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From Concentration to Isoperimetry: Semigroup Proofs

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Abstract. In a remarkable series of works, E. Milman recently showed how to reverse the usual hierarchy, and deduce from measure concentration inequalities, dimension free isoperimetric type inequalities in spaces with non-negative (Ricci) curvature. The results cover two basic instances, linear isoperimetry under arbitrarily slow concentration, and logarithmic strengthenings above the linear case under exponential concentration. The proofs are developed in a Riemannian (with density) context making use of isoperimetric minimizers and refined tools from geometric measure theory. In this note, we present simple semigroup arguments to cover the super-linear case, of potential usefulness in more general settings. A particular emphasis is put on functional inequalities for heat kernel measures.

1. Introduction

In the terminology of [**G2**], a metric measure space (X, d, μ) is a (separable) metric space (X, d) equipped with a Borel measure μ which, throughout this work, will be finite and normalized to be a probability measure. The isoperimetric profile of a metric measure space (X, d, μ) is the function $I_\mu(v)$, $v \in [0,1]$, defined as the infimum of $\mu^+(A)$ over all Borel measurable sets A in X with $\mu(A) = v$. Here, $\mu^+(A)$ is the (exterior) Minkowski boundary measure of the Borel set A defined by

$$\mu^+(A) = \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \big[\mu(A_\varepsilon) - \mu(A)\big]$$

where A_ε denotes the open neighborhood of order $\varepsilon > 0$ of A. A typical example is the case of the standard Gaussian measure γ on $\mathbb{R}^n$ equipped with its Euclidean structure for which the isoperimetric inequality of [**Bo**], [**S-T**] indicates that $I_\gamma = \varphi \circ \Phi^{-1}$ where $\varphi(x) = (2\pi)^{-1/2}\, e^{-x^2/2}$, $x \in \mathbb{R}$, and $\Phi(t) = \int_{-\infty}^t \varphi(x)dx$, $t \in \mathbb{R}$. Another example is the case of the two-sided exponential distribution $d\mu(x) = \frac{1}{2}\, e^{-|x|}dx$ on the line for which $I_\mu(v) = \min(v, 1-v)$, $v \in [0,1]$ [**T**] (see also [**B-H2**]). An isoperimetric type inequality is a lower bound $I_\mu(v) \geq i(v)$ on the isoperimetric profile by some suitable function i. In the standard examples, a set A

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and its complement will have the same boundary measure, so that the description of I_μ or i may be restricted by symmetry to $[0, \frac{1}{2}]$.

Another concept in metric measure spaces is measure concentration ([**Mi-Sc**], [**G2**], [**Le5**]...). The concentration function of a metric measure space (X, d, μ) is defined as

$$\alpha_\mu(r) = \sup\big\{1 - \mu(A_r); A \subset X, \mu(A) \geq \tfrac{1}{2}\big\}, \quad r > 0.$$

It is classical (cf. e.g. [**Le5**]) that α_μ is characterized equivalently in terms of Lipschitz functions on (X, d) in the sense that whenever $F : X \to \mathbb{R}$ is 1-Lipschitz and $\mu(F \geq m) \leq \frac{1}{2}$, then

$$\mu\big(F \geq m + r\big) \leq \alpha_\mu(r), \quad r > 0.$$

A concentration inequality describes the decay of the concentration function α_μ as $r \to \infty$. Typical examples are Gaussian concentration $\alpha_\mu(r) \leq C\, e^{-r^2/C}$, $r > 0$, for some constant $C > 0$ (satisfied by the standard Gaussian measure γ in $\mathbb{R}^n$ with constants independent of n) or exponential concentration $\alpha_\mu(r) \leq C\, e^{-r/C}$, $r > 0$ (satisfied by the exponential distribution and its products [**B-H1**]).

Measure concentration bounds are typically drawn from isoperimetric type inequalities. The first such example is concentration on the sphere [**Mi-Sc**]. In the general framework of a metric measure space (X, d, μ), it is part of the folklore that an isoperimetric type inequality $I_\mu \geq i$ controls the decay of the concentration function α_μ by an easy integration along the isoperimetric differential inequality $\mu^+(A) \geq i(\mu(A))$. For example, as shown in [**Mi-So**] (see also [**B-H2**]), if $I_\mu(v) \geq v\, \beta(\log \frac{1}{v})$, $v \in [0, \frac{1}{2}]$, for some non-negative continuous function β, then $\alpha_\mu(r) \leq \exp(-\delta(r))$ where δ is the inverse function of $\int_{\log 2}^r \frac{dx}{\beta(x)}$, $r > 0$. In the example of the standard Gaussian measure γ on $\mathbb{R}^n$, one may choose $\beta(x) \sim \sqrt{x}$, $x \geq 0$ (since $I_\gamma(v) \sim v(2 \log \frac{1}{v})^{1/2}$ as $v \to 0$), so that we indeed recover Gaussian concentration $\alpha_\gamma(r) \leq C\, e^{-r^2/C}$, $r > 0$.

