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GAFA Geometric And Functional Analysis

FROM BRUNN–MINKOWSKI TO BRASCAMP–LIEB AND TO LOGARITHMIC SOBOLEV INEQUALITIES

S.G. Bobkov and M. Ledoux

Abstract

We develop several applications of the Brunn–Minkowski inequality in the Prékopa–Leindler form. In particular, we show that an argument of B. Maurey may be adapted to deduce from the Prékopa–Leindler theorem the Brascamp–Lieb inequality for strictly convex potentials. We deduce similarly the logarithmic Sobolev inequality for uniformly convex potentials for which we deal more generally with arbitrary norms and obtain some new results in this context. Applications to transportation cost and to concentration on uniformly convex bodies complete the exposition.

1 Introduction

The Prékopa–Leinder theorem [Pr1,2], [Le], is a functional form of the geometric Brunn–Minkowski inequality which indicates that whenever $t, s > 0$, $t + s = 1$, and u, v, w are non-negative measurable functions on $\mathbb{R}^n$ such that for all $x, y \in \mathbb{R}^n$,

$$w(tx + sy) \geq u(x)^t v(y)^s \,,$$

then

$$\int w\,dx \geq \left(\int u\,dx\right)^t \left(\int v\,dx\right)^s . \tag{1.1}$$

Applied to the characteristic functions of bounded measurable sets A and B in $\mathbb{R}^n$, it yields the multiplicative form of the Brunn–Minkowski inequality

$$\mathrm{vol}(tA + sB) \geq \mathrm{vol}(A)^t \,\mathrm{vol}(B)^s$$

for every $t, s > 0$, $t+s = 1$ (cf. e.g. [P2] for a short proof, due to K. Ball, of (1.1)). As is well known, the Brunn–Minkowski inequality may be used to produce a direct proof of the classical isoperimetric inequality in Euclidean space. The Prékopa–Leindler theorem has been strengthened and studied extensively by H. Brascamp and E. Lieb in their paper [BrL]. Its dimension free character makes it a useful tool for infinite dimensional analysis illustrated by Gaussian measures. For example, Maurey used it in [Ma] to give a simple proof of the Gaussian concentration property

$$\int e^{\frac{1}{4}d(\,\cdot\,,A)^2}d\gamma \le \frac{1}{\gamma(A)}$$

where $d(\,\cdot\,,A)$ denotes the (Euclidean) distance to the set A in $\mathbb{R}^n$, as well as of the Poincaré, or spectral gap, inequality

$$\mathrm{Var}_\gamma(f) \le \int |\nabla f|^2 d\gamma \tag{1.2}$$

where $\mathrm{Var}_\gamma(f)$ stands for the variance of $f : \mathbb{R}^n \to \mathbb{R}$ with respect to the canonical Gaussian measure γ on $\mathbb{R}^n$, and where $|\nabla f|$ denotes the Euclidean length of the gradient ∇f of the smooth function f. (In inequalities such as (1.2), smooth usually means locally Lipschitz for example.) For an arbitrary Gaussian measure μ with covariance matrix Γ, (1.2) takes the form

$$\mathrm{Var}_\mu(f) \le \int \langle \Gamma\nabla f, \nabla f\rangle d\mu\,. \tag{1.3}$$

By means of the isoperimetric inequality in Euclidean space and an asymptotic argument, W. Beckner [Be] recently recaptured Gross logarithmic Sobolev inequality [Gr]

$$\mathrm{Ent}_\gamma(f^2) = \int f^2 \log f^2 d\gamma - \int f^2 d\gamma \log \int f^2 d\gamma \le 2\int |\nabla f|^2 d\gamma \tag{1.4}$$

that is well known to contain both the spectral gap inequality and the concentration properties of γ (cf. [L]).

Now, let μ be a log-concave measure on $\mathbb{R}^n$ thus given by

$$\tfrac{d\mu}{dx} = e^{-V(x)}\,, \quad x \in \mathbb{R}^n\,,$$

where V is a convex function on $\mathbb{R}^n$. Brascamp and Lieb proved in their paper (although somewhat independently of their investigation of the Prékopa–Leindler theorem) that whenever V is strictly convex, for every smooth function f on $\mathbb{R}^n$,

$$\mathrm{Var}_\mu(f) \le \int \big\langle V''^{-1}\nabla f, \nabla f\big\rangle d\mu \tag{1.5}$$

where V''^{-1} denotes the inverse of the Hessian of V, an inequality that considerably extends the Gaussian case (1.2), (1.3). In particular, if, as symmetric matrices, $V''(x) \ge c\,\mathrm{Id}$ for some $c > 0$ and every x,

$$\mathrm{Var}_\mu(f) \le \frac{1}{c}\int |\nabla f|^2 d\mu \tag{1.6}$$

(a result known earlier in Riemannian geometry). On the other hand, it has been shown recently in [Bo] that this inequality (1.6) actually holds for any convex V with a constant however depending on the dimension n.

Under the condition $V'' \geq c\,\mathrm{Id}$, $c > 0$, the inequality (1.6) has been strengthened into a logarithmic Sobolev inequality by means of the Γ_2 criterion of Bakry–Emery and semigroup methods (see [B]) as

$$\mathrm{Ent}_\mu(f^2) \leq \frac{2}{c} \int |\nabla f|^2 d\mu \tag{1.7}$$

for any smooth function f on $\mathbb{R}^n$. As developed in the notes [L], such a logarithmic Sobolev inequality entails, by the Herbst argument, Gaussian concentration properties for the measure μ. In particular,

$$\int \mathrm{e}^{\alpha |x|^2} d\mu(x) < \infty \tag{1.8}$$

for some $\alpha > 0$ (actually every $\alpha < 1/2c$). It is further shown in [W], [A], [Bo] (cf. [L]) that a log-concave measure μ such that (1.8) holds for some $\alpha > 0$ satisfies the logarithmic Sobolev inequality (1.7) (again with a constant essentially depending on the dimension).

