PANEITZ-TYPE OPERATORS AND APPLICATIONS

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To the memory of André Lichnerowicz

Given (M, g) a smooth 4-dimensional Riemannian manifold, let S_g be the scalar curvature of g, and let Rc_g be the Ricci curvature of g. The Paneitz operator, discovered in [21], is the fourth-order operator defined by

$$P_g^4 u = \Delta_g^2 u - \operatorname{div}_g \left(\frac{2}{3} S_g g - 2 Rc_g \right) du,$$

where $\Delta_g u = -\operatorname{div}_g \nabla u$ is the Laplacian of u with respect to g. When (M, g) is the 4-dimensional standard unit sphere (S^4, h), we get that

$$P_h^4 u = \Delta_h^2 u + 2\Delta_h u.$$

The Paneitz operator is conformally invariant in the sense that if $\tilde{g} = e^{2\varphi} g$ is a conformal metric to g, then for all $u \in C^\infty(M)$,

$$P_{\tilde{g}}^4 u = e^{-4\varphi} P_g^4(u).$$

The 2-dimensional analogue of this relation is

$$\Delta_{\tilde{g}} u = e^{-2\varphi} \Delta_g u.$$

When the dimension is 2, it is well known that the scalar curvatures of g and $\tilde{g}$ are related by the equation

$$\Delta_g \varphi + \frac{1}{2} S_g = \frac{1}{2} S_{\tilde{g}} e^{2\varphi}.$$

When the dimension is 4, we get that

$$P_g^4 \varphi + Q_g^4 = Q_{\tilde{g}}^4 e^{4\varphi},$$

where

$$Q_g^4 = \frac{1}{6}\left(\Delta_g S_g + S_g^2 - 3|Rc_g|^2\right).$$

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With respect to the Euler-Poincaré characteristic, in dimension 2, we have that

$$\chi(M) = \frac{1}{4\pi}\int_M S_g\, dv_g,$$

while in dimension 4,

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{1}{4}|W_g|^2 + Q_g^4\right) dv_g,$$

where W_g stands for the Weyl tensor of g. Beautiful works on P_g^4 have been developed recently by Beckner [2], Branson-Chang-Yang [4], Chang-Yang [7], Chang-Gursky-Yang [6], and Gursky [12]. Also see the survey Chang [5]. The Paneitz operator was generalized to higher dimensions by Branson [3]. Given (M,g) a smooth compact Riemannian n-manifold, $n \geq 5$, let P_g^n be the operator defined by

$$P_g^n u = \Delta_g^2 u - \operatorname{div}_g\big(a_n S_g g + b_n Rc_g\big)\, du + \frac{n-4}{2} Q_g^n u,$$

where

$$Q_g^n = \frac{1}{2(n-1)}\Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2}|Rc_g|^2$$

and

$$\begin{cases} a_n = \dfrac{(n-2)^2+4}{2(n-1)(n-2)}, \\ b_n = -\dfrac{4}{n-2}. \end{cases}$$

If $\tilde{g} = \varphi^{4/(n-4)} g$ is a conformal metric to g, then for all $u \in C^\infty(M)$,

$$P_g^n(u\varphi) = \varphi^{(n+4)/(n-4)} P_{\tilde{g}}^n u.$$

In particular,

$$P_g^n \varphi = \frac{n-4}{2} Q_{\tilde{g}}^n \varphi^{(n+4)/(n-4)}.$$

These two relations have well-known analogues when dealing with the conformal Laplacian L_g^n, the second order operator whose expression is given by

$$L_g^n u = \Delta_g u + \frac{n-2}{4(n-1)} S_g u.$$

If $\tilde{g} = \varphi^{4/(n-2)} g$ is a conformal metric to g, for all $u \in C^\infty(M)$, we get that

$$L_g^n(u\varphi) = \varphi^{(n+2)/(n-2)} L_{\tilde{g}}^n(u).$$

In particular,

$$L_g^n\varphi = \frac{n-2}{4(n-1)}S_{\tilde{g}}\varphi^{(n+2)/(n-2)}.$$

On the standard unit sphere (S^n, h), $n \geq 5$, the expression of P_h^n is

$$P_h^n u = \Delta_h^2 u + c_n \Delta_h u + d_n u, \tag{0.1}$$

where

$$\begin{cases} c_n = \dfrac{1}{2}(n^2 - 2n - 4), \\ d_n = \dfrac{n-4}{16} n(n^2-4), \end{cases} \tag{0.2}$$

that is, $c_n = n(n-1)a_n + (n-1)b_n$, $d_n = (n-4)Q_h^n/2$. Note that $P_h^n = (L_h^n)^2 - 2L_h^n$. Given (M, g) a smooth n-dimensional compact Riemannian manifold, $n \geq 5$, and $\alpha > 0$ real, we let P_g be the fourth-order operator defined by

$$P_g u = \Delta_g^2 u + \alpha \Delta_g u. \tag{0.3}$$

Keeping in mind the expression of P_g^4 on the standard sphere, we refer to P_g as a Paneitz-type operator. Results and remarks often shift between operators like P_g, and Paneitz-Branson-type operators like

$$\tilde{P}_g u = \Delta_g^2 u + \alpha \Delta_g u + \beta u,$$

where α and β are real numbers. Here we should regard P_g as the essential part of $\tilde{P}_g$, like the Laplacian is the essential part of the conformal Laplacian, and note that the Paneitz-Branson operator P_g^n reduces to $\tilde{P}_g$ when g is Einstein. A natural space when studying P_g is the Sobolev space $H_2^2(M)$ defined as the completion of $C^\infty(M)$ with respect to the norm

$$\|u\|^2 = \|\nabla^2 u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2.$$

Following standard notations, $\|\cdot\|_p$ in the above expression stands for the L^p-norm (with respect to the Riemannian measure dv_g). As is well known and easy to see, for all $u \in C^\infty(M)$,

$$(\Delta_g u)^2 \leq n|\nabla^2 u|^2.$$

Conversely, by the Bochner-Lichnerowicz-Weitzenböck formula we get that

$$\begin{aligned} \int_M |\nabla^2 u|^2 \, dv_g &= \int_M |\Delta_g u|^2 \, dv_g - \int_M Rc_g(\nabla u, \nabla u) \, dv_g \\ &\leq \int_M |\Delta_g u|^2 \, dv_g + k \int_M |\nabla u|^2 \, dv_g, \end{aligned}$$

where k is such that $Rc_g \geq -k$. Hence, $\|\cdot\|_{H_2^2}$ defined by

$$\|u\|_{H_2^2}^2 = \int_M (P_g u) u \, dv_g + \int_M u^2 \, dv_g$$

is a norm on $H_2^2(M)$ that is equivalent to the above more classical one $\|\cdot\|$. Here and in what follows,

$$\int_M (P_g u) u \, dv_g = \int_M (\Delta_g u)^2 \, dv_g + \alpha \int_M |\nabla u|^2 \, dv_g.$$

By the Sobolev embedding theorem (see, e.g., [14] and recall that $n \geq 5$), we get an embedding of $H_2^2(M)$ in $L^{2^\sharp}(M)$, where

$$2^\sharp = \frac{2n}{n-4}.$$

This embedding is critical. It is also continuous, so that there exist $A \in \mathbb{R}$ and $B \in \mathbb{R}$ such that for all $u \in H_2^2(M)$,

$$\|u\|_{2^\sharp}^2 \leq A \int_M (P_g u) u \, dv_g + B \|u\|_2^2. \tag{$S1$}$$

Another possible inequality is that for all $u \in H_2^2(M)$,

$$\|u\|_{2^\sharp}^2 \leq A \int_M (P_g u) u \, dv_g + B \|u\|_{H_1^2}^2, \tag{$S2$}$$

where $\|\cdot\|_{H_1^2}$ is the usual norm of $H_1^2(M)$ given by

$$\|u\|_{H_1^2}^2 = \|\nabla u\|_2^2 + \|u\|_2^2.$$

Given $i = 1, 2$, we define $A_{\mathrm{opt}}^{(i)}(M)$ as the best constant A in (Si). Formally,

$$A_{\mathrm{opt}}^{(i)}(M) = \inf \left\{ A \in \mathbb{R} \text{ s.t. } \exists B \in \mathbb{R} \text{ with the property that } (Si) \text{ is valid} \right\},$$

where by "(Si) is valid", we mean that (Si) holds with A and B for all $u \in H_2^2(M)$. Similarly, we define $B_{\mathrm{opt}}^{(i)}(M)$ as the best constant B in (Si), that is,

$$B_{\mathrm{opt}}^{(i)}(M) = \inf \left\{ B \in \mathbb{R} \text{ s.t. } \exists A \in \mathbb{R} \text{ with the property that } (Si) \text{ is valid} \right\}.$$

Section 1 of this paper is devoted to the study of these best constants. We determine their exact values and investigate the corresponding optimal inequalities. Applications to a fourth-order partial differential equation of critical growth are presented in Section 2, with special attention devoted to the standard sphere and the Paneitz-Branson operator P_h^n given by (0.1). The results we formulate do involve at some point conditions on the parameter $\alpha > 0$ in the definition (0.3) of P_g. It appears that α plays a central role. Results may be expressed equivalently in terms of conditions on α and β for the operator

$$\tilde{P}_g u = \Delta_g^2 u + \alpha \Delta_g u + \beta u$$

(and we do so for the operator P_h^n on the sphere). For the sake of comparison with the classical Sobolev inequalities, and in order to preserve the energy as $\int (P_g u) u \, dv_g$, we deal with definition (0.3) for Paneitz-type operators.

1. Optimal inequalities. As a starting point, let us consider the Euclidean inequality $\forall u \in \mathcal{D}(\mathbb{R}^n)$, $n \geq 5$,

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} \, dx \right)^{2/2^\sharp} \leq K \int_{\mathbb{R}^n} (\Delta u)^2 \, dx,$$

where Δ stands for the Laplacian with respect to the Euclidean metric. The best constant K in this inequality was studied by Edmunds, Fortunato, and Janelli [10], Lieb [18], and Lions [19]. If K_0 stands for this best constant, it was shown that

$$K_0^{-1} = \pi^2 n(n-4)(n^2-4) \Gamma \left(\frac{n}{2} \right)^{4/n} \Gamma(n)^{-4/n}.$$

The sharp Euclidean Sobolev inequality is then

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} \, dx \right)^{2/2^\sharp} \leq K_0 \int_{\mathbb{R}^n} (\Delta u)^2 \, dx. \tag{1.1}$$

Its extremal functions are

$$u_{\epsilon, x_0}(x) = \left(|x - x_0|^2 + \epsilon^2 \right)^{-(n-4)/2},$$

and we have that

$$\Delta^2 u_{1, x_0} = n(n-4)(n^2-4) u_{1, x_0}^{2^\sharp - 1}.$$

Given $\alpha > 0$ real, in what follows, we let P be as in (0.3), with $g = \delta$ being the Euclidean metric. In other words,

$$Pu = \Delta^2 u + \alpha \Delta u.$$

The first result we prove is the following.

LEMMA 1.1. *For all $u \in \mathcal{D}(\mathbb{R}^n)$, $n \geq 5$,*

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} \, dx \right)^{2/2^\sharp} \leq K_0 \int_{\mathbb{R}^n} (Pu) u \, dx, \tag{1.2}$$

and K_0 is the best constant in this inequality.

Proof. Suppose that $A > 0$ is such that for all $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp}\,dx\right)^{2/2^\sharp} \le A \int_{\mathbb{R}^n} (Pu)u\,dx. \tag{1.3}$$

Let $\mathcal{B}$ be the unit ball in $\mathbb{R}^n$. Then for all $u \in \mathcal{D}(\mathcal{B})$,

$$\left(\int_{\mathcal{B}} |u|^{2^\sharp}\,dx\right)^{2/2^\sharp} \le A \int_{\mathcal{B}} (Pu)u\,dx.$$

Also let

$$H^2_{0,2}(\mathcal{B}) = \text{completion of } \mathcal{D}(\mathcal{B}) \text{ w.r.t. } \|\cdot\|_{H^2_2},$$
$$H^2_{0,1}(\mathcal{B}) = \text{completion of } \mathcal{D}(\mathcal{B}) \text{ w.r.t. } \|\cdot\|_{H^2_1},$$

where, as above,

$$\|u\|^2_{H^2_2} = \int_{\mathcal{B}} (Pu)u\,dx + \int_{\mathcal{B}} u^2\,dx.$$

The embedding of $H^2_{0,2}(\mathcal{B})$ in $H^2_{0,1}(\mathcal{B})$ is compact, while the embedding of $H^2_{0,1}(\mathcal{B})$ in $L^2(\mathcal{B})$ is continuous. It easily follows that for any $\epsilon > 0$, there exists $B_\epsilon > 0$ such that for all $u \in \mathcal{D}(\mathcal{B})$,

$$\|u\|^2_{H^2_1} \le \epsilon\|u\|^2_{H^2_2} + B_\epsilon\|u\|^2_2.$$

In particular,

$$\|\nabla u\|^2_2 \le \frac{\epsilon}{1-\alpha\epsilon}\|\Delta u\|^2_2 + \frac{B_\epsilon - 1 + \epsilon}{1-\alpha\epsilon}\|u\|^2_2,$$

and we get that for any $\epsilon > 0$, there exists $B_\epsilon > 0$ such that for all $u \in \mathcal{D}(\mathcal{B})$,

$$\left(\int_{\mathcal{B}} |u|^{2^\sharp}\,dx\right)^{2/2^\sharp} \le (A+\epsilon)\int_{\mathcal{B}} (\Delta u)^2\,dx + B_\epsilon \int_{\mathcal{B}} u^2\,dx.$$

