The concentration of measure phenomenon

M. Ledoux

Institut de Mathématiques de Toulouse, France

concentration of measure phenomenon

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first historical observations

E. Borel 1914

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(geometric) law of large numbers

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μ^n uniform on $[-1, +1]^n$

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$H = (1,\ldots,1)^\perp$ hyperplan

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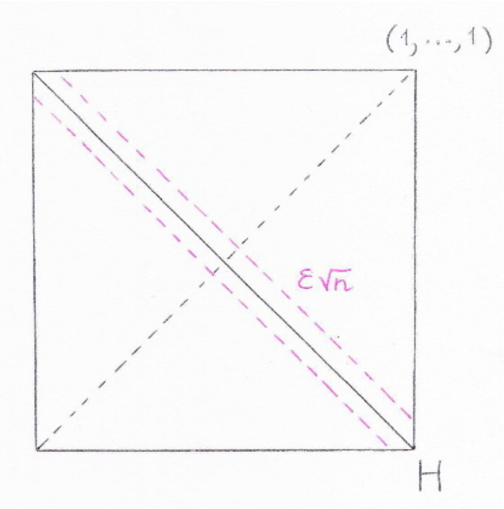
$$\mu^n\big(d(\cdot, H) \leq \varepsilon\sqrt{n}\,\big) \to 1$$



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relevant observable

$$f : (x_1, \dots, x_n) \mapsto \frac{1}{n} \sum_{i=1}^{n} x_i$$

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$$\mu^n\big(d(\cdot, H) \leq \varepsilon\sqrt{n}\,\big) \geq 1 - C\,e^{-n\,\varepsilon^2/C}$$

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high dimensional effect

real birth of **measure concentration**

new proof by **V. Milman 1970**

Dvoretzky's theorem on spherical sections of convex bodies

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for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$

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$K \cap F$

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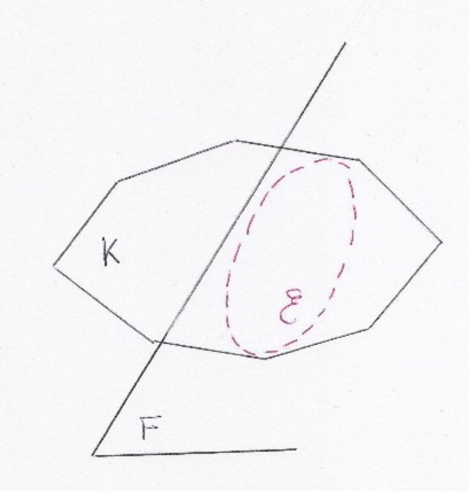

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find F **at random**

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concentration of spherical measures in high dimension

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most sections are spherical

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asymptotic geometric analysis

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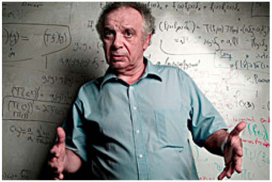
asymptotic geometric analysis

"The concentration of measure phenomenon, ubiquitous
ins probability theory and statistical mechanics,
was brought to geometry (starting from Banach spaces)
by Vitali Milman, following the earlier work by Paul Lévy"

(M. Gromov 1999)

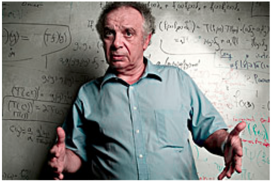
"The idea of concentration of measure,
which was discovered by Vitali Milman,
is arguably one of the great ideas of analysis in our times"

(M. Talagrand 1996)



Vitali Milman

David Milman **(Krein-Milman's** theorem**)**



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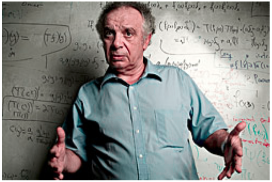

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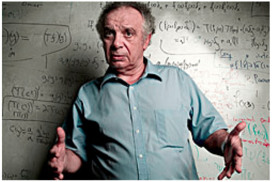

