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Inventiones mathematicae

Lévy–Gromov's isoperimetric inequality for an infinite dimensional diffusion generator

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Abstract. We establish, by simple semigroup arguments, a Lévy–Gromov isoperimetric inequality for the invariant measure of an infinite dimensional diffusion generator of positive curvature with isoperimetric model the Gaussian measure. This produces in particular a new proof of the Gaussian isoperimetric inequality. This isoperimetric inequality strengthens the classical logarithmic Sobolev inequality in this context. A local version for the heat kernel measures is also proved, which may then be extended into an isoperimetric inequality for the Wiener measure on the paths of a Riemannian manifold with bounded Ricci curvature.

1. Introduction

Lévy–Gromov's isoperimetric inequality [Lé], [Gr] (cf. e.g. [G-H-L]) indicates that if M is a (compact) connected Riemannian manifold of dimension n ($\geqq 2$) and of Ricci curvature bounded below by $R > 0$, then its isoperimetric function is larger than or equal to the isoperimetric function of the sphere of dimension n and constant curvature R. In other words, if we denote by $\sigma(r)$ the normalized volume of a geodesic ball of radius $r \geqq 0$ on the n-sphere with curvature R, for every set A in M with smooth boundary ∂A,

$$\sigma' \circ \sigma^{-1}(\mu(A)) \leqq \mu_s(\partial A) \tag{1.1}$$

where μ denotes the normalized Riemannian measure on M and $\mu_s(\partial A)$ stands for the surface measure of the boundary ∂A of A (see below). This holds in particular for $\mu = \sigma$ itself ([Lé], [Sc]).

By Poincaré's limit (cf. e.g. [MK]), spherical measures converge to Gaussian distributions to yield a Gaussian isoperimetric inequality [Bor], [S-T]. Let γ denote the canonical Gaussian measure on $\mathbb{R}^k$ with density with respect to Lebesgue measure $(2\pi)^{-k/2} \exp(-|x|^2/2)$ (or, more generally, a centered

Gaussian measure on some abstract Wiener space). Then, for every Borel set A in $\mathbb{R}^k$ with smooth boundary,

$$\varphi \circ \Phi^{-1}(\gamma(A)) \leqq \gamma_s(\partial A) \tag{1.2}$$

where $\Phi(r) = (2\pi)^{-1/2} \int_{-\infty}^{r} e^{-x^2/2} dx$, $r \in \mathbb{R}$, is the distribution function of the canonical Gaussian measure in dimension one and $\varphi = \Phi'$. In particular, half-spaces satisfy the equality in (1.2) and are the extremal sets of the Gaussian isoperimetric inequality. It is known also (cf. [Bo1], [B-H]) that the infinitesimal versions (1.1) or (1.2) of the isoperimetric statement may easily be integrated. It yields, in the Gaussian case for example, that if A is a Borel set in $\mathbb{R}^k$ with $\gamma(A) \geqq \Phi(a)$, then, for every $r \geqq 0$, $\gamma(A_r) \geqq \Phi(a+r)$ where A_r is the Euclidean (or Hilbertian in case of an abstract Wiener measure) neighborhood of order r of A. (This was actually established directly in [F-L-M], [Bor], [S-T].)

The aim of this work is to establish a version of the Lévy–Gromov isoperimetric inequality for "infinite dimensional generators" with isoperimetric model the Gaussian isoperimetric function $\mathcal{U} = \varphi \circ \Phi^{-1}$. We will work in the abstract framework of Markov diffusion generators on some probability space which allows us to freely speak of curvature and infinite dimension. The proof of our isoperimetric inequalities only relies on elementary semigroup arguments and provides at the same time a new simple proof of Gaussian isoperimetry. Isoperimetric inequalities on spheres or in Gauss space are usually established through delicate symmetrization arguments ([Sc], [F-L-M] – cf. e.g. [B-Z] and the references therein –, and [Eh] for the Gaussian case). Recently, S. Bobkov [Bo2] (after prior contributions by M. Talagrand [Ta]) gave a surprising proof of the Gaussian isoperimetric inequality via a two-point inequality and the central limit theorem, very close in spirit to Gross' approach to logarithmic Sobolev inequalities [Gro]. His proof was inspired by the following functional form of the Gaussian isoperimetric inequality that he studied in a previous work [Bo1] and that strongly influenced this paper: for every smooth function f on $\mathbb{R}^k$ with values on $[0,1]$,

$$\mathcal{U}\left(\int f d\gamma\right) - \int \mathcal{U}(f)\, d\gamma \leqq \int |\nabla f| d\gamma \tag{1.3}$$

where we recall that $\mathcal{U} = \varphi \circ \Phi^{-1}$. Since $\mathcal{U}(0) = \mathcal{U}(1) = 0$, the functional inequality (1.3) reduces in a simple way to the isoperimetric statement (1.2) when f approximates the indicator function of some set A. Rather than isoperimetry itself, we put in this work main emphasis on functional inequalities and will namely establish an inequality such as (1.3) in an abstract infinite dimensional semigroup framework. Following S. Bobkov [Bo2], we will actually prove a strengthened form of (1.3), which, in the Gaussian case, is the form established in [Bo2] via a two-point argument, and reads

$$\mathcal{U}\left(\int f d\gamma\right) \leqq \int \sqrt{\mathcal{U}^2(f) + |\nabla f|^2} d\gamma\,. \tag{1.4}$$

With respect to (1.3), this inequality, like logarithmic Sobolev inequalities, may easily be tensorized.

Our main purpose in this work will thus be to establish (1.3) and (1.4) for the invariant measure of a positively curved diffusion generator. In particular, it will follow that the distributions of Lipschitz functions for this generator are contractions of the canonical Gaussian measure. We will actually deal more generally with heat kernel measures and prove a whole family of functional inequalities of this type under a general lower bound on the curvature. These inequalities may then be tensorized by the Markov property to yield sharp functional inequalities of isoperimetric type for the Wiener measure on the (Brownian) paths of a Riemannian manifold with Ricci curvature bounded above and below. It might be worthwhile pointing out that the inequalities of isoperimetric type we will establish are potentially much stronger than logarithmic Sobolev inequalities. This has been shown in case of Gaussian measures in [Led1] and we will prove a general result in this respect (Theorem 3.2).

The main results are presented in Sect. 2. In Sect. 3, we show the stability by tensorization of the family of inequalities under study and prove that the distributions of Lipschitz functions are contractions of Gaussian measures. In the next section, we briefly discuss several similar results and inequalities for hypercontractive generators. We however observe that curvature assumptions cannot in general be dispensed. In the last section, we extend our functional isoperimetric inequalities to path spaces following the corresponding approach by P.E. Hsu [Hs] for logarithmic Sobolev inequalities.

Before turning to our main conclusions, let us illustrate some of the ideas of this work by a simple direct semigroup proof of (1.3). Recall the Ornstein–Uhlenbeck semigroup $(P_t)_{t \geqq 0}$ with integral representation

$$P_t f(x) = \int_{\mathbb{R}^k} f(\mathrm{e}^{-t}x + (1-\mathrm{e}^{-2t})^{1/2} y)\, d\gamma(y), \quad x \in \mathbb{R}^k ,$$

for every f say in $\mathrm{L}^1(\gamma)$. The operators P_t are contractions on all $\mathrm{L}^p(\gamma)$-spaces, symmetric and invariant with respect to γ and with infinitesimal generator L acting on smooth functions f on $\mathbb{R}^k$ by

$$\mathrm{L} f(x) = \Delta f(x) - \langle x, \nabla f(x) \rangle .$$

Moreover, the integration by parts formula for L indicates that for f and g smooth,

$$\int f(-\mathrm{L}g)\, d\gamma = \int \langle \nabla f, \nabla g \rangle d\gamma .$$

Let f be a fixed smooth function on $\mathbb{R}^k$ with values in [0,1]. It might actually be convenient to assume throughout the argument that $0 < \varepsilon \leqq f \leqq 1 - \varepsilon$ and let then ε tend to zero. Recall $\mathcal{U} = \varphi \circ \Phi^{-1}$ and observe the fundamental differential equation

$$\mathcal{U}'' = -\frac{1}{\mathcal{U}} . \tag{1.5}$$

As in the semigroup proofs of logarithmic Sobolev inequalities ([B-E], [D-S]), we can write, by the semigroup properties and integration by parts,

$$
\begin{aligned}
\mathscr{U}\left(\int f\,d\gamma\right) - \int \mathscr{U}(f)\,d\gamma &= \int_0^\infty \frac{d}{dt}\left(\int \mathscr{U}(P_t f)\,d\gamma\right) dt \\
&= \int_0^\infty \left(\int \mathscr{U}'(P_t f)\,\mathrm{L}P_t f\,d\gamma\right) dt \\
&= \int_0^\infty \left(\int \frac{|\nabla P_t f|^2}{\mathscr{U}(P_t f)} d\gamma\right) dt
\end{aligned} \tag{1.6}
$$

where we used (1.5) in the last step. It is clear on the integral representation of P_t that it satisfies the commutation property $\nabla P_t f = \mathrm{e}^{-t} P_t(\nabla f)$. In particular, $|\nabla P_t f| \leqq \mathrm{e}^{-t} P_t(|\nabla f|)$. The key argument is then to split $|\nabla P_t f|^2$ to get, for every $t \geqq 0$,

$$
\frac{|\nabla P_t f|^2}{\mathscr{U}(P_t f)} \leqq \frac{|\nabla P_t f|}{\mathscr{U}(P_t f)} \mathrm{e}^{-t} P_t(|\nabla f|)\,. \tag{1.7}
$$