The definitive difference between isoperimetric profile and concentration function is that the first one concerns small enlargements A_ε while the second one controls A_r for $r > 0$ reasonably large. That measure concentration conversely controls the isoperimetric profile cannot be true in general as can be seen from instances where μ has small mass between two sets of big measure (cf. [**Mi1, Mi3**]). A remarkable recent achievement by E. Milman [**Mi1, Mi3**] is that this hierarchy reverse actually holds under curvature (convexity) assumptions. These curvature assumptions may be suitably defined and discussed within (weighted) Riemannian manifolds. Let thus (X, g) be a complete Riemannian manifold with Riemannian volume element dx. Equip X with a finite, normalized, probability measure $d\mu = e^{-V} dx$ where V is a smooth potential on X. Endowed with the Riemannian metric d of X, the triple (X, d, μ) is an example of metric measure space. The two main conclusions of the works of E. Milman are the following. Assume the non-negative (generalized) Ricci curvature or convexity assumption

$$\mathrm{Ric}_g + \mathrm{Hess}_g V \geq 0 \tag{1}$$

as tensor fields on X. This setting covers in particular bounded convex domains in $\mathbb{R}^n$ with the uniform measure, and log-concave probability measures on $X = \mathbb{R}^n$.

Theorem 1 ([**Mi1**]). *Under the curvature condition* (1), *if* $\alpha_\mu(r) \to 0$ *as* $r \to \infty$, *then linear isoperimetry holds in the sense that*

$$I_\mu(v) \geq c \min(v, 1 - v), \quad v \in [0, 1],$$

where $c > 0$ only depends on α_μ.

THEOREM 2 ([**Mi3**]). *Under the curvature condition (1), if $\alpha_\mu(r) \to 0$ as $r \to \infty$ and $r(v) > 0$ denotes the smallest $r > 0$ such that $\alpha_\mu(r) < v$, then*

$$I_\mu(v) \geq \frac{c}{r(v)}\, v \log \frac{1}{v}\,, \quad v \in [0, \tfrac{1}{2}],$$

where $c > 0$ only depends on α_μ.

The linear isoperimetric inequality of Theorem 1 may be translated equivalently as a Cheeger type inequality $\int |f| d\mu \leq C \int |\nabla f| d\mu$ for all smooth functions f with zero mean (cf. e.g. [**B-H2**]). Theorem 2 is of course only of interest when the concentration function α_μ decays faster than exponential so that $r(v) < C \log \frac{1}{v}$ (but, as we will see, there is a fundamental difference in the treatment of both statements). As an illustration, if μ has Gaussian concentration $\alpha_\mu(r) \leq C\, e^{-r^2/C}$, $r > 0$, then $r(v) \sim (\log \frac{1}{v})^{1/2}$ so that by Theorem 2, $I_\mu(v) \geq cv(\log \frac{1}{v})^{1/2}$, $v \in [0, \frac{1}{2}]$. If μ has exponential concentration $\alpha_\mu(r) \leq C\, e^{-r/C}$, $r > 0$, then $r(v) \sim \log \frac{1}{v}$ and from Theorem 2, $I_\mu(v) \geq cv$, $v \in [0, \frac{1}{2}]$, which is however not any better than Theorem 1 as already mentioned.

One striking feature of these theorems is their independence with respect to the dimension of the underlying manifold. In both theorems, the dependence of the constant c on the concentration function α_μ can be expressed through a given small enough numerical value $v_0 > 0$ and $r_0 > 0$ such that $\alpha_\mu(r_0) < v_0$. In the terminology of M. Gromov [**G2**] (see also [**Le5**]), $1/c$ may be interpreted as the observable diameter of the metric measure space (X, d, μ). In some instances, the dependence of the constant c in terms of the observable diameter of the metric measure space (X, d, μ) may reflect a dimensional feature. For example, the observable diameter of the standard sphere in $\mathbb{R}^n$ is of the order of $\frac{1}{\sqrt{n}}$, so that Theorems 1 and 2 describe quantitatively the known optimal bounds (cf. [**Le4**]).