Let $\mathcal{B}_\delta$ be the ball of center 0 and radius $\delta > 0$. For any $\delta < 1$, any $\epsilon > 0$, and all $u \in \mathcal{D}(\mathcal{B}_\delta)$,

$$\left(\int_{\mathcal{B}_\delta} |u|^{2^\sharp}\,dx\right)^{2/2^\sharp} \le (A+\epsilon)\int_{\mathcal{B}_\delta} (\Delta u)^2\,dx + B_\epsilon \int_{\mathcal{B}_\delta} u^2\,dx.$$

By Hölder's inequality,

$$\int_{\mathcal{B}_\delta} u^2\,dx \le |\mathcal{B}_\delta|^{1-2/2^\sharp}\left(\int_{\mathcal{B}_\delta} |u|^{2^\sharp}\,dx\right)^{2/2^\sharp},$$

where $|\mathcal{B}_\delta|$ stands for the Euclidean volume of $\mathcal{B}_\delta$. Choosing δ small, we are led to the following: for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $u \in \mathcal{D}(\mathcal{B}_\delta)$,

$$\left(\int_{\mathcal{B}_\delta} |u|^{2^\sharp}\,dx\right)^{2/2^\sharp} \le (A+2\epsilon)\int_{\mathcal{B}_\delta} (\Delta u)^2\,dx.$$

Given $u \in \mathcal{D}(\mathbb{R}^n)$ and $\lambda > 0$, let $u_\lambda(x) = u(\lambda x)$. For λ large, $u_\lambda \in \mathcal{D}(\mathcal{B}_\delta)$. In addition,

$$\left(\int_{\mathcal{B}_\delta} |u_\lambda|^{2^\sharp}\,dx\right)^{2/2^\sharp} = \lambda^{4-n}\left(\int_{\mathcal{B}_\delta} |u|^{2^\sharp}\,dx\right)^{2/2^\sharp},$$

while

$$\int_{\mathcal{B}_\delta} (\Delta u_\lambda)^2\,dx = \lambda^{4-n}\int_{\mathcal{B}_\delta} (\Delta u)^2\,dx.$$

As a consequence, for all $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp}\,dx\right)^{2/2^\sharp} \le (A+2\epsilon)\int_{\mathbb{R}^n} (\Delta u)^2\,dx,$$

so that if A is as in (1.3), then $A + 2\epsilon \ge K_0$. Since ϵ is arbitrary, $A \ge K_0$. Independently, we clearly get from (1.1) that for all $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp}\,dx\right)^{2/2^\sharp} \le K_0\int_{\mathbb{R}^n} (Pu)u\,dx,$$

so that $A = K_0$ in (1.3) is an admissible value. Lemma 1.1 is established. □

The following lemma easily follows from Lemma 1.1.

LEMMA 1.2. *Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold, $n \ge 5$. Any constant A in $(S1)$ or $(S2)$, whatever the constant B, has to be greater than or equal to K_0.*

Proof. Given $x_0 \in M$, we consider a geodesic, normal coordinates system at x_0. We let $B_{x_0}(\delta)$ be the ball on which this coordinates system is defined. Given $\epsilon > 0$, choosing δ sufficiently small, and by local comparison of the Riemannian metric with the Euclidean metric in the above coordinates system, we easily get from $(S1)$ and $(S2)$ that for all $u \in \mathcal{D}(\mathcal{B}_\delta)$,

$$\|u\|^2_{2^\sharp} \le (A+\epsilon)\int_{\mathbb{R}^n} (Pu)u\,dx + \tilde{B}\|u\|^2_2, \tag{1.4}$$

$$\|u\|^2_{2^\sharp} \le (A+\epsilon)\int_{\mathbb{R}^n} (Pu)u\,dx + \tilde{B}\|u\|^2_{H_1^2}, \tag{1.5}$$

where the norms and P in the above expressions are understood with respect to the Euclidean metric. Passing in the coordinates system, we indeed have that

$$(\Delta_g u)^2 \le (\Delta u)^2 + \hat{\epsilon}(\delta)|\nabla^2 u|^2 + \hat{\epsilon}(\delta)|\nabla u|^2,$$

where $\hat{\epsilon}(\delta) \to 0$ as $\delta \to 0$, while by the Bochner-Lichnerowicz-Weitzenböck formula,

$$\int_{\mathbb{R}^n} |\nabla^2 u|^2 \, dx = \int_{\mathbb{R}^n} (\Delta u)^2 \, dx.$$

As in the proof of Lemma 1.1, it is enough to deal with (1.4). Applying Hölder's inequality to the L^2-norm of u, with (1.4) as in the proof of Lemma 1.1, we get that for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $u \in \mathcal{D}(\mathcal{B}_\delta)$,

$$\|u\|_{2^\sharp}^2 \le (A + 2\epsilon) \int_{\mathbb{R}^n} (Pu) u \, dx.$$

For homogeneity reasons, see once more the proof of Lemma 1.1; such an inequality, if valid for functions with small support, has to be valid for all functions with compact support. By Lemma 1.1, this implies that $A + 2\epsilon \ge K_0$. Since ϵ is arbitrary, this proves the result. □

The answer to the first question we asked, dealing with the exact value of $A_{\mathrm{opt}}^{(i)}(M)$, now follows from Lemma 1.2.

THEOREM 1.1. *Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold, $n \ge 5$, and let P_g be as in (0.3), that is,*

$$P_g u = \Delta_g^2 u + \alpha \Delta_g u.$$

For any $\epsilon > 0$, there exists $B_\epsilon \in \mathbb{R}$ such that for all $u \in H_2^2(M)$,

$$\|u\|_{2^\sharp}^2 \le (1 + \epsilon) K_0 \int_M (P_g u) u \, dv_g + B_\epsilon \|u\|_2^2, \tag{1.6}$$

$$\|u\|_{2^\sharp}^2 \le (1 + \epsilon) K_0 \int_M (P_g u) u \, dv_g + B_\epsilon \|u\|_{H_1^2}^2. \tag{1.7}$$

In particular, $A_{\mathrm{opt}}^{(1)}(M) = A_{\mathrm{opt}}^{(2)}(M) = K_0$. Moreover, we can choose B_ϵ depending only on n, ϵ, α, a bound for the Ricci curvature Rc_g of g, and a positive lower bound for the injectivity radius i_g of g.

Proof. It suffices to prove the result for (1.6), since

$$\|u\|_2^2 \le \|u\|_{H_1^2}^2.$$

Let $\Lambda > 0$ be such that $|Rc_g| \le \Lambda$, and let $i > 0$ be such that $i_g \ge i$. By a well-known result of Anderson [1], the $C^{1,1/2}$-harmonic radius of g is bounded from below by a positive constant depending mainly on n, Λ, and i. More precisely, given $\delta > 0$, there exists $r_H = r_H(n, \delta, \Lambda, i)$, depending only on these constants, such that the following holds: for any $x \in M$, there exists a harmonic chart φ on the ball $B_x(r_H)$ with the property that

$$\frac{1}{1+\delta}\delta_{ij} \le g_{ij} \le (1+\delta)\delta_{ij} \text{ as bilinear forms,}$$
$$\sum_{k=1}^{n} r_H \sup_x \left|\partial_k g_{ij}(x)\right| + \sum_{k=1}^{n} r_H^{3/2} \sup_{y\ne z} \frac{\left|\partial_k g_{ij}(y) - \partial_k g_{ij}(z)\right|}{d_g(y,z)^{1/2}} \le \delta,$$

where d_g is the distance with respect to g, and the g_{ij}'s stand for the components of g in the harmonic coordinates system. Without loss of generality, we may assume that $r_H \le 1/2$. Let $\epsilon > 0$ be given. The expression of the Laplacian in harmonic coordinates is

$$\Delta_g u = -g^{ij}\partial_{ij}u.$$

As we easily check, if $\delta = \delta(\epsilon)$ is small enough, for all $x \in M$, and all $u \in \mathcal{D}(B_x(r_H))$, then

$$\int_{\mathbb{R}^n} \big(P(u\circ\varphi^{-1})\big)(u\circ\varphi^{-1})\,dx \le (1+\epsilon)\int_M (P_g u)u\,dv_g.$$

Similarly, for all $x \in M$, and all $u \in \mathcal{D}(B_x(r_H))$,

$$\int_M |u|^{2^\sharp}\,dv_g \le (1+\epsilon)\int_{\mathbb{R}^n} \left|(u\circ\varphi^{-1})\right|^{2^\sharp} dx.$$

It follows from these relations and Lemma 1.1 that for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $u \in \mathcal{D}(B_x(r_H))$,

$$\left(\int_M |u|^{2^\sharp}\,dv_g\right)^{2/2^\sharp} \le \left(1+\frac{\epsilon}{2}\right)K_0\int_M (P_g u)u\,dv_g. \tag{S_{loc}}$$

Since M is compact, it can be covered by a finite number of balls $B_{x_i}(r_H/2)$, $i = 1,\dots,N$. As is by now classical, we may choose these balls such that any point in M has a neighborhood that intersects at most S_0 of the $B_{x_i}(r_H)$'s, where the integer $S_0 = S_0(n,\epsilon,\Lambda,i)$ depends only on n,ϵ,Λ, and i. Let $\alpha_i \in \mathcal{D}(B_{x_i}(r_H))$ be such that $0 \le \alpha_i \le 1$ and $\alpha_i = 1$ in $B_{x_i}(r_H/2)$. We set

$$\eta_i = \frac{\alpha_i^2}{\sum_k \alpha_k^2}.$$

Clearly, (η_i) is a partition of unity subordinated to the covering $(B_{x_i}(r_H))$ such that the $\sqrt{\eta_i}$'s are smooth. As we easily check, we may assume that

$$\begin{cases} \left|\nabla\sqrt{\eta_i}\right| \le H_1, \\ \left|\Delta_g\sqrt{\eta_i}\right| \le H_2, \\ \left|\nabla\Delta_g\sqrt{\eta_i}\right| \le H_3, \end{cases}$$

where H_1, H_2, and H_3 only depend on n, ϵ, Λ, and i. Given $u \in C^\infty(M)$, we write that

$$\|u\|_{2^\sharp}^2 = \|u^2\|_{2^\sharp/2} \le \sum_i \left\|\eta_i u^2\right\|_{2^\sharp/2} = \sum_i \left\|\sqrt{\eta_i} u\right\|_{2^\sharp}^2.$$

With (S_{loc}), it follows that for all $u \in C^\infty(M)$,

$$\left(\int_M |u|^{2^\sharp} dv_g\right)^{2/2^\sharp} \le \left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \left(\Delta_g\left(\sqrt{\eta_i}u\right)\right)^2 dv_g + \alpha\left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \left|\nabla\left(\sqrt{\eta_i}u\right)\right|^2 dv_g.$$

Hence,

$$\begin{aligned}
\left(\int_M |u|^{2^\sharp} dv_g\right)^{2/2^\sharp} \le & \left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \eta_i (\Delta_g u)^2 dv_g \\
&+ \left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \left(\Delta_g \sqrt{\eta_i}\right)^2 u^2 dv_g \\
&+ 4\left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \left\langle\nabla\sqrt{\eta_i}, \nabla u\right\rangle_g^2 dv_g \\
&+ 2\left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \sqrt{\eta_i} u (\Delta_g u)\left(\Delta_g\sqrt{\eta_i}\right) dv_g \\
&- 4\left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \sqrt{\eta_i}(\Delta_g u)\left\langle\nabla\sqrt{\eta_i}, \nabla u\right\rangle_g dv_g \qquad (1.8)\\
&- 4\left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \left(\Delta_g\sqrt{\eta_i}\right) u \left\langle\nabla\sqrt{\eta_i}, \nabla u\right\rangle_g dv_g \\
&+ \alpha\left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \eta_i |\nabla u|^2 dv_g \\
&+ \alpha\left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M \left|\nabla\sqrt{\eta_i}\right|^2 u^2 dv_g \\
&+ 2\alpha\left(1+\frac{\epsilon}{2}\right) K_0 \sum_i \int_M u\sqrt{\eta_i}\left\langle\nabla\sqrt{\eta_i}, \nabla u\right\rangle_g dv_g.
\end{aligned}$$

Since $r_H \le 1/2$, and if δ is sufficiently small, we get that for every $x \in M$, $\varphi(B_x(r_H)) \subset \mathcal{B}$, where $\mathcal{B}$ stands for the unit ball in $\mathbb{R}^n$ with center 0. As in the proof of Lemma 1.1, given $\epsilon_0 > 0$, there exists C_1, depending only on ϵ_0, such that for all $u \in \mathcal{D}(\mathcal{B})$,

$$\int_{\mathcal{B}} |\nabla u|^2 dx \le \frac{\epsilon_0}{2} \int_{\mathcal{B}} (\Delta u)^2 dx + C_1 \int_{\mathcal{B}} u^2 dx.$$

As before, this implies that for all $x \in M$, and all $u \in \mathcal{D}(B_x(r_H))$,

$$\int_M |\nabla u|^2\, dv_g \le \epsilon_0 \int_M (\Delta_g u)^2\, dv_g + C_2 \int_M u^2\, dv_g,$$

where C_2 depends only on ϵ_0. Let us now deal with the various terms in (1.8). As a starting point,

$$\sum_i \int_M \eta_i (\Delta_g u)^2\, dv_g = \int_M (\Delta_g u)^2\, dv_g$$

and

$$\sum_i \int_M \left(\Delta_g \sqrt{\eta_i}\right)^2 u^2\, dv_g \le S_0 H_2^2 \int_M u^2\, dv_g.$$

