Grundlehren der mathematischen Wissenschaften 348

A Series of Comprehensive Studies in Mathematics

For further volumes:
www.springer.com/series/138

Dominique Bakry • Ivan Gentil • Michel Ledoux

Analysis and Geometry of Markov Diffusion Operators

Springer

Dominique Bakry
Institut de Mathématiques de Toulouse
Université de Toulouse
and Institut Universitaire de France
Toulouse, France

Michel Ledoux
Institut de Mathématiques de Toulouse
Université de Toulouse
and Institut Universitaire de France
Toulouse, France

Ivan Gentil
Institut Camille Jordan
Université Claude Bernard Lyon 1
Villeurbanne, France

ISSN 0072-7830 ISSN 2196-9701 (electronic)
Grundlehren der mathematischen Wissenschaften
ISBN 978-3-319-00226-2 ISBN 978-3-319-00227-9 (eBook)
DOI 10.1007/978-3-319-00227-9
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013952461

Mathematics Subject Classification: 39B62, 39B72, 47D07, 53C21

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

How far can you go with the Cauchy-Schwarz inequality and integration by parts?

To Leonard Gross

Preface

Semigroups of operators on a Banach space provide very general models and tools in the analysis of time evolution phenomena and dynamical systems. They have a long history in mathematics and have been studied in a number of settings, from functional analysis and mathematical physics to probability theory, Riemannian geometry, Lie groups, analysis of algorithms, and elsewhere.

The part of semigroup theory investigated in this book deals with Markov diffusion semigroups and their infinitesimal generators, which naturally arise as solutions of stochastic differential equations and partial differential equations. As such, the topic covers a large body of mathematics ranging from probability theory and partial differential equations to functional analysis and differential geometry for operators or processes on manifolds. Within these frameworks, research interests have grown over the years, now encompassing a wide variety of questions such as regularity and smoothing properties of differential operators, Sobolev-type estimates, heat kernel bounds, non-explosion properties, convergence to equilibrium, existence and regularity of solutions of stochastic differential equations, martingale problems, stochastic calculus of variations and so on.

This book is more precisely focused on the concrete interplay between the analytic, probabilistic and geometric aspects of Markov diffusion semigroups and generators involved in convergence to equilibrium, spectral bounds, functional inequalities and various bounds on solutions of evolution equations linked to geometric properties of the underlying structure.

One prototypical example at this interface is simply the standard heat semigroup $(P_t)_{t\geq 0}$ on the Euclidean space $\mathbb{R}^n$ whose Gaussian kernel

$$u = u(t,x) = p_t(x) = \frac{1}{(4\pi t)^{n/2}}\, e^{-|x|^2/4t}, \quad t > 0,\ x \in \mathbb{R}^n,$$

is a fundamental solution of the heat equation

$$\partial_t u = \Delta u, \quad u(0,x) = \delta_0,$$

for the standard Laplace operator Δ, thus characterized as the infinitesimal generator of the semigroup $(P_t)_{t\geq 0}$.

From the probabilistic viewpoint, the family of kernels $p_t(x)$, $t > 0$, $x \in \mathbb{R}^n$, represents the transition probabilities of a standard Brownian motion $(B_t)_{t \geq 0}$ as

$$\mathbb{E}\big(f(x + B_{2t})\big) = \int_{\mathbb{R}^n} f(y)\, p_t(x - y)dy = P_t f(x), \quad t > 0,\ x \in \mathbb{R}^n,$$

for all bounded measurable functions $f : \mathbb{R}^n \to \mathbb{R}$.

The third aspect investigated in this work is geometric, and perhaps less immediately apparent than the analytic and probabilistic aspects. It aims to interpret, in some sense, the commutation of derivation and action of the semigroup as a curvature condition. For the standard Euclidean semigroup example above, the commutation $\nabla P_t f = P_t(\nabla f)$ will express a zero curvature, although this corresponds not only to the curvature of Euclidean space as a Riemannian manifold but rather to the curvature of Euclidean space equipped with the Lebesgue measure, invariant under the heat flow $(P_t)_{t \geq 0}$, and the bilinear operator $\Gamma(f, g) = \nabla f \cdot \nabla g$.