The results in [**Mi3**] are actually more general, allowing for some negative curvature bounds and some freedom on the concentration function. They cover moreover suitable limits of probability densities on a given Riemannian manifold. The arguments developed by E. Milman in the proofs of Theorems 1 and 2 rely on tools from geometric measure theory inspired by M. Gromov's proof of P. Lévy's isoperimetric inequality [**G1**]. In particular, the proofs make use of the existence and regularity of minimizing sets for the isoperimetric profile, variation of area formulas, and a version of the Heintze-Karcher theorem in manifolds with density involving total curvature due to F. Morgan [**Mo**]. While the first set of conclusions by E. Milman [**Mi1**] used at some point tools from semigroup theory (initiated in [**Le1**] and to which we will come back below), the latest developments only rely on geometric measure theory. One important step in the proof of linear isoperimetry under arbitrary decay of the concentration function (Theorem 1) is the fact from Riemannian geometry that, under the preceding convexity assumption, the isoperimetric profile I_μ is concave, or at least that $I_\mu(v)/v$ is non-increasing. We refer to [**Mi1, Mi3**] for detailed discussion and background on this basic property.

The dimension free feature of E. Milman's results suggests the possibility of a framework and a methodology of proof not directly relying on a finite dimensional state space and on geometric tools. In this note, we propose a new proof of the second quantitative part of E. Milman's results (Theorem 2) only based on semigroup tools, the first approach initiated in [**Mi1**]. We will namely show

how families of functional inequalities for heat kernel measures are rather simply and nicely suited to control the isoperimetric profile by means of the concentration function. In particular, inequalities for heat kernel measures produce in a natural way families of Lipschitz functions which are thus controlled by the concentration function. This new abstract semigroup approach is of potential use in more general settings (briefly discussed at the end of this paper). The proof of Theorem 1 in the case of arbitrarily slow decay of the concentration function relies, in its first step, on the same approach but requires apparently to be completed by an additional ingredient, such as concavity or non-increasing property of the isoperimetric profile. It is not clear whether the latter property can hold outside a Riemannian setting, or whether it can be avoided in the proof of Theorem 1. On the other hand, the proof of Theorem 2 presented in this note does not require any particular property of the isoperimetric profile.

We do not discuss in this note the most general results and assumptions, and only concentrate on the methodology of proof. We refer to the papers [**Mi1**]–[**Mi3**] by E. Milman for a careful and detailed exposition of the history of these results, a precise description of the setting and hypotheses, applications and complete references and acknowledgements. Section 2 below reviews the basic (Poincaré and logarithmic Sobolev) heat kernel inequalities under the curvature lower bound (1) in terms of the so-called Γ_2 criterion of D. Bakry and M. Émery [**B-E**], [**Ba1, Ba2**]. In the subsequent section, we address Theorem 1 under quadratic inequalities, while in Section 4 we prove Theorem 2 using entropic inequalities. Improvements under some negative (optimal) curvature bounds as in [**Mi3**] may also be achieved following the semigroup proof although we omit the details here. The final section indicates possible extensions and generalizations of the abstract semigroup arguments in various settings.

2. Heat kernel inequalities

On the complete Riemannian manifold (X, g), denote by $(P_t)_{t\geq 0}$ the semigroup with infinitesimal generator $\mathrm{L} = \Delta - \nabla V \cdot \nabla$. The semigroup $(P_t)_{t\geq 0}$ and generator L are invariant and symmetric with respect to the measure $d\mu = e^{-V}dx$. Set, for smooth functions u, v on X,

$$\Gamma(u,v) = \tfrac{1}{2}\big[\mathrm{L}(uv) - u\mathrm{L}v - v\mathrm{L}u\big] = \nabla u \cdot \nabla v$$

for which the integration by parts formula

$$\int u(-\mathrm{L}v)d\mu = \int \Gamma(u,v)d\mu$$

holds. Set further

$$\Gamma_2(u,u) = \tfrac{1}{2}\,\mathrm{L}\big(\Gamma(u,u)\big) - \Gamma(u, \mathrm{L}u) = \big(\mathrm{Ric}_g + \mathrm{Hess}_g V\big)(\nabla u, \nabla u).$$

Below, we write for simplicity $\Gamma(u) = \Gamma(u,u)$ and $\Gamma_2(u) = \Gamma_2(u,u)$. The generalized Ricci curvature assumption (1) on the manifold with weight (X, g, μ) where $d\mu = e^{-V}dx$ may be thus expressed as $\Gamma_2 \geq 0$ (meaning $\Gamma_2(u) \geq 0$ for every smooth function u). More general lower bounds are described by $\Gamma_2 \geq -\kappa$, $\kappa \in \mathbb{R}$. We refer to [**Ba1**] for a general account.

