Measure concentration, functional inequalities, and curvature of metric measure spaces

M. Ledoux

Institut de Mathématiques de Toulouse, France

circle of ideas

between analysis, geometry and probability theory

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concentration of measure phenomenon

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(Ricci) curvature bounds on metric measure spaces

concentration of measure phenomenon

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Dvoretzky's theorem on spherical sections

of convex bodies in high dimension

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using a concentration (isoperimetric) property
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$$r \sim \frac{1}{\sqrt{n}} \quad (n \to \infty), \qquad \mu(A_r) \approx 1$$

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$$= \inf \big\{ D \geq 0, \exists\, A \subset X, \mathrm{Diam}\,(A) \leq D, \mu(A) \geq 1 - \kappa \big\}, \quad \kappa > 0$$

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less restrictive than isoperimetry, widely shared

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variety of examples and tools

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recent work by

Emanuel Milman (2007-2008)

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$$d\gamma(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{k/2}} \quad \text{on} \quad (\mathbb{R}^k, |\cdot|)$$

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Gauss space : curvature 1 dimension ∞

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infinite dimensional analysis (Wiener space)

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complementary to PDE and calculus of variations viewpoint

geometric description : Brunn-Minkowski inequality

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Prékopa-Leindler theorem

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$u = \chi_A, \ \ v = \chi_B$ multiplicative form of Brunn-Minkowski

$$\mathrm{vol}_n\big(\theta A + (1-\theta)B\big) \geq \mathrm{vol}_n(A)^\theta \mathrm{vol}_n(B)^{1-\theta}$$

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extension : $\quad d\mu = e^{-V} dx, \quad V'' \geq c > 0$

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A. Stam (1959), L. Gross (1975)

concentration via the logarithmic Sobolev inequality **(I. Herbst)**

$F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\int F \, d\gamma = 0$

$f = e^{\lambda F} / \int e^{\lambda F} d\gamma, \quad \lambda \in \mathbb{R}$

differential inequality on $\int e^{\lambda F} d\gamma$

$$\int e^{\lambda F} d\gamma \, \leq \, e^{\lambda^2/2}, \quad \lambda \in \mathbb{R}$$

$$\gamma\big(\{F < r\}\big) \, \geq \, 1 - e^{-r^2/2}, \quad r > 0$$

measure/information theoretic description :

transportation cost inequality

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relative entropy

measure/information theoretic description :

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$$H(\nu \,|\, \gamma) = \int \log \frac{d\nu}{d\gamma}\, d\nu$$

relative entropy

$$W_2(\nu,\gamma)^2 = \inf_{\nu \leftarrow \pi \rightarrow \gamma} \iint \frac{1}{2}\, |x-y|^2 d\pi(x,y)$$

Kantorovich-Rubinstein-Wasserstein distance

measure/information theoretic description :

transportation cost inequality

ν probability measure on $\mathbb{R}^n$, $\nu \ll \gamma$

$$W_2(\nu,\gamma)^2 \leq H(\nu \,|\, \gamma)$$

$$H(\nu \,|\, \gamma) = \int \log \frac{d\nu}{d\gamma}\, d\nu$$

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Kantorovich-Rubinstein-Wasserstein distance

M. Talagrand (1996)

concentration via the transportation cost inequality **(K. Marton)**

$$A, B \subset \mathbb{R}^n, \quad d(A,B) \geq r > 0$$

$$\gamma_A = \gamma(\cdot\,|A), \quad \gamma_B = \gamma(\cdot\,|B)$$

$$W_2(\gamma_A, \gamma_B) \leq \left(\log \frac{1}{\gamma(A)}\right)^{1/2} + \left(\log \frac{1}{\gamma(B)}\right)^{1/2}$$

$$\frac{r}{\sqrt{2}} \leq W_2(\gamma_A, \gamma_B) \leq \left(\log \frac{1}{\gamma(A)}\right)^{1/2} + \left(\log \frac{1}{\gamma(B)}\right)^{1/2}$$

$\gamma(A) \geq 1/2, \quad B =$ complement of A_r

$$\gamma(A_r) \geq 1 - e^{-r^2/4}, \quad r \geq r_0$$

Prékopa-Leindler inequality

logarithmic Sobolev inequality

transportation cost inequality

Prékopa-Leindler inequality

$$\int w\, d\gamma \ \geq \left(\int u\, d\gamma\right)^{\theta} \left(\int v\, d\gamma\right)^{1-\theta}$$

logarithmic Sobolev inequality

$$\int f \log f\, d\gamma \ \leq \ \frac{C}{2} \int \frac{|\nabla f|^2}{f}\, d\gamma$$

transportation cost inequality

$$W_2(\nu, \gamma)^2 \ \leq \ C\, H(\nu \,|\, \gamma)$$

Prékopa-Leindler inequality

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$$W_2(\nu, \mu)^2 \ \leq \ C \, H(\nu \, | \, \mu)$$