In the first part of this work, we show how Maurey's method may be adapted to deduce the Brascamp–Lieb inequality (1.5) from the Prékopa–Leindler theorem. We observe furthermore that one cannot hope for the entropic version of the Brascamp–Lieb inequality

$$\mathrm{Ent}_\mu(f^2) \leq 2 \int \langle V''^{-1} \nabla f, \nabla f \rangle d\mu$$

to hold. However, we prove at the end of section 3 that it actually holds (up to a multiplicative constant) under an additional convexity assumption on V. Next, we show how to deduce the logarithmic Sobolev inequality (1.7) again from the Prékopa–Leindler theorem. With respect to [Ma], we make use of the Prékopa–Leindler inequality for the full range of values of $t, s > 0$, $t + s = 1$ (rather than only $t = s = 1/2$), following in this the recent contribution [Bo]. This approach actually allows us to deal with more general convexity conditions involving arbitrary norms $\| \cdot \|$ on $\mathbb{R}^n$ (or on a finite dimensional vector space). For example, if for some $c > 0$ and all $x, y \in \mathbb{R}^n$,

$$V(x) + V(y) - 2V\Big(\frac{x+y}{2}\Big) \geq \frac{c}{2} \, \|x - y\|^2 ,$$

then, for every smooth function f on $\mathbb{R}^n$,

$$\mathrm{Ent}_\mu(f^2) \leq \frac{2}{c} \int \|\nabla f\|_*^2 d\mu$$

where $\| \cdot \|_*$ denotes the dual norm. We further discuss a similar result for p-convex V's, $p \geq 2$. A typical example is given by $V(x) = \|x\|^p$, $p \geq 2$, for a p-uniformly convex norm. In section 4, we describe a transportation cost version of our results, while in the last section we recover the concentration results of M. Gromov and V. Milman [GM] in uniformly convex spaces. In particular, we describe there logarithmic Sobolev inequalities for the uniform measure on a uniformly convex body. Our result similarly apply to measures with log-concave densities with respect to μ, and thus in particular to the conditional probabilities with respect to arbitrary convex sets (of positive measure).

2 From Prékopa–Leindler to Brascamp–Lieb

In this section, we develop Maurey's method [Ma] to deduce the Brascamp–Lieb inequality (1.5) from the Prékopa–Leindler theorem. Let E be a finite dimensional linear space with a fixed Lebesgue measure dx. Consider a probability measure μ on E with density $\mathrm{e}^{-V(x)}$, $x \in \Omega$, with respect to Lebesgue measure where V is a convex function on some open convex set $\Omega \subset E$ (in particular, μ is supported by Ω.) The following statement is just the Brascamp–Lieb inequality of [BrL].

PROPOSITION 2.1. *Assume that V is twice continuously differentiable and strictly convex on Ω. Then, for every smooth enough function f on Ω,*

$$\mathrm{Var}_\mu(f) \le \int \langle V''^{-1} \nabla f, \nabla f \rangle d\mu$$

where V''^{-1} denotes the inverse of the Hessian of V.

Proof. We assume for simplicity that $E = \mathbb{R}^n$ (equipped with its Euclidean scalar product $\langle \cdot, \cdot \rangle$). We may further assume that f is smooth with compact support and takes non-negative values. Finally, by a simple perturbation argument, we can also assume that $V'' \ge \rho \,\mathrm{Id}$ for some $\rho > 0$. For $t, s > 0$, $t+s = 1$, apply the Prékopa–Leindler theorem to the functions on Ω

$$u(x) = \mathrm{e}^{(f(x)/t) - V(x)}, \quad v(y) = e^{-V(y)}, \quad w(z) = \mathrm{e}^{f_t(z) - V(z)},$$

to get

$$\int \mathrm{e}^{f_t} d\mu \ge \left(\int \mathrm{e}^{f/t} d\mu \right)^t \tag{2.1}$$

for the optimal f_t given by

$$f_t(z) = \sup_{z = tx+sy,\, x,y \in \Omega} \big(f(x) - [tV(x) + sV(y) - V(tx+sy)] \big), \quad z \in \Omega.$$

When $z = tx + sy$, then $x = z + \frac{s}{t}(z-y)$. Hence, setting $z - y = h$,

$$f_t(z) = \sup_h \left(f\left(z + \tfrac{s}{t} h\right) - \left[tV\left(z + \tfrac{s}{t} h\right) + sV(z-h) - V(z) \right] \right) \tag{2.2}$$

where the supremum is running over all h's such that $z+\frac{s}{t}h, z-h\in\Omega$. Now, let us fix $t=s=1/2$. Let further $\delta>0$ and denote by $\tilde f_\delta$ the optimal function in (2.2) for δf. For each $z\in\Omega$, the supremum in (2.2) is attained at some point h_δ such that

$$\delta f'(z+h_\delta)=\tfrac12\big[V'(z+h_\delta)-V'(z-h_\delta)\big]\,,$$

where we write for simplicity f' and V' for the gradients of f and V. Since f is smooth with compact support, and since V is strictly convex, $h_\delta\to0$ as $\delta\to0$. Actually, since $V''\geq\rho\,\mathrm{Id}$ and f' is bounded, $h_\delta=O(\delta)$ as $\delta\to0$. Therefore, by Taylor's formula,

$$\delta f'(z)=V''(z)h_\delta+o(\delta) \tag{2.3}$$

where $o(\delta)$ can be chosen uniform in z. Then, again by Taylor's formula,

$$\begin{aligned}\tilde f_\delta(z)&=\delta f(z+h_\delta)-\big[\tfrac12\,V(z+h_\delta)+\tfrac12\,V(z-h_\delta)-V(z)\big]\\&=\delta f(z)+\delta\big\langle f'(z),h_\delta\big\rangle-\tfrac12\big\langle V''(z)h_\delta,h_\delta\big\rangle+o(\delta^2)\,.\end{aligned}$$

By (2.3), it follows that

$$\tilde f_\delta(z)=\delta f(z)+\frac{\delta^2}{2}\,\big\langle V''(z)^{-1}f'(z),f'(z)\big\rangle+o(\delta^2) \tag{2.4}$$

where, as before, $o(\delta^2)$ is uniform in z. Now, (2.1) reads for $t=s=1/2$ as

$$\int \mathrm{e}^{2\delta f}d\mu\leq\bigg(\int\mathrm{e}^{\tilde f_\delta}d\mu\bigg)^2.$$

Clearly,

$$\int \mathrm{e}^{2\delta f}d\mu=1+2\delta\int f d\mu+2\delta^2\int f^2d\mu+o(\delta^2)$$

while, by (2.4),

$$\begin{aligned}\bigg(\int\mathrm{e}^{\tilde f_\delta}d\mu\bigg)^2=1+2\delta\int f\,d\mu+\delta^2\bigg(\int f\,d\mu\bigg)^2+\delta^2\int f^2d\mu\\+\delta^2\int\langle V''^{-1}f',f'\rangle d\mu+o(\delta^2)\end{aligned}$$

from which the Brascamp–Lieb inequality easily follows as $\delta\to0$. The proof of the proposition is complete. □