In order to carry out the investigation along these lines, the exposition emphasizes the basic structure of a Markov Triple[1] (E, μ, Γ) consisting of a (measurable) state space E, a carré du champ operator Γ and a measure μ invariant under the dynamics induced by Γ. The notion of a carré du champ operator Γ associated with a Markov semigroup $(P_t)_{t \geq 0}$ with infinitesimal generator L given (on a suitable algebra $\mathcal{A}$ of functions on E) by

$$\Gamma(f, g) = \frac{1}{2}\big[\mathrm{L}(fg) - f\mathrm{L}g - g\mathrm{L}f\big],$$

will be a central tool of investigation, the associated Γ-calculus providing, at least at a formal level, a kind of algebraic framework encircling the relevant properties and results.

These analytic, stochastic and geometric features form the basis of the investigation undertaken in this book, describing Markov semigroups via their infinitesimal generators as solutions of second order differential operators and their probabilistic representations as Markov processes, and analyzing them with respect to curvature properties. The investigation is limited to symmetric (reversible in the Markovian terminology) semigroups, although various ideas and techniques go beyond this framework. We also restrict our attention to the diffusion setting, that is when the carré du champ operator is a derivation operator in its two arguments, even in those cases where the result could be extended to a more general setting. These restrictions rule out many interesting fields of applications (discrete Markov chains, models of statistical mechanics, most of the analysis of algorithms of interest in optimization theory or approximations of partial differential equations, for example), but allow us to concentrate on central features in the analysis of semigroups, in the same way that ordinary differential equations are in general easier to handle than discrete sequences. Even within the field of symmetric diffusion semigroups, we have not tried

[1] The terminology "Markov triple" should not, of course, be confused with solutions of the Markov Diophantine equation $x^2 + y^2 + z^2 = 3xyz$!

to cover all the possible interesting cases. In order to keep the monograph within a reasonable size, we have had to omit, among other things, the specific analysis related to hypoelliptic diffusions, special features of diffusions on Lie groups, and many interesting developments arising from infinite interacting particle systems.

In addition, although we have largely been motivated by the analysis of the behavior of diffusion processes (that is, solutions of time homogeneous stochastic differential equations), rather than concentrating on the probabilistic aspects of the subject, such as almost sure convergence of functionals of the trajectories of the underlying Markov processes, recurrence or transience, we instead chose to translate most of the features of interest into functional analytic properties of the Markov structure (E, μ, Γ) under investigation.

Heat kernel bounds, functional inequalities and their applications to convergence to equilibrium and geometric features of Markov operators are among the main topics of interest developed in this monograph. A particular emphasis is placed on families of inequalities relating, on a Markov Triple (E, μ, Γ), functionals of functions $f : E \to \mathbb{R}$ to the energy induced by the invariant measure μ and the carré du champ operator Γ,

$$\mathcal{E}(f, f) = \int_E \Gamma(f, f) d\mu.$$

Typical functionals are the variance, entropy or $\mathbb{L}^p$-norms leading to the main functional inequalities of interest, the Poincaré or spectral gap inequality, the logarithmic Sobolev inequality and the Sobolev inequality. A particular goal is to establish such families of inequalities under suitable curvature conditions which may be described by the carré du champ operator Γ and its iterated Γ_2 operator.

Similar inequalities are investigated at the level of the underlying semigroup $(P_t)_{t \geq 0}$ for the heat kernel measures, comparing $P_t(\varphi(f))$ (for some $\varphi : \mathbb{R} \to \mathbb{R}$) to $P_t(\Gamma(f, f))$ or $\Gamma(P_t f, P_t f)$, which give rise to heat kernel bounds. With this task in mind, we will develop the main powerful tool of heat flow monotonicity, or semigroup interpolation, with numerous illustrative applications and strong intuitive content. To illustrate the principle, as a wink towards what is to come, let us briefly present here a heat flow proof of the classical Hölder inequality which is very much in the spirit of this book. In particular, the reduction to a quadratic bound is typical of the arguments developed in this work. Let f, g be suitable (strictly) positive functions on $\mathbb{R}^n$ and $\theta \in (0, 1)$. For fixed $t > 0$, consider, at any point (omitted), the interpolation

