self-adjoint random matrices $\|X\|_\infty \leq C$ as discussed in [Vo2] and [H-M-P] – cf. [H-P-U2]. For simplicity in the exposition, we leave this out.)

Let now μ be another compactly supported probability measure on $\mathbb{R}$, associated to a potential T with corresponding probability density g_N on $\mathcal{H}_N$. We aim to apply Shannon's entropy power inequality (15) on $\mathcal{H}_N \simeq \mathbb{R}^{N^2}$ to f_N and g_N and to let $N\to\infty$. By (15) thus,

$$\mathrm{e}^{\frac{2}{N^2}[S(f_N * g_N) + \kappa_N]} \geq \mathrm{e}^{\frac{2}{N^2}[S(f_N) + \kappa_N]} + \mathrm{e}^{\frac{2}{N^2}[S(g_N) + \kappa_N]}. \tag{17}$$

By the entropic inequality, for every function φ on $\mathcal{H}_N$ (integrable with respect to $f_N * g_N$),

$$S(f_N * g_N) \leq \int \varphi f_N * g_N dX + \log \int \mathrm{e}^{-\varphi} dX.$$

Choose $\varphi(X) = N\mathrm{Tr}\,(F(X))$ where $F : \mathbb{R} \to \mathbb{R}$ is to be specified below. Hence

$$S(f_N * g_N) \leq \int N\mathrm{Tr}\,\big(F(X)\big) f_N * g_N dX + \log Z_N(F).$$

The density $f_N * g_N$ is the law of the sum of two independent random matrices from $\mathcal{H}_N$ with respective laws f_N and g_N. Under $f_N * g_N$, the normalized trace $\frac{1}{N}\,\mathrm{Tr}\,(F(X))$ is then known (cf. [V-D-N], [H-P]) to converge to $\int F d\nu \boxplus \mu$ where $\nu \boxplus \mu = \nu_R \boxplus \nu_T$ is the free additive convolution of $\nu = \nu_R$ and $\mu = \nu_T$. It thus follows again from the large deviation asymptotics (9) that

$$\limsup_{N\to\infty} \frac{1}{N^2}\,\big[S(f_N * g_N) + \kappa_N\big] \leq \int F d\nu \boxplus \mu + \mathcal{E}_F(\nu_F).$$

Choose finally F smooth enough given by $F(x) = 2\int \log|x-y| d\nu \boxplus \mu(y)$ on the (compact) support of $\nu \boxplus \mu$ so that

$$\int F d\nu \boxplus \mu + \mathcal{E}_F(\nu_F) = \Sigma(\nu \boxplus \mu).$$

Therefore

$$\limsup_{N\to\infty} \mathrm{e}^{\frac{2}{N^2}\,[S(f_N * g_N) - \kappa_N]} \leq \mathrm{e}^{2\Sigma(\nu\boxplus\mu)}.$$

Together with (17) and (16), we thus conclude to the free version of Shannon's entropy power inequality of [S-V1], [S-V2].

Theorem 3. *Let ν, μ be compactly supported probability measures on $\mathbb{R}$. Then*

$$\mathrm{e}^{2\Sigma(\nu\boxplus\mu)} \geq \mathrm{e}^{2\Sigma(\nu)} + \mathrm{e}^{2\Sigma(\mu)}.$$

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