To ease the notations, set $\mathcal{B}_i = B_{x_i}(r_H)$. Then, as is easily checked,

$$\begin{aligned}\sum_i \int_M \left\langle \nabla\sqrt{\eta_i}, \nabla u\right\rangle_g^2 dv_g &\le \sum_i \int_M \left|\nabla\sqrt{\eta_i}\right|^2 |\nabla u|^2\, dv_g \\ &\le H_1^2 \sum_i \int_{\mathcal{B}_i} |\nabla u|^2\, dv_g \\ &\le H_1^2 \epsilon_0 \sum_i \int_{\mathcal{B}_i} (\Delta_g u)^2\, dv_g + H_1^2 C_2 \int_{\mathcal{B}_i} u^2\, dv_g \\ &\le S_0 H_1^2 \epsilon_0 \int_M (\Delta_g u)^2\, dv_g + S_0 H_1^2 C_2 \int_M u^2\, dv_g.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\sum_i \int_M \sqrt{\eta_i}\, u (\Delta_g u)\left(\Delta_g \sqrt{\eta_i}\right) dv_g\right| = \Bigg| &\sum_i \int_M \left\langle \nabla\sqrt{\eta_i}, \nabla u\right\rangle_g u \left(\Delta_g\sqrt{\eta_i}\right) dv_g \\ &+ \sum_i \int_M \sqrt{\eta_i}\, |\nabla u|^2 \left(\Delta_g\sqrt{\eta_i}\right)^2 dv_g \\ &+ \sum_i \int_M \sqrt{\eta_i}\, u \left\langle \nabla u, \nabla\left(\Delta_g\sqrt{\eta_i}\right)\right\rangle_g dv_g \Bigg|.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\sum_i \int_M \sqrt{\eta_i}\, u(\Delta_g u)\left(\Delta_g\sqrt{\eta_i}\right) dv_g\right| \\ &\quad\le (H_1 H_2 + H_3) \sum_i \int_{\mathcal{B}_i} |u||\nabla u|\, dv_g + H_2^2 \sum_i \int_{\mathcal{B}_i} |\nabla u|^2\, dv_g \\ &\quad\le \frac{1}{2}(H_1 H_2 + H_3) \sum_i \int_{\mathcal{B}_i} u^2\, dv_g + \left(\frac{1}{2}(H_1 H_2 + H_3) + H_2^2\right) \sum_i \int_{\mathcal{B}_i} |\nabla u|^2\, dv_g.\end{aligned}$$

As above, this leads to

$$\begin{aligned}\left|\sum_i \int_M \sqrt{\eta_i}\, u(\Delta_g u)\big(\Delta_g \sqrt{\eta_i}\big)\, dv_g\right| &\\ \le \left(\frac{S_0}{2}(H_1H_2+H_3)+S_0H_2^2\right)\epsilon_0 \int_M (\Delta_g u)^2\, dv_g &\\ +\left(\frac{S_0}{2}(H_1H_2+H_3)+\left(\frac{S_0}{2}(H_1H_2+H_3)+S_0H_2^2\right)C_2\right)\int_M u^2\, dv_g.&\end{aligned}$$

On the other hand,

$$\sum_i \int_M \sqrt{\eta_i}(\Delta_g u)\big\langle \nabla\sqrt{\eta_i}, \nabla u\big\rangle_g\, dv_g = \frac{1}{2}\sum_i \int_M (\Delta_g u)\langle \nabla \eta_i, \nabla u\rangle_g\, dv_g = 0,$$

while

$$\begin{aligned}\left|\sum_i \int_M \big(\Delta_g \sqrt{\eta_i}\big)u\big\langle \nabla\sqrt{\eta_i}, \nabla u\big\rangle_g\, dv_g\right| &\\ \le H_1H_2 \sum_i \int_{\mathcal{B}_i} |u||\nabla u|\, dv_g &\\ \le \frac{H_1H_2}{2}\sum_i \int_{\mathcal{B}_i} u^2\, dv_g + \frac{H_1H_2}{2}\sum_i \int_{\mathcal{B}_i} |\nabla u|^2\, dv_g.&\end{aligned}$$

Here again, it follows that

$$\begin{aligned}\left|\sum_i \int_M \big(\Delta_g \sqrt{\eta_i}\big)u\big\langle \nabla\sqrt{\eta_i}, \nabla u\big\rangle_g dv_g\right| &\\ \le \frac{H_1H_2}{2}S_0\epsilon_0 \int_M (\Delta_g u)^2\, dv_g + \frac{H_1H_2}{2}S_0(1+C_2)\int_M u^2\, dv_g.&\end{aligned}$$

Moreover,

$$\sum_i \int_M \eta_i |\nabla u|^2\, dv_g \le \sum_i \int_{\mathcal{B}_i} |\nabla u|^2\, dv_g \le S_0\epsilon_0 \int_M (\Delta_g u)^2\, dv_g + S_0C_2 \int_M u^2\, dv_g,$$

while

$$\sum_i \int_M \big|\nabla\sqrt{\eta_i}\big|^2 u^2\, dv_g \le S_0H_2^2 \int_M u^2\, dv_g$$

and

$$\left|\sum_i \int_M u\sqrt{\eta_i}\big\langle \nabla\sqrt{\eta_i}, \nabla u\big\rangle_g \,dv_g\right|$$
$$\leq H_2 \sum_i \int_{\mathcal{B}_i} |u||\nabla u|\,dv_g \leq \frac{H_2}{2}\sum_i \int_{\mathcal{B}_i} u^2\,dv_g + \frac{H_2}{2}\sum_i \int_{\mathcal{B}_i} |\nabla u|^2\,dv_g$$
$$\leq \frac{H_2}{2} S_0\epsilon_0 \int_M (\Delta_g u)^2\,dv_g + \frac{H_2}{2} S_0(1+C_2)\int_M u^2\,dv_g.$$

Summarizing our various estimates, we get that for all $u \in C^\infty(M)$,

$$\left(\int_M |u|^{2^\sharp}\,dv_g\right)^{2/2^\sharp} \leq \left(\left(1+\frac{\epsilon}{2}\right)K_0 + C_3\epsilon_0\right)\int_M (\Delta_g u)^2\,dv_g + C_4\int_M u^2\,dv_g,$$

where the constant C_3, explicitly known, depends only on $n, \epsilon, \alpha, \Lambda$, and i, and where the constant C_4, explicitly known, depends only on $n, \epsilon, \alpha, \epsilon_0, \Lambda$, and i. Choosing ϵ_0 such that

$$C_3\epsilon_0 = \frac{\epsilon}{2}K_0$$

proves the theorem. □

By Theorem 1.1, $A^{(1)}_{\text{opt}}(M) = A^{(2)}_{\text{opt}}(M) = K_0$. It is natural to ask whether these constants are attained in $(S1)$ and $(S2)$, a question to which we now turn. We do believe that the answer should be positive for what concerns $(S2)$, and a partial result in that direction is given by Theorem 1.2. The answer for $(S1)$ might be more delicate as shown by Proposition 1.2. As a starting point, we state the following elementary lemma that is used in the proof of Theorem 1.2.

LEMMA 1.3. *Given M a smooth, compact, n-dimensional manifold, $n \geq 5$, and g_1, $g_2 = \varphi^{4/(n-4)}g_1$ two conformal metrics on M,*

$$\int_M \big(P_{g_2}u\big)u\,dv_{g_2} = \int_M \big(P_{g_1}(u\varphi)\big)(u\varphi)\,dv_{g_1} + \mathcal{R}$$

for all $u \in C^\infty(M)$, where P_{g_1} and P_{g_2} are as in (0.3) with $g = g_1, g_2$, and where $\mathcal{R}$, explicitly known, is made of lower-order terms in the sense that for all $i = 1, 2$, and all $u \in C^\infty(M)$,

$$|\mathcal{R}| \leq A_i \int_M |\nabla u|^2\,dv_{g_i} + B_i \int_M u^2\,dv_{g_i},$$

the constants A_i and B_i being independent of u.

Proof. The proof is simple. We may see the result as an easy consequence of the relation

$$P^n_{g_2}(u) = \varphi^{-(n+4)/(n-4)}P^n_{g_1}(u\varphi).$$

We may also proceed by direct computation. □

We now prove the following theorem, which yields a partial answer to the question we raised above. As already mentioned, the case where (M, g) is not conformally flat remains open.

THEOREM 1.2. *Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold, $n \geq 5$. If g is conformally flat, then $A_{\mathrm{opt}}^{(2)}(M) = K_0$ is attained in $(S2)$. In other words, there exists $B \in \mathbb{R}$ such that*

$$\|u\|_{2^\sharp}^2 \leq K_0 \int_M (P_g u) u \, dv_g + B \|u\|_{H_1^2}^2$$

for all $u \in H_2^2(M)$.

Proof. Since (M, g) is compact and conformally flat, it can be covered by a finite number of open subsets Ω_i, $i = 1, \ldots, N$, such that

(1) for any i, there exists $\varphi_i \in C^\infty(M)$, $\varphi_i > 0$, with the property that the metric $g_i = \varphi_i^{4/(n-4)} g$ is flat on Ω_i;

(2) for any i, and if g_i is as above, (Ω_i, g_i) is isometric to an open subset of $\mathbb{R}^n$ equipped with the Euclidean metric.

Let (η_i) be a partition of unity subordinated to the covering (Ω_i). Without loss of generality, we may assume that for any i, $\sqrt{\eta_i} \in C^\infty(M)$. Given $u \in C^\infty(M)$, write that

$$\left(\int_M |u|^{2^\sharp} \, dv_g \right)^{2/2^\sharp} \leq \sum_i \left(\int_M \left| \sqrt{\eta_i} u \right|^{2^\sharp} dv_g \right)^{2/2^\sharp}.$$

For any i, let $\psi_i = \varphi_i^{-1}$ so that $g = \psi_i^{4/(n-4)} g_i$. Then for all $u \in C^\infty(M)$,

$$\int_M \left| \sqrt{\eta_i} u \right|^{2^\sharp} dv_g = \int_M \left| \sqrt{\eta_i} u \psi_i \right|^{2^\sharp} dv_{g_i}.$$

It follows from Lemma 1.1 that

$$\left(\int_M \left| \sqrt{\eta_i} u \psi_i \right|^{2^\sharp} dv_{g_i} \right)^{2/2^\sharp} \leq K_0 \int_M \left(P_{g_i} \left(\sqrt{\eta_i} u \psi_i \right) \right) \left(\sqrt{\eta_i} u \psi_i \right) dv_{g_i}.$$

By Lemma 1.3, for any i, and for all $u \in C^\infty(M)$,

$$\begin{aligned} \left(\int_M \left| \sqrt{\eta_i} u \right|^{2^\sharp} dv_g \right)^{2/2^\sharp} \leq{}& K_0 \int_M P_g \left(\sqrt{\eta_i} u \right) \left(\sqrt{\eta_i} u \right) dv_g \\ &+ A_i \int_M |\nabla u|^2 \, dv_g + B_i \int_M u^2 \, dv_g. \end{aligned}$$

Hence, for all $u \in C^\infty(M)$,

$$\left(\int_M |u|^{2^\sharp} dv_g\right)^{2/2^\sharp} \le K_0 \sum_i \int_M P_g\left(\sqrt{\eta_i} u\right)\left(\sqrt{\eta_i} u\right) dv_g + \left(\sum_i A_i\right) \int_M |\nabla u|^2 dv_g + \left(\sum_i B_i\right) \int_M u^2 dv_g.$$

Now, as in the proof of Theorem 1.1,

$$\begin{aligned}
&\sum_i \int_M P_g\left(\sqrt{\eta_i} u\right)\left(\sqrt{\eta_i} u\right) dv_g \\
&\quad \le \sum_i \int_M \left(\Delta_g\left(\sqrt{\eta_i} u\right)\right)^2 dv_g + \alpha \sum_i \int_M \left|\nabla\left(\sqrt{\eta_i} u\right)\right|^2 dv_g \\
&\quad \le \sum_i \int_M \eta_i (\Delta_g u)^2 dv_g + \sum_i \int_M \left(\Delta_g \sqrt{\eta_i}\right)^2 u^2 dv_g \\
&\qquad + 4 \sum_i \int_M \left\langle \nabla \sqrt{\eta_i}, \nabla u \right\rangle_g^2 dv_g + 2 \sum_i \int_M \sqrt{\eta_i} u (\Delta_g u)\left(\Delta_g \sqrt{\eta_i}\right) dv_g \\
&\qquad - 4 \sum_i \int_M \sqrt{\eta_i} (\Delta_g u) \left\langle \nabla \sqrt{\eta_i}, \nabla u \right\rangle_g dv_g \\
&\qquad - 4 \sum_i \int_M \left(\Delta_g \sqrt{\eta_i}\right) u \left\langle \nabla \sqrt{\eta_i}, \nabla u \right\rangle_g dv_g \\
&\qquad + \alpha \sum_i \int_M \eta_i |\nabla u|^2 dv_g + \alpha \sum_i \int_M \left|\nabla \sqrt{\eta_i}\right|^2 u^2 dv_g \\
&\qquad + 2\alpha \sum_i \int_M u \sqrt{\eta_i} \left\langle \nabla \sqrt{\eta_i}, \nabla u \right\rangle_g dv_g.
\end{aligned}$$

We again analyze the various terms in this expression. First,

$$\sum_i \int_M \eta_i (\Delta_g u)^2 dv_g = \int_M (\Delta_g u)^2 dv_g,$$

while

$$\sum_i \int_M \left(\Delta_g \sqrt{\eta_i}\right)^2 u^2 dv_g \le N \left(\max_i \max_M \left|\Delta_g \sqrt{\eta_i}\right|\right)^2 \int_M u^2 dv_g.$$

Similarly,

$$\sum_i \int_M \left\langle \nabla \sqrt{\eta_i}, \nabla u \right\rangle_g^2 dv_g \le N \left(\max_i \max_M \left|\nabla \sqrt{\eta_i}\right|\right)^2 \int_M |\nabla u|^2 dv_g.$$