$$\Lambda(s) = P_s\big(e^{\theta \log P_{t-s} f + (1-\theta) \log P_{t-s} g}\big), \quad s \in [0, t],$$

where $(P_t)_{t \geq 0}$ is the standard heat semigroup on $\mathbb{R}^n$ as recalled above. Together with the heat equation $\partial_s P_s = \Delta P_s = P_s \Delta$, the derivative in s of Λ is given by

$$\Lambda'(s) = P_s\big(\Delta\big(e^H\big) - e^H\big[\theta\, e^{-F} \Delta\big(e^F\big) + (1-\theta) e^{-G} \Delta\big(e^G\big)\big]\big)$$

where $F = \log P_{t-s} f$, $G = \log P_{t-s} g$ and $H = \theta F + (1-\theta)G$. Now by standard calculus,

$$\begin{aligned} & e^{-H}\Delta\big(e^H\big) - \big[\theta\, e^{-F}\Delta\big(e^F\big) + (1-\theta)e^{-G}\Delta\big(e^G\big)\big] \\ & \quad = |\nabla H|^2 - \theta|\nabla F|^2 - (1-\theta)|\nabla G|^2 \end{aligned}$$

which is negative by convexity of the square function. Hence $\Lambda(s)$, $s \in [0,t]$, is decreasing, and thus

$$\Lambda(t) = P_t\big(f^\theta g^{1-\theta}\big) \le (P_t f)^\theta (P_t g)^{1-\theta} = \Lambda(0).$$

Normalizing by $t^{n/2}$ and letting t tend to infinity yields Hölder's inequality for the Lebesgue measure. Actually, the same argument may be performed at the level of a Markov semigroup with invariant finite discrete measure, thus yielding Hölder's inequality for arbitrary measures.

While functional inequalities and their related applications are an important focal point, they also give us the opportunity to discuss a number of issues related to examples and properties of Markov semigroups and operators. One objective of this work is thus also to present the basic tools and ideas revolving around Markov semigroups and to illustrate their usefulness in different contexts.

The monograph comprises three main parts.

The first part, covering Chaps. 1 to 3, presents some of the main features, properties and examples of Markov diffusion semigroups and operators as considered in this work. In a somewhat informal but intuitive way, Chap. 1 introduces Markov semigroups, their infinitesimal generators and associated Markov processes, stochastic differential equations and diffusion semigroups. It also describes a few of the standard operations and techniques while working with semigroups. Chapter 2 develops in detail a number of central geometric models which will serve as references for later developments, namely the heat semigroups and Laplacians on the flat Euclidean space, the sphere and the hyperbolic space. Sturm-Liouville operators on the line, and some of the most relevant examples (Ornstein-Uhlenbeck, Laguerre and Jacobi), are also presented therein. On the basis of these preliminary observations and examples, Chap. 3 then tries to describe a general framework of investigation. While it would not be appropriate to try to cover in a unique formal mould all the cases of interest, it is nevertheless useful to emphasize the basic properties and tools in order to easily and suitably develop the Γ-calculus. In particular, it is necessary to describe with some care the various classes and algebras of functions that we shall be dealing with and to show their relevance in the classical smooth settings. Note that while infinite-dimensional models would require further care in this abstract formalism, the methods and principles emphasized throughout this work are similarly relevant for them. Taking the more classical picture as granted, Chap. 3 may be skipped at first reading (or limited to the summary Sect. 3.4).