Our aim is to obtain a pointwise upper-bound of the ratio

$$
\frac{|\nabla P_t f|}{\mathscr{U}(P_t f)}
$$

again by the commutation property. It is worthwhile mentioning that the preceding splitting and the bound we will obtain are optimal in case f is the indicator function of a half-space. To this aim, fix $t > 0$ and write that

$$
[\mathscr{U}(P_t f)]^2 - [P_t(\mathscr{U}(f))]^2 = -\int_0^t \frac{d}{ds}[P_s(\mathscr{U}(P_{t-s} f))]^2 ds\,.
$$

By (1.5),

$$
\begin{aligned}
-\frac{d}{ds}[P_s(\mathscr{U}(P_{t-s} f))]^2 &= -2P_s(\mathscr{U}(P_{t-s} f))P_s(\mathscr{U}''(P_{t-s} f)|\nabla P_{t-s} f|^2) \\
&= 2P_s(\mathscr{U}(P_{t-s} f))P_s\left(\frac{|\nabla P_{t-s} f|^2}{\mathscr{U}(P_{t-s} f)}\right)\,.
\end{aligned}
$$

Since P_s is given by a kernel, it satisfies a Cauchy–Schwarz inequality, and thus

$$
P_s(Y)P_s\left(\frac{X^2}{Y}\right) \geqq [P_s(X)]^2, \quad X, Y \geqq 0\,.
$$

Hence, with $X = |\nabla P_{t-s} f|$ and $Y = \mathscr{U}(P_{t-s} f)$,

$$
[\mathscr{U}(P_t f)]^2 - [P_t(\mathscr{U}(f))]^2 \geqq 2\int_0^t [P_s(|\nabla P_{t-s} f|)]^2 ds\,.
$$

Since $P_s(|\nabla g|) \geqq \mathrm{e}^s |\nabla P_s g|$, with $g = P_{t-s} f$ it follows that

$$
[\mathscr{U}(P_t f)]^2 - [P_t(\mathscr{U}(f))]^2 \geqq 2\int_0^t \mathrm{e}^{2s} ds |\nabla P_t f|^2 = (\mathrm{e}^{2t} - 1)|\nabla P_t f|^2\,.
$$

In particular (and this is optimal on indicator functions), for every $t > 0$,

$$\frac{|\nabla P_t f|}{\mathcal{U}(P_t f)} \leqq \frac{1}{\sqrt{e^{2t} - 1}} .$$

Together with (1.6) and (1.7), we have therefore obtained that

$$\begin{aligned}\mathcal{U}\left(\int f d\gamma\right) - \int \mathcal{U}(f)\, d\gamma &\leqq \int_0^\infty \frac{e^{-t}}{\sqrt{e^{2t} - 1}} \left(\int P_t(|\nabla f|)\, d\gamma\right) dt \\ &= \int_0^\infty \frac{e^{-t}}{\sqrt{e^{2t} - 1}} dt \int |\nabla f| d\gamma \\ &= \int |\nabla f| d\gamma\end{aligned}$$

since, by invariance, $\int P_t g d\gamma = \int g d\gamma$ for every t. Thus, (1.3) is established in this way. Note that we basically only used the fundamental theorem of calculus and the Cauchy–Schwarz inequality. The preceding argument may also be developed on the basis of the Ornstein–Uhlenbeck semigroup on an abstract Wiener space.

It might be worthwhile mentioning for further purposes that the function $\mathcal{U} = \varphi \circ \Phi^{-1}$ defined on [0,1] is non-negative, concave, symmetric with respect to the vertical line going through $\frac{1}{2}$ with a maximum there equal to $(2\pi)^{-1/2}$ and such that $\mathcal{U}(0) = \mathcal{U}(1) = 0$. Its behavior at 0, or at 1 by symmetry, is given by the equivalence

$$\lim_{x \to 0} \frac{\mathcal{U}(x)}{x\sqrt{2 \log \frac{1}{x}}} = 1 . \tag{1.8}$$

This is easily seen by noting that the derivative of $\mathcal{U}(x)$ is $-\Phi^{-1}(x)$ which is of the order of $\sqrt{2 \log \frac{1}{x}}$ as $x \to 0$. Recall finally that $\mathcal{U}$ satisfies the differential equation (1.5) $\mathcal{U}\mathcal{U}'' = -1$.

2. Infinite dimensional Lévy–Gromov isoperimetric inequalities

To describe our main results, we first introduce the abstract Markov generator setting we will deal with following [Ba3], [Ba4] (to which we refer for further details). On some measure space $(E, \mathcal{E}, \mu)$, let L be a Markov generator associated to some semigroup $(P_t)_{t \geqq 0}$ continuous in $L^2(\mu)$. We will assume that L and $(P_t)_{t \geqq 0}$ are invariant with respect to μ. We assume furthermore that we are given a nice algebra $\mathcal{A}$ of (bounded) functions on E dense in the L^2-domain of L, stable by L and P_t and by the action of C^∞ functions which are zero at zero. The stability by P_t may not be satisfied even in basic examples such as non-degenerate second order differential operators with no constant term on a smooth manifold. This assumption is however not strictly necessary and is mostly only used for convenience (see below). When μ is finite, we always normalize it to a probability measure and assume that $\mathscr{A}$ contains the constants and is stable by C^∞ functions. We furthermore assume in this case that $P_t f \to \int f d\mu$ as $t \to \infty$ for every f in $\mathscr{A}$.

We may introduce, following P.-A. Meyer, the so-called "carré du champ" operator Γ as the symmetric bilinear operator on $\mathscr{A} \times \mathscr{A}$ defined by

$$2\Gamma(f,g) = \mathrm{L}(fg) - f\mathrm{L}g - g\mathrm{L}f, \quad f,g \in \mathscr{A} .$$

Note that $\Gamma(f,f) \geqq 0$. We will say that L is a diffusion if for every C^∞ function Ψ on $\mathbb{R}^k$, and every finite family $F = (f_1, \ldots, f_k)$ in $\mathscr{A}$,

$$\mathrm{L}\Psi(F) = \nabla\Psi(F) \cdot \mathrm{L}F + \nabla\nabla\Psi(F) \cdot \Gamma(F,F) . \tag{2.1}$$

In particular, if Ψ is C^∞ on $\mathbb{R}$, for every f in $\mathscr{A}$,

$$\mathrm{L}\Psi(f) = \Psi'(f)\mathrm{L}f + \Psi''(f)\Gamma(f,f) .$$

This hypothesis essentially expresses that L is a second order differential operator with no constant term and that we have a chain rule formula for Γ, $\Gamma(\Psi(f),g) = \Psi'(f)\Gamma(f,g)$, $f,g \in \mathscr{A}$. By the diffusion and invariance properties,

$$\int \Psi(f)(-\mathrm{L}f)\, d\mu = \int \Psi'(f)\Gamma(f,f)\, d\mu, \quad f \in \mathscr{A} .$$

One basic operator is the Laplace–Beltrami operator Δ on a complete connected Riemannian manifold M (with its Riemannian measure). For $\mathscr{A}$ the class, say, of C_c^∞ functions on M (that is however not stable by the heat semigroup in the non-compact case), $\Gamma(f,f)$ is simply the Riemannian length (squared) $|\nabla f|^2$ of the gradient ∇f of $f \in \mathscr{A}$. The previous abstract framework includes a number of further examples of interest (cf. [Ba2]). For example, one may consider $\mathrm{L} = \Delta + X$ where X is a smooth vector field on M, or more general second order differential operators with no constant term. We may also consider infinite dimensional examples such as the Ornstein–Uhlenbeck generator on $\mathbb{R}^k$

$$\mathrm{L}f(x) = \Delta f(x) - \langle x, \nabla f(x)\rangle$$

(or, more generally, on some abstract Wiener space for an appropriate algebra $\mathscr{A}$) with invariant measure the canonical Gaussian measure γ.

Curvature, and dimension, in this setting are introduced via the iterated carré du champ operator Γ_2 defined by

$$2\Gamma_2(f,g) = \mathrm{L}\Gamma(f,g) - \Gamma(f,\mathrm{L}g) - \Gamma(g,\mathrm{L}f), \quad f,g \in \mathscr{A} .$$

For simplicity, we write below $\Gamma(f) = \Gamma(f,f)$ and similarly for Γ_2. In a Riemannian setting, Bochner's formula (cf. [Ch], [G-H-L]) indicates that

$$\Gamma_2(f) = \Gamma_2(f,f) = \mathrm{Ric}(\nabla f, \nabla f) + \|\mathrm{Hess\,f}\|_2^2$$

where Ric is the Ricci tensor on M and $\|\mathrm{Hess\,f}\|_2$ is the Hilbert–Schmidt norm of the tensor of the second derivatives of f. We will say that L satisfies a curvature-dimension inequality $\mathrm{CD}(R,n)$ of curvature $R \in \mathbb{R}$ and dimension $n \geqq 1$ if, for all functions f in $\mathscr{A}$ (μ-almost everywhere),

$$\Gamma_2(f) \geqq R\Gamma(f) + \frac{1}{n}(\mathrm{L}f)^2 . \tag{2.2}$$

A n-dimensional complete Riemannian manifold (M,g) with Ricci curvature bounded below, or rather the Laplacian Δ on M, satisfies the inequality $\mathrm{CD}(R,n)$ with R the infimum of the Ricci tensor over all unit tangent vectors (recall that $\|\mathrm{Hess}\,\mathrm{f}\|_2^2 \geqq \frac{1}{n}(\Delta f)^2$). If $\mathrm{L} = \Delta + \nabla h$ for a smooth function h, and if (and only if), as symmetric tensors,

$$\nabla h \otimes \nabla h \leqq (m-n)[\mathrm{Ric} - \nabla\nabla h - \rho g] \tag{2.3}$$

with $m \geqq n$, then L satisfies $\mathrm{CD}(\rho,m)$ (cf. [Ba3], Proposition 6.2).