On the other hand,

$$\begin{aligned}\sum_i \int_M \sqrt{\eta_i}u(\Delta_g u)\big(\Delta_g\sqrt{\eta_i}\big)\,dv_g &= \sum_i \int_M \big\langle \nabla\sqrt{\eta_i}, \nabla u\big\rangle_g u\big(\Delta_g\sqrt{\eta_i}\big)\,dv_g \\ &\quad + \sum_i \int_M \sqrt{\eta_i}|\nabla u|^2\big(\Delta_g\sqrt{\eta_i}\big)^2\,dv_g \\ &\quad + \sum_i \int_M \sqrt{\eta_i}u\big\langle \nabla u, \nabla\big(\Delta_g\sqrt{\eta_i}\big)\big\rangle_g\,dv_g,\end{aligned}$$

so that

$$\begin{aligned}&\left|\sum_i \int_M \sqrt{\eta_i}u(\Delta_g u)\big(\Delta_g\sqrt{\eta_i}\big)\,dv_g\right| \\ &\quad\le \frac{N}{2}\Big(\max_i\max_M\big|\nabla\sqrt{\eta_i}\big|\Big)\Big(\max_i\max_M\big|\Delta_g\sqrt{\eta_i}\big|\Big)\int_M u^2\,dv_g \\ &\qquad + \frac{N}{2}\Big(\max_i\max_M\big|\nabla\sqrt{\eta_i}\big|\Big)\Big(\max_i\max_M\big|\Delta_g\sqrt{\eta_i}\big|\Big)\int_M |\nabla u|^2\,dv_g \\ &\qquad + N\Big(\max_i\max_M\big|\Delta_g\sqrt{\eta_i}\big|\Big)^2\int_M |\nabla u|^2\,dv_g \\ &\qquad + \frac{N}{2}\Big(\max_i\max_M\big|\nabla\Delta_g\sqrt{\eta_i}\big|\Big)\int_M u^2\,dv_g \\ &\qquad + \frac{N}{2}\Big(\max_i\max_M\big|\nabla\Delta_g\sqrt{\eta_i}\big|\Big)\int_M |\nabla u|^2\,dv_g.\end{aligned}$$

Furthermore,

$$\sum_i \int_M \sqrt{\eta_i}(\Delta_g u)\big\langle \nabla\sqrt{\eta_i}, \nabla u\big\rangle_g\,dv_g = \frac{1}{2}\sum_i \int_M (\Delta_g u)\langle \nabla\eta_i, \nabla u\rangle_g\,dv_g = 0,$$

while

$$\begin{aligned}&\left|\sum_i \int_M \big(\Delta_g\sqrt{\eta_i}\big)u\big\langle \nabla\sqrt{\eta_i}, \nabla u\big\rangle_g\,dv_g\right| \\ &\quad\le \frac{N}{2}\Big(\max_i\max_M\big|\Delta_g\sqrt{\eta_i}\big|\Big)\Big(\max_i\max_M\big|\nabla\sqrt{\eta_i}\big|\Big)\int_M u^2\,dv_g \\ &\qquad + \frac{N}{2}\Big(\max_i\max_M\big|\Delta_g\sqrt{\eta_i}\big|\Big)\Big(\max_i\max_M\big|\nabla\sqrt{\eta_i}\big|\Big)\int_M |\nabla u|^2\,dv_g.\end{aligned}$$

For the next terms, note that

$$\sum_i \int_M \eta_i|\nabla u|^2\,dv_g = \int_M |\nabla u|^2\,dv_g$$

and

$$\sum_i \int_M \left|\nabla\sqrt{\eta_i}\right|^2 u^2\,dv_g \le N\Big(\max_i \max_M \left|\nabla\sqrt{\eta_i}\right|\Big)^2 \int_M u^2\,dv_g.$$

Finally,

$$\left|\sum_i \int_M u\sqrt{\eta_i}\langle\nabla\sqrt{\eta_i},\nabla u\rangle_g\,dv_g\right| \le \frac{N}{2}\Big(\max_i \max_M \left|\nabla\sqrt{\eta_i}\right|\Big)\int_M u^2\,dv_g + \frac{N}{2}\Big(\max_i \max_M \left|\nabla\sqrt{\eta_i}\right|\Big)\int_M |\nabla u|^2\,dv_g.$$

As a conclusion, for all $u \in C^\infty(M)$,

$$\left(\int_M |u|^{2^\sharp}\,dv_g\right)^{2/2^\sharp} \le K_0 \int_M (\Delta_g u)^2\,dv_g + A\int_M |\nabla u|^2\,dv_g + B\int_M u^2\,dv_g,$$

where A and B are constants that do not depend on u. In particular, and for all $u \in C^\infty(M)$,

$$\|u\|_{2^\sharp}^2 \le K_0 \int_M (P_g u)u\,dv_g + C\|u\|_{H_1^2}^2,$$

where C does not depend on u. Theorem 1.2 is thus proved. □

When dealing with conformally flat manifolds, Theorem 1.2 indicates that $A_{\mathrm{opt}}^{(2)}(M) = K_0$ is attained in $(S2)$. The situation for $(S1)$ is more complicated. In particular, α in the definition (0.3) of P_g has to play a role. This is what we prove in Proposition 1.2 below. As a first result, we establish the following.

PROPOSITION 1.1. *Let (S^n, h) be the standard unit sphere of $\mathbb{R}^{n+1}$, $n \ge 5$, and let P_h^n be the Paneitz-Branson operator given by (0.1), that is,*

$$P_h^n u = \Delta_h^2 u + c_n \Delta_h u + d_n u.$$

Then

$$\inf_{u \in C^\infty(S^n)\setminus\{0\}} \frac{\int_{S^n}\left(P_h^n u\right)u\,dv_h}{\left(\int_{S^n}|u|^{2^\sharp}\,dv_h\right)^{2/2^\sharp}} = \frac{1}{K_0} \tag{1.9}$$

with the additional property that for any $\beta > 1$, and any $x_0 \in S^n$, if

$$u_\beta = \frac{(\beta^2-1)^{(n-4)/4}}{(\beta - \cos r)^{(n-4)/2}},$$

where r is the distance on S^n to x_0, then

$$P_h^n u_\beta = d_n u_\beta^{2^\sharp - 1},$$

and u_β realizes the infimum in the left-hand side of (1.9).

Proof. We first prove (1.9). For that purpose, let x_0 be some point on S^n, and let $\Phi : S^n \backslash \{x_0\} \to \mathbb{R}^n$ be the stereographic projection of pole x_0. If δ stands for the Euclidean metric of $\mathbb{R}^n$, then

$$(\Phi^{-1})^\star h = \varphi^{4/(n-4)}\delta,$$

where

$$\varphi(x) = \left(\frac{4}{(1+|x|^2)^2}\right)^{(n-4)/4}.$$

By conformal invariance of the Paneitz-Branson operator P_g^n, we get that for all $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\frac{\int_{\mathbb{R}^n}\left((\Delta_{\tilde{h}}u)^2 + c_n|\nabla u|^2 + d_n u^2\right) dv_{\tilde{h}}}{\left(\int_{\mathbb{R}^n}|u|^{2^\sharp}\, dv_{\tilde{h}}\right)^{2/2^\sharp}} = \frac{\int_{\mathbb{R}^n}(\Delta(u\varphi))^2\, dx}{\left(\int_{\mathbb{R}^n}|u\varphi|^{2^\sharp}\, dx\right)^{2/2^\sharp}}, \tag{1.10}$$

where $\tilde{h} = (\Phi^{-1})^\star h$. Suppose now that

$$\inf_{u \in C^\infty(S^n)\backslash\{0\}} \frac{\int_{S^n}\left((\Delta_h u)^2 + c_n|\nabla u|^2 + d_n u^2\right) dv_h}{\left(\int_{S^n}|u|^{2^\sharp}\, dv_h\right)^{2/2^\sharp}} < \frac{1}{K_0}, \tag{1.11}$$

and let $u_0 \in C^\infty(S^n)$, $u_0 \not\equiv 0$, be such that

$$\frac{\int_{S^n}\left((\Delta_h u_0)^2 + c_n|\nabla u_0|^2 + d_n u_0^2\right) dv_h}{\left(\int_{S^n}|u_0|^{2^\sharp}\, dv_h\right)^{2/2^\sharp}} < \frac{1}{K_0}.$$

We let (η_s), $s > 0$ small, be a family of smooth functions on S^n having the property that $0 \le \eta_s \le 1$, $\eta_s = 0$ on $B_P(s)$, $\eta_s = 1$ on $S^n \backslash B_P(2s)$, and

$$\begin{cases} |\nabla \eta_s| \le \dfrac{C_1}{s}, \\ |\Delta_h \eta_s| \le \dfrac{C_2}{s^2}, \end{cases}$$

where C_1, C_2 are positive constants that do not depend on s. In order to get such a family, we might fix some η_{s_0} as above, for instance, radially symmetric, and set then, for $s \le s_0$, $\eta_s = \eta_{s_0}(r/s)$. As we easily check,

$$\lim_{s\to 0} \frac{\int_{S^n}\left((\Delta_h u_s)^2 + c_n|\nabla u_s|^2 + d_n u_s^2\right) dv_h}{\left(\int_{S^n}|u_s|^{2^\sharp}\, dv_h\right)^{2/2^\sharp}} = \frac{\int_{S^n}\left((\Delta_h u_0)^2 + c_n|\nabla u_0|^2 + d_n u_0^2\right) dv_h}{\left(\int_{S^n}|u_0|^{2^\sharp}\, dv_h\right)^{2/2^\sharp}},$$

where $u_s = \eta_s u_0$. Just note here that

$$\lim_{s\to 0} \frac{1}{s^2} V_h\left(B_P(2s)\backslash B_P(s)\right) = 0,$$

where $V_h(\Omega)$ stands for the volume of Ω with respect to h. Choosing s sufficiently small, it follows from (1.10) and (1.11) that there exists $\tilde{u}_s \in \mathcal{D}(\mathbb{R}^n)$ of the form

$$\tilde{u}_s = \left(u_s \circ \Phi^{-1}\right)\varphi,$$

such that

$$\frac{\int_{\mathbb{R}^n}\left(\Delta \tilde{u}_s\right)^2 dx}{\left(\int_{\mathbb{R}^n}|\tilde{u}_s|^{2^\sharp}dx\right)^{2/2^\sharp}} < \frac{1}{K_0}.$$

This contradicts (1.1) so that (1.9) follows. Now let u_β be as in the statement of the proposition. As is well known, see, for instance, [15], there exists a conformal diffeomorphism φ_β of (S^n, h) such that

$$\varphi_\beta^\star h = u_\beta^{4/(n-4)} h.$$

By conformal invariance of the Paneitz-Branson operator P_g^n, this implies that

$$P_h^n u_\beta = d_n u_\beta^{2^\sharp - 1}.$$

On the other hand, it is easily seen that

$$d_n K_0 = \omega_n^{-4/n},$$

where ω_n stands for the volume of S^n with respect to h. This follows from the relations

$$4^{n/2}\omega_{n-1} = \frac{2\Gamma(n)}{\omega_{n-1}\Gamma(n/2)^2}$$

and

$$\omega_{n-1}\Gamma\left(\frac{n}{2}\right) = 2\pi^{n/2}.$$

Noting that

$$\int_{S^n} u_\beta^{2^\sharp} dv_h = \omega_n,$$

we get that

$$\int_{S^n}\left(P_h^n u_\beta\right)u_\beta \, dv_h = \frac{1}{K_0}\left(\int_{S^n} u_\beta^{2^\sharp} dv_h\right)^{2/2^\sharp}.$$

This ends the proof of the proposition. □

As an ending result in the study of the best first constant, we now prove the following. Since (S^n, h) is conformally flat, this result has to be compared to Theorem 1.2. Due to the lack of concentration, the approach we use does not allow us to conclude when $n = 5$.

PROPOSITION 1.2. *Let (S^n, h) be the standard unit sphere of $\mathbb{R}^{n+1}$, $n \geq 6$, and let P_h be as in (0.3) with $g = h$, that is,*

$$P_h u = \Delta_h^2 u + \alpha \Delta_h u.$$

There exists $B \in \mathbb{R}$ such that for all $u \in H_2^2(S^n)$,

$$\|u\|_{2^\sharp}^2 \leq K_0 \int_{S^n} (P_h u) u \, dv_h + B \|u\|_2^2$$

if and only if $\alpha \geq c_n$, where c_n is as in (0.2).