In the more general abstract framework of diffusion operators, the preceding Γ_2 criterion has been put forward by D. Bakry and M. Émery in the study of functional inequalities such as logarithmic Sobolev inequalities [**B-E**], [**Ba1, Ba2**]. Their basic argument has in turn proved useful to study functional inequalities with respect to the heat kernel measures $P_tf(x)$, $t \geq 0$, $x \in X$. (Dependence on $x \in X$, arbitrary, will be omitted below.) In particular, the following families have been put forward through a simple Γ_2 calculus (cf. [**Ba2**], [**S-Z**], [**Le4**]). Natural classes of functions f for which the subsequent inequalities may be established include, for example, compactly supported C^∞ functions (or compactly supported C^∞ plus a constant) in order to deal with the basic commutation property $\sqrt{\Gamma(P_sf)} \leq P_s(\sqrt{\Gamma(f)})$.

Proposition 3 (Poincaré and reverse Poincaré inequalities). *Under the curvature condition $\Gamma_2 \geq 0$, for every suitable function f, and every $t \geq 0$,*

$$2t\,\Gamma(P_tf) \leq P_t(f^2) - (P_tf)^2 \leq 2t\,P_t\big(\Gamma(f)\big).$$

Proposition 4 (Logarithmic Sobolev and reverse logarithmic Sobolev inequalities). *Under the curvature condition $\Gamma_2 \geq 0$, for every suitable function $f > 0$, and every $t \geq 0$,*

$$t\,\frac{\Gamma(P_tf)}{P_tf} \leq P_t(f\log f) - P_tf\log P_tf \leq t\,P_t\Big(\frac{\Gamma(f)}{f}\Big).$$

It is classical that the logarithmic Sobolev inequalities improve upon the Poincaré inequalities. To briefly recall the argument leading to Propositions 1 and 2, write (at any point x thus),

$$P_t(f^2) - (P_tf)^2 = -2\int_0^t \frac{d}{ds}P_s\big[\big(P_{t-s}f\big)^2\big]ds = 2\int_0^t P_s\big(\Gamma(P_{t-s}f)\big)ds.$$

Now, the condition $\Gamma_2 \geq 0$ expresses equivalently that $\Gamma(P_su) \leq P_s(\Gamma(u))$ (since similarly $P_s(\Gamma(u)) - \Gamma(P_su) = 2\int_0^s P_r(\Gamma_2(P_{s-r}u))dr$). Therefore,

$$P_t(f^2) - (P_tf)^2 \leq 2\int_0^t P_sP_{t-s}\big(\Gamma(f)\big)ds = 2t\,P_t\big(\Gamma(f)\big).$$

The reverse Poincaré inequality is proved similarly. The proof of the logarithmic Sobolev inequalities is entirely similar on the basis however of the stronger commutation $\sqrt{\Gamma(P_su)} \leq P_s(\sqrt{\Gamma(u)})$ still equivalent to $\Gamma_2 \geq 0$ [**Ba1, Ba2**]. Propositions 1 and 2 admit simple variations when $\Gamma_2 \geq -\kappa$ for some real κ.

An interesting byproduct of the reverse Poincaré and logarithmic Sobolev inequalities is that P_tf for $t > 0$ satisfies some Lipschitz properties in terms of the Γ operator. Namely, whenever $0 \leq f \leq 1$, it follows from the reverse Poincaré inequality of Proposition 1 that

$$\Gamma(P_tf) \leq \frac{1}{2t}\,. \tag{2}$$

Similarly, the reverse logarithmic Sobolev inequality of Proposition 2 ensures that whenever $0 < f \leq 1$,

$$\Gamma(P_tf) \leq \frac{1}{t}\,(P_tf)^2\log\frac{1}{P_tf} \leq \frac{1}{t}\,\log\frac{1}{P_tf}\,. \tag{3}$$

In particular, from (3), it was observed by M. Hino in [**H**] that the function $\varphi = \big(\log\frac{1}{P_tf}\big)^{1/2}$ is $(1/2\sqrt{t})$-Lipschitz in the sense that $\Gamma(\varphi) \leq 1/4t$.

In this sense, the curvature hypothesis $\Gamma_2 \geq 0$ may be nicely combined with concentration properties of the Lipschitz functions P_tf and $(\log\frac{1}{P_tf})^{1/2}$ for $t > 0$. (Note that when $\Gamma(u) \leq C$, then u is Lipschitz with respect to the Riemannian metric d.) Namely, from the first Lipschitz property (2) and by definition of the concentration function α_μ, if $0 \le f \le 1$, for every $t > 0$ and $r > 0$,

$$\mu\Big(P_t f \ge m + \frac{r}{\sqrt{2t}}\Big) \le \alpha_\mu(r) \tag{4}$$

where m is such that $\mu(P_t f \ge m) \le \frac{1}{2}$. In particular, one may choose $m = 2\int f d\mu$. Similarly, from the Lipschitz property of the function $\varphi = (\log \frac{1}{P_t f})^{1/2}$ above, if $\mu(\varphi \ge m) \ge \frac{1}{2}$, for every $r > 0$ and $t > 0$,

$$\mu\Big(\varphi \le m - \frac{r}{2\sqrt{t}}\Big) \le \alpha_\mu(r).$$

Choosing for example $m = \big(\log \frac{1}{2\int f d\mu}\big)^{1/2}$ provided $2\int f d\mu \le 1$, we get that when $0 \le f \le 1$, for every $r > 0$ and $t > 0$,

$$\mu\Big(P_t f \ge \big(2\textstyle\int f d\mu\big)^{1/2}\, e^{r^2/4t}\Big) \le \alpha_\mu(r). \tag{5}$$