We will say more simply that L is of curvature $R \in \mathbb{R}$ if, for all functions f in $\mathscr{A}$,

$$\Gamma_2(f) \geqq R\Gamma(f) \tag{2.4}$$

(and write sometimes $\Gamma_2 \geqq R\Gamma$.) This definition thus corresponds to an infinite dimensional generator. The Ornstein–Uhlenbeck generator, even on a finite dimensional state space, is of curvature 1, and of no finite dimension. Actually, the Ornstein–Uhlenbeck generator is of no finite dimension for any curvature $R \in \mathbb{R}$. Indeed, in this example, for every smooth function f on $\mathbb{R}^k$ say, $\Gamma(f) = |\nabla f|^2$, $\Gamma_2(f) = |\nabla f|^2 + \|\mathrm{Hess}\,\mathrm{f}\|_2^2$ and

$$\|\mathrm{Hess}\,\mathrm{f}\|_2^2 \geqq (R-1)|\nabla f|^2 + c(\mathrm{L}f)^2$$

is impossible for some $c > 0$, $R \in \mathbb{R}$ and every f as can be seen by choosing for example $f(x) = |x|^2$ and by letting $x \to \infty$. The Laplacian Δ on a manifold (M,g) with Ricci curvature bounded below by R, $R \in \mathbb{R}$, is of curvature R. The operator $\mathrm{L} = \Delta + \nabla h$ on (M,g) is of curvature ρ as soon as (cf. (2.3))

$$\mathrm{Ric} - \nabla\nabla h \geqq \rho g .$$

Conditions for more general differential operators of the form

$$\mathrm{L}f(x) = \sum_{i,j=1}^{k} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{k} b_i(x)\frac{\partial f}{\partial x_i}(x) ,$$

where $A(x) = (a_{ij}(x))_{1 \leqq i,j \leqq k}$ is symmetric positive definite at every point, to be of some curvature may be given in the same spirit.

Let us mention that, in these examples, although the natural algebra $\mathscr{A}$ may not be stable by P_t, it is clear that $\Gamma(P_t f)$ and $\Gamma_2(P_t f)$ may be defined for large classes of smooth functions (on Riemannian manifolds, cf. e.g. [El], [Ba2]). This is basically only what is really needed in the proofs below concerning $\mathscr{A}$. We actually put emphasis in this work on the structure of the algebraic methods we develop. In this regard, the stability by P_t removes all kinds of analysis questions which are of a different nature. The question of extending the results presented in this work to large classes of functions in the domain of given generators is thus another issue not addressed here.

The curvature assumption (2.4) $\Gamma_2 \geqq R\Gamma$ on the infinitesimal generator L may be translated equivalently on the semigroup $(P_t)_{t \geqq 0}$. It is known indeed and is easy to prove that L is of curvature R if and only if, for every f in $\mathscr{A}$ and every $t \geqq 0$,

$$\Gamma(P_t f) \leqq \mathrm{e}^{-2Rt} P_t(\Gamma(f)) . \tag{2.5}$$

For the proof, let, for $f \in \mathscr{A}$ and $t > 0$ fixed, $F(s) = \mathrm{e}^{-2Rs} P_s(\Gamma(P_{t-s} f))$, $0 \leqq s \leqq t$. Now, by definition of Γ_2,

$$F'(s) = 2\mathrm{e}^{-2Rs}[-RP_s(\Gamma(P_{t-s} f)) + P_s(\Gamma_2(P_{t-s} f))] .$$

Hence, by (2.4) applied to $P_{t-s} f$ for every s, F is non-decreasing and (2.5) follows. The converse is similar.

Now, when L is a diffusion, it is also true that the curvature condition (2.4) is equivalent to saying that, for every f in $\mathscr{A}$ and every $t \geqq 0$,

$$\sqrt{\Gamma(P_t f)} \leqq \mathrm{e}^{-Rt} P_t \left(\sqrt{\Gamma(f)} \right) . \tag{2.6}$$

Actually, the equivalent infinitesimal version of (2.6) indicates that for every f in $\mathscr{A}$,

$$\Gamma(f)(\Gamma_2(f) - R\Gamma(f)) \geqq \tfrac{1}{4}\Gamma(\Gamma(f)) . \tag{2.7}$$

This is well-known in a Riemannian setting (see e.g. [D-L], [El], [D-S], [Ba4]...) but is less familiar in the preceding abstract framework and has been established in [Ba1], p. 148–149, in the study of Riesz' transforms. More precisely, it was shown there that (2.7) follows from the curvature assumption $\Gamma_2 \geqq R\Gamma$ by the diffusion property (cf. also [Ba4]). Parts of what will be accomplished below may also be seen as extensions of the relation between (2.6) and (2.4) (and (2.7)).

It is well-known [B-E] that a Markov diffusion generator L of curvature $R > 0$ with finite (normalized) invariant measure μ is hypercontractive, that is, equivalently, satisfies the logarithmic Sobolev inequality

$$\int f^2 \log f^2 d\mu - \int f^2 d\mu \log \int f^2 d\mu \leqq \frac{2}{R} \int f(-\mathrm{L} f)\, d\mu = \frac{2}{R} \int \Gamma(f)\, d\mu \tag{2.8}$$

(for every f in $\mathscr{A}$). Observe that under the curvature-dimension hypothesis $\mathrm{CD}(R, n)$, the constant may be improved to $2(n-1)/nR$ [B-E]. Our purpose here will actually be to prove that under the same curvature assumption we also have an isoperimetric inequality for the measure μ in the form of the functional inequalities (1.3) and (1.4). We will deal more generally with heat kernel measures and will establish the following rather general statement. Although it may appear difficult to grasp at a first reading, we decided to present in a synthetic form the various inequalities we want to emphasize since they basically all follow from the same scheme of proof. Recall the Gaussian isoperimetric function $\mathscr{U} = \varphi \circ \Phi^{-1}$.

Theorem 2.1. *Let* L *be a Markov diffusion generator such that, for some* $R \in \mathbb{R}$, $\Gamma_2(f) \geqq R\Gamma(f)$ *for every* f *in* $\mathscr{A}$. *Then, for every* f *in* $\mathscr{A}$ *with values in* $[0,1]$, *every* $\alpha \geqq 0$ *and every* $t \geqq 0$,

$$\sqrt{\mathscr{U}^2(P_t f) + \alpha\Gamma(P_t f)} \leqq P_t\left(\sqrt{\mathscr{U}^2(f) + c_\alpha(t)\Gamma(f)}\right)$$

where

$$c_\alpha(t) = \frac{1 - \mathrm{e}^{-2Rt}}{R} + \alpha \mathrm{e}^{-2Rt}, \quad t \geqq 0 .$$

Notice that since $c_{c_\alpha(s)}(t) = c_\alpha(s+t)$, the inequalities of Theorem 2.1 are stable by P_t. We will see in Section 3 that these inequalities are also stable by tensorization.

Theorem 2.1 admits several corollaries according to the choice of α. If $\alpha \to \infty$, we first recover (2.6). If $R > 0$, we may let $\alpha = 1/R$ ($c_\alpha(t) = 1/R$ for every $t \geqq 0$) to get, for every f and $t \geqq 0$,

$$\sqrt{R\mathscr{U}^2(P_t f) + \Gamma(P_t f)} \leqq P_t\left(\sqrt{R\mathscr{U}^2(f) + \Gamma(f)}\right) .$$

When μ is finite (and normalized), we can let $t \to \infty$ to get the following announced corollary (that extends the Gaussian inequalities (1.4) and (1.3)).

Corollary 2.2. *Let* L *be a Markov diffusion generator of curvature* $R > 0$ *with invariant probability measure* μ. *Then, for every* f *in* $\mathscr{A}$ *with values in* $[0,1]$,

$$\sqrt{R}\,\mathscr{U}\left(\int f d\mu\right) \leqq \int \sqrt{R\mathscr{U}^2(f) + \Gamma(f)}\, d\mu \tag{2.9}$$

and

$$\sqrt{R}\left[\mathscr{U}\left(\int f d\mu\right) - \int \mathscr{U}(f)\, d\mu\right] \leqq \int \sqrt{\Gamma(f)}\, d\mu . \tag{2.10}$$

As another choice, we can take $\alpha = 0$ so that

$$c_0(t) = \frac{1 - \mathrm{e}^{-2Rt}}{R} \quad (= 2t \text{ if } R = 0) .$$

Hence we may state the following.

Corollary 2.3. *Let* L *be a Markov diffusion generator of some curvature* $R \in \mathbb{R}$. *Then for every* f *in* $\mathscr{A}$ *with values in* $[0,1]$, *and every* $t \geqq 0$,

$$\mathscr{U}(P_t f) \leqq P_t\left(\sqrt{\mathscr{U}^2(f) + c_0(t)\Gamma(f)}\right) .$$

Note that when $R > 0$, μ is finite and $t \to \infty$, the inequality of Corollary 2.3 reads as (2.9).