Proof. If $\alpha \geq c_n$, the result follows from (1.9), and we may take $B = K_0 d_n$. Suppose, on the contrary, that $\alpha < c_n$. For $\beta > 1$ real, and r the distance on S^n to a given point, we let u_β be as in Proposition 1.1, that is,

$$u_\beta = \frac{(\beta^2 - 1)^{(n-4)/4}}{(\beta - \cos r)^{(n-4)/2}}.$$

Then, for any $\beta > 1$,

$$\frac{\int_{S^n} \left(P_h^n u_\beta\right) u_\beta \, dv_h}{\left(\int_{S^n} u_\beta^{2^\sharp} \, dv_h\right)^{2/2^\sharp}} = \frac{1}{K_0}.$$

Let B be given. Writing that

$$P_h u + B u = P_h^n u + (\alpha - c_n) \Delta_h u + (B - d_n) u,$$

and since

$$\int_{S^n} u_\beta^{2^\sharp} \, dv_h = \omega_n,$$

it follows that for any $\beta > 1$,

$$\begin{aligned} &\frac{\int_{S^n} \left(P_h u_\beta\right) u_\beta \, dv_h + B \|u_\beta\|_2^2}{\|u_\beta\|_{2^\sharp}^2} \\ &\quad = \frac{1}{K_0} + \frac{1}{\omega_n^{2/2^\sharp}} \left((\alpha - c_n) \int_{S^n} \left|\nabla u_\beta\right|^2 dv_h + (B - d_n) \int_{S^n} u_\beta^2 \, dv_h \right). \end{aligned} \tag{1.12}$$

Performing the changes of variables $x = \tan(r/2)$ and $y = \sqrt{(\beta+1)/(\beta-1)}\,x$, it is easily seen that

$$\int_{S^n} \left|\nabla u_\beta\right|^2 dv_h = C_1(\beta, n)(\beta - 1) \int_0^{+\infty} \frac{y^{n+1} \, dy}{\left(1 + ((\beta-1)/(\beta+1)) y^2\right)^4 (1 + y^2)^{n-2}},$$

and that

$$\int_{S^n} u_\beta^2\,dv_h = C_2(\beta,n)(\beta-1)$$
$$\times\int_0^{+\infty}\frac{y^{n-1}\,dy}{\big(1+((\beta-1)/(\beta+1))y^2\big)^3\big(1/(\beta-1)+(1/(\beta+1))y^2\big)(1+y^2)^{n-4}},$$

where

$$C_1(\beta,n)=\frac{2^n(n-4)^2\omega_{n-1}}{(\beta+1)^3},\qquad C_2(\beta,n)=\frac{2^n\omega_{n-1}}{(\beta+1)^2}.$$

By the dominated convergence theorem, if $n>6$,

$$\lim_{\beta\to1^+}\int_0^{+\infty}\frac{y^{n-1}\,dy}{\big(1+((\beta-1)/(\beta+1))y^2\big)^3\big(1/(\beta-1)+(1/(\beta+1))y^2\big)(1+y^2)^{n-4}}=0,$$

while

$$\lim_{\beta\to1^+}\int_0^{+\infty}\frac{y^{n+1}\,dy}{\big(1+((\beta-1)/(\beta+1))y^2\big)^4(1+y^2)^{n-2}}=\int_0^{+\infty}\frac{y^{n+1}\,dy}{(1+y^2)^{n-2}}.$$

The latter integral is a finite positive constant. It follows that if $\alpha<c_n$ and $n>6$, then for $\beta>1$ sufficiently close to 1,

$$(\alpha-c_n)\int_{S^n}\left|\nabla u_\beta\right|^2dv_h+(B-d_n)\int_{S^n}u_\beta^2\,dv_h<0.$$

We then get with (1.12) that for $\beta>1$ sufficiently close to 1,

$$\int_{S^n}(P_h u_\beta)u_\beta\,dv_h+B\|u_\beta\|_2^2<\frac{1}{K_0}\|u_\beta\|_{2^\sharp}^2.$$

This proves the proposition when $n>6$. If $n=6$, we decompose our integrals into three pieces by writing that $\int_0^{+\infty}=\int_0^1+\int_1^{1/\sqrt{\beta-1}}+\int_{1/\sqrt{\beta-1}}^{+\infty}$. Easy computations then give us that

$$A_1(\beta-1)\ln\left(\frac{1}{\beta-1}\right)\le\int_{S^6}|\nabla u_\beta|^2\,dv_h\le A_2(\beta-1)\ln\left(\frac{1}{\beta-1}\right)$$

for some positive constants $A_1<A_2$ independent of β, while

$$\lim_{\beta\to1^+}\frac{\int_{S^6}u_\beta^2\,dv_h}{(\beta-1)\ln(1/(\beta-1))}=0.$$

As above, this gives that if $\alpha<c_6$, then for $\beta>1$ sufficiently close to 1,

$$(\alpha-c_6)\int_{S^6}\left|\nabla u_\beta\right|^2dv_h+(B-d_6)\int_{S^6}u_\beta^2\,dv_h<0.$$

The proposition is proved. □

Parallel with the study of the best first constant, we can ask similar questions on the best second constant. We state our results on this parallel program without any proofs. Details on these proofs can be found in Djadli-Hebey-Ledoux [9]. As a starting point, it is easily seen that whatever (M, g) smooth and compact of dimension $n \geq 5$ is,

$$B_{\mathrm{opt}}^{(1)}(M) = B_{\mathrm{opt}}^{(2)}(M) = V_g^{-4/n},$$

where V_g is the volume of (M, g). Moreover, we can prove that these constants are attained in the sense that there always exists $A \in \mathbb{R}$ such that for all $u \in H_2^2(M)$,

$$\|u\|_{2^\sharp}^2 \leq A \int_M (P_g u) u \, dv_g + V_g^{-4/n} \|u\|_2^2, \tag{1.13}$$

and such that for all $u \in H_2^2(M)$,

$$\|u\|_{2^\sharp}^2 \leq A \int_M (P_g u) u \, dv_g + V_g^{-4/n} \|u\|_{H_1^2}^2.$$

There A can be chosen such that it depends only on n, α, a lower bound on the Ricci curvature of g, a lower bound on the volume of M with respect to g, and an upper bound on the diameter of M with respect to g. Looking for more precise information on the remaining constant A, an easy statement is that A in (1.13) has to be such that

$$A \geq \frac{2^\sharp - 2}{\lambda_1(\lambda_1 + \alpha)} V_g^{-4/n}, \tag{1.14}$$

where λ_1 is the first nonzero eigenvalue of Δ_g. In the specific case of the standard unit sphere (S^n, h), as proved by Beckner [2], the Sobolev inequality

$$\|u\|_p^2 \leq \frac{p-2}{n} \omega_n^{2/p-1} \|\nabla u\|_2^2 + \omega_n^{2/p-1} \|u\|_2^2$$

holds for all $p \in [2, 2^\star]$. By the variational characterization of the first nonzero eigenvalue λ_1 of Δ_h, and the Bochner-Lichnerowicz-Weitzenböck formula, it follows that for every $u \in H_2^2(S^n)$,

$$\|u\|_p^2 \leq \frac{p-2}{n(n+\alpha)} \omega_n^{2/p-1} \int_{S^n} (P_h u) u \, dv_h + \omega_n^{2/p-1} \|u\|_2^2.$$

It is natural to question whether or not this Beckner's type inequality extends to real numbers p such that $p > 2^\star$. Assuming that this is the case, and, in particular, that the inequality holds for all $p \in [2, 2^\sharp]$, we would get that

$$\|u\|_{2^\sharp}^2 \leq \frac{8}{n(n-4)(n+\alpha)} \omega_n^{-4/n} \int_{S^n} (P_h u) u \, dv_h + \omega_n^{-4/n} \|u\|_2^2. \tag{1.15}$$

Observe that the first constant in this inequality is the constant given by (1.14) when the manifold considered is the standard sphere. Let c_n be as in (0.2). We can prove (see [9] for details) that if $\alpha \le c_n$, then for all $u \in H_2^2(S^n)$,

$$\|u\|_{2^\sharp}^2 \le \frac{8}{n(n-4)(n+\alpha)}\omega_n^{-4/n}\int_{S^n}(P_h u)u\,dv_h + \omega_n^{-4/n}\|u\|_2^2,$$

and the two constants in this inequality cannot be lowered. In a similar way, we can prove that if $\alpha > c_n$, then for all $u \in H_2^2(S^n)$,

$$\|u\|_{2^\sharp}^2 \le \frac{16}{n(n-4)(n^2-4)}\omega_n^{-4/n}\int_{S^n}(P_h u)u\,dv_h + \omega_n^{-4/n}\|u\|_2^2,$$

and again the two constants in this inequality cannot be lowered. In particular, (1.15) is true if $\alpha \le c_n$, but false if $\alpha > c_n$. It follows that Beckner's inequality does not extend to $p = 2^\sharp$. As an ending remark, coming back to an arbitrary, smooth, compact, Riemannian manifold of dimension $n \ge 5$, we mention that it is possible to prove that if g is such that $Rc_g \ge n-1$, then for all $u \in H_2^2(M)$,

$$\|u\|_{2^\sharp}^2 \le A V_g^{-4/n}\int_M (P_g u)u\,dv_g + V_g^{-4/n}\|u\|_2^2, \tag{1.16}$$

where $A = A(n,\alpha)$, explicitly known, depends only on n and α. Let $\hat{A}(n,\alpha)$ be the constant involved in the above inequalities on the sphere

$$\hat{A}(n,\alpha) = \begin{cases} \dfrac{8}{n(n-4)(n+\alpha)} & \text{if } \alpha \le c_n, \\[2ex] \dfrac{16}{n(n-4)(n^2-4)} & \text{if } \alpha > c_n. \end{cases}$$

With respect to what was proved by Ilias [16] when dealing with the Sobolev space H_1^2, an open question we are left with is whether or not (1.16) holds with $A = \hat{A}(n,\alpha)$ when g is such that $Rc_g \ge n-1$.

2. On a fourth-order partial differential equation. Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold, $n \ge 5$, and let α, a be two positive real numbers. Let f be a smooth real-valued function on M. We are here concerned with the fourth-order partial differential equation

$$P_g u + au = f u^{2^\sharp - 1}, \tag{E}$$

where, as in (0.3),

$$P_g u = \Delta_g^2 u + \alpha\Delta_g u.$$

When referring to a solution of (E), we assume that the solution is positive and smooth. Multiplying (E) by u, and integrating over M, a necessary condition for (E) to have a solution is that f is positive somewhere on M. For u in $H_2^2(M)$, we let

$$I_g(u) = \int_M (P_g u) u \, dv_g + a \int_M u^2 \, dv_g.$$

We also let

$$\mathcal{H}_f = \left\{ u \in H_2^2(M) \Big/ \int_M f |u|^{2^\sharp} \, dv_g = 1 \right\}.$$

Our first result here is the following theorem.

THEOREM 2.1. *Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold, $n \geq 5$, let P_g be the operator given by (0.3), let a be some positive real number, and let f be a smooth positive function on M. Then the inequality*

$$\inf_{u \in \mathcal{H}_f} I_g(u) \leq \frac{1}{(\max_M f)^{2/2^\sharp} K_0} \tag{2.1}$$

always holds, with the additional property that if the inequality in (2.1) is strict, and if $a \leq \alpha^2/4$, then the infimum in the left-hand side of (2.1) is attained by a smooth positive function. In particular, if the inequality in (2.1) is strict and $a \leq \alpha^2/4$, then (E) possesses a smooth positive solution.

Proof. We start by proving (2.1). Suppose on the contrary that

$$\inf_{u \in \mathcal{H}_f} I_g(u) > \frac{1}{(\max_M f)^{2/2^\sharp} K_0}.$$

Then there exists $\epsilon > 0$ such that for all $u \in H_2^2(M)$,

$$\frac{1}{(\max_M f)^{2/2^\sharp}} \left(\int_M f |u|^{2^\sharp} \, dv_g \right)^{2/2^\sharp} \leq K_0 (1 - \epsilon) \int_M (P_g u) u \, dv_g + B \|u\|_2^2,$$

where, for instance, $B = a K_0$. If x_0 is a point where f is maximum, for $r > 0$ sufficiently small, and all $x \in B_{x_0}(r)$,

$$f(x) \geq f(x_0) \left(1 - \frac{\epsilon}{2} \right)^{2^\sharp/2}.$$

Let $\hat{\epsilon} = \epsilon/(2 - \epsilon)$ and $\hat{B} = B(1 - (\epsilon/2))^{-1}$. Then for all $u \in \mathcal{D}(B_{x_0}(r))$,

$$\|u\|_{2^\sharp}^2 \leq K_0 (1 - \hat{\epsilon}) \int_M (P_g u) u \, dv_g + \hat{B} \|u\|_2^2.$$

The same arguments as the ones used in the proof of Lemma 1.2 then lead to a contradiction. It follows that (2.1) holds. Let us now prove the second part of the theorem. Let $q \in (2, 2^\sharp)$. Set

$$\mu_q = \inf_{u \in \mathcal{H}_f^q} I_g(u),$$

where

$$\mathcal{H}_f^q = \left\{ u \in H_2^2(M) \Big/ \int_M f|u|^q \, dv_g = 1 \right\}.$$

Since the embedding of $H_2^2(M)$ in $L^q(M)$ is compact, we know from classical variational arguments that μ_q is attained. In other words, there exists $u_q \in \mathcal{H}_f^q$ such that $I_g(u_q) = \mu_q$. In particular, u_q is a weak solution of

$$P_g u_q + a u_q = \mu_q f |u_q|^{q-2} u_q.$$

By classical bootstrap, $u_q \in L^s(M)$ for all s. It easily follows that u_q is in fact C^3. Mimicking what is done in Van der Vorst [23], let $\tilde{u}_q$ be the solution of

$$\Delta_g \tilde{u}_q + \frac{\alpha}{2} \tilde{u}_q = \left| \Delta_g u_q + \frac{\alpha}{2} u_q \right|.$$

Clearly, $\tilde{u}_q$ is C^2, and

$$\Delta_g \left(\tilde{u}_q \pm u_q \right) + \frac{\alpha}{2} \left(\tilde{u}_q \pm u_q \right) \geq 0.$$

It follows from the maximum principle that $\tilde{u}_q \geq |u_q|$, and that $\tilde{u}_q > 0$. Noting that

$$\int_M \left(\Delta_g \tilde{u}_q + \frac{\alpha}{2} \tilde{u}_q \right)^2 dv_g = \int_M \left(\Delta_g u_q + \frac{\alpha}{2} u_q \right)^2 dv_g,$$

it follows from the assumption $a \leq \alpha^2/4$ that

$$\begin{aligned} I_g(\tilde{u}_q) &= \int_M \left(\Delta_g \tilde{u}_q + \frac{\alpha}{2} \tilde{u}_q \right)^2 dv_g + \left(a - \frac{\alpha^2}{4} \right) \int_M \tilde{u}_q^2 \, dv_g \\ &= \mu_q + \left(a - \frac{\alpha^2}{4} \right) \left(\int_M \tilde{u}_q^2 \, dv_g - \int_M u_q^2 \, dv_g \right) \leq \mu_q. \end{aligned}$$

