SOBOLEV INEQUALITIES AND MYERS'S DIAMETER THEOREM FOR AN ABSTRACT MARKOV GENERATOR

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1. Introduction. The classical theorem of (Bonnet-) Myers on the diameter [M] (see [C], [GHL]) states that if (M, q) is a complete, connected Riemannian manifold of dimension $n \ (\geq 2)$ such that $\operatorname{Ric} \geq (n-1)g$, then its diameter D = D(M) is less than or equal to π (and, in particular, M is compact). Equivalently, after a change of scale, if $Ric \ge Rg$ with R > 0, and if S_r^n is the sphere of dimension n and constant curvature $R = (n-1)/r^2$ where r > 0 is the radius of S_r^n , then the diameter of M is less than or equal to the diameter of S_r^n , that is,

$$(1.1) D \leq \pi r = \pi \sqrt{\frac{n-1}{R}}.$$

The aim of this work is to prove an analogue of Myers's theorem for an abstract Markov generator and to provide at the same time a new analytic proof of this result based on Sobolev inequalities. In particular, we will show how to get exact bounds on the diameter in terms of the Sobolev constant. As an introduction, let us describe the framework, referring to [B2] for further details. On some probability space (E, \mathscr{E}, μ) , let L be a Markov generator associated to some semigroup $(P_t)_{t \ge 0}$ continuous in $L^2(\mu)$. We will assume that L is invariant and symmetric with respect to μ , as well as ergodic. We assume furthermore that we are given a nice algebra \mathcal{A} of bounded functions on E, containing the constants and stable by L and P_t (though this last hypothesis is not really needed but is used here for convenience) and by the action of C^{∞} functions. We may then introduce, following P.-A. Meyer, the so-called *carré du champ* operator Γ as the symmetric bilinear operator on $\mathscr{A} \times \mathscr{A}$ defined by

$$2\Gamma(f,g) = \mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f, \qquad f,g \in \mathscr{A},$$

as well as the iterated carré du champ operator Γ_2

$$2\Gamma_2(f,g) = \mathrm{L}\Gamma(f,g) - \Gamma(f,\mathrm{L}g) - \Gamma(g,\mathrm{L}f), \qquad f,g \in \mathscr{A}.$$

Finally, we assume that L is a diffusion: for every C^{∞} function Ψ on \mathbb{R}^k , and every finite family $F = (f_1, \ldots, f_k)$ in \mathcal{A} ,

$$\mathbf{L}\Psi(F) = \nabla\Psi(F) \cdot \mathbf{L}F + \nabla\nabla\Psi(F) \cdot \Gamma(F,F).$$

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This hypothesis essentially expresses that L is a second-order differential operator with no constant term and that we have a chain rule formula for Γ .

Recall that, by invariance of L, the following integration by parts formula is satisfied

$$\int f(-\mathbf{L}g) \, d\mu = \int \Gamma(f,g) \, d\mu, \qquad f,g \in \mathscr{A}.$$

We denote below by $\|\cdot\|_p$, $1 \le p \le \infty$, the L^{*p*}-norm with respect to μ .

Given such a Markov generator L, we define its diameter D = D(L) (relative to \mathcal{A}) as

(1.2)
$$D = D(L) = \sup\{\|\bar{f}\|_{L^{\infty}(\mu \otimes \mu)}; f \in \mathscr{A}, \|\Gamma(f, f)\|_{\infty} \leq 1\}$$

where $\tilde{f}(x, y) = f(x) - f(y), x, y \in E$.

The basic operator we have in mind, is of course, the Laplace-Beltrami operator Δ on a compact, connected Riemannian manifold M. For \mathscr{A} the class, say, of Lipschitz or C^1 functions on M, $\Gamma(f, f)$ is simply the Riemannian length (squared) $|\nabla f|^2$ of the gradient ∇f of $f \in \mathscr{A}$, and the diameter $D = D(\Delta)$ of (1.2) coincides with the usual notion of diameter associated to the Riemannian distance on M. One may also mention that Bochner's formula (cf. [C], [GHL]) indicates in this example that

(1.3)
$$\Gamma_2(f, f) = \operatorname{Ric}(\nabla f, \nabla f) + \|\operatorname{Hess} f\|_2^2,$$

where Ric is the Ricci tensor on M and $\|\text{Hess } f\|_2$ is the Hilbert-Schmidt norm of the tensor of the second derivatives of f.

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The previous abstract framework includes a number of further examples of interest (cf. [B2]). For example, one may consider $L = \Delta + X$ where X is a smooth vector field on M. We may also consider infinite-dimensional examples such as the Ornstein-Uhlenbeck generator. Of basic inspiration for this work is the class of ultraspheric generators on]-1,+1[given by

(1.4)
$$L_n f(x) = (1 - x^2) f''(x) - nx f'(x)$$

for every f smooth enough, where n > 0. In this example, $\Gamma(f, f) = (1 - x^2) f'(x)^2$, and the invariant measure is given by $d\mu_n = c_n(1 - x^2)^{(n/2)-1} dx$ on]-1, +1[.

The analogue of Myers's theorem requires geometric notions of curvature and dimension on the abstract Markov generator L. We will say that L satisfies a curvature-dimension inequality CD(R, n) of curvature R and dimension $n \ge 1$ if, for all functions f in \mathcal{A} ,

$$\operatorname{CD}(R,n)$$
 $\Gamma_2(f,f) \ge R \Gamma(f,f) + \frac{1}{n} (Lf)^2.$

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According to Bochner's formula (1.3), an *n*-dimensional compact Riemannian manifold (M, g), or rather the Laplacian Δ on M, satisfies the inequality CD(R, n) with R the infimum of the Ricci tensor over all unit tangent vectors. If $L = \Delta + \nabla h$ for a smooth function h, and if, as symmetric tensors,

$$\nabla h \otimes \nabla h \leq (m-n)[\operatorname{Ric} - \nabla \nabla h - \rho g]$$

with $m \ge n$, then L satisfies $CD(\rho, m)$ (cf. [B2], Proposition 6.2). The ultraspheric generators L_n , which, for n an integer, may be obtained as projections of the Laplacian of S_1^n on a diameter, satisfy the same curvature-dimension inequalities CD(n-1,n) ($n \ge 1$) as the spheres S_1^n themselves, although n need not be an integer anymore.