Before turning to the proof of Theorem 2.1, let us comment the isoperimetric aspects of the preceding inequalities, especially (2.9) or (2.10) with say $R = 1$ for simplicity. It namely gives rise to a Lévy–Gromov isoperimetric inequality in this infinite dimensional setting. On more concrete spaces, (2.9) and (2.10) indeed really turn into a geometric inequality. We may define a pseudo-metric d on E by setting

$$d(x,y) = \operatorname{ess\,sup} |f(x) - f(y)|, \quad x, y \in E ,$$

the supremum being running over all f's in $\mathscr{A}$ with $\Gamma(f) \leqq 1$ almost surely. Assume actually for what follows that functions f in $\mathscr{A}$ are true functions (rather than classes), and that d is a true metric and μ a separable non-atomic Borel probability measure on (E,d). Assume furthermore that $\sqrt{\Gamma(f)}$ may be identified to a modulus of gradient as

$$\sqrt{\Gamma(f)}(x) = \limsup_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x,y)}, \quad x \in E .$$

These requirements are in particular fulfilled in differentiable structures such as Riemannian manifolds with Riemannian measures. Then, when f approximates the indicator function of some closed set A in E, $\int \sqrt{\Gamma(f)}d\mu$ approaches the lower-outer Minkowski content of the boundary of A

$$\mu_s(\partial A) = \liminf_{r\to 0} \frac{1}{r}[\mu(A_r) - \mu(A)] ,$$

where $A_r = \{x \in E\,; d(x,A) < r\}$. Since $\mathscr{U}(0) = \mathscr{U}(1) = 0$, (2.9) or (2.10) (with thus $R = 1$) therefore read on sets as

$$\mathscr{U}(\mu(A)) \leqq \mu_s(\partial A) . \tag{2.11}$$

Hence, the isoperimetric function of μ is larger than or equal to the Gaussian isoperimetric function $\mathscr{U}$, which is the analogue of Lévy–Gromov's result.

This differential inequality may also be integrated to yield that whenever A is a Borel set in (E,d) with $\mu(A) \geqq \Phi(a)$ for some real number a, for every $r \geqq 0$,

$$\mu(A_r) \geqq \Phi(a + r) . \tag{2.12}$$

For example, if f is such that $\Gamma(f) \leqq 1$ and if $\mu(\{f \leqq m\}) \geqq \frac{1}{2} = \Phi(0)$, for every $r \geqq 0$,

$$\mu(\{f \leqq m + r\}) \geqq \Phi(r) \geqq 1 - \tfrac{1}{2}\mathrm{e}^{-r^2/2} . \tag{2.13}$$

This property may also be seen as an infinite dimensional analogue of the Riemannian comparison theorems of volumes of balls (cf. [Ch]). We will come back to it from another point of view in Theorem 3.2. We thus recover the full strength of the Gaussian isoperimetric inequality with the half-spaces as extremal sets [Bor], [S-T]. We refer to [Bo1] and [B-H] for a proof of the equivalence between (2.11) and (2.12) (and (2.10) for the Gaussian measure [Bo1]) and for further general comments and results on these geometric aspects of the functional inequalities (2.9) and (2.10).

We now prove Theorem 2.1. The argument is very similar to the proof of (2.6) (cf. [Ba1], [Ba4]) and makes basic use of the equation $\mathscr{U}\mathscr{U}'' = -1$.

Proof of Theorem 2.1. Let $f \in \mathscr{A}$ with values in $[0,1]$ and $t \geqq 0$ be fixed. (As in the introduction, we may assume preferably that $0 < \varepsilon \leqq f \leqq 1-\varepsilon$ and let then ε tend to zero.) For every $0 \leqq s \leqq t$, set

$$F(s) = P_s\left(\sqrt{\mathscr{U}^2(P_{t-s}f) + c_\alpha(s)\Gamma(P_{t-s}f)}\right) .$$

Since $c_\alpha(0) = \alpha$, it will be enough to show that F is non-decreasing. We state and prove a general result in this regard.

Lemma 2.4. *Let Ψ be smooth on $\mathbb{R}^3$, $f \in \mathscr{A}$ and $t > 0$ be fixed. Then*

$$\frac{d}{ds}P_s(\Psi(s, P_{t-s}f, \Gamma(P_{t-s}f))) = P_s(K)$$

with

$$K = \partial_1\Psi + 2\partial_3\Psi\Gamma_2(f) + \partial_2^2\Psi\Gamma(f) + 2\partial_2\partial_3\Psi\Gamma(f,\Gamma(f)) + \partial_3^2\Psi\Gamma(\Gamma(f))$$

where we wrote on the right-hand side f for $P_{t-s}f$ and Ψ for $\Psi(s,f,\Gamma(f)) = \Psi(s,P_{t-s}f,\Gamma(P_{t-s}f))$.

Proof. We have

$$\begin{aligned}&\frac{d}{ds}P_s(\Psi(s,P_{t-s}f,\Gamma(P_{t-s}f)))\\ &\quad= P_s\left(\mathrm{L}\Psi(s,P_{t-s}f,\Gamma(P_{t-s}f)) + \frac{d}{ds}\Psi(s,P_{t-s}f,\Gamma(P_{t-s}f))\right) .\end{aligned}$$

By the diffusion property (2.1), and with the notation of the statement,

$$\begin{aligned}&\mathrm{L}\Psi(s,P_{t-s}f,\Gamma(P_{t-s}f)) + \frac{d}{ds}\Psi(s,P_{t-s}f,\Gamma(P_{t-s}f))\\ &\quad= \partial_2\Psi\mathrm{L}f + \partial_3\Psi\mathrm{L}\Gamma(f) + \partial_2^2\Psi\Gamma(f) + 2\partial_2\partial_3\Psi\Gamma(f,\Gamma(f))\\ &\qquad+ \partial_3^2\Psi\Gamma(\Gamma(f)) + \partial_1\Psi - \partial_2\Psi\mathrm{L}f - 2\partial_3\Psi\Gamma(f,\mathrm{L}f)) .\end{aligned}$$

Since $2\Gamma_2(f) = \mathrm{L}\Gamma(f) - 2\Gamma(f,\mathrm{L}f)$, the lemma follows. $\square$

We apply Lemma 2.4 with

$$\Psi(s,x,y) = \sqrt{\mathscr{U}^2(x) + c_\alpha(s)y} .$$

It is immediate that

$$\begin{aligned}\Psi\partial_1\Psi &= \frac{c_\alpha'}{2}y ,\\ \Psi\partial_2\Psi &= \mathscr{U}\mathscr{U}' ,\\ \Psi\partial_3\Psi &= \frac{c_\alpha}{2} ,\end{aligned}$$

and that

$$\Psi^3 \partial_2^2 \Psi = -\mathscr{U}^2 \mathscr{U}'^{\,2} + \Psi^2(\mathscr{U}'^{\,2} - 1)\,,$$

$$\Psi^3 \partial_2 \partial_3 \Psi = -\frac{c_\alpha}{2} \mathscr{U}\mathscr{U}'\,,$$

$$\Psi^3 \partial_3 \Psi = -\frac{c_\alpha^2}{4}\,.$$

Therefore, K of Lemma 2.4 satisfies

$$\begin{aligned}\Psi^3 K = {} & \Psi^2 \frac{c'_\alpha(s)}{2} \Gamma(f) + \Psi^2 c_\alpha(s) \Gamma_2(f) - \mathscr{U}^2(f)\mathscr{U}'^2(f)\Gamma(f) \\ & + \Psi^2(\mathscr{U}'^2(f) - 1)\Gamma(f) - c_\alpha(s)\mathscr{U}(f)\mathscr{U}'(f)\Gamma(f, \Gamma(f)) \\ & - \frac{c_\alpha(s)^2}{4} \Gamma(\Gamma(f))\,.\end{aligned}$$

With the short notation of Lemma 4.2, $\Psi = \Psi(s, f, \Gamma(f)) = \sqrt{\mathscr{U}^2(f) + c_\alpha(s)\Gamma(f)}$, so that

$$\begin{aligned}\Psi^3 K = {} & [\mathscr{U}^2(f) + c_\alpha(s)\Gamma(f)] \frac{c'_\alpha(s)}{2} \Gamma(f) + [\mathscr{U}^2(f) + c_\alpha(s)\Gamma(f)] c_\alpha(s) \Gamma_2(f) \\ & - \mathscr{U}^2(f)\mathscr{U}'^2(f)\Gamma(f) + [\mathscr{U}^2(f) + c_\alpha(s)\Gamma(f)](\mathscr{U}'^2(f) - 1)\Gamma(f) \\ & - c_\alpha(s)\mathscr{U}(f)\mathscr{U}'(f)\Gamma(f, \Gamma(f)) - \frac{c_\alpha(s)^2}{4}\Gamma(\Gamma(f))\,.\end{aligned}$$