On the other hand,

$$\int_M f \tilde{u}_q^q \, dv_g \geq \int_M f |u_q|^q \, dv_g.$$

Hence,

$$\hat{u}_q = \frac{1}{\left(\int_M f \tilde{u}_q^q \, dv_g \right)^{1/q}} \tilde{u}_q$$

realises μ_q. Here again, $\hat{u}_q$ is a solution of

$$P_g\hat{u}_q + a\hat{u}_q = \mu_q f\hat{u}_q^{q-1}.$$

By classical regularity, $\hat{u}_q$ is in fact C^∞. The family $(\hat{u}_q)$ is obviously bounded in $H_2^2(M)$. Up to the extraction of a subsequence, and for $q \to 2^\sharp$, it converges weakly to some nonnegative u in $H_2^2(M)$. The embedding $H_2^2(M) \subset H_1^2(M)$ being compact, we may also assume that it converges strongly to u in $H_1^2(M)$. It follows from classical arguments that u is a weak solution of

$$P_g u + au = \mu f u^{2^\sharp - 1},$$

where μ is given by

$$\mu = \limsup_{q\to 2^\sharp} \mu_q.$$

By Lemma 2.1 below, $u \in L^s(M)$ for all s. It easily follows that u is C^4. From the maximum principle, and noting that

$$\left(\Delta_g + \frac{\alpha}{2}\right)^2 u \geq 0,$$

we get that u is either positive or the zero function. In both cases, it is actually C^∞. Let

$$\mu_0 = \inf_{u\in\mathcal{H}_f} I_g(u).$$

It is easily seen that $\mu \leq \mu_0$. Coming back to the family $(\hat{u}_q)$, we have that

$$\begin{aligned} 1 &= \left(\int_M f\hat{u}_q^q\,dv_g\right)^{2/q} \\ &\leq \left(\max_M f\right)^{2/q}\left(\int_M \hat{u}_q^q\,dv_g\right)^{2/q} \\ &\leq \left(\max_M f\right)^{2/q} V_g^{2(1/q-1/2^\sharp)}\left(\int_M \hat{u}_q^{2^\sharp}\,dv_g\right)^{2/2^\sharp}, \end{aligned}$$

where V_g stands for the volume of M with respect to g. Under the assumption that inequality (2.1) is strict, there exists $\epsilon > 0$ such that

$$\mu_0(K_0+\epsilon) \leq \frac{1-\epsilon}{(\max_M f)^{2/2^\sharp}}.$$

We fix such an ϵ. It follows from Theorem 1.1 that there exists a constant B, independent of q, such that

$$\|\hat{u}_q\|_{2^\sharp}^2 \leq (K_0+\epsilon)\int_M \left(P_g\hat{u}_q\right)\hat{u}_q\,dv_g + B\|\hat{u}_q\|_2^2.$$

Therefore,

$$1 \le \left(\max_M f\right)^{2/q} V_g^{2(1/q-1/2^\sharp)}(K_0+\epsilon)\big(\mu_q + C\|\hat{u}_q\|_2^2\big),$$

where C is independent of q. As $q \to 2^\sharp$, and since $\mu \le \mu_0$, we get that

$$\epsilon \le C\|u\|_2^2.$$

In particular, $u \not\equiv 0$. It thus follows that u is a smooth positive solution of (E). We are left with the proof that μ_0 is attained. As we easily check,

$$\int_M f u^{2^\sharp}\, dv_g \le 1.$$

Besides,

$$\int_M (P_g u)u\, dv_g + a\int_M u^2\, dv_g = \mu \int_M f u^{2^\sharp}\, dv_g,$$

while according to the definition of μ_0,

$$\int_M (P_g u)u\, dv_g + a\int_M u^2\, dv_g \ge \mu_0 \left(\int_M f u^{2^\sharp}\, dv_g\right)^{2/2^\sharp}.$$

It follows that

$$\mu\left(\int_M f u^{2^\sharp}\, dv_g\right)^{1-2/2^\sharp} \ge \mu_0.$$

Hence, $\mu = \mu_0$ and

$$\int_M f u^{2^\sharp}\, dv_g = 1.$$

In particular, u achieves the infimum of the definition of μ_0. This ends the proof of the theorem. □

The following lemma, based on ideas developed in Van der Vorst [23], has been used in the proof of Theorem 2.1.

LEMMA 2.1. *Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold, let α be a positive real number, let b be a real-valued function defined on M, and let $u \in H_2^2(M)$ be a weak solution of*

$$\Delta_g^2 u + \alpha\Delta_g u + \frac{\alpha^2}{4}u = bu.$$

If $b \in L^{n/4}(M)$, then $u \in L^s(M)$ for all $s \ge 1$.

Proof. We proceed as in Van der Vorst [23]. As a starting point, we claim that for any $\epsilon > 0$, there exists $q_\epsilon \in L^{n/4}(M)$, $f_\epsilon \in L^\infty(M)$, and a constant $K_\epsilon > 0$ such that

$$bu = q_\epsilon u + f_\epsilon, \qquad \|q_\epsilon\|_{n/4} < \epsilon, \qquad \|f_\epsilon\|_\infty \le K_\epsilon.$$

Here we may assume that $b \not\equiv 0$, and we let

$$\Omega_k = \left\{x \in M / |b| < k\right\},$$
$$\Omega_l = \left\{x \in M / |u| < l\right\},$$

where if $\hat{\epsilon}$ is such that $(2\hat{\epsilon})^{4/n} = \epsilon/2$, k and l are chosen such that

$$\|b\|_{L^{n/4}(M\backslash\Omega_k)} < \hat{\epsilon} \qquad \text{and} \qquad \|b\|_{L^{n/4}(M\backslash\Omega_l)} < \hat{\epsilon},$$
$$\Omega_k \cap \Omega_l \neq \emptyset \qquad \text{and} \qquad b \not\equiv 0 \quad \text{on } \Omega_k \cap \Omega_l.$$

Given $p \geq 1$ an integer that we fix below, let

$$q_\epsilon = \begin{cases} \frac{1}{p} b & \text{in } \Omega_k \cap \Omega_l, \\ b & \text{in } (M\backslash\Omega_k) \cup (M\backslash\Omega_l), \end{cases}$$

and

$$f_\epsilon = \left(b - q_\epsilon\right) u.$$

Clearly, $f_\epsilon = 0$ on $M\backslash(\Omega_k \cap \Omega_l)$. On the other hand,

$$\begin{aligned}
\|q_\epsilon\|_{n/4}^{n/4} &= \int_{\Omega_k\cap\Omega_l} |q_\epsilon|^{n/4}\, dv_g + \int_{M\backslash(\Omega_k\cap\Omega_l)} |q_\epsilon|^{n/4}\, dv_g \\
&\leq \int_{\Omega_k\cap\Omega_l} |q_\epsilon|^{n/4}\, dv_g + \int_{M\backslash\Omega_k} |q_\epsilon|^{n/4}\, dv_g + \int_{M\backslash\Omega_l} |q_\epsilon|^{n/4}\, dv_g \\
&\leq \left(\frac{1}{p}\right)^{n/4} \int_{\Omega_k\cap\Omega_l} |b|^{n/4}\, dv_g + 2\hat{\epsilon}
\end{aligned}$$

so that

$$\|q_\epsilon\|_{n/4} \leq \frac{1}{p}\|b\|_{n/4} + \frac{1}{2}\epsilon.$$

Choosing p such that $p > 2\|b\|_{n/4}/\epsilon$, we get that

$$\|q_\epsilon\|_{n/4} < \epsilon.$$

Now, since $f_\epsilon = 0$ on $M\backslash(\Omega_k \cap \Omega_l)$,

$$\|f_\epsilon\|_\infty \leq \left|1 - \frac{1}{p}\right| kl,$$

and this proves the above claim. The equation

$$\Delta_g^2 u + \alpha \Delta_g u + \frac{\alpha^2}{4} u = bu$$

may now be written as

$$L_g^2 u = q_\epsilon u + f_\epsilon,$$

where

$$L_g u = \Delta_g u + \frac{\alpha}{2} u.$$

For any $s > 1$ and any $f \in L^s(M)$, there exists one and only one $u \in H_4^s(M)$ such that $L_g^2 u = f$. We let $\mathcal{H}_\epsilon$ be the operator

$$\mathcal{H}_\epsilon u = (L_g)^{-2}(q_\epsilon u).$$

The preceding equation becomes

$$u - \mathcal{H}_\epsilon u = (L_g)^{-2}(f_\epsilon).$$

Let $v \in L^s(M)$, let $s \geq 2^\sharp$, and let u_ϵ be such that

$$L_g^2 u_\epsilon = q_\epsilon v.$$

Set $\hat{s} = ns/(n+4s)$. Clearly, $q_\epsilon v \in L^{\hat{s}}(M)$, and it follows from elliptic-type arguments that

$$\|u_\epsilon\|_s \leq C\|q_\epsilon v\|_{\hat{s}}.$$

By Hölder's inequality,

$$\|q_\epsilon v\|_{\hat{s}} \leq \|q_\epsilon\|_{n/4}\|v\|_s$$

so that

$$\|u_\epsilon\|_s \leq C\epsilon\|v\|_s.$$

In other words, for all $s \geq 2^\sharp$, $\mathcal{H}_\epsilon$ acts from $L^s(M)$ into $L^s(M)$, and its norm is less than or equal to $C\epsilon$. Let $s \geq 2^\sharp$ be given. For $\epsilon > 0$ sufficiently small,

$$\|\mathcal{H}_\epsilon\|_{L^s \to L^s} < \frac{1}{2},$$

and the operator

$$\big(I - \mathcal{H}_\epsilon\big) : L^s(M) \longrightarrow L^s(M)$$

has an inverse. Since

$$\big(I - \mathcal{H}_\epsilon\big)u = (L_g)^{-2}(f_\epsilon)$$

and since $u \in L^{2^\sharp}(M)$ and $f_\epsilon \in L^\infty(M)$, we get that $u \in L^s(M)$. The lemma is proved. $\square$

On what concerns Theorem 2.1, the equality in (2.1) holds when (M,g) is the standard unit sphere (S^n,h), $P_g+a=P_h^n$, and f is a positive constant. This is just equation (1.9) of Proposition 1.1:

$$\inf_{u\in C^\infty(S^n)\setminus\{0\}} \frac{\int_{S^n}\left(P_h^n u\right)u\,dv_h}{\left(\int_{S^n}|u|^{2^\sharp}\,dv_h\right)^{2/2^\sharp}}=\frac{1}{K_0}. \tag{2.2}$$

Independently, let

$$u_0\equiv\left(\int_M f\,dv_g\right)^{-1/2^\sharp}.$$

Clearly, $u_0\in\mathcal{H}_f$, and

$$I_g(u_0)=\frac{aV_g}{\left(\int_M f\,dv_g\right)^{2/2^\sharp}}.$$

We then get the following result from Theorem 2.1.

COROLLARY 2.1. *Let (M,g) be a smooth, compact, n-dimensional Riemannian manifold, $n\geq 5$, let P_g be the operator given by (0.3), let $a>0$ be real, and let f be a smooth, positive function defined on M. If $a\leq\alpha^2/4$ and if*

$$\frac{\int_M f\,dv_g}{V_g\max_M f}>(aK_0)^{2^\sharp/2}V_g^{2^\sharp/2-1}, \tag{2.3}$$

where V_g stands for the volume of M with respect to g, then (E) possesses a smooth positive solution.

Here again, the standard unit sphere plays a particular role in this result. As already mentioned in the proof of Proposition 1.1,

$$d_nK_0=\omega_n^{-4/n}, \tag{2.4}$$

where ω_n is the volume of S^n with respect to h. It follows that if $(M,g)=(S^n,h)$ and $P_g+a=P_h^n$, then the right-hand side in (2.3) is 1. On the contrary, the left-hand side is always less than or equal to 1. The strict inequality (2.3) is therefore never satisfied when $(M,g)=(S^n,h)$ and $P_g+a=P_h^n$. On the other hand, the condition $a\leq\alpha^2/4$ does hold for P_h^n, and we indeed do have that $d_n\leq c_n^2/4$. As we easily check, the difficulty mentioned above disapears when considering quotients of S^n. The volume there becomes smaller, and the following result holds. In the particular case $n\leq 7$, see also Theorem 2.2.

COROLLARY 2.2. *Let (S^n,h) be the standard n-dimensional unit sphere, $n\geq 5$. For any $\epsilon\in(0,1)$, there exists an integer k_ϵ with the following property: if f smooth on S^n is invariant under the action of a subgroup G of $O(n+1)$ acting freely on S^n and of order $k\geq k_\epsilon$, and if f is such that $\|f-1\|_{C^0}<\epsilon$, then the equation*

$$\Delta_h^2u+c_n\Delta_hu+d_nu=fu^{2^\sharp-1}$$

possesses a smooth, positive, G-invariant solution. In particular, there exists a metric g in the conformal class of h for which $Q_g^n = f$, where

$$Q_g^n = \frac{1}{2(n-1)}\Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}S_g^2 - \frac{2}{(n-2)^2}|Rc_g|^2,$$

and for which $G \subset \mathrm{Isom}_g(S^n)$, where $\mathrm{Isom}_g(S^n)$ stands for the isometry group of S^n with respect to g.

