

The geometry of convolution inequalities

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Brascamp-Lieb inequalities

for multiple integrals

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$$T_i : H = \mathbb{R}^n \rightarrow H_i = \mathbb{R}^{n_i}, \quad i = 1, \dots, m$$

linear surjective maps

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linear surjective maps

$$c_i > 0$$

Brascamp-Lieb inequalities

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linear surjective maps

$$c_i > 0$$

find (best) $C > 0$ such that for all $f_i : H_i \rightarrow \mathbb{R}_+$

$$\int_H \prod_{i=1}^m f_i^{c_i}(T_i(x)) dx \leq C \prod_{i=1}^m \left(\int_{H_i} f_i(y) dy \right)^{c_i}$$

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examples

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examples

$$T_i(x) = x_i, \quad i = 1, \dots, n$$

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Fubini's theorem $c_i = 1, \quad C = 1$

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examples

$$T_i(x) = x_j, \quad i = 1, \dots, n$$

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$$T_i = \text{Id} : H \rightarrow H$$

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Fubini's theorem $c_i = 1, \quad C = 1$

$$T_i = \text{Id} : H \rightarrow H$$

Hölder's inequality $c_1 + \dots + c_m = 1, \quad C = 1$

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Loomis-Whitney inequality

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$$\begin{aligned} \int_{\mathbb{R}^3} f_1(x_2, x_3)^{1/2} f_2(x_1, x_3)^{1/2} f_3(x_1, x_2)^{1/2} dx \\ \leq \left(\int_{\mathbb{R}^2} f_1 dy \right)^{1/2} \left(\int_{\mathbb{R}^2} f_2 dy \right)^{1/2} \left(\int_{\mathbb{R}^2} f_3 dy \right)^{1/2} \end{aligned}$$

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$$H = \mathbb{R}^d \times \mathbb{R}^d, \quad H_i = \mathbb{R}^d, \quad i = 1, 2, 3$$

$$T_1(x, y) = x, \quad T_2(x, y) = y, \quad T_3(x, y) = x - y$$

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$$c_i = \frac{1}{p_i}, \quad f_i \rightarrow f_i^{p_i} \quad \text{duality} \quad \|f_2 * f_3\|_{p'_1} \leq C \|f_2\|_{p_2} \|f_3\|_{p_3}$$

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$$1 + \frac{1}{p_1'} = \frac{1}{p_2} + \frac{1}{p_3} \quad (c_1 + c_2 + c_3 = 2)$$

Lieb's theorem

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best $C > 0$

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achieved for f_i Gaussian

$f_i = e^{-Q_i}$, Q_i non-negative quadratic form on H_i

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optimal (Lieb's theorem) $C = C(p_1, p_2, p_3) = \prod_{i=1}^3 \left(\frac{p_i^{1/p_i}}{p_i'^{1/p_i'}} \right)^{d/2}$

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Brascamp-Lieb (1976) symmetrization methods ($\dim(H_i) = 1$)

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Brascamp-Lieb (1976) symmetrization methods ($\dim(H_i) = 1$)

Lieb (1990) Gaussian kernels

Lieb's theorem

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Brascamp-Lieb (1976) symmetrization methods ($\dim(H_i) = 1$)

Lieb (1990) Gaussian kernels

Barthe (1998) optimal transportation

recent new proof

Carlen-Lieb-Loss (2004)

Bennett-Carbery-Christ-Tao (2008)

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evolution along heat kernel

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geometric characterizations, extremals

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geometric characterizations, extremals

change of variables around the Gaussian maximizer

reduction to **geometric formulation**

geometric Brascamp-Lieb inequalities

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K. Ball (1989) (convex geometry)

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one-dimensional version $\dim(H_i) = 1$

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decomposition of the identity of \mathbb{R}^n

$$\text{Id}_{\mathbb{R}^n} = \sum_{i=1}^m c_i u_i \otimes u_i$$

$u_i \in \mathbb{R}^n$ unit vectors, $c_i > 0$

geometric Brascamp-Lieb inequalities

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Brascamp-Lieb inequality holds with $C = 1$

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i^{c_i}(\langle u_i, x \rangle) dx \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(y) dy \right)^{c_i}$$

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$(u_i)_{1 \leq i \leq n}$ orthonormal basis : $c_i = 1$ (Brascamp-Lieb = Fubini)

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higher dimensional projections $\text{Id}_{\mathbb{R}^n} = \sum_{i=1}^m c_i P_{E_i}$

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Loomis-Whitney : $E_i = e_i^\perp$ $c_i = \frac{1}{n-1}$

geometric Brascamp-Lieb inequality

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sketch of proof by evolution equations

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$(P_t)_{t \geq 0}$ heat semigroup on \mathbb{R}^n , Δ Laplacian

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sketch of proof by evolution equations