It is worthwhile mentioning that the latter inequality (5) may also be deduced from Wang's Harnack inequality under Γ_2 lower bounds ([**W1, W2**]). Actually, Hino's observation leads to an alternate proof of (a version of) Wang's Harnack inequality. F.-Y. Wang [**W1**] used his Harnack inequality to establish a variant of Theorem 2 (under Gaussian concentration), however with dimensional dependence (see [**Mi3**] for a discussion).

3. Quadratic bounds: linear isoperimetry

In this section, we address the conclusion of Theorem 1 with the quadratic heat kernel inequalities from Proposition 1. For a given smooth function f on X, and every $t \ge 0$,

$$\int f^2 d\mu - \int (P_t f)^2 d\mu = -\int_0^t \frac{d}{ds}\Big(\int (P_s f)^2 d\mu\Big) ds = 2\int_0^t \int \Gamma(P_s f) d\mu\, ds. \tag{6}$$

As a consequence of (2), whenever $0 \le f \le 1$, for every $s > 0$,

$$\int \Gamma(P_s f) d\mu \le \frac{1}{\sqrt{2s}} \int \sqrt{\Gamma(P_s f)}\, du \le \frac{1}{\sqrt{2s}} \int \sqrt{\Gamma(f)}\, du$$

where the last inequality follows from the fact that (under $\Gamma_2 \ge 0$), $\sqrt{\Gamma(P_s f)} \le P_s\big(\sqrt{\Gamma(f)}\big)$. Therefore, for every $t \ge 0$ and every (smooth) function f such that $0 \le f \le 1$,

$$\int f^2 d\mu - \int (P_t f)^2 d\mu \le 2\sqrt{2t} \int \sqrt{\Gamma(f)}\, d\mu. \tag{7}$$

As such, the inequality extends to all locally Lipschitz functions in $\mathrm{L}^2(\mu)$.

This inequality (first emphasized in [**Le1**] in a slightly different form and exploited in [**Mi1**]) is the starting point of the analysis. Apply it namely to $f = \mathbf{1}_A$ for some measurable set $A \subset X$ (actually use first some smooth approximation of the type $(1 - \frac{1}{\varepsilon} d(\cdot, A))^+)$, $\varepsilon > 0$. For this choice of f, (7) yields, for every $t \ge 0$,

$$2\sqrt{2t}\, \mu^+(A) \ge \mu(A) - \int (P_t \mathbf{1}_A)^2 d\mu.$$

Now, from the concentration inequality (4) with $m = 2\int P_t \mathbf{1}_A d\mu = 2\mu(A)$,

$$\int (P_t \mathbf{1}_A)^2 d\mu \leq \Big(2\mu(A) + \frac{r}{\sqrt{2t}}\Big)^2 + \alpha_\mu(r),$$

so that

$$2\sqrt{2t}\,\mu^+(A) \geq \mu(A) - \Big(2\mu(A) + \frac{r}{\sqrt{2t}}\Big)^2 - \alpha_\mu(r).$$

It remains to suitably optimize in $t > 0$ and $r > 0$. Using that $\alpha_\mu(r) \to 0$ as $r \to \infty$, one then basically gets that whenever $\mu(A) \leq a$, $\mu^+(A) \geq i(\mu(A))$ for some small enough $a > 0$ and some function $i > 0$ tending to 0 at 0. Hence $I_\mu(v) \geq i(v)$, $v \leq a$.

But to reach linear isoperimetry in Theorem 1, that is $I_\mu(v) \geq c \min(v, 1-v)$, seems to require a further argument, such as that the isoperimetric function I_μ is concave, or at least that $I_\mu(v)/v$ is decreasing. These properties hold in a Riemannian setting and are extensively discussed in [**Mi1, Mi3**], relying on isoperimetric minimizers and tools from geometric measure theory, leading thus to the conclusion. In view of the preceding semigroup argument, such an additional ingredient seems indeed unavoidable to conclude to the full statement of Theorem 1.