With these notations, Myers's theorem states that if L is the Laplacian Δ on a Riemannian manifold (M, g) of dimension *n* such that $\text{Ric} \ge (n-1)g$ —in other words, if $L = \Delta$ satisfies CD(n-1,n)—then $D(M) = D(\Delta) \le \pi$. The geometric proofs of this result (cf. [C], [GHL]) rely on minimizing geodesics and Jacobi fields and cannot therefore be extended to the preceding abstract framework. The main advantage of the method we will develop is that it relies only on measure-theoretic arguments and makes no use of the distance function, the diameter being defined only in terms of the Γ operator.

As in earlier attempts [BM], [B2], our approach in this work will be functional analytic. We will namely translate the curvature-dimension inequality into a functional Sobolev inequality and then bound the diameter only on the basis of this inequality. Of course, that the diameter is bounded under a (local) Sobolev inequality is a relatively easy fact. The point is that we are looking here for optimal bounds. In particular, we will only deal with inequalities involving firstorder derivatives and not second-order derivatives such as in Bochner's formula. Actually, Bochner's formula and the curvature-dimension inequality are only used to establish the functional inequalities of Sobolev type with sharp constants.

In this direction, it was shown in [B2] that if L satisfies the following entropyenergy, or logarithmic Sobolev, inequality

(1.5)
$$\int f^2 \log f^2 d\mu \leq \Phi\left(\int \Gamma(f, f) d\mu\right)$$

for every $f \in \mathscr{A}$ with $\int f^2 du = 1$ where $\Phi \colon [0, \infty) \to [0, \infty)$ is increasing, then

(1.6)
$$D \leqslant \int_0^\infty \frac{1}{u^2} \Phi(u^2) \, du.$$

One therefore is led to ask for the optimal inequality (1.5) under CD(R, n). It was shown in [B1], [B2] that if CD(n-1, n) holds, one may choose in (1.5)

(1.7)
$$\Phi(u) = \frac{n}{2} \log \left(1 + \frac{4u}{n(n-1)} \right)$$

yielding together with (1.6)

$$D\leqslant \pi\sqrt{\frac{n}{n-1}}.$$

This partial result is fairly close to the optimal one and, strangely enough, is as better as n increases to infinity.

The first proofs of (1.6) [BM], [B2] were based on delicate heat-kernel minorations depending on (families of) logarithmic Sobolev inequalities. A recent proof [L], inspired by [AMS], consists in applying inequality (1.5) to the family of functions $e^{\lambda f}$, $\lambda \in \mathbb{R}$, where $f \in \mathcal{A}$, $\int f d\mu = 0$, and $\|\Gamma(f, f)\|_{\infty} \leq 1$. One then obtains a simple differential inequality on λ which may easily be integrated to yield (1.6). Now, and this was actually our starting point, exponential functions are classically the extremal functions of the logarithmic Sobolev inequality for the Ornstein-Uhlenbeck generator that corresponds formally in our abstract framework precisely to $n = \infty$! On the other hand, it may be shown (cf. [B2]) that the function Φ of (1.7) is not optimal on the sphere S_1^n and that, anyhow, the knowledge of the best Φ would not yield Myers's theorem. Indeed, if this were the case, the method of proof of [L] applied to the distance function d to a fixed point would show that functions of the form $e^{\lambda d}$ are extremal for logarithmic Sobolev inequalities of the type

$$\int f^2 \log f^2 d\mu \leq a(\lambda) + b(\lambda) \int \Gamma(f, f) d\mu, \qquad \int f^2 d\mu = 1,$$

which is clearly not the case on spheres. Notice also that the proof of [L] actually shows that $||f||_{\infty} \leq (\pi/2)\sqrt{n/(n-1)}$ for every f in \mathscr{A} with $\int f d\mu = 0$ and $||\Gamma(f, f)||_{\infty} \leq 1$. We will not improve this result to $\pi/2$ here, and we actually do not know whether this could be true.

The preceding observations strongly suggest that logarithmic Sobolev inequalities are not well adapted to sharp estimates on the diameter. Now, while the best entropy-energy inequality on spheres is still unknown, the best Sobolev inequality as well as its extremal functions are known. Following Th. Aubin [A], on S_1^n , n > 2, with $L = \Delta$ the conformal Laplacian and μ the normalized Haar measure, for every smooth function f,

(1.8)
$$||f||_{2n/n-2}^2 \leq ||f||_2^2 + \frac{4}{n(n-2)} \int f(-\Delta f) \, d\mu$$

Furthermore, the functions $f_{\lambda} = (1 + \lambda \sin(d))^{1-(n/2)}$, $-1 < \lambda < +1$, where d is the distance to a fixed point, are solutions of the nonlinear equation

$$f_{\lambda}^{(n+2)/(n-2)} - f_{\lambda} = -\frac{4}{n(n-2)} \Delta f_{\lambda}$$

and satisfy the equality in inequality (1.8). One therefore is led to develop the preceding approach, but now on the basis of a Sobolev inequality and with these extremal functions instead of exponentials. This is the method we will follow.

The first step in this program is to establish a Sobolev inequality (1.8) with best constant for an abstract Markov generator L under curvature-dimension hypotheses. The following theorem, which was presented in [B2], fully answers this question. On a Riemannian manifold with L the Laplacian and μ the normalized Riemannian measure, the result goes back to S. Ilias [I].