Proof. Let M be the quotient manifold S^n/G, and let g_0 be its standard metric induced by h. We still denote by f the quotient of f on M. The Paneitz-Branson operator on M is given by

$$P_{g_0}^n u = \Delta_{g_0}^2 u + c_n \Delta_{g_0} u + d_n u,$$

and as already mentioned, $d_n \le c_n^2/4$. If f is such that $\|f-1\|_{C^0} < \epsilon$, then

$$\frac{\int_M f\, dv_{g_0}}{V_{g_0} \max_M f} > \frac{1-\epsilon}{1+\epsilon}.$$

Now,

$$(d_n K_0)^{2^\sharp/2} V_{g_0}^{2^\sharp/2-1} = \frac{1}{k^{2^\sharp/2-1}}\left((d_n K_0)^{2^\sharp/2}\omega_n^{2^\sharp/2-1}\right).$$

Then set

$$k_\epsilon = \left[\left(\frac{1+\epsilon}{1-\epsilon}\right)^{2/(2^\sharp-2)}(d_n K_0)^{2^\sharp/(2^\sharp-2)}\omega_n\right]+1,$$

where $[x]$ stands for the largest integer not exceeding x. If $k \ge k_\epsilon$, then

$$\frac{\int_M f\, dv_{g_0}}{V_{g_0} \max_M f} > (d_n K_0)^{2^\sharp/2} V_{g_0}^{2^\sharp/2-1}.$$

Noticing that the existence of a solution to the equation on M gives the existence of a G-invariant solution of the equation on S^n, the result follows from Theorem 2.1. □

We now concentrate on the study of (E) when (M, g) is the standard sphere and $P_g + a = P_h^n$. The following result, together with Theorem 2.3, shows that c_n in the definition of P_h^n is critical. In the study of (E) on the standard sphere, we do get obstructions by Theorem 2.3 when $\alpha = c_n$ and $a = d_n$. These obstructions disappear according to Corollary 2.3 if $\alpha < c_n$. When studying (E) on the standard sphere, both the medium term $\alpha = c_n$ and the nonlinear growth $p = 2^\sharp - 1$ are critical. For more details on such assertions, we refer to the remark after the proof of Theorem 2.3.

COROLLARY 2.3. *Let (S^n, h) be the standard n-dimensional unit sphere, $n \geq 6$, let α and a be two positive real numbers, and let f be a smooth, positive function on S^n. If $a \leq \alpha^2/4$, $\alpha < c_n$, and if*

$$(n-6)\frac{\Delta_h f(x_0)}{f(x_0)} < \frac{8n(n-1)}{(n-4)(n+2)}(c_n - \alpha)$$

for at least one x_0 where f is maximum, then the equation

$$\Delta_h^2 u + \alpha \Delta_h u + au = f u^{2^\sharp - 1}$$

possesses a smooth, positive solution.

Proof. Let x_0 be a point where f is maximum, and r be the distance on S^n to x_0. For $\beta > 1$, we let u_β be the function

$$u_\beta = \frac{(\beta^2-1)^{(n-4)/4}}{(\beta - \cos r)^{(n-4)/2}}.$$

As already mentioned in the proof of Proposition 1.1, h and $u_\beta^{4/(n-4)} h$ are isometric. It follows that

$$P_h^n u_\beta = d_n u_\beta^{2^\sharp - 1},$$

where P_h^n is the Paneitz-Branson operator on the sphere, as defined in (0.1). According to the developments made in the proof of Proposition 1.2,

$$\begin{aligned}
&\int_{S^n} (P_h u_\beta) u_\beta \, dv_h + a \int_{S^n} u_\beta^2 \, dv_h \\
&\qquad = d_6 \omega_6 + A\,(\alpha - c_6)\,(\beta - 1) \ln\left(\frac{1}{\beta - 1}\right) + o\left((\beta - 1)\ln\left(\frac{1}{\beta-1}\right)\right) \quad \text{if } n = 6 \\
&\qquad = d_n \omega_n + 2^{n-3}(n-4)^2(\alpha - c_n)(\beta - 1)\omega_{n-1} I + o(\beta - 1) \quad \text{if } n > 6,
\end{aligned}$$

where A is some positive constant and

$$I = \int_0^{+\infty} \frac{y^{n+1}\,dy}{(1+y^2)^{n-2}}.$$

We now write

$$f = f(x_0) + (1 - \cos r)\hat{f}.$$

It is easily seen that

$$\lim_{t \to 0^+} \frac{\int_{\partial B_{x_0}(t)} \hat{f}\, d\sigma}{V_h\big(\partial B_{x_0}(t)\big)} = -\frac{\omega_{n-1}}{n} \Delta_h f(x_0),$$

where $V_h(\partial B_{x_0}(t))$ stands for the area of $\partial B_{x_0}(t)$ with respect to the metric induced by h. By the changes of variables $x = \tan(r/2)$, and then $y = \sqrt{(\beta+1)/(\beta-1)}x$, we get that

$$
\begin{aligned}
&(\beta-1)^{-1}\int_{B_{x_0}(t)}(1-\cos r)u_\beta^{2^\sharp}\,dv_h\\
&\qquad = \frac{2^{n+1}\omega_{n-1}}{\beta+1}\int_0^{\sqrt{(\beta+1)/(\beta-1)}T}\frac{y^{n+1}\,dy}{\big(1+((\beta-1)/(\beta+1))y^2\big)(1+y^2)^n}
\end{aligned}
$$

for all $t \in (0,\pi)$, where $T = \tan(t/2)$. It easily follows that

$$
\lim_{\beta\to 1^+}(\beta-1)\int_{S^n}(1-\cos r)\hat{f}u_\beta^{2^\sharp}\,dv_h = -\frac{2^n\omega_{n-1}}{n}J\,\Delta_h f(x_0),
$$

where

$$
J = \int_0^{+\infty}\frac{y^{n+1}\,dy}{(1+y^2)^n}.
$$

As a consequence,

$$
\int_{S^n} f u_\beta^{2^\sharp}\,dv_h = f(x_0)\omega_n\left(1-\frac{2^n\omega_{n-1}}{n\omega_n}(\beta-1)J\frac{\Delta_h f(x_0)}{f(x_0)}+o(\beta-1)\right).
$$

Since $d_n K_0 = \omega_n^{-4/n}$, and since x_0 is a point where f is maximum, we get that

$$
\begin{aligned}
&\frac{\int_{S^n}(P_h u_\beta)u_\beta\,dv_h + a\int_{S^n}u_\beta^2\,dv_h}{\left(\int_{S^n} f u_\beta^{2^\sharp}\,dv_h\right)^{2/2^\sharp}}\\
&\qquad = \frac{1}{K_0(\max_M f)^{2/2^\sharp}}\big(1+B\,(\alpha-c_6)\,\varepsilon_\beta+o(\varepsilon_\beta)\big) \quad \text{if } n=6\\
&\qquad = \frac{1}{K_0(\max_M f)^{2/2^\sharp}}\big(1+C(\beta-1)+o(\beta-1)\big) \quad \text{if } n>6,
\end{aligned}
$$

where $\varepsilon_\beta = (\beta-1)\ln(1/(\beta-1))$, $B>0$ does not depend on β, and

$$
C = \frac{2^{n+1}\omega_{n-1}}{2^\sharp n\omega_n}J\left(\frac{\Delta_h f(x_0)}{f(x_0)}+\frac{2nI}{(n^2-4)J}(\alpha-c_n)\right).
$$

As we easily check, see, for instance, Demengel-Hebey [8],

$$
I = \frac{1}{2}\frac{\Gamma((n+2)/2)\Gamma((n-6)/2)}{\Gamma(n-2)} \qquad \text{and} \qquad J = \frac{1}{2}\frac{\Gamma((n+2)/2)\Gamma((n-2)/2)}{\Gamma(n)}.
$$

Hence,

$$
\frac{2nI}{(n^2-4)J} = \frac{8n(n-1)}{(n-6)(n-4)(n+2)}.
$$

Under our assumptions $C < 0$, so that for $\beta > 1$ sufficiently close to 1,

$$\frac{\int_{S^n}(P_h u_\beta)u_\beta \, dv_h + a\int_{S^n} u_\beta^2 \, dv_h}{\left(\int_{S^n} f u_\beta^{2^\sharp} \, dv_h\right)^{2/2^\sharp}} < \frac{1}{K_0(\max_M f)^{2/2^\sharp}}.$$

The result now follows from Theorem 2.1. □

The equation involved in the study of the prescribed scalar curvature problem on the sphere, also referred to as the Kazdan-Warner problem or the Nirenberg problem, is the equation

$$\Delta_h u + \frac{n(n-2)}{4} u = f u^{2^\star - 1},$$

where $2^\star = 2n/(n-2)$. In the study of this equation, a celebrated result of Escobar and Schoen [11] states that if $n = 3$ and if f is invariant under the action of a nontrivial subgroup G of $O(4)$ acting freely on S^3, then the above equation possesses a smooth, positive G-invariant solution. In particular, under these assumptions, f is the scalar curvature of a G-invariant conformal metric to h. The same result was proved by Moser [20] when $n = 2$ and f is assumed to be invariant under the action of the antipodal group $G = \{Id, -Id\}$, the only group acting freely on S^n when the dimension n is even. A natural question is whether such types of results do hold for the equation

$$P_h^n u = f u^{2^\sharp - 1}.$$

This is the subject of the following theorem. As a first remark, note that by Edmunds-Fortunato-Janelli [10] and Pucci-Serrin [22], low dimensions for the Euclidean biharmonic operator are $n = 5, 6, 7$. As another remark, we mention that there should be an analogue of our result when G acts without fixed points (i.e., for any x, the G-orbit of x has at least two elements). Concerning the above mentioned scalar curvature problem on the sphere, this was proved by Hebey [13].

THEOREM 2.2. *Let (S^n, h) be the standard n-dimensional unit sphere, $n = 5, 6$, or 7, and let f be a smooth positive function on S^n. We assume that f is invariant under the action of a nontrivial subgroup G of $O(n+1)$ acting freely on S^n, and if $n = 6$ or 7, we assume that $\Delta_h f(x) = 0$ for at least one x where f is maximum. Then the equation*

$$P_h^n u = f u^{2^\sharp - 1}$$

possesses a smooth, positive G-invariant solution. In particular, there exists a metric g in the conformal class of h for which $Q_g^n = f$, where

$$Q_g^n = \frac{1}{2(n-1)}\Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2}|Rc_g|^2,$$

and for which $G \subset \mathrm{Isom}_g(S^n)$, where $\mathrm{Isom}_g(S^n)$ stands for the isometry group of S^n with respect to g.

Proof. Let M be the quotient manifold S^n/G, and let g_0 be its standard metric induced by h. Also let $u \in H_2^2(M)$, $u \not\equiv 0$, and let $\tilde{u}$ be the function on S^n induced by u. As we easily check,

$$\frac{\int_{S^n} \left(P_h^n \tilde{u}\right)\tilde{u}\, dv_h}{\left(\int_{S^n} f|\tilde{u}|^{2^\sharp}\, dv_h\right)^{2/2^\sharp}} = k^{1-2/2^\sharp} \frac{\int_M \left(P_{g_0}^n \tilde{u}\right)\tilde{u}\, dv_{g_0}}{\left(\int_M f|\tilde{u}|^{2^\sharp} dv_{g_0}\right)^{2/2^\sharp}},$$

where k is the number of elements in G, and f in the right-hand side of this relation stands for the quotient of f on M. The existence of a solution to the equation on M leads to the existence of a G-invariant solution to the equation on S^n. Therefore, as a consequence of Theorem 2.1, it suffices to show that

$$\inf_{u\in\Lambda} \frac{\int_{S^n} \left(P_h^n u\right)u\, dv_h}{\left(\int_{S^n} f|u|^{2^\sharp}\, dv_h\right)^{2/2^\sharp}} < \frac{k^{1-2/2^\sharp}}{(\max_{S^n} f)^{2/2^\sharp} K_0}, \tag{2.5}$$

where Λ stands for the subset of $H_2^2(S^n)$ consisting of nonzero G-invariant functions. Now let x_1 be a point where f is maximum, and denote by

$$O_G(x_1) = \{x_1, \ldots, x_k\}$$

the G-orbit of x_1. If r_i stands for the distance on S^n to x_i, let $u_{i,\beta}$, $\beta > 1$ be the functions on S^n defined by

$$u_{i,\beta} = \frac{(\beta^2-1)^{(n-4)/4}}{(\beta - \cos r_i)^{(n-4)/2}}.$$

As already mentioned, h and $u_{i,\beta}^{4/(n-4)}h$ are isometric. In particular,

$$P_h^n u_{i,\beta} = d_n u_{i,\beta}^{2^\sharp - 1}$$

and

$$\int_{S^n} u_{i,\beta}^{2^\sharp}\, dv_h = \omega_n.$$

Then let

$$u_\beta = \sum_{i=1}^{k} u_{i,\beta}.$$

On the one hand, u_β is G-invariant. On the other hand,

$$\int_{S^n} \left(P_h^n u_\beta\right) u_\beta\, dv_h = k d_n \omega_n + k d_n \int_{S^n} u_{1,\beta}^{2^\sharp - 1} \left(\sum_{i=2}^{k} u_{i,\beta}\right) dv_h.$$

Set

$$A(n) = 2^{1+3n/4}\omega_{n-1}\int_0^{+\infty}\frac{y^{n-1}\,dy}{(1+y^2)^{(n+4)/2}}.$$

We claim that for all t in $(0,\pi)$,

$$\lim_{\beta\to 1^+}(\beta-1)^{1-n/4}\int_{B_{x_1}(t)} u_{1,\beta}^{2^\sharp-1}\,dv_h = A(n).$$