Part II, forming the core of the text, includes Chaps. 4 to 6 and covers the three main functional inequalities of interest, Poincaré or spectral gap inequalities, logarithmic Sobolev inequalities and Sobolev inequalities. For each family, some basic properties and tools are detailed, in tight connection with the reference examples of Chap. 2 and their geometric properties. Stability, perturbation and comparison properties, characterization in dimension one, concentration bounds and convergence to equilibrium are thus addressed for each family. The discussion then distinguishes between inequalities for the heat kernel measures (local) and for the invariant measure (global) which are analyzed and established under curvature hypotheses. Chapter 4 is thus devoted to Poincaré or spectral gap inequalities, closely related to spectral decompositions. Chapter 5 deals with logarithmic Sobolev inequalities, emphasized as the natural substitute for classical Sobolev-type inequalities in infinite dimension, and their equivalent hypercontractive smoothing properties. Sobolev inequalities form a main family of interest for which Chap. 6 provides a number of equivalent descriptions (entropy-energy, Nash or Gagliardo-Nirenberg inequalities) and associated heat kernel bounds. A significant proportion of this chapter is devoted to the rich geometric content of Sobolev inequalities, their conformal invariance, and the curvature-dimension conditions.

On the basis of the main functional inequalities of Part II, Part III, consisting of Chaps. 7 to 9, addresses several variations, extensions and related topics of interest. Chapter 7 deals with general families of functional inequalities, each of them having their own interest and usefulness. The exposition mainly emphasizes entropy-energy (on the model of logarithmic Sobolev inequalities) and Nash-type inequalities. In addition, the tightness of functional inequalities is studied by employing the tool of weak Poincaré inequalities. Chapter 8 is an equivalent description of the various families of inequalities for functions presented so far in terms of sets and capacities for which co-area formulas provide the suitable link. The second part of this chapter is concerned with isoperimetric-type inequalities for which semigroup tools again prove most useful. Chapter 9 briefly presents some of the recent important developments in optimal transportation in connection with the semigroup and Γ-calculus, including in particular a discussion of the relationships between functional and transportation cost inequalities (in a smooth Riemannian setting).

The last part of the monograph consists of three appendices, on semigroups of operators on a Banach space, elements of stochastic calculus and the basics of differential and Riemannian geometry. At the interface between analysis, probability and geometry, these appendices aim to possibly supplement the reader's knowledge depending on his own background. They are not strictly necessary for the comprehension of the core of the text, but may serve as a support for the more specialized parts. It should be mentioned, however, that the last two sections of the third appendix on the basics of Riemannian geometry actually contain material on the Γ-calculus (in a Riemannian context) which will be used in a critical way in some parts of the book.

This book has been designed to be both an introduction to the subject, intended to be accessible to non-specialists, and an exposition of both basic and more advanced results of the theory of Markov diffusion semigroups and operators. Indeed we chose to concentrate on those points where we felt that the techniques and ideas are central and may be used in a wider context, even though we have not attempted to reach the widest generality. Every chapter starts at a level which is elementary for the notions developed in it, but may evolve to more specialized topics which in general may be skipped at first reading. It should be stressed that the level of exposition throughout the book fairly non-uniform, sometimes putting emphasis on facts or results which may appear as obvious or classical for some readers while developing at the same time more sophisticated issues. This choice is motivated by our desire to make the text accessible to readers with different backgrounds, and also by our aim to provide tools and methods to access more difficult parts of the theory or to be applied in different contexts. This delicate balance is not always reached but we nevertheless hope that the chosen style of exposition is helpful.

The monograph is intended for students and researchers interested in the modern aspects of Markov diffusion semigroups and operators and their connections with analytic functional inequalities, probabilistic convergence to equilibrium and geometric curvature. Selected chapters may be used for advanced courses on the topic. Readers who wish to get a flavor of Markov semigroups and their applications should concentrate on Part I (with the exception of Chap. 3) and Part II. Via an appropriate selection of topics, Part III tries to synthesise the developments of the last decade. The book demands from the reader only a reasonable knowledge of basic functional analysis, measure theory and probability theory. It is also expected that it may be read in a non-linear way, although the various chapters are not completely independent. The reader not familiar with the main themes (analysis, probability and geometry) will find some of the basic material collected together in the appendices.

Each Chapter is divided into Sections, often themselves divided in Sub-Sections. Section 1.8 is the eighth section in Chap. 1. Theorem 4.6.2 indicates a theorem in Chap. 4, Sect. 4.6, and (3.2.2) is a formula in Sect. 3.2. An item of a given chapter is also referred to in other chapters by the page on which it appears. There are no references to articles or books within the exposition of a given chapter. The Sections "Notes and References" at the end of each chapter briefly describe some historical developments with pointers to the literature. The references are far from exhaustive and in fact are rather limited. There is no claim for completeness and we apologize for omissions and errors. For books and monographs, we have tried to present the references in historical order with respect to original editions (although the links point toward the latest editions).