THEOREM 1. Let L be a Markov generator satisfying CD(R, n) for some R > 0and n > 2. Then, for every $f \in A$,

$$\|f\|_{2n/n-2}^2 \leq \|f\|_2^2 + \frac{4(n-1)}{n(n-2)R} \int f(-Lf) \, d\mu.$$

Note that if R = n - 1, then

$$\frac{4(n-1)}{n(n-2)R} = \frac{4}{n(n-2)} \; .$$

The proof actually shows (cf. [B2]) that under CD(R, n) with R > 0 and $n \ge 1$ (with the convention that, when n = 1, $R/(n - 1) = \lambda_1$, the spectral gap of -L; see below), then, for every $1 \le p \le 2n/n - 2$ if n > 2 and every $p \ge 1$ if $1 \le n \le 2$,

(1.9)
$$\frac{nR}{n-1} \cdot \frac{\|f\|_p^2 - \|f\|_2^2}{p-2} \leq \int f(-Lf) \, d\mu.$$

This result extends the case of spheres due to W. Beckner [Be]. In [F], a somewhat sharper bound than (1.9) is obtained involving λ_1 , R, and n as a convex combination. Denote by λ_1 the spectral gap of -L, that is, the largest s > 0 such that for every f in \mathcal{A} ,

(1.10)
$$s \int \left[f - \int f \, d\mu \right]^2 d\mu \leqslant \int f(-\mathbf{L}f) \, d\mu = \int \Gamma(f,f) \, d\mu.$$

Then

(1.11)
$$c \; \frac{\|f\|_p^2 - \|f\|_2^2}{p-2} \leq \int f(-\mathbf{L}f) \, d\mu$$

with

$$c = \alpha \, \frac{nR}{n-1} + (1-\alpha)\lambda_1$$

where

$$\alpha = \alpha(p) = \frac{(n-1)^2(p-1)}{(p-2) + (n+1)^2}.$$

Note that $\alpha(p) = 1$ for the critical exponent p = 2n/n - 2. In (1.11), it is assumed more precisely that we already know that some Sobolev inequality (of dimension *n*) holds for L (as, for example, on compact Riemannian manifolds). Inequality (1.11) will be used below to bound above λ_1 in terms of *n*, *R*, and the diameter *D*.

It might be worthwhile noting that the proof of Theorem 1 relies on the study of the nonlinear equation

$$c(f^{p-1}-f) = -\mathbf{L}f.$$

Lower bounds on c > 0 are then obtained by a comparison with the CD(R, n) inequality following ideas introduced by O. S. Rothaus [R] in the context of logarithmic Sobolev inequalities (actually, (1.9) or (1.11) for p = 2 have to be understood in the limit as such logarithmic Sobolev inequalities). The method consists in performing the change of variables $f \rightarrow f^r$, $r \neq 0$, (f > 0), in this nonlinear equation as well as another change $f \rightarrow f^s$, $s \neq 0$, into the CD(R, n) inequality. By the diffusion property and integration by parts, the result may then be shown to follow from optimal choices of the parameters r and s (see [B2] for the details).

We thus have optimal Sobolev inequalities for an abstract Markov generator under a curvature-dimension condition. According to our preceding observations, we will now make use of the form of the extremal functions of this Sobolev inequality on spheres to deduce some differential inequality. The miracle of the approach is that this inequality may be integrated to exactly bound the diameter of L by its Sobolev constant. To emphasize the fact that we will be working here only with functional inequalities involving first-order derivatives, we state our main result only in terms of the carré du champ operator. On (E, \mathscr{E}, μ) , let \mathscr{A} be a subspace of L^{∞} containing the constants and stable by composition with smooth functions φ with all derivatives bounded. We assume that we are given a bilinear map

$$\Gamma : \mathscr{A} \times \mathscr{A} \to L^{\infty}$$

such that $\Gamma(f, f) \ge 0$ and

$$\Gamma(\varphi(f),g)) = \varphi'(f)\Gamma(f,g)$$

for every f, g in \mathscr{A} and every φ smooth. We will call Γ a carré du champ. We may then define as before the "diameter" of such a carré du champ Γ as

$$D = D(\Gamma) = \sup\{\|\tilde{f}\|_{L^{\infty}(\mu \otimes \mu)}; f \in \mathscr{A}, \|\Gamma(f, f)\|_{\infty} \leq 1\}$$

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where we recall that $\tilde{f}(x, y) = f(x) - f(y)$, $x, y \in E$. Of course, $D(\Gamma)$ is relative to \mathscr{A} (and is as smaller as \mathscr{A} is small) and $D(\Gamma) = D(L)$ if Γ is the carré du champ of the Markov generator L. The following theorem is the main result of this work.

THEOREM 2. Let Γ be a carré du champ satisfying the Sobolev inequality

(1.12)
$$||f||_{2n/n-2}^2 \leq ||f||_2^2 + \frac{4}{n(n-2)} \int \Gamma(f,f) \, d\mu, \quad f \in \mathscr{A},$$

for some n > 2. Then

$$D=D(\Gamma)\leqslant\pi.$$

If Γ is changed in $a\Gamma$ for some a > 0, then $D(a\Gamma) = a^{-1/2}D(\Gamma)$. Therefore, if Γ satisfies the inequality

(1.13)
$$\|f\|_p^2 \leq \|f\|_2^2 + A \int \Gamma(f, f) \, d\mu, \qquad f \in \mathscr{A},$$

for some p > 2, then

(1.14)
$$D = D(\Gamma) \leq \pi \frac{\sqrt{2pA}}{p-2}$$

Together with Theorem 1, Theorem 2 answers our initial question.

THEOREM 3. Let L be a Markov generator satisfying the curvature-dimension inequality CD(R, n) for some R > 0 and $n \ge 2$. Then

$$D=D(\mathrm{L})\leqslant \pi\sqrt{rac{n-1}{R}}.$$

The case n = 2, and by extension $1 \le n \le 2$, is somewhat particular. Since (1.9) holds for every $p \ge 1$ in this case, we get together with (1.14)

$$D \leqslant \pi \left(\frac{n-1}{nR} \cdot \frac{2p}{p-2}\right)^{1/2}$$

for every p > 2. When $p \to \infty$,

$$D \leqslant \pi \left(\frac{2(n-1)}{nR}\right)^{1/2}.$$

This is optimal however only for n = 2.