Therefore, after some algebra,

$$\begin{aligned}\Psi^3 K = {} & c_\alpha(s)\Gamma(f) \left[c_\alpha(s)\Gamma_2(f) - \left(1 - \frac{c'_\alpha(s)}{2}\right) \Gamma(f) \right] - \frac{c_\alpha(s)^2}{4} \Gamma(\Gamma(f)) \\ & + \mathscr{U}^2(f) \left[c_\alpha(s)\Gamma_2(f) - \left(1 - \frac{c'_\alpha(s)}{2}\right) \Gamma(f) \right] \\ & - c_\alpha(s)\mathscr{U}'(f)\mathscr{U}(f)\Gamma(f, \Gamma(f)) + c_\alpha(s)\mathscr{U}'^{\,2}(f)\Gamma(f)^2\,.\end{aligned}$$

By the very definition of $c_\alpha(s)$,

$$1 - \frac{c'_\alpha(s)}{2} = R c_\alpha(s)$$

for every s. Hence

$$\begin{aligned}\Psi^3 K = {} & c_\alpha(s)^2 \left(\Gamma(f)(\Gamma_2(f) - R\Gamma(f)) - \tfrac{1}{4}\Gamma(\Gamma(f)) \right) \\ & + c_\alpha(s) \left(\mathscr{U}^2(f)(\Gamma_2(f) - R\Gamma(f)) - \mathscr{U}'(f)\mathscr{U}(f)\Gamma(f, \Gamma(f)) \right. \\ & \left. + \mathscr{U}'^{\,2}(f)\Gamma(f)^2 \right)\,.\end{aligned}$$

Now, as a consequence of (2.7) applied twice,

$$\Psi^3 K \geqq c_\alpha(s) \left(\mathscr{U}^2(f) \frac{\Gamma(\Gamma(f))}{4\Gamma(f)} - \mathscr{U}'(f)\mathscr{U}(f)\Gamma(f, \Gamma(f)) + \mathscr{U}'^{\,2}(f)\Gamma(f)^2 \right)\,.$$

The right-hand side of this inequality is a quadratic form in $\mathcal{U}(f)$ and $\mathcal{U}'(f)$ that is non-negative since, as $\Gamma(f, f) \geqq 0$ for every f,

$$\Gamma(f, \Gamma(f)) \leqq \Gamma(f)^2 \Gamma(\Gamma(f)) .$$

Hence $K \geqq 0$ and thus $F'(s) = P_s(K) \geqq 0$ for every $0 \leqq s \leqq t$. The proof of Theorem 2.1 is complete. □

The preceding proof and statements may also be given with a function $\mathcal{U}$ such that $\mathcal{U}\mathcal{U}'' = 1$. This would correspond to an isoperimetric model with density $\mathrm{e}^{x^2/2}$.

When $\alpha > 0$, the family of inequalities of Theorem 2.1 when t varies (actually as $t \to 0$) is equivalent to the curvature assumption $\Gamma_2 \geqq R\Gamma$. The argument is very similar to the proof of the theorem. The function

$$H(t) = P_t\left(\sqrt{\mathcal{U}^2(f) + c_\alpha(t)\Gamma(f)}\right) - \sqrt{\mathcal{U}^2(P_t f) + \alpha\Gamma(P_t f)}, \quad t \geqq 0 ,$$

is non-negative and $H(0) = 0$. Therefore,

$$\begin{aligned}
&[\mathcal{U}^2(f) + \alpha\Gamma(f)]^{3/2} H'(0) \\
&\quad = \alpha^2 \left(\Gamma(f)(\Gamma_2(f) - R\Gamma(f)) - \tfrac{1}{4}\Gamma(\Gamma(f))\right) \\
&\qquad + \alpha\left(\mathcal{U}^2(f)(\Gamma_2(f) - R\Gamma(f)) - \mathcal{U}(f)\mathcal{U}'(f)\Gamma(f, \Gamma(f)) + \mathcal{U}'^2(f)\Gamma(f)^2\right) \\
&\quad \geqq 0 .
\end{aligned}$$

For some $0 < a < 1$ and g bounded in $\mathcal{A}$, apply the latter to $f = a + \varepsilon g \in [0, 1]$ for every $\varepsilon > 0$ small enough. Dividing by ε^2 and letting ε go to zero yields $\Gamma_2(g) - R\Gamma(g) \geqq 0$. This observation proves the sharpness of Theorem 2.1. It is not clear however how to move between the various inequalities of Theorem 2.1 besides going back to their infinitesimal version expressed by the curvature hypothesis $\Gamma_2 \geqq R\Gamma$.

To conclude this section, let us mention that we of course would like to follow a similar procedure in case of the original Lévy–Gromov inequality involving the dimension parameter n of the Markov generator. This however turns out to be more involved since, at this point, there is no equivalent formulation of the $\mathrm{CD}(R, n)$ hypothesis on the semigroup $(P_t)_{t \geqq 0}$ similar to (2.5) or (2.6).

3. Contractions of Gaussian measures and logarithmic Sobolev inequalities

In this paragraph, we first show, following [Bo2], that the functional inequalities of the preceding section are stable by tensorization. For simplicity, we assume that for two carré du champ operators Γ^1 and Γ^2 on two independent spaces E^1 and E^2, we have the inequalities

$$\sqrt{\mathcal{U}^2(P^i f) + \alpha_i \Gamma^i(P^i f)} \leqq P^i\left(\sqrt{\mathcal{U}^2(f) + \beta_i \Gamma^i(f)}\right), \quad i = 1, 2 , \tag{3.1}$$

where $\alpha_i, \beta_i \geqq 0$ and where P^1, P^2 are Markov operators (which essentially represent $P^1_{t_1}$, $P^2_{t_2}$, t_1, $t_2 \geqq 0$).

Proposition 3.1. *If* $f : E^1 \times E^2 \to [0,1]$ *in the domain of* $\Gamma^1 \otimes \Gamma^2$ *satisfies* (3.1), *then*

$$\begin{aligned}\sqrt{\mathscr{U}^2(P^1P^2f) + \alpha_1\Gamma^1(P^1P^2f) + \alpha_2\Gamma^2(P^1P^2f)}\\ \leqq P^1P^2\left(\sqrt{\mathscr{U}(f) + \beta_1\Gamma^1(f) + \beta_2\Gamma^2(f)}\right) .\end{aligned}$$

Proof. We concentrate on the convexity argument and skip several details on the underlying tensorization of the generators and the carrés du champ. It is plain that these aspects are fulfilled in more concrete spaces such as Riemannian manifolds. If we apply (3.1), $i = 1$, to $P^2 f$, we get

$$\begin{aligned}\sqrt{\mathscr{U}^2(P^1P^2f) + \alpha_1\Gamma^1(P^1P^2f) + \alpha_2\Gamma^2(P^1P^2f)}\\ \leqq \sqrt{\left[P^1\left(\sqrt{\mathscr{U}^2(P^2f) + \beta_1\Gamma^1(P^2f)}\right)\right]^2 + \alpha_2\Gamma^2(P^1P^2f)} \,.\end{aligned}$$

Now, a carré du champ Γ is a non-negative symmetric bilinear operator so that it satisfies the triangle inequality $\sqrt{\Gamma(f+g)} \leqq \sqrt{\Gamma(f)} + \sqrt{\Gamma(g)}$. By standard approximations of integrals by sums,

$$\Gamma^2(P^1P^2f) \leqq \left[P^1\left(\sqrt{\Gamma^2(P^2f)}\right)\right]^2 \tag{3.2}$$

(assuming the proper domain considerations). Using Minkowski's inequality

$$\sqrt{\left(\int X\right)^2 + \left(\int Y\right)^2} \leqq \int\sqrt{X^2+Y^2}, \quad X, Y \geqq 0\,, \tag{3.3}$$

for the kernel P^1 and $X = \sqrt{\mathscr{U}^2(P^2f) + \beta_1\Gamma^1(P^2f)}, Y = \sqrt{\alpha_2\Gamma^2(P^2f)}$, it follows that

$$\begin{aligned}\sqrt{\mathscr{U}^2(P^1P^2f) + \alpha_1\Gamma^1(P^1P^2f) + \alpha_2\Gamma^2(P^1P^2f)}\\ \leqq P^1\left(\sqrt{\mathscr{U}^2(P^2f) + \beta_1\Gamma^1(P^2f) + \alpha_2\Gamma^2(P^2f)}\right) .\end{aligned}$$

Repeating the procedure with P^2 and f easily concludes the argument. The proof is complete. □

The next statement clarifies some of the relationships between the inequalities that we are studying and logarithmic Sobolev inequalities. It also describes an important contraction property. In what follows, μ is a probability measure. We will say that L, or rather the carré du champ operator Γ, satisfies a logarithmic Sobolev inequality with constant $\rho_0 > 0$ if for all f in $\mathscr{A}$,

$$\begin{aligned}\rho_0 E(f^2) = \rho_0\left[\int f^2\log f^2 d\mu - \int f^2 d\mu \log\int f^2 d\mu\right]\\ \leqq 2\int f(-Lf)\,d\mu = 2\int\Gamma(f)\,d\mu\,.\end{aligned}$$

For f in $\mathscr{A}$, denote by ν its distribution on the real line with respect to the underlying invariant probability measure μ.

Theorem 3.2. *Let Γ be a carré du champ satisfying the inequality*

$$\mathscr{U}\left(\int f d\mu\right) \leqq \int \sqrt{\mathscr{U}^2(f) + \Gamma(f)}\, d\mu \tag{3.4}$$

for every f in $\mathscr{A}$ with values in $[0,1]$. *Then Γ satisfies the logarithmic Sobolev inequality*

$$E(f^2) \leqq 2\int \Gamma(f)\, d\mu \tag{3.5}$$

for every $f \in \mathscr{A}$. Furthermore, if $\Gamma(f) \leqq 1$ almost everywhere, the distribution ν of f is a contraction of the canonical Gaussian measure γ on $\mathbb{R}$.