As announced in the introduction, we now ask for the entropic version of the Brascamp–Lieb inequality, that is, does the inequality

$$\mathrm{Ent}_\mu(f^2)\leq2\int\langle V''^{-1}\nabla f,\nabla f\rangle d\mu \tag{2.5}$$

hold for every smooth function f? It is classical (replacing f by $1+\varepsilon f$ and letting $\varepsilon\to0$) that (2.5) would strengthen the Brascamp–Lieb inequality. On the other hand, such a logarithmic Sobolev inequality would imply by the Herbst argument (cf. [L]) that every f with $\langle {V''}^{-1}\nabla f, \nabla f\rangle \leq 1$ almost everywhere is such that

$$\int \mathrm{e}^{\alpha f^2} d\mu < \infty$$

for every $\alpha < 1/2$. However this cannot hold in general. Let us consider for example, in dimension 1, $V(x) = -\log(2x)$ on $\Omega = (0,1)$ and $f(x) = \log(x)$. Then V is strictly convex and $|f'|^2/V'' = 1$. However,

$$\int \mathrm{e}^{\alpha f^2} d\mu = 2\int_0^1 \mathrm{e}^{\alpha(\log x)^2} x\, dx = \infty$$

for every $\alpha > 0$ so that, by the preceding, (2.5) cannot hold for this V, even with some constant instead of 2 on the right-hand side. It is easy to modify this example to define V on all $\mathbb{R}$. It might be worthwhile mentioning that this example does not contradict the exponential integrability result under a Poincaré inequality (cf. e.g. [L]) since $\int \mathrm{e}^{\alpha|f|} d\mu < \infty$ for every $\alpha < 2$.

Although (2.5) does not hold in general, we will show at the end of the next section that it holds under a further (rather strong) convexity condition on V.

3 From Prékopa–Leindler to Logarithmic Sobolev Inequalities

In this section, we present an approach to logarithmic Sobolev inequalities based on the Prékopa–Leindler theorem. It might be helpful to start with the Gaussian case. However, the general case is no more difficult so that we directly turn to it. Moreover, we can formulate our main result with respect to arbitrary norms yielding some new inequalities in this context. Let thus $(E, \|\cdot\|)$ be a finite dimensional normed space (with a fixed Lebesgue measure dx). Denote by $(E^*, \|\cdot\|_*)$ the dual normed space. Consider an arbitrary probability measure μ on E with density $e^{-V(x)}$, $x \in \Omega$, with respect to Lebesgue measure where V is a convex function on some open convex set $\Omega \subset E$. Moreover, assume that, for all $t, s > 0$ with $t + s = 1$ and all $x, y \in \Omega$,

$$tV(x) + sV(y) - V(tx + sy) \geq \tfrac{cts}{2}\, \|x - y\|^2 \tag{3.1}$$

for some $c > 0$. As mentioned in the introduction, when the norm is Euclidean, the proposition below follows from the Bakry–Emery criterion [BE], [B] (cf. [L]). It does not seem that the semigroup methods developed in [BE] allow the extension to arbitrary norms.

Proposition 3.1. *Under* (3.1), *for every smooth function f on Ω,*

$$\mathrm{Ent}_\mu(f^2) \le \frac{2}{c} \int \|\nabla f\|_*^2 d\mu \,.$$

Proof. We start as in the proof of Proposition 2.1. We may and do assume that $f^2 = e^g$ where g is a smooth function on Ω with compact support. Let $t, s > 0$, $t + s = 1$, and apply the Prékopa–Leindler theorem to the functions on Ω

$$u(x) = \mathrm{e}^{(g(x)/t) - V(x)} \,, \quad v(y) = e^{-V(y)} \,, \quad w(z) = \mathrm{e}^{g_t(z) - V(z)} \,,$$

to get

$$\int \mathrm{e}^{g_t} d\mu \ge \left(\int \mathrm{e}^{g/t} d\mu \right)^t \tag{3.2}$$

where the optimal function g_t is given by

$$g_t(z) = \sup_{z = tx + sy,\, x, y \in \Omega} \big(g(x) - [tV(x) + sV(y) - V(tx + sy)] \big) \,, \quad z \in \Omega \,.$$

Since for every $x, y \in \Omega$,

$$tV(x) + sV(y) - V(tx + sy) \ge \tfrac{cts}{2} \|x - y\|^2 \,,$$

we see that

$$g_t(z) \le \sup_{z = tx + sy,\, x, y \in \Omega} \big(g(x) - \tfrac{cts}{2} \|x - y\|^2 \big) \,, \quad z \in \Omega \,.$$

The idea of the proof is to let $t \to 1$ ($s \to 0$) in (3.2) to deduce the logarithmic Sobolev inequality. To this task, it might be helpful to notice that entropy naturally arises as the derivative of L^p-norms so that by Taylor's formula,

$$\left(\int \mathrm{e}^{g/t} d\mu \right)^t = \int \mathrm{e}^g d\mu + s \, \mathrm{Ent}_\mu(\mathrm{e}^g) + O(s^2) \tag{3.3}$$

as $t \to 1$ ($s \to 0$).

Now, when $z = tx + sy$, then $x = z + \frac{s}{t}(z - y)$, $x - y = \frac{1}{t}(z - y)$, and setting $h = z - y$ and $\eta = s/t$, the above inequality yields

$$g_t(z) \le \sup_{h \in E} \left[g(z + \eta h) - \tfrac{c\eta}{2} \|h\|^2 \right] \,.$$

By a Taylor expansion,

$$g(z + \eta h) = g(z) + \eta \langle \nabla g(z), h \rangle + \|h\|^2 O(\eta^2)$$

as $\eta \to 0$ (equivalently as $s \to 0$). Noticing that $|O(\eta^2)| \le C\eta^2$ with a constant C independent of z, we get that

$$g_t(z) \le g(z) + \eta \sup_{h \in E} \left[\langle \nabla g(z), h \rangle - \left(\tfrac{c}{2} - C\eta \right) \|h\|^2 \right] \,.$$