$(P_t)_{t \geq 0}$ heat semigroup on \mathbb{R}^n , Δ Laplacian

Brascamp-Lieb inequality along P_t , $t \geq 0$

$$P_t \left(\prod_{i=1}^m f_i^{c_i} \circ T_i \right) \leq \prod_{i=1}^m P_t(f_i \circ T_i)^{c_i}$$

(at any point)

$$P_t\left(\prod_{i=1}^m f_i^{c_i} \circ T_i\right) \leq \prod_{i=1}^m P_t(f_i \circ T_i)^{c_i}$$

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$$\alpha'(s) = P_s \left(\Delta(e^H) - e^H \sum_{i=1}^m c_i e^{-G_i} \Delta(e^{G_i}) \right)$$

$$G_i = \log P_{t-s}(f_i \circ T_i), \quad H = \sum_{i=1}^m c_i G_i$$

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infinitesimal Brascamp-Lieb condition

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equivalent to

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commutation property

$$T_i(x) = \langle u_i, x \rangle, \quad i = 1, \dots, m$$

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for all (smooth) $g_i : \mathbb{R} \rightarrow \mathbb{R}$

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duality

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decomposition of the identity $\text{Id}_{\mathbb{R}^n} = \sum_{i=1}^m c_i u_i \otimes u_i$

Brascamp-Lieb inequality along $P_t, t \geq 0$

$$P_t \left(\prod_{i=1}^m f_i^{c_i} \circ T_i \right) \leq \prod_{i=1}^m P_t(f_i \circ T_i)^{c_i}$$

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general principle (geometric Brascamp-Lieb)

F. Barthe, D. Cordero-Erausquin, M. L., B. Maurey (2009)

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$$\Delta_i f(x_i) = (1 - x_i^2) f''(x_i) - (n-1) x_i f'(x_i)$$

ultraspherical operator

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$$\int_{\mathbb{S}^{n-1}} \prod_{i=1}^n f_i(x_i) d\mu \leq \prod_{i=1}^n \left(\int_{\mathbb{S}^{n-1}} f_i^2(x_i) d\mu \right)^{1/2}$$

μ uniform probability measure on \mathbb{S}^{n-1}

Carlen-Lieb-Loss (2004)

multidimensional extensions on \mathbb{S}^{n-1}

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(multidimensional extensions, Cayley graphs)

multidimensional extensions

Riemannian manifolds, Lie groups

G compact Riemannian Lie group, $(G_i)_{i=1}^m$ Lie subgroups

$$\mathcal{E}_i = \mathcal{G}_i^\perp \quad (\text{Lie algebra})$$

$$\text{if } \sum_{i=1}^m c_i P_{\mathcal{E}_i} \leq \text{Id}_G$$

$$\int_G \prod_{i=1}^m f_i^{c_i} d\mu \leq \prod_{i=1}^m \left(\int_G f_i d\mu \right)^{c_i}$$

$$f_i : G \rightarrow \mathbb{R}^+ \quad G_i\text{-right-invariant}$$

geometric Brascamp-Lieb inequality

decomposition of the identity of \mathbb{R}^n

$$\text{Id}_{\mathbb{R}^n} = \sum_{i=1}^m c_i u_i \otimes u_i$$

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Young's inequality, hypercontractivity...

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sharp constants : without the Lieb theorem

Lieb's theorem

best $C > 0$

$$\int_H \prod_{i=1}^m f_i^{c_i}(T_i(x)) dx \leq C \prod_{i=1}^m \left(\int_{H_i} f_i(y) dy \right)^{c_i}$$

achieved for f_i Gaussian

$f_i = e^{-Q_i}$, Q_i non-negative quadratic form on H_i

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elementary question : \mathbb{R}^2 ($n = 2$)

given $0 < c_1, c_2, c_3 < 1$, $c_1 + c_2 + c_3 = 2$

decomposition in 3 unit vectors u_1, u_2, u_3 of $\text{Id}_{\mathbb{R}^2}$

$$\text{Id}_{\mathbb{R}^2} = c_1 u_1 \otimes u_1 + c_2 u_2 \otimes u_2 + c_3 u_3 \otimes u_3$$

all solutions (after isometry)

$$u_1 = (1, 0)$$

$$u_2 = \left(\left[\frac{(1 - c_1)(1 - c_2)}{c_1 c_2} \right]^{1/2}, \left(\left[\frac{1 - c_3}{c_1 c_2} \right]^{1/2} \right) \right)$$

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duality

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tensorization

further inequality : Shannon's inequality

$$S\left(\frac{X+Y}{\sqrt{2}}\right) \leq \frac{S(X)+S(Y)}{2}$$

X, Y independent random variables in \mathbb{R}^n

$$S(X) = \int f \log f \, dx$$

f density of X