Vitali Milman, Pierre Milman **Emanuel Milman**

new proof by **V. Milman 1970**

main tool : **Lévy's inequality**

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origin : **isoperimetric inequalities**

isoperimetric inequalities

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Queen Dido

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E. Schmidt 1948

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extremal sets are geodesic balls

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integrated form

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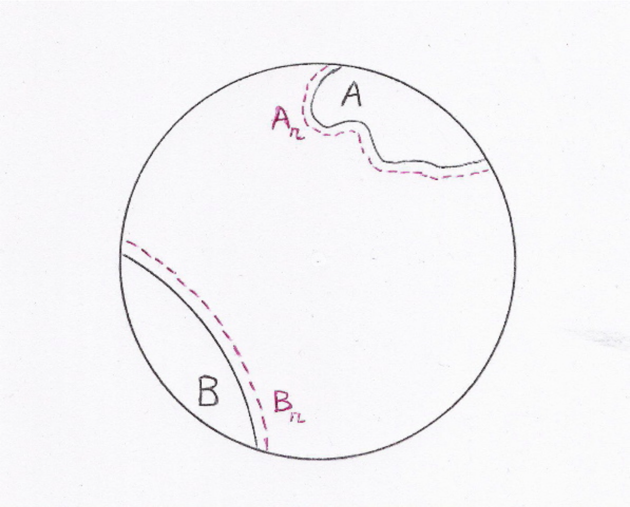
$$A_r = \{x \in \mathbb{S}^n ; d(x, A) < r\}$$



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infinitesimal form

$r \to 0 \qquad \mathrm{vol}\,(\partial A) \geq \mathrm{vol}\,(\partial B)$

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normalized volume $\mu(\cdot) = \dfrac{\mathrm{vol}\,(\cdot)}{\mathrm{vol}\,(\mathbb{S}^n)}$

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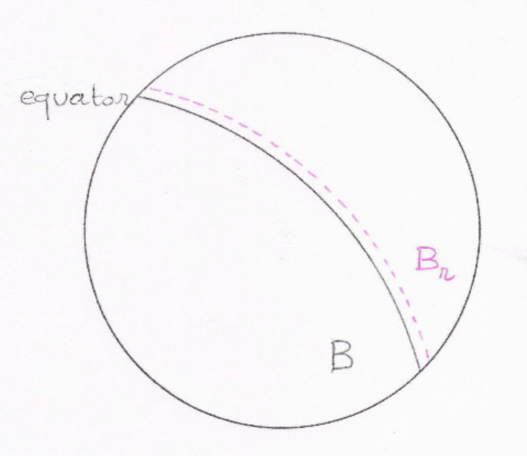
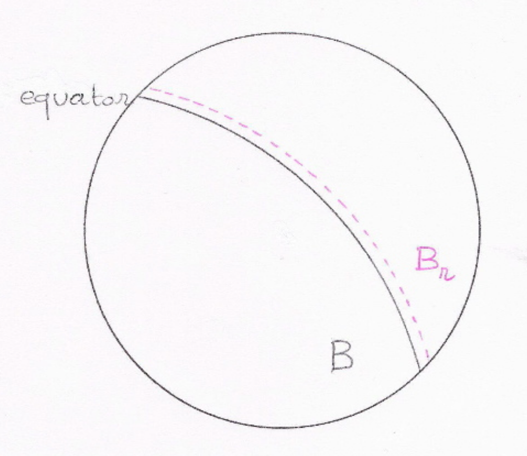