To identify the last supremum, consider a function ψ on E of the form $\psi(h) = \langle v, h \rangle - \frac{\theta}{2} \|h\|^2$ where $h \in E$, $v \in E^*$ and $\theta > 0$. Writing $h = \lambda e$

with $\|e\| = 1$, we see that

$$\sup_{h\in E} \psi(h) = \sup_{\lambda\in\mathbb{R}} \sup_{\|e\|=1} \Big(\lambda\langle v, e\rangle - \tfrac{\theta\lambda^2}{2} \Big) = \sup_{\lambda\in\mathbb{R}} \Big(\lambda\|v\|_* - \tfrac{\theta\lambda^2}{2} \Big) = \tfrac{1}{2\theta} \, \|v\|_*^2 \, .$$

Applying this observation to $v = \nabla g(z)$, $\theta = c - 2C\eta > 0$ for small enough η's, we conclude that, uniformly over all $z \in \Omega$,

$$g_t(z) \le g(z) + \tfrac{\eta}{2c} \, \big\|\nabla g(z)\big\|_*^2 + O(\eta^2) \, .$$

Hence, uniformly over all $z \in \Omega$,

$$\mathrm{e}^{g_t(z)} \le \mathrm{e}^{g(z)} + \tfrac{\eta}{2c} \big\|\nabla g(z)\big\|_*^2 \, \mathrm{e}^{g(z)} + O(\eta^2)$$

and thus

$$\int \mathrm{e}^{g_t} d\mu \le \int \mathrm{e}^{g} d\mu + \frac{\eta}{2c} \int \|\nabla g\|_*^2 \mathrm{e}^{g} d\mu + O(\eta^2) \, .$$

Together with (3.2) and (3.3), we thus get the logarithmic Sobolev inequality

$$\mathrm{Ent}_\mu(\mathrm{e}^g) \le \frac{1}{2} \int \|\nabla g\|_*^2 \, e^g d\mu \, .$$

Since $f^2 = \mathrm{e}^g$, the proof is complete. □

We now collect some remarks on the convexity condition (3.1). If V is twice continuously differentiable on Ω, the condition (3.1) is equivalent to the property that for all $x \in \Omega$, $h \in E$,

$$\big\langle V''(x)h, h\big\rangle \ge c\|h\|^2 \, . \tag{3.4}$$

It is actually enough that (3.1) holds for some $t, s > 0$, $t+s = 1$. However, it might be worthwhile noting that the proof of Proposition 3.1 only requires that for all $t, s > 0$ with $t + s = 1$, and all $x, y \in \Omega$,

$$tV(x) + sV(y) - V(tx + sy) \ge \tfrac{c}{2}\big(s + o(s)\big)\|x - y\|^2$$

where $o(s)$ is a function of $s \in (0, 1)$ such that $o(s)/s \to 0$ as $s \to 0$.

Applying Proposition 3.1 to $1 + \varepsilon f$ and letting $\varepsilon \to 0$ yields classically the Poincaré inequality

$$\mathrm{Var}_\mu(f^2) \le \frac{1}{c} \int \|\nabla f\|_*^2 d\mu \tag{3.5}$$

for every smooth f on Ω. By (3.4), this actually follows directly from the Brascamp–Lieb inequality (1.5).

Proposition 3.1 may be extended to p-convex potentials V, $p \ge 2$. Again, let $(E, \|\cdot\|)$ be a finite-dimensional normed space with a fixed Lebesgue measure dx, and let $(E^*, \|\cdot\|_*)$ be the dual normed space. Consider a probability measure μ on E with density e^{-V}, where V is a convex function on some open convex set $\Omega \subset E$, such that, for some $c > 0$, all $t, s > 0$ with $t + s = 1$ and all $x, y \in \Omega$,

$$tV(x) + sV(y) - V(tx + sy) \ge \tfrac{c}{p}\big(s + o(s)\big)\|x - y\|^p \tag{3.6}$$

where $p \geq 2$, and where $o(s)$ is a function of $s \in (0,1)$ such that $o(s)/s \to 0$ as $s \to 0$. Denote by $q = \frac{p}{p-1}$ the conjugate number of p.

Before stating our result, it is worthwhile mentioning that the following property of a (continuous) function V is sufficient for (3.6) to hold: for all $x, y \in \Omega$, $p \geq 2$,

$$V(x) + V(y) - 2V\left(\frac{x+y}{2}\right) \geq \frac{c}{p}\|x-y\|^p . \tag{3.7}$$

To derive (3.6) from (3.7), first note that since V is convex on Ω, for all $x, y \in \Omega$,

$$tV(x) + sV(y) - V(tx+sy) \geq \frac{c\min(t,s)}{p}\|x-y\|^p .$$

Indeed, this inequality holds for $s = 0$ and $s = 1/2$. In addition, its left-hand side is concave in $s \in [0,1/2]$, while the right-hand side is a linear function on that interval. Hence, the above inequality holds for all $s \in [0,1/2]$ (and for all $s \in [1/2,1]$ after replacing x by y). It remains to note that $\min(t,s) \geq ts$.

Proposition 3.2. *Under* (3.6), *for every smooth non-negative function f on Ω,*

$$\mathrm{Ent}_\mu(f^q) \leq \left(\tfrac{q}{c}\right)^{q-1} \int \|\nabla f\|_*^q d\mu .$$

Proof. Let g be a smooth function on Ω with compact support, and let $t, s > 0$ with $t + s = 1$. As in the proof of Proposition 3.1, the inequality (3.2) holds true with optimal g_t given by

$$g_t(z) = \sup_{z=tx+sy,\, x,y\in\Omega} \big(g(x) - [tV(x) + sV(y) - V(tx+sy)]\big) , \quad z \in \Omega .$$

By the assumption on V,

$$g_t(z) \leq \sup_{z=tx+sy,\, x,y\in\Omega} \left(g(x) - \tfrac{c}{p}\big(s + o(s)\big)\|x-y\|^p\right) , \quad z \in \Omega .$$

Again, since $x = z + \frac{s}{t}(z-y)$, $x - y = \frac{1}{t}(z-y)$, setting $h = z - y$, $\eta = \frac{s}{t}$, we have that $s + o(s) = \eta + o(\eta)$, and the above inequality reads, for every $z \in \Omega$,

$$g_t(z) \leq \sup_{h\in E} \left(g(z+\eta h) - \left(\tfrac{c\eta}{p} + o(\eta)\right)\|h\|^p\right) .$$

By Taylor's formula,

$$g(z+\eta h) = g(z) + \eta\big\langle\nabla g(z), h\big\rangle + \|h\|^2 O(\eta^2) ,$$

as $\eta \to 0$ (equivalently as $s \to 0$). Using that $|O(\eta^2)| \leq C\,\eta^2$, $|o(\eta)| \leq D\eta$ with $D = D(\eta) \to 0$ as $\eta \to 0$, we get that

$$g_t(z) \leq g(z) + \eta \sup_{h\in E}\left(\big\langle\nabla g(z), h\big\rangle - \left(\tfrac{c}{p} - D\right)\|h\|^p + C\eta\|h\|^2\right) .$$

The next lemma will allow us to estimate the last supremum and to show that the term $C\eta\|h\|^2$ does not make any influence when η is small.