μ probability measure on $\mathbb{R}^n$

or more general spaces

hierarchy

Prékopa-Leindler inequality

$$\Downarrow$$

logarithmic Sobolev inequality

$$\Downarrow$$

transportation cost inequality

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transportation cost inequality

F. Otto, C. Villani (1999)

PDE and transportation methods

hypercontractivity of Hamilton-Jacobi equations

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PDE and transportation methods

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P. Cattiaux, A. Guillin (2004), N. Gozlan, P.-M. Samson (2009)

transportation cost inequality is **equivalent**

to logarithmic Sobolev inequality for semi-convex functions

hierarchy

Prékopa-Leindler inequality

$$\Downarrow$$

logarithmic Sobolev inequality

$$\Downarrow$$

transportation cost inequality

hierarchy

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dimension free Gaussian concentration

hierarchy

Prékopa-Leindler inequality

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stability by products

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logarithmic Sobolev, or transportation cost inequality

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then $\mu^{\otimes k}$ (Euclidean structure)

also satisfies these inequalities (with the same constant)

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N . Gozlan (2008) large deviations techniques

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hierarchy

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tools to establish these inequalities

hierarchy

Prékopa-Leindler inequality

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dimension free measure concentration

tools to establish these inequalities

parametrisation methods

heat kernel parametrisation

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generator $Lf = \Delta f - x \cdot \nabla f,$

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equivalent to a curvature condition

extensions

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◇ $d\mu = e^{-V} dx$ on $\mathbb{R}^n$, $V'' \geq c > 0$

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$$\frac{1}{2}\,\Delta\big(|\nabla f|^2\big) - \nabla f \cdot \nabla(\Delta f) \,=\, \mathrm{Ric}(\nabla f, \nabla f) + \|\mathrm{Hess}\,(f)\|^2$$

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equivalent to $|\nabla P_t f| \, \leq \, e^{-ct} P_t\big(|\nabla f|\big)$ **D. Bakry (1986)**

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◇ manifolds with weights

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extensions

◇ second order differential operators

D. Bakry, M. Emery (1985)

extensions

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Γ_2 curvature principle

analogue of Bochner's formula for Markov operator

extensions

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Markov chains, discrete structures

Y. Ollivier (2008)

parametrisation by optimal transportation

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Riemannian geometry of $(\mathcal{P}_2, W_2)$ **F. Otto (2001), C. Villani (2005)**

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optimal : $W_2(\gamma, \nu)^2 = \int \frac{1}{2} |x - T(x)|^2 d\gamma(x)$

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Y. Brenier, S. T. Rachev - L. Rüschendorf (1990) $T = \nabla \phi$, ϕ convex

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manifold case **R. McCann (1995)**

parametrisation $T_\theta = (1-\theta)\,\mathrm{Id} + \theta\, T, \quad \theta \in [0,1]$

$$(T_0\gamma = \gamma, \quad T_1\gamma = T\gamma = \nu)$$

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Monge-Ampère equation

$$e^{-|x|^2/2} = f_\theta \circ T_\theta \, e^{-|T_\theta|^2/2} \mathrm{det}\big((1-\theta)\,\mathrm{Id} + \theta \phi''\big)$$

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non-smooth analysis, PDE methods

mass transportation method

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Prékopa-Leindler theorem on a Riemannian manifold X

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for every $z_\theta(x,y)$ *theta*-barycenter of x, y

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characterizes curvature $\quad \mathrm{Ric} + \mathrm{Hess}(V) \geq c$

K. Bacher (2008), E. Hillion (2009)

optimal parametrisation and entropy

J. Lott - C. Villani, K. Th. Sturm (2006-09)

optimal parametrisation and entropy

J. Lott - C. Villani, K. Th. Sturm (2006-09)

μ_0, μ_1 probability measures on $\mathbb{R}^n$, $T : \mu_0 \to \mu_1$ optimal

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$$c = 0 \qquad H(\mu_\theta \,|\, \mu) \leq (1-\theta) H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu)$$

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$$H(\mu_\theta \,|\, \mu) \leq (1-\theta) H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu) - c\,\theta(1-\theta) W_2(\mu_0, \mu_1)^2$$

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K. Th. Sturm (2005)

notion of Ricci curvature bound

in a metric measure space (length space) (X, d, μ)

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J. Lott - C. Villani, K.-Th. Sturm (2006-09)

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analysis on singular spaces (limits of Riemannian manifolds)