When (1.11) holds, we also see that

$$D^2 \leq \pi^2 \frac{2p}{p-2} \cdot \frac{1}{\alpha(p)(nR/n-1) + (1-\alpha(p))\lambda_1}$$

for every $2 . Optimizing over p, we can obtain upper bounds on <math>\lambda_1$. For example, if R = 0, we get

$$\lambda_1 \leqslant \frac{\pi^2 n^2 (n+2)}{D^2}$$

(the constant is not sharp). If $R \ge -K$, $K \ge 0$, then

$$\lambda_1 \leqslant \frac{(n-1)K}{4} + \frac{C(n)}{D^2}$$

for some C(n) > 0 only depending on *n*. We thus recover, with these functional tools, geometric bounds first established in [Ch], [Che].

If (M, g) is a Riemannian manifold with dimension n and Ric $\ge (n-1)g$, and if the diameter of M is equal to π , S.-Y. Cheng [Che] showed that M is isometric to S_1^n , generalizing the Topogonov theorem [T] that was dealing with the sectional curvature (cf. [C] for a modern geometric proof of the Topogonov-Cheng result). Our next theorem is an analogue of this result. It is again formulated in terms of the Sobolev constant and shows that if $D(\Gamma) = \pi$, the constant in (1.12) is reached on functions of the form $f_{\lambda} = (1 + \lambda \sin(f))^{1-(n/2)}, -1 < \lambda < +1$, for some nonconstant function f with $\int \sin(f) d\mu = 0$. In particular, we include in this way the example of the spheres themselves.

THEOREM 4. Let Γ be a carré du champ satisfying the Sobolev inequality (1.12) for some n > 2. If there is a function f in \mathscr{A} such that $\|\Gamma(f, f)\|_{\infty} \leq 1$ and $\|\tilde{f}\|_{\infty} = \pi$, then there exist nonconstant extremal functions of (1.12). More precisely, if we translate f such that $\int \sin(f) d\mu = 0$, for every $-1 < \lambda < +1$,

$$\|f_{\lambda}\|_{2n/n-2}^{2} = \|f_{\lambda}\|_{2}^{2} + \frac{4}{n(n-2)} \int \Gamma(f_{\lambda}, f_{\lambda}) d\mu$$

where $f_{\lambda} = (1 + \lambda \sin(f))^{1-(n/2)}$. Furthermore, if we set $X = \sin(f)$, L agrees on the functions of X with the ultraspheric generator of dimension n; that is, for every smooth function φ on \mathbb{R} ,

$$L\varphi(X) = (1 - X^2)\varphi''(X) - nX\varphi'(X).$$

In particular, if $L = \Delta$ is the Laplace-Beltrami operator on an n-dimensional compact manifold (M, g) with $\text{Ric} \ge (n - 1)g$ and with diameter equal to π , then M is isometric to the sphere S_1^n .

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Applying the Sobolev inequality (1.12) to $f = 1 + \varepsilon \varphi$ where $\int \varphi d\mu = 0$ and using a Taylor expansion at $\varepsilon = 0$ shows that $\lambda_1 \ge n$ (that is, the analogue of the Lichnerowicz minoration, cf. [GHL]). In the same way, a Taylor expansion on the functions f_{λ} at $\lambda = 0$ in Theorem 4 shows that $\lambda_1 = n$. Therefore, if L is the Laplace-Beltrami operator on an *n*-dimensional Riemannian manifold (M, g)with Ric $\ge (n-1)g$ and if $D = \pi$, then $\lambda_1 = n$, and therefore, by Obata's theorem [O] (see [GHL]), *M* is isometric to the sphere S_1^n , proving the last assertion of Theorem 4. This functional approach thus provides a new proof of the Topogonov-Cheng theorem.

The next sections of this paper are devoted to the proofs of the preceding statements. In Section 2 we establish Theorem 2, and in Section 3, we prove Theorem 4.

2. Myers's diameter theorem. The scheme of the proof of Theorem 2 is the following. Let f be a function in \mathscr{A} such that $\|\Gamma(f, f)\|_{\infty} \leq 1$. We first apply Sobolev's inequality (1.12) to the family of functions $f_{\lambda} = (1 + \lambda \sin(f))^{1-(n/2)}$, $-1 < \lambda < +1$, to deduce a (nonlinear) differential inequality on

(2.1)
$$F(\lambda) = \int (1+\lambda\sin(f))^{2-n} d\mu, \quad -1 < \lambda < +1.$$

The crucial argument of the proof then consists in showing that when $\int \sin(f) d\mu > 0$ (resp., <0), then (essentially)

$$F(1) = \int \left(1 + \sin(f)\right)^{2-n} d\mu < \infty$$

(resp., $F(-1) < \infty$). Iterating the result on the basis again of (1.12), we actually have that

$$\|(1\pm\sin(f))^{-1}\|_{\infty}<\infty,$$

from which the conclusion then easily follows.

The first step of this program is summarized in the next proposition. Throughout this proof, we thus fix f in \mathscr{A} with $\|\Gamma(f, f)\|_{\infty} \leq 1$ and define F by (2.1). For every k > 0, let D_k be the differential operator on]-1,+1[defined by

$$D_k = \frac{1}{k} \lambda \frac{\partial}{\partial \lambda} + I.$$

Set also $\alpha = (n - 2)/n$ (<1).