LEMMA 3.3. *Consider a function ψ of the form*

$$\psi(h) = \langle v, h\rangle - \tfrac{a}{p}\|h\|^p + b\|h\|^2\,, \quad h \in E\,,$$

where $v \in E^*$, $a, b > 0$. Then, if $b \le \frac{1}{2}\big(\frac{a}{1+\|v\|_*}\big)^{q-1}$,

$$\sup_{h\in E} \psi(h) \le \frac{\|v\|_*^q}{qa^{q-1}} + b\bigg(\frac{1+\|v\|_*}{a}\bigg)^{2(q-1)}\,.$$

Proof. Writing $h = \lambda e$ with $\|e\| = 1$,

$$\sup_{h\in E}\psi(h) = \sup_{\lambda\in\mathbb{R}}\sup_{\|e\|=1}\Big(\lambda\langle v, e\rangle - \tfrac{a}{p}|\lambda|^p + b\lambda^2\Big) = \sup_{\lambda\in\mathbb{R}}\Big(\lambda\|v\|_* - \tfrac{a}{p}|\lambda|^p + b\lambda^2\Big)\,.$$

The value $\lambda\|v\|_* - \frac{a}{p}|\lambda|^p$ is maximized at $\lambda_0 = \big(\frac{\|v\|_*}{a}\big)^{q-1}$ and is equal to $\|v\|_*^q/qa^{q-1}$ at this point. On the other hand, consider the function

$$\varphi(\lambda) = \lambda\|v\|_* - \tfrac{a}{p}\lambda^p + b\lambda^2\,, \quad \lambda \ge 0\,,$$

and assume that $\|v\|_* > 0$. Clearly, φ is maximized for some $\lambda_1 > 0$ at which point

$$\varphi'(\lambda_1) = \|v\|_* - a\,\lambda_1^{p-1} + 2b\lambda_1 = 0.$$

Moreover, $\varphi''(\lambda) = -a(p-1)\lambda^{p-2} + 2b$, so that, for some $\lambda_2 \ge 0$, φ is convex on $[0, \lambda_2]$ and concave on $[\lambda_2, \infty)$. Since $\varphi'(0) = \|v\|_* > 0$, we may conclude that φ increases on $[0, \lambda_1]$ and decreases on $[\lambda_1, \infty)$. Now, for a fixed $\theta > 1$ to be chosen later on, we have

$$\varphi'(\theta^{q-1}\lambda_0) = -(\theta-1)\|v\|_* + 2b\left(\theta\tfrac{\|v\|_*}{a}\right)^{q-1} \le 0$$

provided that

$$b \le b_0 = \tfrac{1}{2}\left(\tfrac{a}{\theta}\right)^{q-1}(\theta-1)\|v\|_*^{2-q}\,.$$

Hence, for $b \le b_0$, $\lambda_1 \le \theta^{q-1}\lambda_0$. Choose than $\theta = 1 + \frac{1}{\|v\|_*}$ so that $b_0 = \frac{1}{2}\big(\frac{a}{1+\|v\|_*}\big)^{q-1}$. Thus, for $b \in (0, b_0]$, $\varphi'(\theta^{q-1}\lambda_0) \le 0$. This implies that

$$\lambda_1 \le \theta^{q-1}\lambda_0 = \bigg(\frac{1+\|v\|_*}{a}\bigg)^{q-1}.$$

Therefore, for $b \le b_0$,

$$\begin{aligned}\sup_{h\in E}\psi(h) = \varphi(\lambda_1) &\le \sup_{\lambda\in\mathbb{R}}\Big(\lambda\|v\|_* - \tfrac{a}{p}|\lambda|^p\Big) + b\lambda_1^2\\ &= \tfrac{\|v\|_*^q}{qa^{q-1}} + b\lambda_1^2\\ &\le \tfrac{\|v\|_*^q}{qa^{q-1}} + b\,(1+\|v\|_*a)^{2(q-1)}\end{aligned}$$

which is the result. Lemma 3.3 is proved. □

We may now conclude the proof of the proposition. In our case $v = \nabla g(z)$, $a = c - Dp$, $b = C\eta$. In particular, the condition $b \leq \frac{1}{2}\big(\frac{a}{1+\|v\|_*}\big)^{q-1}$ is satisfied for sufficiently small η's. Therefore, since $\|\nabla g(z)\|_*$ is bounded above and since $D = o(1)$, we get that

$$\sup_{h \in E} \psi(h) \leq q^{-1} c^{1-q} \big\|\nabla g(z)\big\|_*^q + o(1)\,,$$

and thus, uniformly over all $z \in \Omega$,

$$g_t(z) \leq g(z) + \eta q^{-1} c^{1-q} \big\|\nabla g(z)\big\|_*^q + o(\eta)\,.$$

Hence, uniformly in $z \in \Omega$,

$$\mathrm{e}^{g_t(z)} \leq \mathrm{e}^{g(z)} + \eta q^{-1} c^{1-q} \big\|\nabla g(z)\big\|_*^q \mathrm{e}^{g(z)} + o(\eta)\,.$$

Arguing then as in the proof of Proposition 3.1, we get as $\eta \to 0$ that

$$\mathrm{Ent}_\mu(\mathrm{e}^g) \leq q^{-1} c^{1-q} \int \|\nabla g\|_*^q \mathrm{e}^g d\mu$$

which amounts to the inequality of the proposition, replacing e^g by f^q. Proposition 3.2 is thus established. □

As was mentioned to us by M. Schmuckenschläger, one may consider, as a more general condition than (3.6), potentials V such that for all $t, s > 0$ with $t + s = 1$ and all $x, y \in \Omega$,

$$tV(x) + sV(y) - V(tx + sy \geq \big(s + o(s)\big)\, \Phi\big(\|x - y\|\big)$$

where Φ is a convex non-negative function on $\mathbb{R}^+$. Under some technical assumptions on Ψ in order to achieve the analogue of Lemma 3.3, Proposition 3.2 extends to yield that, for every smooth function f on Ω,

$$\mathrm{Ent}_\mu(\mathrm{e}^f) \leq \int \Phi^*\big(\|\nabla f\|_*\big) \mathrm{e}^f d\mu \tag{3.8}$$

where $\Phi^*(u) = \sup_{v>0}[uv - \Phi(v)]$ is the Legendre transform of Φ. Inequality (3.8) is of particular interest for its formal relation to large deviations. For the applications below to transportation inequalities and uniformly convex bodies, we however only consider the classical case $\Phi(u) = u^p$, $p \geq 2$.