$\mu(B_r)$



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another example : **Gaussian concentration**

limit of spherical concentration

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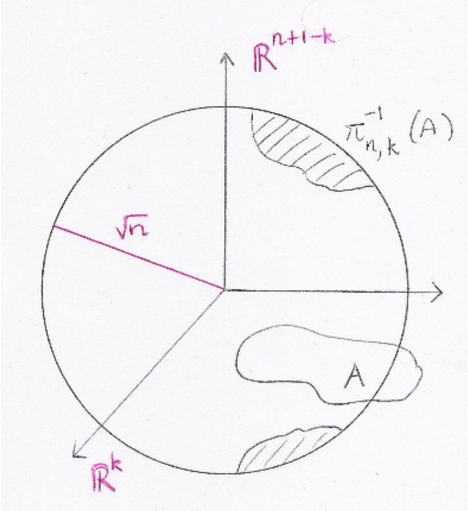

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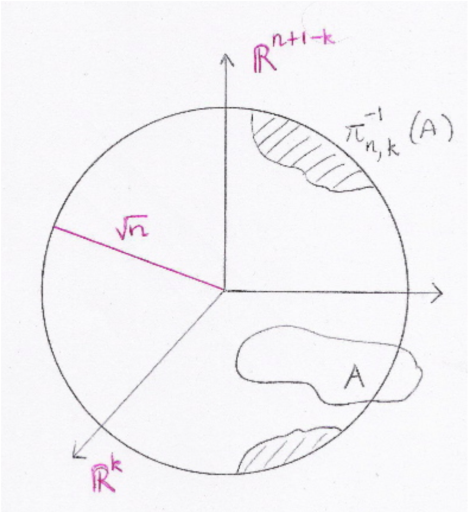

$$\mu\left(\pi_{n,k}^{-1}(A) \cap \mathbb{S}^n(\sqrt{n})\right) \to \gamma(A)$$

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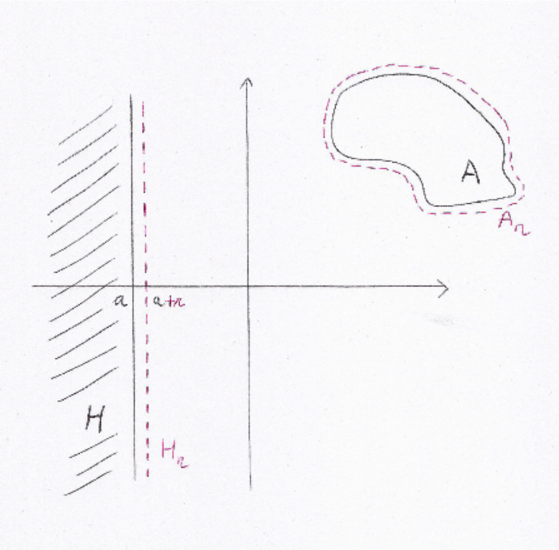
extremal sets are half-spaces $H = \{x \in \mathbb{R}^k ; \, x_1 \leq a\}$

(balls with centers at infinity)



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A_r Euclidean neighbourhood

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C. Borell, V. Sudakov - B. Tsirelson 1974

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extension to Wiener space

framework for measure concentration

metric measure space (X, d, μ)

framework for measure concentration

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(X, d) metric space

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μ Borel measure on X, $\mu(X) = 1$

framework for measure concentration

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concentration function

$$\alpha_\mu(r) = \alpha_{(X,d,\mu)}(r) = \sup\left\{1 - \mu(A_r);\, \mu(A) \geq \tfrac{1}{2}\right\}, \quad r > 0$$

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framework for measure concentration

metric measure space (X, d, μ)

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variety of examples and tools

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- geometric, functional, transportation inequalities

illustration

simple (functional) proof of Gaussian concentration $\alpha_\gamma(r) \leq e^{-r^2/2}$

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interpolation $\phi(t) = P_t(e^F)$ $\phi(0) = e^{P_t F}$

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a first **application** (of Gaussian concentration)

the **Johnson-Lindenstrauss 1984** **flatening lemma**

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$$(1-\varepsilon)|p_i - p_j| \leq \big|\pi(p_i) - \pi(p_j)\big| \leq (1+\varepsilon)|p_i - p_j|$$