PROPOSITION 5. If $G = D_{n-1}F$,

(2.2)
$$(D_{n-2}G)^{\alpha} + \alpha(1-\lambda^2)D_{n-2}G \leq (1+\alpha)G.$$

Proof. Let $f_{\lambda} = (1 + \lambda \sin(f))^{1-(n/2)}$, $-1 < \lambda < +1$. By the chain-rule formula for the operator Γ and the hypothesis $\|\Gamma(f, f)\|_{\infty} \leq 1$,

$$\int \Gamma(f_{\lambda}, f_{\lambda}) d\mu \leq \left(\frac{n}{2} - 1\right)^2 \lambda^2 \int (1 + \lambda \sin(f))^{-n} (1 - \sin^2(f)) d\mu.$$

Hence, by (1.12) applied to f_{λ} , for every λ ,

(2.3)
$$\left(\int (1+\lambda\sin(f))^{-n}d\mu\right)^{\alpha} \leq \int (1+\lambda\sin(f))^{2-n}d\mu + \alpha\lambda^{2}\int (1+\lambda\sin(f))^{-n}(1-\sin^{2}(f))d\mu.$$

Now, observe that

$$F(\lambda) = \int (1 + \lambda \sin(f))^{2-n} d\mu = \int (1 + \lambda \sin(f))^{-n} A(\lambda) d\mu,$$
$$\frac{F'(\lambda)}{n-2} = \int (1 + \lambda \sin(f))^{-n} B(\lambda) d\mu,$$
$$\frac{F''(\lambda)}{(n-1)(n-2)} = \int (1 + \lambda \sin(f))^{-n} C(\lambda) d\mu,$$

with $A(\lambda) = 1 + 2\lambda \sin(f) + \lambda^2 \sin^2(f)$, $B(\lambda) = -\sin(f) - \lambda \sin^2(f)$, $C(\lambda) = C = \sin^2(f)$. Clearly,

$$A(\lambda) + 2\lambda B(\lambda) = 1 - \lambda^2 \sin^2(f) = 1 - \lambda^2 C,$$

so that (2.3) reads as

$$\left(\int (1+\lambda\sin(f))^{-n}d\mu\right)^{\alpha} + \alpha(1-\lambda^2)\int (1+\lambda\sin(f))^{-n}d\mu$$
$$\leqslant 2\left(1-\frac{1}{n}\right)F(\lambda) + \frac{2}{n}\lambda F'(\lambda)$$
$$= (1+\alpha)G(\lambda).$$

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The conclusion follows since

$$\int (1+\lambda\sin(f))^{-n} d\mu = F(\lambda) + \frac{2}{n-2}\lambda F'(\lambda) + \frac{\lambda^2}{(n-1)(n-2)}F''(\lambda)$$
$$= D_{n-2}G(\lambda).$$

In the second part of the proof, we will be interested in the solutions H of the equation

(2.4)
$$(D_{n-2}H)^{\alpha} + \alpha(1-\lambda)^2 D_{n-2}H = (1+\alpha)H, \quad -1 < \lambda < +1.$$

The proposed solutions are given by

$$H(\lambda) = H_c(\lambda) = \frac{1}{1+\alpha} U(\lambda)^{2\alpha/1-\alpha} + \frac{\alpha}{1+\alpha} (1-\lambda^2) U(\lambda)^{2/1-\alpha}$$

where $c \in \mathbb{R}$ and

$$U(\lambda) = U_c(\lambda) = rac{c\lambda + \sqrt{c^2\lambda^2 + (1 - \lambda^2)}}{1 - \lambda^2}$$

Although we will not use these remarks later on, it might be useful to give a hint on how this solution may be obtained. First, it should be clear that if

$$\tilde{F}(\lambda) = \int_{-1}^{+1} (1 + \lambda x)^{2-n} d\mu_n(x), \qquad -1 < \lambda < +1,$$

where $d\mu_n(x) = c_n(1-x^2)^{(n/2)-1} dx$ is the invariant measure of the ultraspheric generator L_n of (1.4), then $\tilde{G} = D_{n-1}\tilde{F}$ is a particular solution of (2.4). Indeed, by Aubin's result (and since L_n is the projection on a diameter of the conformal Laplacian on S_1^n), the functions $(1 + \lambda x)^{1-(n/2)}$ are extremal functions of the Sobolev inequality (1.12) for L_n . Since in this case $\Gamma(f, f) = 1 - x^2$ when f(x) = x, all the inequalities of the proof of Proposition 5 are actually equalities for \tilde{F} (x playing the role of $\sin(f)$). It may be shown directly (see [Fo]) that $D_{n-2}\tilde{G} = (1 - \lambda^2)^{-n/2}$, $-1 < \lambda < +1$, so that $\tilde{G} = H_0$. However, we will find again this observation in the process of the proof of Theorem 4.

To describe the general solutions, let, for every $t \ge 0$ and $-1 < \lambda < +1$, $v = v(t, \lambda)$ be the unique nonnegative solution of the equation

(2.5)
$$v^{\alpha} + \alpha(1-\lambda^2)v = (1+\alpha)t.$$

Note that v(1,0) = 0. If we set $V(\lambda) = v(H(\lambda), \lambda)$, (2.4) reads

$$(2.6) D_{n-2}H = V, V^{\alpha} + \alpha(1-\lambda^2)V = (1+\alpha)H.$$

Taking the derivative of the equation $V^{\alpha} + \alpha(1 - \lambda^2)V = (1 + \alpha)H$, we see that the equation $D_{n-2}H = V$ amounts to

$$V' = \frac{2V}{\lambda(1-\alpha)} \cdot \frac{(1+\lambda^2) - V^{\alpha-1}}{(1-\lambda^2) + V^{\alpha-1}} \,.$$

Letting $W = V^{1-\alpha}$, it remarkably follows that this equation is actually independent of α (that is, of *n*) and may be written

$$W' = rac{2W}{\lambda} \cdot rac{(1+\lambda^2) - W^{-1}}{(1-\lambda^2) + W^{-1}} \, .$$

One may then show that $W = U^2 = U_c^2$, $c \in \mathbb{R}$, are solutions of the latter equation.

The next lemma compares the solutions G of the differential inequality (2.2) to the solutions $H_c, c \in \mathbb{R}$, of (2.4).

LEMMA 6. Assume that $G(\lambda_0) < H_c(\lambda_0)$ for some $\lambda_0 \in [0, 1)$. Then, for every $\lambda_0 \leq \lambda < 1$,

$$G(\lambda) \leq H_c(\lambda)$$
.