We would like to mention that the main results, Propositions 3.1 and 3.2, still hold for measures with log-concave densities with respect to μ, μ satisfying the hypotheses of either Proposition 3.1 or 3.2. In particular, one may choose for h the characteristic function of some convex set B (of positive measure), and the results apply to the conditional probability

$$\mu_B(A) = \frac{\mu(A \cap B)}{\mu(B)}\,, \quad A \text{ Borel set in } E\,.$$

For example, if γ is the canonical Gaussian measure on $\mathbb{R}^n$, each measure γ_B will satisfy the logarithmic Sobolev inequality

$$\mathrm{Ent}_{\gamma_B}(f^2) \leq 2 \int |\nabla f|^2 d\gamma_B\,,$$

with thus a constant uniform in B. Such a result has to be compared with the corresponding inequalities for the conditional restriction with respect to Lebesgue measures that essentially depends on the size and shape of B.

To conclude this section, we show how the preceding techniques may be used to produce a version of the Brascamp–Lieb inequality for entropy under some restriction on V.

Proposition 3.4. *As in Proposition 2.1, assume that V is twice continuously differentiable and strictly convex on Ω. Assume furthermore that, for any $h \in E$, the function $x \to \langle V''(x)h, h\rangle$ is concave on Ω. Then, for every smooth function f on Ω,*

$$\mathrm{Ent}_\mu(f^2) \le 3 \int \langle V''^{-1} \nabla f, \nabla f\rangle d\mu\,.$$

Proof. We start as in the proof of Propositions 2.1 and 3.1, and let $E = \mathbb{R}^n$ with its Euclidean structure. Let g be a smooth function with compact support in Ω. We also assume that $V'' \ge \rho\,\mathrm{Id}$ for some $\rho > 0$. We start from (3.2) of the proof of Proposition 3.1, but now bound below the expression $L(s) = tV(x) + sV(y) - V(tx+sy)$, $t, s > 0$, $t+s = 1$, $x, y \in \Omega$, $z = tx+sy$, with the help of the representation

$$L(s) = \frac{ts}{2}\int_0^1 \big[s\langle V''(rz+(1-r)x)k,k\rangle + t\langle V''(rz+(1-r)y)k,k\rangle\big]\,dr^2$$

where $z = tx + sy$ and $k = x - y$. By concavity of V'' and convexity of V,
$\big\langle V''(rz+(1-r)x)k,k\big\rangle \ge r\big\langle V''(z)k,k\big\rangle + (1-r)\big\langle V''(x)k,k\big\rangle \ge r\big\langle V''(z)k,k\big\rangle$
and similarly with y instead of x. Therefore

$$L(s) \ge \frac{ts}{2}\int_0^1 r\,dr^2 \big\langle V''(z)k,k\big\rangle = \frac{ts}{3}\big\langle V''(z)k,k\big\rangle\,.$$

Thus,

$$g_t(z) \le \sup_{z=tx+sy,\,x,y\in\Omega} \Big(g(x) - \tfrac{ts}{3}\big\langle V''(z)k,k\big\rangle\Big)\,, \quad z \in \Omega\,, \quad k = y - x\,.$$

Since $x = z + sk$, by Taylor's formula,

$$\begin{aligned} g_t(z) &\le \sup_{k;z+sk,z-tk\in\Omega} \Big(g(z+sk) - \tfrac{ts}{3}\big\langle V''(z)k,k\big\rangle\Big) \\ &\le g(z) + s\sup_k \Big(\big\langle g'(z),k\big\rangle - \tfrac{t}{3}\big\langle V''(z)k,k\big\rangle + |k|^2 O(s)\Big) \end{aligned}$$

where $O(s)$ is uniform in z. Since $V'' \ge \rho\,\mathrm{Id}$, for s small enough, it is easily seen that

$$g_t(z) \le g(z) + \tfrac{3s}{4}\big\langle V''(z)^{-1}g'(z), g'(z)\big\rangle + O(s^2)$$

where, again, the constant in $O(s^2)$ can be chosen independent of z. Hence, uniformly over all $z \in \Omega$

$$\mathrm{e}^{g_t(z)} \le \mathrm{e}^{g(z)} + \tfrac{3s}{4}\big\langle V''(z)^{-1}g'(z), g'(z)\big\rangle\,\mathrm{e}^{g(z)} + O(s^2)\,,$$

and thus

$$\int e^{g_t} d\mu \le \int e^g d\mu + \frac{3s}{4} \int \langle V''^{-1} g', g' \rangle \, e^g d\mu + O(s^2) \, .$$

As in the proof of Proposition 3.1, it then follows from (3.2) and (3.3) as $s \to 0$ ($t \to 1$), that

$$\mathrm{Ent}_\mu(e^g) \le \frac{3}{4} \int \langle V''^{-1} g', g' \rangle e^g d\mu \, .$$

Applying this inequality to g such that $e^g = f^2$, we get that

$$\mathrm{Ent}_\mu(f^2) \le 3 \int \langle V''^{-1} f', f' \rangle d\mu \, .$$

This inequality readily extends to all locally Lipschitz function f on Ω. Proposition 3.4 is thereby established. □

The following example illustrates the applicability of Proposition 3.4. Let $E = \mathbb{R}$, $\Omega = (0, \infty)$, and let μ with density $C \, e^{-x^p/p}$, $x > 0$, with respect to Lebesgue measure on Ω. Proposition 3.4 can be applied in the case $2 \le p \le 3$ since then $V''(x) = (p-1)x^{p-2}$ is a concave function in $x > 0$. Thus, for every smooth function f on $(0, \infty)$,

$$\mathrm{Ent}_\mu(f^2) \le \frac{3}{p-1} \int_0^\infty x^{2-p} f'(x)^2 d\mu(x).$$

On the other hand, it is plain that the example developed after (2.5) does not satisfy the requirements of Proposition 3.4.