Taken the concavity of the profile as granted, it of course suffices that $I_\mu(v_0) > 0$ for some $v_0 > 0$. As was shown by E. Milman [**Mi3**], the minimal hypothesis on the concentration function α_μ in order to reach the latter, and thus linear isoperimetry, is that $\alpha_\mu(r_0) < \frac{1}{2}$ for some $r_0 > 0$. This may also be recovered from the preceding semigroup argument, analyzing $\int f^p d\mu - \int (P_t f)^p d\mu$ for some large enough $p \geq 2$. We omit the details.

4. Entropic bounds: super-linear isoperimetry

In this section, we investigate Theorem 2. With respect to the linear case, the bounds need to be strengthened with a logarithmic factor which will be achieved through the entropic inequalities of Proposition 2. We follow exactly the same steps as in the quadratic case above. For every smooth positive function f, and every $t \geq 0$,

$$\begin{aligned}\int f \log f d\mu - \int P_t f \log P_t f d\mu &= -\int_0^t \frac{d}{ds}\Big(\int P_s f \log P_s f d\mu\Big) ds \\ &= \int_0^t \int \frac{\Gamma(P_s f)}{P_s f}\, d\mu\, ds.\end{aligned}$$

As a consequence of (3), when $\eta \leq f \leq 1$, $0 < \eta < 1$, for every $s > 0$,

$$\sqrt{\Gamma(P_s f)} \leq \Big(\frac{1}{s}\log\frac{1}{\eta}\Big)^{1/2} P_s f.$$

Hence, for every $s > 0$ and $0 < \eta < 1$,

$$\int \frac{\Gamma(P_s f)}{P_s f}\, d\mu \leq \Big(\frac{1}{s}\log\frac{1}{\eta}\Big)^{1/2} \int \sqrt{\Gamma(P_s f)}\, d\mu \leq \Big(\frac{1}{s}\log\frac{1}{\eta}\Big)^{1/2} \int \sqrt{\Gamma(f)}\, d\mu$$

where the last inequality follows from the fact that (under $\Gamma_2 \geq 0$), $\sqrt{\Gamma(P_s f)} \leq P_s\big(\sqrt{\Gamma(f)}\big)$. It thus follows from the preceding inequalities that for every $t > 0$ and $0 < \eta < 1$, and every (smooth) function f such that $\eta \leq f \leq 1$,

$$\int f \log f d\mu - \int P_t f \log P_t f d\mu \leq 2\Big(t \log\frac{1}{\eta}\Big)^{1/2} \int \sqrt{\Gamma(f)}\, d\mu. \tag{8}$$

This inequality may be considered as the entropic analogue of the quadratic inequality (7).

For a measurable set $A \subset X$, apply now the preceding to $f = \max(\mathbf{1}_A, \eta)$, $0 < \eta < 1$ (actually first some smooth approximation). For this choice of f, (8) implies that

$$2\Big(t \log \frac{1}{\eta}\Big)^{1/2} \mu^+(A) \geq \eta \log \eta - \int P_t f \log P_t f d\mu.$$

From the concentration inequality (5) with $\beta = \big[2(\eta + (1-\eta)\mu(A))\big]^{1/2} e^{r^2/4t}$ assumed to be less than 1,

$$\begin{aligned} \int P_t f \log \frac{1}{P_t f}\, d\mu &\geq \log \frac{1}{\beta} \int_{\{P_t f \leq \beta\}} P_t f d\mu \\ &\geq \log \frac{1}{\beta} \bigg[\int f d\mu - \int_{\{P_t f > \beta\}} P_t f d\mu \bigg] \\ &\geq \log \frac{1}{\beta} \big[\eta + (1-\eta)\mu(A) - \alpha_\mu(r)\big]. \end{aligned}$$

Therefore,

$$2\Big(t \log \frac{1}{\eta}\Big)^{1/2} \mu^+(A) \geq -\eta \log \frac{1}{\eta} + \big[(1-\eta)\mu(A) - \alpha_\mu(r)\big] \log \frac{1}{\beta}\,.$$

It remains to suitably optimize the various parameters. Make the simple choices of (for example) $\eta = \mu(A)^2$ and $t \log \frac{1}{\eta} = 2\, r^2$ to see that, whenever $\mu(A) < \frac{1}{16}$, for every $r > 0$,

$$2\sqrt{2}\, r \mu^+(A) \geq -2\mu(A)^2 \log \frac{1}{\mu(A)} + \frac{1}{4}\Big[\frac{\mu(A)}{2} - \alpha_\mu(r)\Big] \log \frac{1}{16\mu(A)}\,.$$