Proof. We write $H = H_c$. First note that since

$$D_{n-2}(G-H) \leq v(G,\lambda) - v(H,\lambda)$$

and since $v(t, \lambda)$ is increasing in t, $D_{n-2}(G-H) \leq 0$ on the set $\{G \leq H\}$. We know that $G(\lambda_0) < H(\lambda_0)$ for some $\lambda_0 \in [0, 1)$ and aim to show that $G \leq H$ on the whole interval $[\lambda_0, 1)$. Suppose that this is not the case, and let

$$\overline{\lambda} = \inf \{\lambda > \lambda_0; G(\lambda) = H(\lambda) \}.$$

Then $\lambda_0 < \overline{\lambda} < 1$ and $G \leq H$ on $[\lambda_0, \overline{\lambda}]$, so that $D_{n-2}(G - H) \leq 0$ on this interval. But this differential inequality may easily be integrated to yield that $\lambda^{n-2}(G - H)(\lambda)$ is nonincreasing on $[\lambda_0, \overline{\lambda}]$ so that

$$\lambda_0^{n-2}(G-H)(\lambda_0) \ge \overline{\lambda}^{n-2}(G-H)(\overline{\lambda}).$$

Since $G(\lambda_0) < H(\lambda_0)$, this contradicts the fact that, by continuity, $G(\overline{\lambda}) = H(\overline{\lambda})$. Lemma 6 is established. One may note that $H_c(0) = 1$ and $H'_c(0) = 4\alpha c/1 - \alpha^2$. Furthermore, a Taylor expansion shows that, when c < 0, $H_c(1) < \infty$, while when c > 0, $H_c(-1) < \infty$. Note also that G(0) = 1 and that

$$G'(0) = \frac{n}{n-1} F'(0) = -\frac{n(n-2)}{n-1} \int \sin(f) \, d\mu.$$

In particular, the sign of G'(0) is determined by the sign of $\int \sin(f) d\mu$.

Now, if $\int \sin(f) d\mu > 0$, we may choose c < 0 so that $G'(0) < H'_c(0) < 0$. Since $G(0) = H_c(0) = 1$, $G < H_c$ in a neighborhood of 0. Therefore, as a consequence of Lemma 6, if $\int \sin(f) d\mu > 0$, and as $\lambda \to 1$,

$$G(1) = \int (1 + \sin(f))^{1-n} \left(1 + \frac{1}{n-1} \sin(f) \right) d\mu \leq H_c(1) < \infty.$$

Replacing f by -f shows that if $\int \sin(f) d\mu < 0$, then $G(-1) < \infty$. Observe furthermore that since

$$\int (1+\lambda\sin(f))^{1-n} d\mu = G(\lambda) + \left(\frac{n-2}{n-1}\cdot\frac{1}{\lambda}-\lambda\right) \int (1+\lambda\sin(f))^{1-n}\sin(f) d\mu,$$

we have that

(2.7)
$$\int (1 \pm \sin(f))^{1-n} d\mu \leq \frac{n-1}{n-2} G(\pm 1).$$

Therefore,

(2.8)
$$\int (1 \pm \sin(f))^{1-n} d\mu < \infty$$

according as $\int \sin(f) d\mu > 0$ or < 0.

With the help of the next proposition, we now improve upon (2.8) and prove, using again the Sobolev inequality (1.12), that actually $(1 \pm \sin(f))^{-1}$ is μ -almost surely bounded in this setting.

PROPOSITION 7. Let f be such that $\|\Gamma(f, f)\|_{\infty} \leq 1$. If $\int (1 \pm \sin(f))^{1-n} d\mu < \infty$, then

$$\|(1\pm\sin(f))^{-1}\|_{\infty}<\infty.$$

Proof. It is enough to deal with the case $\int (1 + \sin(f))^{1-n} d\mu < \infty$. Let us apply inequality (1.12) to the family of functions $(1 + \sin(f))^{-p/2}$, $p \ge n-2$.

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Letting $\beta = 1/\alpha = n/(n-2) > 1$, and $\hat{F}(p) = \int (1 + \sin(f))^{-p} d\mu$, we have

(2.9)
$$\hat{F}(\beta p)^{1/\beta} \leq \hat{F}(p) + Cp^2 \hat{F}(p+1)$$

where C = 2/n(n-2). Since $\beta p > p+1$ when $p \ge n-2$, it already follows by iteration of (2.9) that $\hat{F}(p) < \infty$ for every $p \ge n-1$. We aim to prove that $\sup_{p \ge n-1} \hat{F}(p)^{1/p} < \infty$ (from which the conclusion follows). It may be assumed that $\hat{F}(p) \ge 1$ for some p large enough; otherwise, there is nothing to prove. But then, by Jensen's inequality,

$$\hat{F}(\beta p)^{1/\beta} \leq (1+Cp^2)\hat{F}(p+1).$$

A simple iteration procedure then yields the result. The proof is complete. \Box

It is worthwhile mentioning here that, in the preceding proposition, we only used that $\|\Gamma(f, f)\|_{\infty} < \infty$, and did not use the explicit value of the Sobolev constant.

We may now conclude the proof of Theorem 2. As a consequence of (2.8) and Proposition 7, we thus obtained that

$$\|(1\pm\sin(f))^{-1}\|_{\infty}<\infty$$

according as $\int \sin(f) d\mu > 0$ or <0. Let us first recall (cf. the comments following Theorem 4) that the Sobolev inequality (1.12) implies the spectral gap inequality

(2.10)
$$n \int \varphi^2 \, d\mu \leqslant \int \, \Gamma(\varphi, \varphi) \, d\mu$$

for every φ in \mathscr{A} with $\int \varphi \, d\mu = 0$. (Take $f = 1 + \varepsilon \varphi$ in (1.12) and use a Taylor expansion at $\varepsilon = 0$.)

Recall that we fixed f in \mathscr{A} with $\|\Gamma(f, f)\|_{\infty} \leq 1$. Applying (2.10) to both $\varphi = \sin(f)$ and $\varphi = \cos(f)$ easily shows that $\int \sin(f) d\mu$ and $\int \cos(f) d\mu$ cannot vanish at the same time. It is then easy to see that there exists $\theta \in \mathbb{R}$ such that

$$\int \sin(f+\theta) d\mu = 0$$
 and $\int \cos(f+\theta) d\mu > 0$.