4 Infimum-convolution and Transportation Inequalities

The logarithmic Sobolev inequality of Proposition 3.2 may be used to produce in an easy way sharp concentration inequalities for the measure μ by means of the Herbst argument. The argument consists in applying Proposition 3.2 to $f^q = e^{\lambda g}$, $\lambda \in \mathbb{R}$, where g is Lipschitz and mean-zero to deduce a differential inequality on the Laplace transform of g. We refer to [AMS], [L] for details.

COROLLARY 4.1. *Under* (3.6), *for every Lipschitz function g on $\Omega \subset E$ with respect to $\| \cdot \|$ and with Lipschitz semi-norm less than or equal to* 1 (*equivalently $\|\nabla g\|_* \le 1$ almost everywhere*), *and every $t \ge 0$,*

$$\mu\bigg(g \ge \int g \, d\mu + t \bigg) \le \exp\bigg(- \frac{ct^p}{p(p-1)^{p-1}} \bigg) \, .$$

As is classical (cf. [L]), this concentration property on functions implies (is essentially equivalent to the fact) that for every measurable set A with $\mu(A) \ge 1/2$, and every $\varepsilon > 0$,

$$\mu(A_\varepsilon) \ge 1 - e^{-c_p \varepsilon^p} \tag{4.1}$$

where A_ε is the ε-neighborhood of A with respect to the norm on E and where $c_p > 0$ only depends on p and c.

There is however an other way to concentration via transportation inequalities put forward recently in the work of K. Marton [M]. This approach has been further developed in the papers [T], [D], [BoG] etc (cf. [L]).

Let as before $(E, \|\cdot\|)$ be a finite dimensional normed space (with a fixed Lebesgue measure). Given $p \geq 1$ and two probability measures μ and ν on E, define the Kantorovich–Rubinstein distance

$$W_p(\mu,\nu) = \inf \left(\iint \|x-y\|^p d\pi(x,y) \right)^{1/p},$$

where the infimum is taken over all probability measures π on $E \times E$ with marginal distributions μ and ν. As is well known, W_p represents a metric on the space of all probability measures on E with finite moment of order p. The value $W_p^p(\mu,\nu)$ may be viewed as the minimal cost needed to transport the measure μ into ν provided that the cost to transport the point x into y is equal to $\|x-y\|^p$.

Let us take again the framework of Proposition 3.2, and consider a Borel probability measure μ on E with density e^{-V} with respect to Lebesgue measure, where V is a convex function on E satisfying (3.6) that is, for all $t, s > 0$ with $t + s = 1$ and all $x, y \in E$,

$$tV(x) + sV(y) - V(tx+sy) \geq \tfrac{c}{p}\big(s + o(s)\big)\|x-y\|^p,$$

where $p \geq 2$, $c > 0$, and where $o(s)$ is a function of $s \in (0,1)$ such that $o(s)/s \to 0$ as $s \to 0$. (One may consider more generally V defined on some open convex subset of E. We leave this to the interested reader.)

If ν is a probability measure on E that is absolutely continuous with respect to μ with density $d\nu/d\mu$, we denote by

$$D\big(\nu \,||\, \mu\big) = \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu}\, d\mu$$

the informational divergence or relative entropy of ν with respect to μ.

Proposition 4.2. *For every probability measure ν on E with a finite moment of order p which is absolutely continuous with respect to μ,*

$$W_p^p(\mu,\nu) \leq \tfrac{p}{c} D\big(\nu||\mu\big). \tag{4.2}$$

Proposition 4.2 provides a kind of extension of a recent observation by M. Talagrand [T] in the case $E = \mathbb{R}^n$, $p = 2$ and $\mu = \gamma$ the canonical Gaussian measure for which he establishes (4.2) by induction over the dimension with best constant $c = 1$. The proof we present extends the proof given in [BoG] in the Gaussian case. To view the concentration content of Proposition 4.2, consider, following [M], two measurable sets A and B in E with

$\mu(A), \mu(B) > 0$ at distance $\varepsilon > 0$. Apply then (4.2) to the conditional probabilities $\mu_A(\,\cdot\,) = \mu(\,\cdot \cap A)/\mu(A)$ and $\mu_B(\,\cdot\,) = \mu(\,\cdot \cap B)/\mu(B)$. By definition, $D(\mu_A||\mu) = \log 1/\mu(A)$, $D(\mu_B||\mu) = \log 1/\mu(B)$, while $W_p(\mu_A, \mu_B) \geq \varepsilon$ (since every measure π on $E \times E$ with marginals μ_A and μ_B is supported by $A \times B$). Hence, by (4.2) and the triangle inequality for W_p,

$$\begin{aligned}\varepsilon &\leq W_p(\mu_A, \mu_B)\\ &\leq W_p(\mu_A, \mu) + W_p(\mu, \mu_B)\\ &\leq \left(\tfrac{p}{c}\right)^{1/p} \left(\left(\log \tfrac{1}{\mu(A)}\right)^{1/p} \left(\log \tfrac{1}{\mu(B)}\right)^{1/p} \right).\end{aligned}$$

Choosing for B the complement of the neighborhood of order $\varepsilon > 0$ of a set A with $\mu(A) \geq 1/2$ then easily yields a concentration result of the type (4.1) (even with sharper numerical constants). As emphasized by Marton, transportation inequalities describe in this way a more symmetric version of concentration.

In the recent contribution [OV], F. Otto and C. Villani remarkably showed that the Euclidean transportation inequality (with $p = 2$) may actually be deduced from the corresponding logarithmic Sobolev inequality. Their argument is based on dissipation of entropy along the associated heat semigroup. Moreover, as was recently mentioned to us by G. Blower [Bl], an alternate proof of Proposition 4.2 may be given using the Brenier–McCann theorem [Bre], [Mc] about monotone measure preserving maps.