Proof. Let f in $\mathscr{A}$. Assume first that ν is absolutely continuous with respect to Lebesgue's measure on $\mathbb{R}$ with a strictly positive density. We set for simplicity $\nu(r) = \nu((-\infty, r])$ so that ν has density ν'. For $r \in \mathbb{R}$, apply (3.4) to $\psi(f)$ where $\psi(x) = 1$ on $(-\infty, r]$, $\psi(x) = 0$ on $[r+\varepsilon, \infty)$ and linear elsewhere. (We of course assume here the corresponding stability of $\mathscr{A}$ or of the domain of Γ.) As $\varepsilon \to 0$, by Lebesgue's theorem, for almost every r,

$$\mathscr{U}(\nu(r)) \leqq \theta(r)\nu'(r) \tag{3.6}$$

where θ is a regular version of the conditional expectation of $\sqrt{\Gamma(f)}$ with respect to the sub-σ-field of $\mathscr{E}$ generated by f. Set $k = \nu^{-1} \circ \Phi$ and $x = \Phi^{-1} \circ \nu(r)$. Since $\mathscr{U}(\Phi(x)) = \varphi(x)$, (3.6) indicates that, for almost every x,

$$\varphi(x) \leqq \theta \circ \nu^{-1} \circ \Phi(x)\nu' \circ \nu^{-1} \circ \Phi(x)\,.$$

In other words, almost everywhere,

$$k'(x) \leqq \theta \circ k(x)\,. \tag{3.7}$$

By definition of k, the distribution of k under γ is ν. If we now apply the classical logarithmic Sobolev inequality for γ [Gro] to k, we get that

$$E(k^2) \leqq 2\int k'^2 d\gamma \leqq 2\int (\theta \circ k)^2 d\gamma\,,$$

that is

$$\int x^2 \log x^2 d\nu(x) - \int x^2 d\nu(x) \log \int x^2 d\nu(x) \leqq 2\int \theta^2(x) d\nu(x)\,.$$

By definition of ν and θ, this is slightly better than the logarithmic Sobolev inequality $E(f^2) \leqq 2\int \Gamma(f) d\mu$.

In general, we need simply tensorize μ with the canonical Gaussian measure γ on $\mathbb{R}$ (that satisfies (3.4)), and work with the function

$$\tilde{f}(x,u) = (1-\varepsilon^2)^{1/2} f(x) + \varepsilon u, \;\; (x,u) \in E \times \mathbb{R},$$

whose carré du champ on the product space is $(1-\varepsilon^2)\Gamma(f) + \varepsilon^2$. The distribution ν of $\tilde{f}$ on $E \times \mathbb{R}$ is smooth and the preceding applies to yield the result as $\varepsilon \to 0$. The second part of Theorem 3.2 is an immediate consequence, in the regular case, of (3.6) and (3.7) in which $\theta \leqq 1$, with k the contraction we are looking for. The general case follows from the previous regularization procedure together with a simple weak-compactness argument. The proof of Theorem 3.2 is complete. □

The last part of Theorem 3.2 expresses equivalently that if ν is a measure on $\mathbb{R}$ such that for every smooth function ψ

$$\mathcal{U}\left(\int \psi \, d\nu\right) \leqq \int \sqrt{\mathcal{U}^2(\psi) + \psi'^2} d\nu , \tag{3.8}$$

then ν is the image by a contraction of the canonical Gaussian measure γ. (Of course, if ν is a contraction of γ on $\mathbb{R}$ or $\mathbb{R}^k$, it will clearly satisfy (3.8), as well as the corresponding logarithmic Sobolev inequality.) We refer to [B-H] for an analogous result on the exponential distribution. Note furthermore that Theorem 3.2 also holds under the inequality

$$\mathcal{U}(\int f d\mu) - \int \mathcal{U}(f) d\mu \leqq \int \sqrt{\Gamma(f)}\, d\mu .$$

In the proof of Theorem 3.2, we used the logarithmic Sobolev inequality for the Gaussian measure γ. This might not be completely satisfactory. As an alternate approach to this proof, one may follow the Gaussian case [Led1] and use the tensorization Proposition 3.1 to conclude. Note that Theorem 3.2 provides another proof of the distributional inequalities (2.13).

Finally, we may observe that Corollary 2.2 together with Theorem 3.2 yield a proof of the logarithmic Sobolev inequality (2.8). As for the Lévy–Gromov theorem (cf. the end of Sect. 2), we would not know however how to improve the inequalities of Corollary 2.2 in order to reach the better (and optimal) constant $2(n-1)/nR$ in (2.8) under the $\mathrm{CD}(R, n)$ hypothesis (2.2) for some finite n.

4. Some isoperimetric statements for hypercontractive generators

In this section, the invariant measure μ is assumed to be finite and normalized into a probability measure. In Corollary 2.2 and the subsequent comments, we have seen how a positively curved diffusion generator satisfies an infinite dimensional Lévy–Gromov isoperimetric inequality. On the other hand, we also know that these generators satisfy a logarithmic Sobolev inequality (2.8) (and we have seen also in Theorem 3.2 how logarithmic Sobolev inequalities follow from the family of inequalities of Sect. 2). One may therefore ask what kind of isoperimetric result still holds for hypercontractive generators. In this last paragraph, we briefly show, following [Led2], [Led3], that if L is hypercontractive and of some curvature R (not necessarily strictly positive), then there is still a form of isoperimetry, with a constant depending on R and on the hypercontractivity constant. Finally, we prove that one cannot hope under hypercontractivity only for an inequality $c\,\mathcal{U}(\mu(A)) \leqq \mu_s(\partial A)$ for some $0 < c \leqq 1$ to hold, even for sets A of the form $A = \{f \leqq r\}$ where $\Gamma(f) \leqq 1$ μ-almost everywhere which correspond formally to balls. In particular, the isoperimetric inequality (3.4) is in general strictly stronger than the corresponding logarithmic Sobolev inequality (3.5).

Let us first recall some definitions about hypercontractivity. Recall from Sect. 3 that L satisfies a logarithmic Sobolev inequality with constant $\rho_0 > 0$ if for all f in $\mathscr{A}$,

$$\rho_0 E(f^2) = \rho_0[\int f^2 \log f^2 d\mu - \int f^2 d\mu \log \int f^2 d\mu]$$
$$\leqq 2 \int f(-\mathrm{L} f)\, d\mu = 2 \int \Gamma(f)\, d\mu\,.$$

Equivalently [Gro], $(P_t)_{t \geqq 0}$ is hypercontractive of constant ρ_0 in the sense that whenever $1 < p < q < \infty$ and $\mathrm{e}^{2\rho_0 t} \geqq [(q-1)/(p-1)]$, for every f in $\mathscr{A}$,

$$\|P_t f\|_q \leqq \|f\|_p$$

where the norms are understood with respect to the measure μ. According to (2.8), a generator L of curvature $R > 0$ is hypercontractive with constant $\rho_0 = R$.

Theorem 4.1. *Let* L *be a Markov diffusion generator with hypercontractivity constant $\rho_0 > 0$ and curvature R ($\in \mathbb{R}$). Then, for every f in $\mathscr{A}$ with values in $[0,1]$ and every $0 \leqq t \leqq 1$,*

$$\|f\|_2^2 - \|f\|_{p(t)}^2 \leqq C\sqrt{t} \int \sqrt{\Gamma(f)}\, d\mu \tag{4.1}$$

where $C > 0$ only depends on R and $p(t) = 1 + \mathrm{e}^{-\rho_0 t}$.

To describe the isoperimetric content of this statement, assume we are in a setting allowing the operations described next to Corollary 2.3. Then, for every Borel set A with $0 < \mu(A) \leqq \frac{1}{2}$, if f approximates the indicator function of A, (4.1) yields

$$\mu(A)[1 - \mu(A)^{(2/p(t))-1}] \leqq C\sqrt{t}\mu_s(\partial A), \quad 0 \leqq t \leqq 1\,.$$

Now, when $0 \leqq t \leqq 1$,

$$\frac{2}{p(t)} - 1 \geqq \frac{1}{2}\rho_0 \mathrm{e}^{-\rho_o} t\,.$$

Hence,

$$\mu(A)\left[1 - \exp\left(-\frac{1}{2}\rho_0 \mathrm{e}^{-\rho_0} t \log \frac{1}{\mu(A)}\right)\right] \leqq C\sqrt{t}\mu_s(\partial A)\,.$$

Choose then $t = (2\log(1/\mu(A)))^{-1}$ ($\leqq 1$ since $0 < \mu(A) \leqq \frac{1}{2}$) to get

$$\mu(A)\sqrt{\log \frac{1}{\mu(A)}} \leqq C' \mu_s(\partial A)$$

for some $C' > 0$ only depending on R and ρ_0. Due to the equivalence (1.8), and by symmetry of $\mathscr{U}$, this amounts to

$$c\mathscr{U}(\mu(A)) \leqq \mu_s(\partial A)$$

for some $0 < c < 1$, that is, a weak form of (2.11).

The following lemma, in the spirit of the methods developed in Sect. 2 (and the introduction), will be needed in the proof of Theorem 4.1.