This book began its life in the form of lectures presented by the first author at Saint-Louis du Sénégal in April 2009. He thanks the organizers of this school for the opportunity to give this course and the participants for their interest. This work presents results and developments which have emerged during the last three decades. Over the years, we have benefited from the vision, expertise and help of a number of friends and colleagues, among them M. Arnaudon, F. Barthe, W. Beckner, S. Bobkov, F. Bolley, C. Borell, E. Carlen, G. Carron, P. Cattiaux, D. Chafaï, D. Cordero-Erausquin, T. Coulhon, J. Demange, J. Dolbeault, K. D. Elworthy, M. Émery, A. Farina, P. Fougères, N. Gozlan, L. Gross, A. Guillin, E. Hebey, B. Helffer, A. Joulin, C. Léonard, X. D. Li, P. Maheux, F. Malrieu, L. Miclo, E. Milman, B. Nazaret, V. H. Nguyen, Z.-M. Qian, M.-K. von Renesse, C. Roberto, M. de la Salle, L. Saloff-Coste, K.-T. Sturm, C. Villani, F.-Y. Wang, L. Wu and B. Zegarliński. We wish to thank them for their helpful remarks and constant support.

F. Bolley, S. Campese and C. Léonard went through parts of the manuscript at several stages of the preparation, and we warmly thank them for all their corrections and comments that helped to improve the exposition.

We sincerely thank the Springer Editors C. Byrne and M. Reizakis and the production staff for a great editing process.

We apologize for all the errors, and invite the readers to report any remarks, mistakes and misprints. A list of errata and comments will be maintained online.

Lyon, Toulouse
June 2013

Dominique Bakry
Ivan Gentil
Michel Ledoux

Basic Conventions

Here are some classical and basic conventions used throughout the book.

$\mathbb{N}$ is the set of integers $\{0, 1, 2, \dots\}$. The set of real numbers is denoted by $\mathbb{R}$. Functions (on some state space E) are always real-valued. Points in $\mathbb{R}$ are usually denoted by x (if $\mathbb{R}$ is the underlying state space) or by r.

An element $r \in \mathbb{R}$ is positive if $r \geq 0$, strictly positive if $r > 0$, negative if $r \leq 0$ and strictly negative if $r < 0$. Moreover, $\mathbb{R}_+ = [0, \infty)$ is the set of positive real numbers while $(0, \infty)$ denotes the set of strictly positive numbers. For $r, s \in \mathbb{R}$, $r \wedge s = \min(r, s)$ and $r \vee s = \max(r, s)$. We agree that $0 \log 0 = 0$.

In the same way (and somewhat against the current), a positive (respectively negative) function f (on E) is such that $f(x) \geq 0$ (respectively $f(x) \leq 0$) for every $x \in E$. The function is strictly positive or strictly negative whenever the inequalities are strict. Similarly, an increasing (respectively decreasing) function f on $\mathbb{R}$ or some interval of $\mathbb{R}$ satisfies $f(x) \leq f(y)$ (respectively $f(x) \geq f(y)$) for every $x \leq y$. The function f is said to be strictly increasing or strictly decreasing whenever the preceding inequalities are strict. A function is monotone if it is increasing or decreasing.

Points in $\mathbb{R}^n$ are denoted by $x = (x_1, \dots, x_n) = (x_i)_{1 \leq i \leq n}$ (or sometimes $x = (x^1, \dots, x^n) = (x^i)_{1 \leq i \leq n}$ depending on the geometric context). The scalar product and Euclidean norm in $\mathbb{R}^n$ are given by

$$x \cdot y = \sum_{i=1}^{n} x_i \, y_i, \quad |x| = (x \cdot x)^{1/2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

The notation $|\cdot|$ is used throughout to denote the Euclidean norm of vectors and of tensors.

The constant function equal to 1 on a state space E is denoted by $\mathbb{1}$. If $A \subset E$, $\mathbb{1}_A$ is the characteristic or indicator function of A.

