Exponential tail inequalities for eigenvalues of random matrices

M. Ledoux

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exponential tail inequalities

classical theme in probability and statistics

exponential tail inequalities

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quantify the asymptotic statements

exponential tail inequalities

classical theme in probability and statistics

quantify the asymptotic statements

central limit theorems

large deviation principles

classical exponential inequalities

sum of independent random variables

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

classical exponential inequalities

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$0 \leq X_i \leq 1$ independent

$$\mathbb{P}\big(S_n \geq \mathbb{E}(S_n) + t\big) \leq e^{-t^2/2}, \quad t \geq 0$$

Hoeffding's inequality

classical exponential inequalities

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Hoeffding's inequality

same as for X_i standard Gaussian

central limit theorem

measure concentration ideas

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asymptotic geometric analysis

V. D. Milman (1970)

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$$S_n = \frac{1}{\sqrt{n}}\,(X_1 + \cdots + X_n)$$

$F(X) = F(X_1, \ldots, X_n), \quad F : \mathbb{R}^n \to \mathbb{R}$ Lipschitz

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Gaussian sample

independent random variables **(M. Talagrand 1995)**

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M. Talagrand (1995)

concentration inequalities

numerous applications

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- geometric functional analysis

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- discrete and combinatorial probability

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- random matrix theory

recent studies of

random matrix and random growth models

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new asymptotics

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common, non central, rate $(\text{mean})^{1/3}$

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universal limiting **Tracy-Widom** distribution

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universal limiting **Tracy-Widom** distribution

random matrices, longest increasing subsequence,

random growth models, last passage percolation...

random matrix models

Wigner matrix

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random matrix models

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asymptotics of the eigenvalues as the size $N \to \infty$

scaling $\widehat{X}^N = X^N / \sqrt{N}$

Wigner's theorem (1955)

asymptotic behavior of the spectral measure $(\widehat{\lambda}_k^N = \lambda_k^N / \sqrt{N})$

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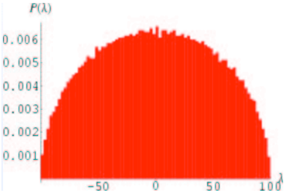
Wigner's semi-circle law



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$$\sum_{k=1}^{N}\left[f\big(\widehat{\lambda}_k^N\big) - \int f\,d\nu\right] \ \to\ G$$ Gaussian variable

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extremal eigenvalues (edge behavior)

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Gaussian Orthogonal or Unitary Ensemble (GOE, GUE)

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GUE $\quad F_{\mathrm{TW}}(s) = \exp\left(- \int_s^\infty (x-s) u(x)^2 dx \right), \quad s \in \mathbb{R}$

$u'' = 2u^3 + xu$ Painlevé II equation

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large deviation principles (Laplace methods)

fluctuations of the spectral measure

law of $(\lambda_1^N, \dots, \lambda_N^N)$: $P^N(dx) = \frac{1}{Z} \left|\Delta_N(x)\right|^\beta \prod_{k=1}^N e^{-\beta x_k^2/4}\, dx_k$

fluctuations of extremal eigenvalues

$N^{2/3} \left[\, \widehat{\lambda}_N^N - 2 \,\right] \;\to\; F_{\mathrm{TW}}$ **Tracy-Widom** distribution law of $(\lambda_1^N, \ldots, \lambda_N^N)$: $P^N(dx) = \frac{1}{Z} \, |\Delta_N(x)|^\beta \prod_{k=1}^N e^{-\beta x_k^2/4} \, dx_k$

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marginals of $(\lambda_1^N, \ldots, \lambda_N^N)$ in terms of the kernel law of $(\lambda_1^N, \ldots, \lambda_N^N)$: $P^N(dx) = \frac{1}{Z} \left| \Delta_N(x) \right|^\beta \prod_{k=1}^N e^{-\beta x_k^2/4} \, dx_k$