Indeed, if $\int \cos(f) d\mu = 0$, we may take either $\theta = \pi/2$ or $\theta = 3\pi/2$. If $\int \cos(f) d\mu \neq 0$, let $\theta_1 \in]-\pi/2, \pi/2[$ be such that $\tan(\theta_1) = -\int \sin(f) d\mu/\int \cos(f) d\mu$ so that $\int \sin(f + \theta_1) d\mu = 0$. Then take either $\theta = \theta_1$ or $\theta = \theta_1 + \pi$. We aim to show that for μ -almost every x, y in E, $|f(x) - f(y)| \leq \pi$. Hence, taking if necessary some translate of f, we may and do assume that

(2.11)
$$\int \sin(f) \, d\mu = 0 \quad \text{and} \quad \int \cos(f) \, d\mu > 0.$$

We thus fix f in \mathscr{A} with $\|\Gamma(f, f)\|_{\infty} \leq 1$ and (2.11). Let also $0 < \varepsilon < \pi$. Set $h = f + \varepsilon$ so that

$$\int \sin(h) \, d\mu = \sin(\varepsilon) \int \cos(f) \, d\mu > 0.$$

Then, (2.8) and Proposition 7 applied to h indicate that

$$\|(1 + \sin(h))^{-1}\|_{\infty} = \|(1 + \sin(f + \varepsilon))^{-1}\|_{\infty} < \infty$$

Similarly,

$$\|(1-\sin(f-\varepsilon))^{-1}\|_{\infty}<\infty.$$

Let, for every $k \in \mathbb{Z}$, $x_k = (4k-1)\pi/2$, $y_k = (4k+1)\pi/2$. The preceding proves that there exists $\delta = \delta(\varepsilon) > 0$ such that, for every $k \in \mathbb{Z}$,

(2.12)
$$\mu(f + \varepsilon \in (x_k - \delta, x_k + \delta)) = \mu(f - \varepsilon \in (y_k - \delta, y_k + \delta)) = 0.$$

The conclusion will then follow from the next elementary lemma, which may be considered as a kind of mean value theorem for functions in the domain of a carré du champ operator Γ . We assume here that Γ is ergodic in the sense that if $\Gamma(f, f) = 0$, then f is constant μ -almost everywhere. This is clearly the case if Γ satisfies a spectral gap inequality.

LEMMA 8. Assume that Γ is ergodic, and let φ be such that for some $x \in E$ and $\delta > 0$, $\mu(\varphi \in (x - \delta, x + \delta)) = 0$. Then, $\mu(\varphi \ge x + \delta) = 0$ or 1.

Proof. Let Ψ be C^1 with $\Psi = 0$ on $(-\infty, x - \delta)$ and $\Psi = 1$ on $(x + \delta, \infty)$. Then $\Gamma(\Psi(\varphi), \Psi(\varphi)) = 0$, and since Γ is ergodic, the conclusion immediately follows.

In summary, and according to this lemma and (2.10), (2.12), we have thus obtained that for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that for every $k \in \mathbb{Z}$,

 $\mu(f + \varepsilon \ge x_k + \delta)$ and $\mu(f - \varepsilon \ge y_k + \delta) = 0 \text{ or } 1.$

Let k_0 be the smallest k in \mathbb{Z} such that $\mu(f + \varepsilon \ge x_{k+1} + \delta) = 0$. Then

$$x_{k_0} + \delta \leq f + \varepsilon < x_{k_0+1} + \delta$$

 μ -almost everywhere. Now, $\mu(f - \varepsilon \ge y_{k_0} + \delta) = 0$ or 1. If $f - \varepsilon \ge y_{k_0} + \delta$, then, μ -almost everywhere,

$$(4k_0+1)\frac{\pi}{2}+\delta+\varepsilon \leqslant f < (4k_0+3)\frac{\pi}{2}+\delta-\varepsilon,$$

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that certainly implies $|f(x) - f(y)| \le \pi$ for μ -almost all x, y in E but is unlikely to happen since ε is arbitrary in $(0, \pi)$. If $f - \varepsilon < y_{k_0} + \delta$, then, μ -almost everywhere,

$$(4k_0-1)\frac{\pi}{2}+\delta-\varepsilon\leqslant f<(4k_0+1)\frac{\pi}{2}+\delta+\varepsilon,$$

so that $|f(x) - f(y)| \le \pi + 2\varepsilon$ for μ -almost all x, y in E. Since $\varepsilon > 0$ is arbitrary, the proof of Theorem 2 is complete.

Note for further purposes that our conclusion is actually that, up to some translation, $\int \sin(f) d\mu = 0$ and $-\pi/2 \le f \le \pi/2 \mu$ -almost everywhere.

3. The Topogonov-Cheng sphere theorem. In this last section, we establish Theorem 4, which implies, as we have seen, an abstract version of the Topogonov-Cheng sphere theorem.

We assume therefore that the Sobolev inequality (1.12) holds and that there exists a nonconstant function f in \mathscr{A} such that $\|\Gamma(f, f)\|_{\infty} \leq 1$ and $\|\tilde{f}\|_{\infty} = \pi$. As in the previous section, we may if necessary take a translate of f and assume, as shown by the proof of Theorem 2, that $\int \sin(f) d\mu = 0$ and, for example, $\int \cos f d\mu > 0$, and $\|f\|_{\infty} = \pi/2$.

The main argument of the proof is contained in the following lemma. Let f be as before. With this function f, we define F as in (2.1) and set $G = D_{n-1}F$, the solution of the differential inequality (2.2) of Proposition 5. Recall also the solutions H_c , $c \in \mathbb{R}$, of (2.4).

LEMMA 9. If f is as before, $G = H_0$.