π **quasi-isometry**

find π **at random**

$$\mathbb{R}^{nk} \ni X = (X_{ij})_{1 \le i \le k, 1 \le j \le n}, \qquad u \in \mathbb{R}^n$$

$$F(X) = |Xu|, \quad |u|\text{-Lipschitz}$$

$$\int_{\mathbb{R}^{nk}} F\, d\gamma \sim \sqrt{k}\, |u|$$

$$\gamma\Big(\big| Xu - \sqrt{k}\, |u| \big| \ge r\, |u| \Big) \le 2\, e^{-r^2/2}$$

$$r = \varepsilon \sqrt{k}, \quad \ell \text{ points } \quad u_1, \dots, u_\ell \qquad (u = p_i - p_j)$$

$$\gamma\bigg(\bigcup_{i=1}^{\ell} \Big\{ \frac{|Xu_i|}{\sqrt{k}} \notin \big[(1-\varepsilon)|u_i|, (1+\varepsilon)|u_i| \big] \Big\} \bigg) \le 2\, \ell\, e^{-\varepsilon^2 k/2} < 1$$

$$k \sim \frac{1}{\varepsilon^2} \log \ell, \qquad \ell \sim N^2$$

further concentration examples

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discrete cube

$$X = \{0, 1\}^n$$

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symmetric group

S^n symmetric group over n objects

$$d(x, y) = \#\big\{1 \leq i \leq n;\, x(i) \neq y(i)\big\}, \quad x, y \in S^n$$

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μ uniform $\mu(\{x\}) = \dfrac{1}{n!}$

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$$\mathrm{ObsDiam}_\mu(S^n) = O(\sqrt{n}\,)$$

B. Maurey 1979

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same **on any** product space

$$X = X_1 \times \cdots \times X_n$$

$$d(x,y) = \#\left\{1 \leq i \leq n;\, x_i \neq y_i\right\}$$ Hamming metric

$$\mu = \mu_1 \otimes \cdots \otimes \mu_n$$

principle of proof

$n = 1$

$F \quad L$-Lipschitz, $\quad \int_{X_1} F \, d\mu_1 = 0$

Jensen $\quad \displaystyle\int_{X_1} e^F \, d\mu_1 \leq \int_{X_1 \times X_1} e^{F(x)-F(y)} d\mu_1(x) d\mu_1(y)$

$$\big|F(x) - F(y)\big| \leq L \, d(x,y) = L \, \mathbf{1}_{x \neq y}$$

$$\int_{X_1 \times X_1} e^{F(x)-F(y)} d\mu_1(x) d\mu_1(y) \leq e^{L^2}$$

induction over n

symmetric group : martingale arguments

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$$d_w(x,y) = \sum_{i=1}^n w_i \, \mathbf{1}_{x_i \neq y_i}, \quad w_i \geq 0$$

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same as for Gaussian

illustration

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empirical processes

$X_1, \ldots, X_n$ independent identically distributed in $(S, \mathcal{S})$

$\mathcal{F}$ collection of functions $f : S \to [0,1]$

$$Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(X_i)$$

Z Lipschitz and convex

concentration inequalities on

$$\mathbb{P}\Big(\big|Z - \mathbb{E}(Z)\big| \geq r\Big), \quad r > 0$$

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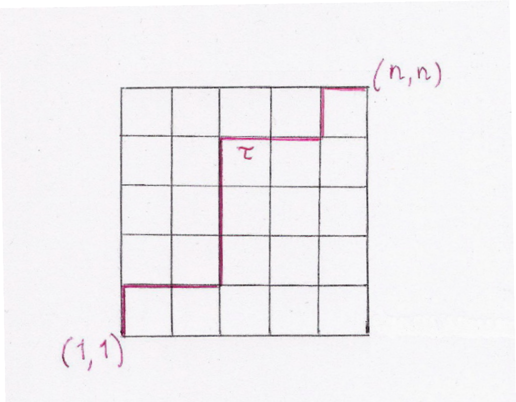

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new challenges of measure concentration