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$N^{2/3} \left[\, \widehat{\lambda}_N^N - 2 \, \right] \rightarrow F_{\mathrm{TW}}$ **Tracy-Widom** distribution

orthogonal polynomial method $(\beta = 2)$

marginals of $(\lambda_1^N, \ldots, \lambda_N^N)$ in terms of the kernel

$$K_N(x, y) = \sum_{\ell=0}^{N-1} P_\ell(x) P_\ell(y)$$

P_ℓ Hermite orthogonal polynomials for $d\mu(x) = e^{-\beta x^2/4} \frac{dx}{Z}$

law of $(\lambda_1^N, \dots, \lambda_N^N)$: $P^N(dx) = \frac{1}{Z} \left| \Delta_N(x) \right|^\beta \prod_{k=1}^N e^{-\beta x_k^2/4} \, dx_k$

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orthogonal polynomial asymptotics

fluctuations of **extremal eigenvalues**

recent years : extensions to (**non Gaussian**) Wigner matrices

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comparison with the Gaussian case

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A. Soshnikov (1999) combinatorial moment method

$$\mathbb{E}\bigg(\sum_{k=1}^{N}(\lambda_k^N)^p\bigg) = \mathbb{E}\Big(\mathrm{Tr}\big((X^N)^p\big)\Big), \quad p = O(N^{2/3})$$

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T. Tao, V. Vu (2009) Lindeberg method

survey of recent approaches to

non asymptotic exponential inequalities

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extremal eigenvalues

catch the **new rate** $(\text{mean})^{1/3}$

two main questions and **objectives**

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tail inequalities for the spectral measure

$$\mathbb{P}\left(\sum_{k=1}^{N} f(\widehat{\lambda}_k^N) \geq t\right)$$

Wigner's theorem

$$\frac{1}{N}\sum_{k=1}^{N}\delta_{\widehat{\lambda}_k^N} \;\to\; d\nu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx \quad \text{semi-circle law}$$

global regime

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measure concentration tool : $\quad F = F(X^N) = F(X_{ij}^N)$

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f smooth measure concentration

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convex if f is convex

concentration inequalities

$$S_n = \frac{1}{\sqrt{n}}\,(X_1 + \cdots + X_n)$$

$F(X) = F(X_1, \ldots, X_n), \quad F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz

$X_1, \ldots, X_n$ independenty standard Gaussian

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}(F(X)) + t\Big) \leq e^{-t^2/2}, \quad t \geq 0$$

$0 \leq X_i \leq 1$ independent, F 1-Lipschitz and convex

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}(F(X)) + t\Big) \leq 2\,e^{-t^2/4}, \quad t \geq 0$$

M. Talagrand (1995)

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Wigner's theorem (1955)

$$\frac{1}{N}\sum_{k=1}^{N}\delta_{\widehat{\lambda}_k^N} \ \to \ d\nu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx$$ semi-circle law

global regime

large deviation asymptotics of the spectral measure

fluctuations of the spectral measure

$$\sum_{k=1}^{N}\Big[f\big(\widehat{\lambda}_k^N\big) - \int f\,d\nu\Big] \ \to \ G$$ Gaussian variable

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non Gaussian Wigner matrices ?

partial results (Stieltjes transform)

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F. Götze, A. Tikhomirov

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Wigner's law at small scales $|A| \sim \frac{1}{N}$

two main questions and **objectives**

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$$\mathbb{P}\big(\widehat{\lambda}_N^N \geq 2+\varepsilon\big), \quad \mathbb{P}\big(\widehat{\lambda}_N^N \leq 2-\varepsilon\big)$$

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fluctuations of the largest eigenvalue

$$N^{2/3}\big[\,\widehat{\lambda}_N^N - 2\,\big] \;\to\; F_{\mathrm{TW}}$$ **Tracy-Widom** distribution

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$u'' = 2u^3 + xu$ Painlevé II equation