As a consequence, there exist numerical $a > 0$ sufficiently small and $K > 0$ large enough such that if $r_{\mu(A),\alpha_\mu}$ denotes the smallest $r > 0$ such that $\alpha_\mu(r) \leq \frac{\mu(A)}{4}$, then for every set A with $0 < \mu(A) \leq a$,

$$r_{\mu(A),\alpha_\mu}\, \mu^+(A) \geq \frac{1}{K}\, \mu(A) \log \frac{1}{\mu(A)}\,.$$

The preceding amounts to the conclusion of Theorem 2, however only for sets A such that $0 < \mu(A) \leq a$ for some small enough (numerical) $a > 0$. To conclude, we briefly indicate how to cover the situation when $a \leq \mu(A) \leq \frac{1}{2}$. To this task, it is enough to work at the quadratic level. Recall first the following easy observation from measure concentration [**Le5**]: whenever B is a set such that $\mu(B) \geq \rho$ for some $\rho > 0$, then for all $r > 0$,

$$1 - \mu(B_{r+r_0}) \leq \alpha_\mu(r)$$

where $r_0 > 0$ is such that $\alpha_\mu(r_0) < \rho$.

Let then A be a measurable set with $a \leq \mu(A) \leq \frac{1}{2}$. Given $\tau > 0$ to be specified below, set $B = \{P_t \mathbf{1}_A \leq (1+\tau)\mu(A)\}$ so that $\mu(B) \geq \tau/(1+\tau) = \rho > 0$. By the preceding, the concentration inequality (4) thus takes the form

$$\mu\Big(P_t \mathbf{1}_A \geq (1+\tau)\mu(A) + \frac{r + r_0}{\sqrt{2t}}\Big) \leq \alpha_\mu(r).$$

Repeating the argument at the end of Section 3 yields

$$2\sqrt{2t}\, \mu^+(A) \geq \mu(A) - \Big((1+\tau)\mu(A) + \frac{r + r_0}{\sqrt{2t}}\Big)^2 - \alpha_\mu(r).$$

Choose then for example $\tau = \frac{1}{10}$ from which r_0 is determined. Let then $r > 0$ be large enough so that $\alpha_\mu(r) \leq \frac{a}{8}$. Finally choose $t > 0$ such that $\frac{r+r_0}{\sqrt{2t}} \leq \frac{a}{10}$. It follows that

$$2\sqrt{2t}\,\mu^+(A) \geq \mu(A) - \frac{144}{100}\,\mu(A)^2 - \alpha_\mu(r) \geq \frac{\mu(A)}{4} - \alpha_\mu(r) \geq \frac{\mu(A)}{8}\,.$$

The claim is thus established and thereby Theorem 2.

As announced above, the proof of Theorem 2 does not require any specific property of the isoperimetric profile under non-negative curvature as opposed to the proof of Theorem 1. In particular, the borderline case of exponential concentration and isoperimetry is covered by Theorem 2. While this case only concerns Cheeger or Poincaré type inequalities, the above proof had however to jump to entropic inequalities.

It is easy to modify the preceding argument to include E. Milman's extension [**Mi3**] to the case of some possible negative curvature $\mathrm{Ric}_g + \mathrm{Hess}_g V \geq -\kappa$, $\kappa > 0$, whenever $\alpha_\mu(r) \leq C\, e^{-\delta\kappa r^2}$, $r > 0$, for some $\delta > \frac{1}{2}$. This is reached through the slight improvement of (5) into

$$\mu\Big(P_t f \geq \big(2\textstyle\int f d\mu\big)^{(\lambda-1)/\lambda}\, e^{(\lambda-1)r^2/4\rho(t)}\Big) \leq \alpha_\mu(r)$$

where $\rho(t) = (1 - e^{-2\kappa t})/2\kappa$ and $\lambda > 1$. The condition $\delta > \frac{1}{2}$ is then achieved for t large enough and λ close to one. This condition is optimal as shown in [**C-W**].

5. Extensions and consequences

The preceding proofs are presented in a standard Riemannian framework but only rely on specific abstract semigroup tools. As such, they are of potential use in more general settings. This however only concerns Theorem 2 since the proof of Theorem 1 uses a special property of the isoperimetric profile only known so far on manifolds or manifolds with densities.

A first instance of illustration is precisely the framework in which the semigroup ideas and techniques were developed, namely the case of diffusion operators in the sense of D. Bakry and M. Émery [**B-E**], [**Ba1, Ba2**], dealing with second order Markov diffusions L on some state space X with invariant probability distribution μ. The metric in this case is defined by the Γ operator associated to L as

$$d(x,y) = \sup\big\{ f(x) - f(y);\, \Gamma(f) \leq 1 \ \ \mu\text{-almost everywhere}\big\}, \quad x, y \in X.$$

These operators nicely include the examples of the Laplacian on a Riemannian manifold and of Laplacian plus drift on a manifold with density. We refer to [**Ba1**] for the basic properties of these operators, and various examples and applications.