Proof. Replacing f by -f, it is enough to work on [0, 1]. According to Lemma 6, and since G'(0) = 0 $(\int \sin(f) d\mu = 0)$, for every c > 0, $G \leq H_c$, so that by continuity $G \leq H_0$. Now assume that there is $\lambda_0 > 0$ such that $G(\lambda_0) < H_0(\lambda_0)$. Then there exists c < 0 such that $G(\lambda_0) < H_c(\lambda_0)$, and, again by Lemma 6, we will have that $G(\lambda) \leq H_c(\lambda)$ for every $\lambda \geq \lambda_0$. Letting $\lambda \to 1$, $G(1) < \infty$. By (2.7) and Proposition 7, we would then conclude that $\|(1 + \sin(f))^{-1}\|_{\infty} < \infty$. This, however, contradicts the fact that $\|f\|_{\infty} = \pi/2$. Lemma 9 is established. \Box

We thus have obtained that (some translate of) f satisfies the equality in the differential inequality (2.2). In particular, for every $-1 < \lambda < +1$,

$$\|f_{\lambda}\|_{2n/n-2}^2 = \|f_{\lambda}\|_2^2 + \frac{4}{n(n-2)} \int \Gamma(f_{\lambda}, f_{\lambda}) d\mu,$$

where $f_{\lambda} = (1 + \lambda \sin(f))^{1 - (n/2)}$, and the first claim of Theorem 4 is proved.

It is worthwhile pointing out that if f is the function we just considered, and if $h_a = (\sqrt{1+a^2} + a \sin(f))^{1-(n/2)}, a \in \mathbb{R}$, then

$$||h_a||_{2n/n-2} = 1$$

for every $a \in \mathbb{R}$. Indeed, since $G = H_0 = (1 - \lambda^2)^{-\alpha/1 - \alpha} = (1 - \lambda^2)^{1 - (n/2)}$,

$$\int (1+\lambda \sin f)^{-n} d\mu = D_{n-2}G = (1-\lambda^2)^{-n/2},$$

and this is equal to $(1 + a^2)^{n/2}$ if $\lambda = a/\sqrt{1 + a^2}$. As we have seen while studying the solutions of (2.4), this is in particular the case for the extremal functions of Sobolev's inequalities on spheres or with respect to the ultraspheric generators.

We may also note to conclude that since h_a is extremal in Sobolev's inequality (1.12),

$$h_a^{(n+2)/(n-2)} - h_a = -\frac{4}{n(n-2)} Lh_a.$$

After a change of variables, we get, with $X = \sin(f)$,

$$-2\sqrt{1+a^2} X - a(1+X^2) = \frac{2}{n} \left[(\sqrt{1+a^2} + aX) LX - \frac{n}{2} a\Gamma(X,X) \right]$$

When a = 0, we recover that -LX = nX, and if we then replace LX by -nX and simplify by a, we see that $\Gamma(X, X) = 1 - X^2$. These observations thus indicate that on the functions of X, L coincide with the ultraspheric generator L_n of dimension n(see (1.4)) and conclude therefore the proof of Theorem 4. In a Riemannian setting, we can use Obata's theorem to conclude that L is "isometric" to the Laplacian of a sphere. In general, however, we do not know exactly what kind of rigidity can be expected.

References

- [AMS] S. AIDA, T. MASUDA, AND I. SHIGEKAWA, Logarithmic Sobolev inequalities and exponential integrability, J. Funct. Anal. 126 (1994), 83–101.
- [A] TH. AUBIN, Nonlinear Analysis on Manifolds, Monge-Ampère Equations, Grundlehren Math. Wiss. 252, Springer-Verlag, Berlin, 1982.
- [B1] D. BAKRY, "Inégalités de Sobolev faibles: un critère Γ₂" in Séminaire de Probabilités, Vol. 25, Lecture Notes in Math. 1485, Springer-Verlag, Berlin, 1991, 234–261.
- [B2] ——, "L'hypercontractivité et son utilisation en théorie des semigroupes" in Ecole d'Eté de Probabilités de Saint-Flour, 1992, Lecture Notes in Math. 1581, Springer-Verlag, Berlin, 1994, 1-114.
- [BM] D. BAKRY AND D. MICHEL, Inégalités de Sobolev et minorations du semigroupe de la chaleur, Ann. Fac. Sci. Toulouse Math. (5) 11 (1990), 23–66.
- [Be] W. BECKNER, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, Ann. of Math. (2) 138 (1993), 213-242.
- [C] I. CHAVEL, Riemannian Geometry: A Modern Introduction, Cambridge Tracts in Math., Cambridge Univ. Press, Cambridge, 1993.
- [Ch] J. CHEEGER, The relation between the Laplacian and the diameter for manifolds of non-negative curvature, Arch. Math. (Basel) 19 (1968), 558–560.
- [Che] S.-Y. CHENG, Eigenvalue comparison theorems and its geometric applications, Math. Z. 143 (1975), 289–297.

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- [F] E. FONTENAS, Sur les constantes de Sobolev des variétés riemanniennes compactes et les fonctions extrémales des sphères, to appear in Bull. Sci. Math. (2).
- [GHL] S. GALLOT, D. HULIN, AND J. LAFONTAINE, Riemannian Geometry, 2nd ed., Universitext, Springer-Verlag, Berlin, 1990.
- S. ILIAS, Constantes explicites pour les inégalités de Sobolev sur les variétés riemanniennes compactes, Ann. Inst. Fourier (Grenoble) 33, Fasc. 2 (1983), 151–165.
- [L] M. LEDOUX, Remarks on logarithmic Sobolev constants, exponential integrability, and bounds on the diameter, J. Math. Kyoto Univ. 35 (1995), 211–220.
- [M] S. B. MYERS, Connections between differential geometry and topology, Duke Math. J. 1 (1935), 376-391.
- [O] M. OBATA, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan 14 (1962), 333-340.
- [R] O. S. ROTHAUS, Hypercontractivity and the Bakry-Emery criterion for compact Lie groups, J. Funct. Anal. 65 (1986), 358–367.
- [T] V. A. TOPOGONOV, Riemannian spaces having their curvature bounded below by a positive number, Uspekhi Mat. Nauk. 14 (1959), 87–135.

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