Lemma 4.2. *Let* L *with curvature* $R \in \mathbb{R}$. *Then, for every* f *in* $\mathscr{A}$ *and every* $t \geqq 0$, *almost everywhere,*

$$P_t(f^2) - (P_t f)^2 \geqq \frac{e^{2Rt} - 1}{R} \Gamma(P_t f).$$

For the proof, write

$$P_t(f^2) - (P_t f)^2 = 2 \int_0^t P_s(\Gamma(P_{t-s} f))\, ds$$

and, by (2.5),

$$P_t(f^2) - (P_t f)^2 \geqq 2 \int_0^t e^{2Rs} ds\, \Gamma(P_t f) = \frac{e^{2Rt} - 1}{R} \Gamma(P_t f).$$

Lemma 4.2 is actually close in spirit to some aspects of the Li–Yau inequality [L-Y]. It namely implies that for every f and $t > 0$,

$$\Gamma(P_t f) \leqq \frac{R}{e^{2Rt} - 1} \|f\|_\infty^2. \tag{4.2}$$

Proof of Theorem 4.1. Fix f in $\mathscr{A}$ with values in $[0, 1]$. For every $t \geqq 0$, write

$$\begin{aligned}
\int f(f - P_t f)\, d\mu &= -\int f\Big(\int_0^t \mathrm{L} P_s f\, ds\Big)\, d\mu \\
&= -\int_0^t \Big(\int f \mathrm{L} P_s f\, d\mu\Big)\, ds \\
&= \int_0^t \Big(\int \Gamma(f, P_s f) d\mu\Big)\, ds \\
&\leqq \int_0^t \Big(\int \sqrt{\Gamma(P_s f)} d\mu\Big)\, ds \int \sqrt{\Gamma(f)} d\mu .
\end{aligned}$$

Now, if $0 \leqq s \leqq 1$, by (4.2),

$$\sqrt{\Gamma(P_s f)} \leqq \frac{C}{\sqrt{s}} \|f\|_\infty \leqq \frac{C}{\sqrt{s}}$$

where $C > 0$ only depends on R. Therefore, if $0 \leqq t \leqq 1$,

$$\int f(f - P_t f) d\mu \leqq 2C\sqrt{t} \int \sqrt{\Gamma(f)}\, d\mu . \tag{4.3}$$

Now, by symmetry of P_t,

$$\int f(f - P_t f) d\mu = \int f^2 d\mu - \int f P_t f\, d\mu = \|f\|_2^2 - \|P_{t/2} f\|_2^2 .$$

By the hypercontractivity property,

$$\|P_{t/2} f\|_2^2 \leqq \|f\|_{p(t)}^2$$

from which the conclusion follows together with (4.3). The proof is complete. $\square$

We complete this section with an example showing that one cannot hope for much under hypercontractivity only and that the curvature assumption is essential in the previous statement. We will actually show that, for any $c > 0$, there exist generators L with hypercontractivity constant $\rho_0 = 1$ such that

$$c\mathcal{U}(\mu(A)) \leqq \mu_s(\partial A) \tag{4.4}$$

may fail even for sets of the form $A = \{f \leqq m\}$ where f is such that $\Gamma(f) \leqq 1$. In particular, the distribution of f (under μ) is not a contraction of the canonical one-dimensional Gaussian measure. These sets correspond formally to balls in this infinite dimensional setting (half-spaces are the extremal sets of the Gaussian isoperimetric inequality). The example we will exhibit thus indicates that one cannot hope, for hypercontractive generators, for comparison theorems similar to the Riemannian comparisons of volumes of balls (cf. e.g. [Ch]). We simply use the ultraspheric generators for the small values of $n > 0$. The class of ultraspheric generators on $]-1,+1[$ is given by

$$\mathrm{L}_n f(x) = (1-x^2)f''(x) - nxf'(x)$$

for every f smooth enough, where $n > 0$. When n is an integer, L_n may be obtained as the projection of the Laplacian on the unit sphere S^n. In this example, $\Gamma(f) = (1-x^2)f'(x)^2$ and the invariant measure is given by $d\mu_n(x) = c_n(1-x^2)^{(n/2)-1}dx$ on $]-1,+1[$. The generators L_n satisfy $\mathrm{CD}(n-1,n)$ $(n \geqq 1)$, as the spheres S^n themselves actually when n is an integer.

It is known since [M-W] that, for every $n > 0$, L_n is hypercontractive with constant $\rho_0 = 1/n$, that is we have the logarithmic Sobolev inequality for μ_n on $]-1,+1[$

$$E(f^2) \leqq \frac{2}{n}\int_{-1}^{+1}(1-x^2)f'(x)^2 d\mu_n(x) \tag{4.5}$$

where $E(f^2)$ is the entropy of f^2 with respect to μ_n. We change the variables in (4.5) and set $x = \sin(u/\sqrt{n})$, $u \in]-\sqrt{n}\pi/2, +\sqrt{n}\pi/2[$, and $d\nu_n(u) = c_n n^{-1/2}\cos(\sqrt{n}u)^{n-1}du$. We also set as usual $\nu_n(u) = \nu_n(]-\sqrt{n}\pi/2, u])$. Then, ν_n satisfies a logarithmic Sobolev inequality with constant 1, that is

$$E(f^2) \leqq 2\int_{-\sqrt{n}\,\pi/2}^{+\sqrt{n}\,\pi/2} f'(u)^2 d\nu_n(u)$$

where now $E(f^2)$ is understood with respect to ν_n. Assume that (4.4) holds for some $c > 0$ independent of n. Then, setting $k_n = \nu_n^{-1}\circ\Phi$, the proof of Theorem 3.2, applied to $f(x) = x$, yields $ck_n' \leqq 1$. At $x = 0$, $\Phi(0) = \nu_n(0) = \frac{1}{2}$, so that this inequality reads

$$c\Phi(0) = \frac{c}{\sqrt{2\pi}} \leqq \nu_n'(0)\,. \tag{4.6}$$

Now $\nu_n'(0) = c_n n^{-1/2}$ and a simple asymptotic as $n \to 0$ shows that $\nu_n'(0)$ is actually of the order of $\sqrt{n}/2$. Therefore (4.6) and (4.4) cannot hold for the small values of n. With respect to Theorem 4.1, this is due to the fact that, after the change of variables, $\widetilde{\mathrm{L}}_n$ on $]-\sqrt{n}\pi/2, +\sqrt{n}\pi/2[$ associated to ν_n and to the Dirichlet form $\int f'(u)^2 d\nu_n(u)$ with Neumann boundary conditions can be written

$$\widetilde{\mathrm{L}}_n f(u) = f''(u) - \frac{n-1}{\sqrt{n}} \tan\left(\frac{u}{\sqrt{n}}\right) f'(u) .$$

The condition $\Gamma_2 \geqq R\Gamma$ reads in this case

$$\left[\frac{n-1}{\sqrt{n}} \tan\left(\frac{u}{\sqrt{n}}\right)\right]' \geqq R .$$

In another words,

$$\frac{n-1}{n}\left[1 + \tan^2\left(\frac{u}{\sqrt{n}}\right)\right] \geqq R$$

which, when $0 < n < 1$, is impossible as u approaches $-\sqrt{n}\pi/2$ or $+\sqrt{n}\pi/2$. Actually (cf. [B-E]), the ultraspheric generators L_n for $0 < n < 1$ satisfies a reverse $\mathrm{CD}(R,n))$ inequality in the form

$$\Gamma_2(f) \leqq (n-1)\Gamma(f) + \frac{1}{n}(\mathrm{L}f)^2 .$$

This inequality is known to imply a logarithmic Sobolev inequality [Be]. It might be that small dimension is the explanation of the failure of (4.4) in general.

5. Inequalities on path spaces

In this last section, we turn to path spaces and extend the family of inequalities of Sect. 2 to the Wiener measure on the paths of Brownian motion with values in a Riemannian manifold with bounded Ricci curvature. This type of extension on logarithmic Sobolev inequalities has been studied in [A-E] and [Hs]. We simply follow here P.E. Hsu's approach [Hs] on the basis of the tensorization argument of Proposition 3.1. (One may wonder whether the alternate approach by S. Aida and K.D. Elworthy [A-E], that however only deals with manifolds embedded in an Euclidean space, may also be followed.) We however only present these results as an illustration of our basic inequalities for heat kernel measures. In particular, we do not enter the various questions related to gradient and parallel transport on path spaces. We use the notation of [Hs] and refer to this paper for further details. Let thus M be a complete connected Riemannian manifold with Riemannian measure dx. We say M has Ricci curvature bounded above and below if $\sup|\mathrm{Ric}(v,v)| < \infty$ where the supremum is running over all unit tangent vectors v. Let x_0 be a fixed point on M and let

$$W_{x_0}(M) = \{w : [0,1] \to M, \text{ continuous}, \ w(0) = x_0\}$$

be the space of continuous paths starting at x_0. Denote by μ the Wiener measure on $W_{x_0}(M)$. If $f : W_{x_0}(M) \to \mathbb{R}$ is in the domain of the gradient operator D, denote by $|Df|$ its norm as an operator from $L^2(\mu)$ into $L^2(\mu)\otimes\mathscr{H}$, where $\mathscr{H}$ is the Cameron–Martin Hilbert space (cf. [Hs]).