Indeed, by the change of variables $x = \tan(r_1/2)$ and $y = \sqrt{(\beta+1)/(\beta-1)}x$, we get that

$$\begin{aligned}&\int_{B_{x_1}(t)} u_{1,\beta}^{2^\sharp-1}\,dv_h\\ &\quad = C_1(n,\beta)(\beta-1)^{n/4-1}\\ &\quad\quad \times\int_0^{\sqrt{(\beta+1)/(\beta-1)}T}\frac{y^{n-1}\,dy}{\big(1+((\beta-1)/(\beta+1))y^2\big)^{(n-4)/2}(1+y^2)^{(n+4)/2}},\end{aligned}$$

where

$$C_1(n,\beta) = 2^n(\beta+1)^{1-n/4}\omega_{n-1}$$

and $T = \tan(t/2)$. The above claim then easily follows. With easier arguments, for all $h\in C^0(S^n)$, and all t in $(0,\pi)$,

$$\lim_{\beta\to 1^+}(\beta-1)^{2-n/2}\int_{S^n\setminus B_{x_1}(t)} h u_{1,\beta}^{2^\sharp-1}\left(\sum_{i=2}^k u_{i,\beta}\right)dv_h = 0.$$

It follows from these two relations that for all $h\in C^0(S^n)$, all t in $(0,\pi)$, and all open subset Ω of S^n that contains x_1,

$$\lim_{\beta\to 1^+}(\beta-1)^{2-n/2}\int_\Omega h u_{1,\beta}^{2^\sharp-1}\left(\sum_{i=2}^k u_{i,\beta}\right)dv_h = A(n)\left(\sum_{i=2}^k \tilde u_{i,1}(x_1)\right)h(x_1), \tag{2.6}$$

where $\tilde u_{i,1} = (1-\cos r_i)^{2-n/2}$. Then let $t_0>0$ be such that $B_{x_i}(t_0)\cap B_{x_j}(t_0) = \emptyset$ for $i\neq j$. Since

$$(a+b)^{2^\sharp}\ge a^{2^\sharp} + 2^\sharp a^{2^\sharp-1}b,$$

we may write that

$$\begin{aligned}\int_{S^n} f u_\beta^{2^\sharp}\,dv_h &\geq \sum_{i=1}^{k}\int_{B_{x_i}(t_0)} f u_\beta^{2^\sharp}\,dv_h\\ &\geq \sum_{i=1}^{k}\int_{B_{x_i}(t_0)} f u_{i,\beta}^{2^\sharp}\,dv_h + 2^\sharp\sum_{i=1}^{k}\sum_{j\neq i}\int_{B_{x_i}(t_0)} f u_{i,\beta}^{2^\sharp-1}u_{j,\beta}\,dv_h\\ &= k\int_{B_{x_1}(t_0)} f u_{1,\beta}^{2^\sharp}\,dv_h + 2^\sharp k\int_{B_{x_1}(t_0)} f u_{1,\beta}^{2^\sharp-1}\left(\sum_{i=2}^{k}u_{i,\beta}\right)dv_h.\end{aligned}$$

It follows that

$$\begin{aligned}\int_{S^n} f u_\beta^{2^\sharp}\,dv_h &\geq kf(x_1)\omega_n - kf(x_1)\int_{S^n\setminus B_{x_1}(t_0)} u_{1,\beta}^{2^\sharp}\,dv_h\\ &\quad + k\int_{B_{x_1}(t_0)}\big(f-f(x_1)\big)u_{1,\beta}^{2^\sharp}\,dv_h + 2^\sharp k\int_{B_{x_1}(t_0)} f u_{1,\beta}^{2^\sharp-1}\left(\sum_{i=2}^{k}u_{i,\beta}\right)dv_h.\end{aligned}$$

It is easily seen that

$$\lim_{\beta\to 1^+}(\beta-1)^{2-n/2}\int_{S^n\setminus B_{x_1}(t_0)} u_{1,\beta}^{2^\sharp}\,dv_h = 0. \tag{2.7}$$

On the other hand, for any $5\leq n\leq 7$,

$$\lim_{\beta\to 1^+}(\beta-1)^{2-n/2}\int_{B_{x_1}(t_0)}\big(f-f(x_1)\big)u_{1,\beta}^{2^\sharp}\,dv_h = 0. \tag{2.8}$$

Indeed, suppose that $n=5$. Since x_1 is a critical point for f, there exists a constant $C>0$ such that for all $x\in B_{x_1}(t_0)$,

$$\big|f(x)-f(x_1)\big|\leq C(1-\cos r_1).$$

With the change of variables $x=\tan(r_1/2)$ and $y=\sqrt{(\beta+1)/(\beta-1)}\,x$, we get that

$$(\beta-1)^{2-n/2}\int_{B_{x_1}(t_0)}(1-\cos r_1)u_{1,\beta}^{2^\sharp}\,dv_h = O\big((\beta-1)^{3-n/2}\big),$$

from which (2.8) follows. Suppose then that $n=6$ or 7 and that x_1 is such that $\Delta_h f(x_1)=0$. We may write that there exists a constant $C>0$ such that for all $x\in B_{x_1}(t_0)$,

$$\big|f(x)-f(x_1)\big|\leq C(1-\cos r_1)^2.$$

As above, we get that

$$(\beta-1)^{2-n/2}\int_{B_{x_1}(t_0)}(1-\cos r_1)^2 u_{1,\beta}^{2^\sharp}\,dv_h = O\big((\beta-1)^{4-n/2}\big),$$

from which (2.8) also follows. Now, by (2.4), (2.6), (2.7), and (2.8),

$$\frac{\int_{S^n}\big(P_h^n u_\beta\big)u_\beta\,dv_h}{\big(\int_{S^n} f u_\beta^{2^\sharp}\,dv_h\big)^{2/2^\sharp}} = \frac{k^{1-2/2^\sharp}}{f(x_1)^{2/2^\sharp}K_0}\times\frac{1+(\beta-1)^{n/2-2}A_k(n)+o\big((\beta-1)^{n/2-2}\big)}{1+2(\beta-1)^{n/2-2}A_k(n)+o\big((\beta-1)^{n/2-2}\big)},$$

where $A_k(n)>0$ is given by

$$A_k(n)=\frac{A(n)}{\omega_n}\sum_{i=2}^{k}\tilde{u}_{i,1}(x_1).$$

Hence, for every $\beta>1$ sufficiently close to 1,

$$\frac{\int_{S^n}\big(P_h^n u_\beta\big)u_\beta\,dv_h}{\big(\int_{S^n} f u_\beta^{2^\sharp}\,dv_h\big)^{2/2^\sharp}} < \frac{k^{1-2/2^\sharp}}{f(x_1)^{2/2^\sharp}K_0}.$$

In particular, since u_β is G-invariant, and $f(x_1)=\max_{S^n} f$,

$$\inf_{u\in\Lambda}\frac{\int_{S^n}\big(P_h^n u\big)u\,dv_h}{\big(\int_{S^n} f|u|^{2^\sharp}\,dv_h\big)^{2/2^\sharp}} < \frac{k^{1-2/2^\sharp}}{(\max_{S^n} f)^{2/2^\sharp}K_0}.$$

This is exactly inequality (2.5). The theorem is thus proved. □

A celebrated result of Kazdan and Warner [17] states that the scalar curvature equation on the sphere (S^n,h) possesses obstructions. We prove here that such obstructions hold similarly for the equation

$$P_h^n u = f u^{2^\sharp-1}.$$

In the statement of Theorem 2.3, $(\nabla f\nabla\varphi)$ stands for the pointwise scalar product with respect to h of ∇f and $\nabla\varphi$.

THEOREM 2.3. *Let (S^n,h) be the standard n-dimensional unit sphere, $n\geq 5$, and let f be a smooth function on S^n, positive somewhere on S^n. If u is a smooth positive solution of the equation*

$$P_h^n u = f u^{2^\sharp-1}, \tag{2.9}$$

where P_h^n is as in (0.1), then for any eigenfunction φ of Δ_h associated to the first nonzero eigenvalue $\lambda_1=n$,

$$\int_{S^n}(\nabla f\nabla\varphi)u^{2^\sharp}\,dv_h = 0.$$

In particular, for any $\epsilon > 0$ and any eigenfunction $\varphi \not\equiv 0$ of Δ_h associated to the first nonzero eigenvalue $\lambda_1 = n$, (2.9) with $f = 1 + \epsilon\varphi$ does not possess a smooth, positive solution.

Proof. The proof mainly follows what was done in Kazdan and Warner [17]. Let φ be an eigenfunction of Δ_h associated to the first nonzero eigenvalue $\lambda_1 = n$, and let u be a smooth function on S^n. As it is easy to see,

$$\left(\Delta_h^2 u\right)(\nabla u \nabla\varphi) \simeq (\Delta_h u)(\nabla\Delta_h u\nabla\varphi) - (n-2)(\Delta_h u)(\nabla u\nabla\varphi) - 2(\Delta_h u)^2\varphi,$$

where the sign "$\simeq$" means that the relation holds modulo terms in divergence form. Clearly,

$$(\Delta_h u)(\nabla\Delta_h u\nabla\varphi) \simeq \frac{1}{2}n\varphi(\Delta_h u)^2$$

so that

$$(\Delta_h u)^2(\nabla u\nabla\varphi) \simeq \frac{n-4}{2}(\Delta_h u)^2\varphi - (n-2)(\Delta_h u)(\nabla u\nabla\varphi).$$

Suppose now that u is a solution of (2.9). Then

$$(\Delta_h u)^2\varphi \simeq 2(\Delta_h u)(\nabla u\nabla\varphi) - nu\varphi(\Delta_h u) + f\varphi u^{2^\sharp} - d_n u^2\varphi - c_n u\varphi(\Delta_h u)$$

so that

$$\begin{aligned}(c_n-2)(\Delta_h u)(\nabla u\nabla\varphi) - \frac{n-4}{2}(n+c_n)u\varphi(\Delta_h u) + d_n u(\nabla u\nabla\varphi)\\ + \frac{n-4}{2}f\varphi u^{2^\sharp} - \frac{n-4}{2}d_n u^2\varphi \simeq fu^{2^\sharp-1}(\nabla u\nabla\varphi).\end{aligned}$$

Since

$$(\Delta_h u)(\nabla u\nabla\varphi) \simeq -\frac{n-2}{2}\left(\frac{1}{2}nu^2\varphi - u\varphi(\Delta_h u)\right)$$

and

$$\frac{n-2}{2}(c_n-2) = \frac{n-4}{2}(n+c_n),$$

the terms $u\varphi(\Delta_h u)$ disappear. We then get that

$$-\frac{n-2}{4}(c_n-2)nu^2\varphi + d_n u(\nabla u\nabla\varphi) + \frac{n-4}{2}f\varphi u^{2^\sharp} - \frac{n-4}{2}d_n u^2\varphi \simeq fu^{2^\sharp-1}(\nabla u\nabla\varphi).$$

Now it is easily seen that

$$u(\nabla u\nabla\varphi) \simeq \frac{n}{2}u^2\varphi$$

and that

$$fu^{2^\sharp-1}(\nabla u\nabla\varphi) \simeq -\frac{1}{2^\sharp}u^{2^\sharp}(\nabla f\nabla\varphi) + \frac{n}{2^\sharp}fu^{2^\sharp}\varphi.$$

Therefore,

$$\left(2d_n - \frac{n(n-2)}{4}(c_n - 2)\right)u^2\varphi + \frac{1}{2^\sharp}u^{2^\sharp}(\nabla f \nabla\varphi) \simeq 0.$$

Since

$$2d_n = \frac{n(n-2)}{4}(c_n - 2),$$

we find that

$$u^{2^\sharp}(\nabla f \nabla\varphi) \simeq 0.$$

This ends the proof of the theorem. □

To conclude, we collect a few remarks on Corollary 2.3 and Theorem 2.3. First, let φ be an eigenfunction of Δ_h associated to the first nonzero eigenvalue $\lambda_1 = n$, and, for $\varepsilon > 0$, set $f_\varepsilon = 1 + \varepsilon\varphi$. Given $\alpha > 0$, consider the equation

$$\Delta_h^2 u + \alpha\Delta_h u + d_n u = f_\varepsilon u^{2^\sharp - 1}. \tag{E_α^ε}$$

According to Corollary 2.3, if $n > 6$ and $\alpha \in [2\sqrt{d_n}, c_n)$, then there exists $\varepsilon_\alpha > 0$ such that if $\varepsilon \le \varepsilon_\alpha$, (E_α^ε) possesses a smooth positive solution. On the contrary, by Theorem 2.3, for all $n \ge 5$ and all $\varepsilon > 0$, $(E_{c_n}^\varepsilon)$ does not possess any smooth positive solution. This is one of the possible illustrations of the criticality of c_n we mentioned before stating Corollary 2.3. As another remark, note that Theorem 2.3, together with Theorem 2.1, gives another proof of (1.9). Indeed, suppose by contradiction that (1.9) is false. In other words, assume that

$$\inf_{u \in C^\infty(S^n)\setminus\{0\}} \frac{\int_{S^n} \left(P_h^n u\right) u \, dv_h}{\left(\int_{S^n} |u|^{2^\sharp} \, dv_h\right)^{2/2^\sharp}} < \frac{1}{K_0}.$$

Then for any f sufficiently close to 1 in the C^0-topology,

$$\inf_{u \in C^\infty(S^n)\setminus\{0\}} \frac{\int_{S^n} \left(P_h^n u\right) u \, dv_h}{\left(\int_{S^n} f|u|^{2^\sharp} \, dv_h\right)^{2/2^\sharp}} < \frac{1}{(\max_{S^n} f)^{2/2^\sharp} K_0}.$$

It follows from Theorem 2.1 that for such an f, the equation

$$P_h^n u = f u^{2^\sharp - 1}$$

has a smooth, positive solution. This is in contradiction to the last part of Theorem 2.3 and thus proves (1.9).

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