Another observation is that the relevant property throughout the argument is the commutation inequality

$$\sqrt{\Gamma(P_t f)} \leq K\, P_t\big(\sqrt{\Gamma(f)}\big)$$

for some constant $K \geq 1$. This property for $K = 1$ is actually equivalent to the curvature condition $\Gamma_2 \geq 0$ ([**Ba1, Ba2**], [**Le4**]), but its validity for $K > 1$ may be established and used in non elliptic contexts where there is actually no lower bound on the Ricci curvature, such as on the Heisenberg group [**Li**]. We refer to the recent [**B-B-B-C**] for a discussion of functional inequalities for heat kernel measures of hypoelliptic models, and examples of illustrations where the results of this note may be applied.

In settings where the main results of E. Milman are satisfied, several consequences, in particular to functional inequalities, may be developed and expanded. The recent paper [**Mi4**], as well as the former ones [**Mi1**]–[**Mi2**], give a full account on these various consequences and applications and we refer to them for further developments. The idea is that concentration inequalities are typically derived from suitable functional inequalities (such as Poincaré or logarithmic Sobolev inequalities) while isoperimetric type inequalities ensure the validity of functional inequalities. Theorems 1 and 2 may thus be formulated (and probably are more useful) as equivalences between concentration and functional inequalities under curvature conditions. To illustrate this principle, and to conclude this paper, we briefly present, following [**Mi4**], one such application to logarithmic Sobolev inequalities. For simplicity in the notation, we keep the Riemannian framework although the conclusions are similar in the more general settings alluded to above.

A Borel probability measure μ on (X, g) is said to satisfy a logarithmic Sobolev inequality if there is a constant $a > 0$ such that for all smooth enough functions f on X such that $\int f^2 d\mu = 1$,

$$\int f^2 \log f^2 d\mu \leq a \int |\nabla f|^2 d\mu. \tag{9}$$

It is said further to satisfy a defective logarithmic Sobolev inequality if there are constants $a, b > 0$ such that for all such f's,

$$\int f^2 \log f^2 d\mu \leq a \int |\nabla f|^2 d\mu + b. \tag{10}$$

It is a classical result, going back to I. Herbst (cf. [**Le3, Le5**]), that under a logarithmic Sobolev inequality (9), the measure μ has Gaussian concentration $\alpha_\mu(r) \leq C\, e^{-r^2/C}$, $r > 0$, where C only depends on a. Now, provided the convexity assumption (1) holds (with $d\mu = e^{-V} dx$), Theorem 2 ensures that the measure μ satisfies a Gaussian isoperimetric type inequality. Following [**Le1**], the latter then implies a defective logarithmic Sobolev inequality (10), with constants a, b only depending on C (in particular independent of the dimension of the underlying state space). Actually, since we also have then linear isoperimetry, and thus a Cheeger and in turn a Poincaré inequality, the defective logarithmic Sobolev inequality may be tightened into a full logarithmic Sobolev inequality (cf. e.g. [**Ba1**]). We summarize the conclusion into the following statement (due to E. Milman [**Mi4**]).

Proposition 5. *If a probability measure $d\mu = e^{-V} dx$ on (X, g) satisfies the logarithmic Sobolev inequality* (9) *for some constant $a > 0$, then μ has Gaussian concentration $\alpha_\mu(r) \leq C\, e^{-r^2/C}$, $r > 0$, where $C > 0$ only depends on a. Conversely, if μ has Gaussian concentration $\alpha_\mu(r) \leq C\, e^{-r^2/C}$, $r > 0$, and if the convexity assumption* (1) *holds, then μ satisfies the logarithmic Sobolev inequality* (9) *with $a > 0$ only depending on C.*

Parts of the result actually extend to defective logarithmic Sobolev inequalities. Namely, under a defective logarithmic Sobolev inequality (10), μ satisfies Gaussian concentration in the sense that, for some (large) constant $C > 0$ only depending on a, b, $\mu(A_r) \geq 1 - C\, e^{-r^2/C}$, $r > 0$, at least for every set A in X with $\mu(A) \geq 1 - \frac{1}{C}$ (cf. [**Le2**]). Under the convexity assumption (1), the proof of Theorem 2 then similarly shows that a Gaussian isoperimetric type inequality holds, at least for every set A with a small measure. This property implies back a defective logarithmic

Sobolev inequality (with constants independent of the dimension). In a manifold context, where the isoperimetric profile is concave under non-negative curvature, the Gaussian isoperimetry type inequality then holds for all sets, and not only small ones. In this case therefore, a defective logarithmic Sobolev inequality implies a true logarithmic Sobolev inequality.

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