All measures on a measurable space $(E, \mathcal{F})$ considered here are positive measures. Positive (measurable) functions on $(E, \mathcal{F})$ may take the value $+\infty$. If μ is a (positive) measure on $(E, \mathcal{F})$, and if f is a function on E which is integrable with respect to μ, its integral with respect to μ is denoted by $\int_E f d\mu$ or $\int_E f(x) d\mu(x)$, or sometimes as $\int_E f(x)\mu(dx)$. The Lebesgue measure on the Borel sets of $\mathbb{R}^n$ is denoted by dx. If B is a Borel set in $\mathbb{R}^n$, its Lebesgue measure is sometimes denoted by $\mathrm{vol}_n(B)$.

The terminology "change of variables" is used in the broad sense of changing a variable x into $h(x)$ and a function f into $\psi(f)$. "Chain rule" is understood more in an algebraic sense when h and ψ are polynomials.

The notations are supposed to be reasonably stable throughout the monograph. Further definitions and conventions will be given in the text when they are needed. A list of symbols and notations with the corresponding reference pages is given on p. 523.

Contents

Part I Markov Semigroups, Basics and Examples

1 Markov Semigroups . . . 3
1.1 Markov Processes and Associated Semigroups . . . 7
1.2 Markov Semigroups, Invariant Measures and Kernels . . . 9
1.3 Chapman-Kolmogorov Equations . . . 16
1.4 Infinitesimal Generators and Carré du Champ Operators . . . 18
1.5 Fokker-Planck Equations . . . 23
1.6 Symmetric Markov Semigroups . . . 24
1.7 Dirichlet Forms and Spectral Decompositions . . . 29
1.8 Ergodicity . . . 32
1.9 Markov Chains . . . 33
1.10 Stochastic Differential Equations and Diffusion Processes . . . 38
1.11 Diffusion Semigroups and Operators . . . 42
1.12 Ellipticity and Hypo-ellipticity . . . 49
1.13 Domains . . . 52
1.14 Summary of Hypotheses (Markov Semigroup) . . . 53
1.15 Working with Markov Semigroups . . . 56
1.16 Curvature-Dimension Condition . . . 70
1.17 Notes and References . . . 74

2 Model Examples . . . 77
2.1 Euclidean Heat Semigroup . . . 78
2.2 Spherical Heat Semigroup . . . 81
2.3 Hyperbolic Heat Semigroup . . . 88
2.4 The Heat Semigroup on a Half-Line and the Bessel Semigroup . . . 92
2.5 The Heat Semigroup on the Circle and on a Bounded Interval . . . 96
2.6 Sturm-Liouville Semigroups on an Interval . . . 97
2.7 Diffusion Semigroups Associated with Orthogonal Polynomials . . . 102
2.8 Notes and References . . . 118

3 Symmetric Markov Diffusion Operators . . . 119
3.1 Markov Triples . . . 120
3.2 Second Order Differential Operators on a Manifold . . . 137
3.3 Heart of Darkness . . . 151
3.4 Summary of Hypotheses (Markov Triple) . . . 168
3.5 Notes and References . . . 173

Part II Three Model Functional Inequalities

4 Poincaré Inequalities . . . 177
4.1 The Example of the Ornstein-Uhlenbeck Semigroup . . . 178
4.2 Poincaré Inequalities . . . 181
4.3 Tensorization of Poincaré Inequalities . . . 185
4.4 The Example of the Exponential Measure, and Exponential Integrability . . . 187
4.5 Poincaré Inequalities on the Real Line . . . 193
4.6 The Lyapunov Function Method . . . 201
4.7 Local Poincaré Inequalities . . . 206
4.8 Poincaré Inequalities Under a Curvature-Dimension Condition . . . 211
4.9 Brascamp-Lieb Inequalities . . . 215
4.10 Further Spectral Inequalities . . . 220
4.11 Notes and References . . . 230