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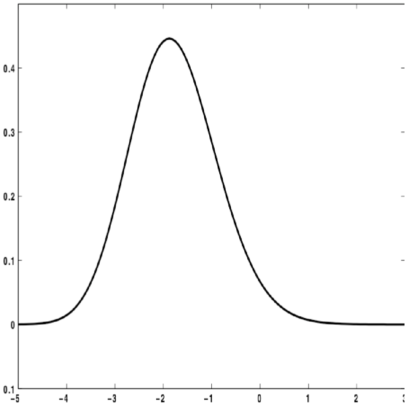

density

mean $\simeq -1.77$

$$F_{\mathrm{TW}}(s) \sim e^{-s^3/12} \quad \text{as} \quad s \to -\infty$$

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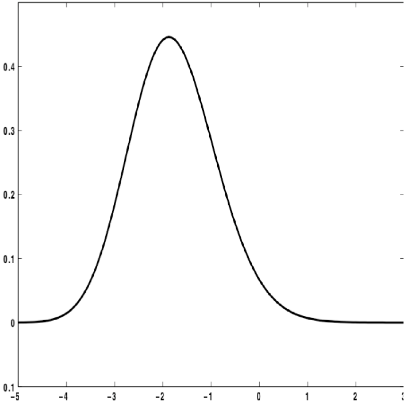

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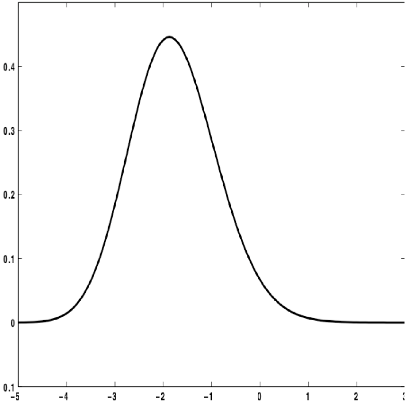

density (similar for GOE)

non asymptotic small deviation inequalities

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expected bounds

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correct large deviation bounds $(t \geq 1)$

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variety of techniques

to the non asymptotic inequalities

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$$(p+1)a_p^N = (4p-2)Na_{p-1}^N + (p-1)(2p-1)(2p-3)a_{p-2}^N$$

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real Gaussian Orthogonal Ensemble (GOE)

five term recurrence equation

combinatorial moment method

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extension to sample covariance matrices

sample covariance matrices

multivariate statistical inference

principal component analysis

sample covariance matrices

multivariate statistical inference

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population $(Y_1, \ldots, Y_N)$

Y_j vector in $\mathbb{R}^M$ (characters)

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(Gaussian) Wishart matrix models

sample covariance matrices

$$Y = Y^{M,N} = (Y_{ij})_{1 \leq i \leq M, 1 \leq j \leq N} \quad M \times N \quad \text{matrix} \quad (M \geq N)$$

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$$\mathbb{E}(Y_{ij}) = 0, \quad \mathbb{E}\big((Y_{ij})^2\big) = 1$$

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ρ **Marchenko-Pastur** distribution

Tracy-Widom theorem for the largest eigenvalue

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O. Feldheim, S. Sodin (2009)

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largest eigenvalue (symmetric, subGaussian entries)

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smallest eigenvalue

at the soft edge $M \sim \kappa\, N, \quad \kappa > 1$

$$\mathbb{P}\big(\widehat{\lambda}_1^N \leq (\sqrt{\kappa} - 1)^2 - \varepsilon \big) \leq C\, e^{-N\, \varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

tridiagonal models (β-ensembles)

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GOE, GUE, Wishart models

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joint law of the eigenvalues $(\lambda_1^N \leq \cdots \leq \lambda_N^N)$ of X^N

$$P^N(dx) = \frac{1}{Z} \left|\Delta_N(x)\right|^\beta \prod_{k=1}^N e^{-\beta x_k^2/4} \, dx_k$$

$\beta = 1$: GOE $\quad$ $\beta = 2$: GUE

tridiagonal representation

$$\begin{pmatrix} g_1 & \chi_{N-1} & 0 & \cdots & \cdots & 0 \\ \chi_{N-1} & g_2 & \chi_{N-2} & 0 & \cdots & \vdots \\ 0 & \chi_{N-2} & g_3 & \chi_{N-3} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \chi_2 & g_{N-1} & \chi_1 \\ 0 & \cdots & \cdots & 0 & \chi_1 & g_N \end{pmatrix}$$

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Painlevé equation ?

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finite N variance bounds

$$\mathrm{Var}(\widehat{\lambda}_N^N) \leq C\,N^{-4/3}$$

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open for general Wigner matrices