Theorem 5.1. *Let M be a Riemannian manifold with Ricci curvature bounded above and below and let μ be Wiener measure on $W_{x_0}(M)$. Then, for every f with values in $[0,1]$ in the domain of the gradient operator D on $W_{x_0}(M)$,*

$$\mathscr{U}\left(\int f d\mu\right) \leqq \int \sqrt{\mathscr{U}^2(f)+C|Df|^2}\, d\mu \tag{5.1}$$

where $C>0$ only depends on the bound on the Ricci curvature.

According to the comments following Corollaries 2.2 and 2.3, (5.1) may be interpreted into an isoperimetric inequality on the path space $W_{x_0}(M)$ that extends the classical isoperimetric inequality on abstract (flat) Wiener spaces (cf. [Bor]). Due to Theorem 3.2, Theorem 5.1 also improves the recent logarithmic Sobolev inequalities in this context [A-E], [Hs].

Proof. As in [Hs], we perform a Markovian tensorization of the finite dimensional inequalities of Sect. 2, similar to the proof of Proposition 3.1. A noticeable difference however with the independent case is that the iteration procedure involves derivatives of the heat kernel. We bound the various gradient terms by $|Df|^2$ according to the coupling Proposition 2.2 and Lemmas 4.1 and 4.3 of [Hs]. To sketch the argument, let g be smooth on $M\times M$ and consider f on $W_{x_0}(M)$ defined by $f(w)=g(w(s),w(t))$ where $0\leqq s<t\leqq 1$ are fixed. Denote by $p_t(x,y)$ the heat kernel on M so that, by the Markov property,

$$\begin{aligned}\mathscr{U}\left(\int f d\mu\right) &= \mathscr{U}\left(\int\int g(x,y)\,p_s(x_0,x)\,p_{t-s}(x,y)dx\,dy\right)\\ &= \mathscr{U}\left(\int G(x)\,p_s(x_0,x)dx\right)\end{aligned}$$

where

$$G(x)=\int g(x,y)\,p_{t-s}(x,y)\,dy\,.$$

Since Ricci curvature is bounded below, by Corollary 2.3 at time s,

$$\begin{aligned}\mathscr{U}\left(\int f d\mu\right) &= \mathscr{U}\left(\int G(x)\,p_s(x_0,x)dx\right)\\ &\leqq \int\sqrt{\mathscr{U}^2(G(x))+c_0(s)|\nabla_x G(x)|^2}\,p_s(x_0,x)\,dx\,.\end{aligned}$$

Similarly at $t-s$, for every x,

$$\mathscr{U}(G(x)) \leqq \int\sqrt{\mathscr{U}^2(g(x,y))+c_0(t-s)|\nabla_y g(x,y)|^2}\,p_{t-s}(x,y)\,dy\,.$$

Note that $c_0(u)\leqq Cu$ for every $0\leqq u\leqq 1$ for some $C>0$ only depending on the lower bound on the Ricci curvature. The gradient term $\nabla_x G(x)$ involves parallel transport along the Brownian paths. Actually, summarizing some of the main conclusions of [Hs] (cf. his Proposition 2.2 and the proof of his Lemma 4.3),

$$|\nabla_x G(x)|^2 \leqq \int Z(x,y)^2\,p_{t-s}(x,y)\,dy$$

and Z is such that

$$sZ^2 + (t-s)|\nabla_y g|^2 \leqq C'|Df|^2 .$$

This is the second use of the curvature assumption and note that, according to Lemma 4.3 in [Hs], the constant C' depends here on both a lower and an upper bound on the Ricci curvature. Hence, by Minkowski's inequality (3.3) applied to the integral with respect to the kernel $p_{t-s}(x,y)\,dy$,

$$\begin{aligned}
&\mathcal{U}\left(\int f d\mu\right) \\
&\quad \leqq \int\int \sqrt{\mathcal{U}^2(g(x,y)) + C(t-s)|\nabla_y g(x,y)|^2 + CsZ(x,y)^2}\, p_{t-s}(x,y)\, p_s(x_0,x) dy dx \\
&\quad \leqq \int \sqrt{\mathcal{U}^2(f) + CC'|Df|^2} d\mu .
\end{aligned}$$

This proof is easily extended to all smooth cylindrical functions on the path space (with constants C and C' independent of the number of coordinates). The dependence of the constants C and C' upon the bound on the Ricci curvature may actually easily be described (cf. again [Hs]). The result then follows by well-known density arguments. The proof of Theorem 5.1 is finished. □

References

[A-E] S. Aida, K.D. Elworthy: Logarithmic Sobolev inequalities on path and loop spaces. (Preprint 1994)

[Ba1] D. Bakry: Transformations de Riesz pour les semigroupes symétriques. Seconde partie: Etude sous la condition $\Gamma_2 \geqq 0$. Séminaire de Probabilités XIX. Lecture Notes in Math. **1123**, 145–174 (1985) Springer, Berlin Heidelberg New York

[Ba2] D. Bakry: Un critère de non-explosion pour certaines diffusions sur une variété riemannienne complète. C.R. Acad. Sci. Paris **303**, 23–26 (1986)

[Ba3] D. Bakry: L'hypercontractivité et son utilisation en théorie des semigroupes. Ecole d'Eté de Probabilités de St-Flour. Lecture Notes in Math. **1581**, 1–114 (1994). Springer, Berlin Heidelberg New York

[Ba4] D. Bakry: On Sobolev and logarithmic Sobolev inequalities for Markov semigroups, in the Proceedings of the Taniguchi Symposium, Warwick 1994 (to appear)

[B-E] D. Bakry, M. Emery: Diffusions hypercontractives. Séminaire de Probabilités XIX. Lecture Notes in Math. **1123**, 175–206 (1985) Springer, Berlin Heidelberg New York

[Be] A. Bentaleb: Analyse des semi-groupes ultrasphériques. Thèse, Toulouse (1993)

[Bob1] S. Bobkov: A functional form of the isoperimetric inequality for the Gaussian measure. (Preprint 1993)

[Bob2] S. Bobkov: An isoperimetric inequality on the discrete cube and an elementary proof of the isoperimetric inequality in Gauss space. (Preprint 1995)

[B-H] S. Bobkov, Ch. Houdré: Some connections between Sobolev-type inequalities and isoperimetry. (Preprint 1995)

[Bor] C. Borell: The Brunn-Minkowski inequality in Gauss space. Invent. Math. **30**, 207–216 (1975)

[B-Z] Y.D. Burago, V.A. Zalgaller: Geometric inequalities. Springer, Berlin Heidelberg New York (1988). Russian Edition Nauka (1980)

[Ch] I. Chavel: Riemannian geometry: a modern introduction. Cambridge University Press, 1993

[D-S] J.-D. Deuschel, D. Stroock: Large deviations. Academic Press, New York 1989

[D-L] H. Donnely, P. Li: Lower bounds for the eigenvalues of Riemannian manifolds. Michigan Math. J. **29**, 149–161 (1982)

[Eh] A. Ehrhard: Symétrisation dans l'espace de Gauss. Math. Scand. **53**, 281–301 (1983)

[El] K.D. Elworthy: Stochastic flows on Riemannian manifolds. Diffusion processes and related problems in analysis, vol. II. Progress in Probability **27**, 37–72 (1992). Birkäuser, Bosel Boston

[F-L-M] T. Figiel, J. Lindenstrauss, V.D. Milman: The dimensions of almost spherical sections of convex bodies. Acta Math. **139**, 52–94 (1977)

[G-H-L] S. Gallot, D. Hulin, J. Lafontaine: Riemannian Geometry. Second Edition. Springer, Berlin Heidelberg New York, 1990

[Gr] M. Gromov. Paul Lévy's isoperimetric inequality. Preprint I.H.E.S. (1980)

[Gro] L. Gross: Logarithmic Sobolev inequalities. Am. J. Math. **97**, 1061–1083 (1975)

[Hs] E. P. Hsu: Logarithmic Sobolev inequalities on path spaces. (Preprint 1994)

[Led1] M. Ledoux: Isopérimétrie et inégalités de Sobolev logarithmiques gaussiennes. C.R. Acad. Sci. Paris **306**, 79–82 (1988)

[Led2] M. Ledoux: A simple analytic proof of an inequality by P. Buser. Proc. Amer. Math. Soc. **121**, 951–959 (1994)

[Led3] M. Ledoux: Semigroup proofs of the isoperimetric inequality in Euclidean and in Gauss space. Bull. Sci. math. **118**, 485–510 (1994)

[Lé] P. Lévy: Problèmes concrets d'analyse fonctionnelle. Gauthier–Villars, Paris 1951

[L-Y] P. Li, S.-T. Yau: On the parabolic kernel of the Schrödinger operator. Acta Math. **156**, 153–201 (1986)

[MK] H.P. McKean: Geometry of differential space. Ann. Probability **1**, 197–206 (1973)

[M-W] C. Mueller, F. Weissler: Hypercontractivity for the heat semigroup for ultraspherical polynomials and on the n-sphere. J. Funct. Anal. **48**, 252–283 (1982)

[Sc] E. Schmidt: Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie. Math. Nach. **1**, 81–157 (1948)

[S-T] V.N. Sudakov, B. S. Tsirel'son: Extremal properties of half-spaces for spherically invariant measures. J. Soviet. Math. **9**, 9–18 (1978); translated from Zap. Nauch. Sem. L.O.M.I. **41**, 14–24 (1974)

[Ta] M. Talagrand: Isoperimetry, logarithmic Sobolev inequalities on the discrete cube, and Margulis' graph connectivity theorem. Geom. Funct. Anal. **3**, 295–314 (1993)