5 Logarithmic Sobolev Inequalities . . . 235
5.1 Logarithmic Sobolev Inequalities . . . 236
5.2 Entropy Decay and Hypercontractivity . . . 243
5.3 Integrability of Eigenvectors . . . 250
5.4 Logarithmic Sobolev Inequalities and Exponential Integrability . . . 252
5.5 Local Logarithmic Sobolev Inequalities . . . 257
5.6 Infinite-Dimensional Harnack Inequalities . . . 265
5.7 Logarithmic Sobolev Inequalities Under a Curvature-Dimension Condition . . . 268
5.8 Notes and References . . . 273

6 Sobolev Inequalities . . . 277
6.1 Sobolev Inequalities on the Model Spaces . . . 278
6.2 Sobolev and Related Inequalities . . . 279
6.3 Ultracontractivity and Heat Kernel Bounds . . . 286
6.4 Ultracontractivity and Compact Embeddings . . . 290
6.5 Tensorization of Sobolev Inequalities . . . 291
6.6 Sobolev Inequalities and Lipschitz Functions . . . 293
6.7 Local Sobolev Inequalities . . . 296
6.8 Sobolev Inequalities Under a Curvature-Dimension Condition . . . 305
6.9 Conformal Invariance of Sobolev Inequalities . . . 313
6.10 Gagliardo-Nirenberg Inequalities . . . 323
6.11 Fast Diffusion Equations and Sobolev Inequalities . . . 330
6.12 Notes and References . . . 340

Part III Related Functional, Isoperimetric and Transportation Inequalities

7 Generalized Functional Inequalities 347
7.1 Inequalities Between Entropy and Energy 348
7.2 Off-diagonal Heat Kernel Bounds 355
7.3 Examples 362
7.4 Beyond Nash Inequalities 364
7.5 Weak Poincaré Inequalities 373
7.6 Further Families of Functional Inequalities 382
7.7 Summary for the Model Example μ_α 386
7.8 Notes and References 387

8 Capacity and Isoperimetric-Type Inequalities 391
8.1 Capacity Inequalities and Co-area Formulas 392
8.2 Capacity and Sobolev Inequalities 396
8.3 Capacity and Poincaré and Logarithmic Sobolev Inequalities . . 399
8.4 Capacity and Further Functional Inequalities 403
8.5 Gaussian Isoperimetric-Type Inequalities Under a Curvature Condition 411
8.6 Harnack Inequalities Revisited 421
8.7 From Concentration to Isoperimetry 425
8.8 Notes and References 429

9 Optimal Transportation and Functional Inequalities 433
9.1 Optimal Transportation 434
9.2 Transportation Cost Inequalities 438
9.3 Transportation Proofs of Functional Inequalities 442
9.4 Hamilton-Jacobi Equations 451
9.5 Hypercontractivity of Solutions of Hamilton-Jacobi Equations . . 454
9.6 Transportation Cost and Logarithmic Sobolev Inequalities 458
9.7 Heat Flow Contraction in Wasserstein Space 462
9.8 Curvature of Metric Measure Spaces 464
9.9 Notes and References 466

Appendices

Appendix A Semigroups of Bounded Operators on a Banach Space . . 473
A.1 The Hille-Yosida Theory 473
A.2 Symmetric Operators 475
A.3 Friedrichs Extension of Positive Operators 477
A.4 Spectral Decompositions 478
A.5 Essentially Self-adjoint Operators 481
A.6 Compact and Hilbert-Schmidt Operators 483
A.7 Notes and References 485

Appendix B Elements of Stochastic Calculus 487
B.1 Brownian Motion and Stochastic Integrals 487
B.2 The Itô Formula 491
B.3 Stochastic Differential Equations 493
B.4 Diffusion Processes 495
B.5 Notes and References 498

Appendix C Basic Notions in Differential and Riemannian Geometry 499
C.1 Differentiable Manifolds 500
C.2 Some Elementary Euclidean Geometry 502
C.3 Basic Notions in Riemannian Geometry 504
C.4 Riemannian Distance 509
C.5 The Riemannian Γ and Γ_2 Operators 511
C.6 Curvature-Dimension Conditions 513
C.7 Notes and References 518

Afterword 521
Chicken “Gaston Gérard” 521

Notation and List of Symbols 523

Bibliography 527

Index 547