Proof of Proposition 4.2. Given $t, s > 0$ with $t + s = 1$, and $x, y \in E$, let

$$L_s(x,y) = \tfrac{1}{ts}\big[tV(x) + sV(y) - V(tx+sy)\big]\,.$$

Applying Prékopa–Leindler's theorem to

$$u(x) = \mathrm{e}^{-sg(x)-V(x)}\,, \quad v(y) = \mathrm{e}^{tf(y)-V(y)}\,, \quad w(z) = \mathrm{e}^{-V(z)}\,,$$

we get

$$1 \geq \left(\int \mathrm{e}^{-sg}\,d\mu\right)^t \left(\int \mathrm{e}^{tf}\,d\mu\right)^s \tag{4.3}$$

provided the functions f and g satisfy

$$f(y) \leq g(x) + L_s(x,y)\,, \quad x, y \in E\,. \tag{4.4}$$

Given g, the optimal function $f = L_s g$ in (4.4) is defined by

$$L_s g(y) = \inf_{x \in E}\big[g(x) + L_s(x,y)\big]$$

so that (4.3) becomes

$$\left(\int \mathrm{e}^{-sg} d\mu\right)^{1/s} \left(\int \mathrm{e}^{tL_s g} d\mu\right)^{1/t} \leq 1\,. \tag{4.5}$$

According to Maurey's terminology [Ma], the function $L_s g$ is the infimum-convolution of the function g with $L_s(x,y)$, and nowadays inequalities of the form (4.5) are often called infimum-convolution inequalities. In general, the function $L_s g$ does not need to be measurable, and the left integral in (4.5) should then be understood as the lower integral. If g is upper semicontinuous, then $L_s g$ is also upper semicontinuous (as an infimum of upper semicontinuous functions). In particular, $L_s g$ is measurable in this case. By a continuity argument, (4.5) extends to measurable functions possibly taking the value $+\infty$.

Now, as a consequence of the convexity assumption on V,

$$\liminf_{s\to 0} L_s(x,y) \geq \tfrac{c}{p}\|x-y\|^p .$$

As a result, letting $s \to 0$ in (4.5), we arrive at an infimum-convolution inequality of another form,

$$\int \mathrm{e}^f \, d\mu \leq \mathrm{e}^{\int g\, d\mu} \tag{4.6}$$

which thus holds true for at least all bounded measurable functions f and g satisfying

$$f(y) \leq g(x) + \tfrac{c}{p}\|x-y\|^p , \quad x,y \in E . \tag{4.7}$$

Now, we can write (4.6) equivalently with the help of the entropy functional as

$$\int \left(f - \int g\, d\mu \right) u \, d\mu \leq \mathrm{Ent}_\mu(u)$$

where u is an arbitrary non-negative μ-integrable function on E. We used here that

$$\mathrm{Ent}_\mu(u) = \sup \int uv \, d\mu$$

where the supremum is running over all functions v with $\int \mathrm{e}^v d\mu \leq 1$. For $u = d\nu/d\mu$, $\mathrm{Ent}_\mu(u) = D(\nu||\mu)$ so that the above inequality takes the form

$$\int f\, d\nu - \int g\, d\mu \leq D\big(\nu \,||\, \mu\big) . \tag{4.8}$$

We should therefore maximize the left-hand side of (4.8) subject to (4.7). To this end, we may apply the duality relations put forward in [K], [Lev], [R] that show in particular that

$$\sup\left\{ \int f\, d\nu - \int g\, d\mu ; f(y) - g(x) \leq \frac{c}{p}\|x-y\|^p \text{ for all } x,y \in E \right\} = \frac{c}{p}\, W_p^p(\mu,\nu).$$

Proposition 4.2 then immediately follows. □

When V is twice continuously differentiable, Proposition 4.2 can be generalized as follows. Note that

$$\begin{aligned} T(x,y) \equiv \lim_{s\to 0} L_s(x,y) &= V(y) - V(x) + \big\langle V'(x), x-y \big\rangle \\ &= \frac{1}{2}\int_0^1 \big\langle V''(rx + (1-r)y)(x-y), (x-y) \big\rangle dr^2 . \end{aligned}$$

As above, we may therefore conclude that, for all bounded measurable functions f and g on E satisfying $f(x) \le g(y) + T(x,y)$ for all $x, y \in E$, we have

$$\int \mathrm{e}^f d\mu \le \mathrm{e}^{\int g\, d\mu} .$$

According to [K], and under the mild condition that for some measurable functions a and b defined on E, $T(x,y) \le a(x) + b(y)$, this general infimum-convolution inequality is equivalent to the transportation inequality

$$\inf\left\{ \int\!\!\int T(x,y) d\pi(x,y); \pi \text{ has marginals } \nu \text{ and } \mu \right\} \le D\big(\nu \,||\, \mu\big) .$$

This holds in particular as soon as $\|V''\|$ is bounded on E.

It is worthwhile mentioning that the transportation inequality of Proposition 4.2 with $p = 2$ may also be seen to imply the Poincaré inequality (3.5). Start indeed from the equivalent formulation (4.6) holding for all functions f and g satisfying (4.7) (with thus $p = 2$), and assume for simplicity that $\int g\, d\mu = 0$. Then, apply (4.6) to δg, where g is smooth with compact support, and denote by f_δ the optimal function f satisfying (4.7) with δg. As in the proof of Proposition 2.2, it is easily seen that

$$f_\delta(y) = \delta g(y) - \tfrac{\delta^2}{2c} \big\|\nabla g(y)\big\|_*^2 + o(\delta^2)$$

uniformly over the support of g. Then (4.6) yields

$$1 \ge \int \mathrm{e}^{f_\delta} d\mu = 1 + \frac{\delta^2}{2} \int g^2 d\mu - \frac{\delta^2}{2c} \int \|\nabla g\|_*^2 d\mu + o(\delta^2)$$

from which (3.5) follows as $\delta \to 0$.

5 Logarithmic Sobolev Inequalities for Uniformly Convex Bodies

Proposition 3.2 and Corollary 4.1 may be applied to norms of uniformly p-convex Banach spaces. Concentration in the form of Corollary 4.1 was actually established directly from the Maurey argument in [S]. The modulus of convexity δ_E of a Banach space $(E, \|\cdot\|)$ is defined as the function of $0 < \varepsilon \le 2$

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|; \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\right\}.$$

E is uniformly convex if $\delta_E(\varepsilon) > 0$ for each $\varepsilon > 0$. It is said to have a modulus of convexity of power type $p \ge 2$ (with constant $C > 0$) if $\delta(\varepsilon) \ge C\varepsilon^p$ for every ε. Equivalently (see for example [P1]), for every x, y in E,

$$\|x\|^p + \|y\|^p - 2\left\|\frac{x+y}{2}\right\|^p \ge \frac{c}{p}\, \|x-y\|^p \tag{5.1}$$

for some $c > 0$. Our result will thus applicable to $V(x) = \|x\|^p$ for $p \geq 2$. ℓ_p-norm on $\mathbb{R}^n$, $1 < p < \infty$,

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

have moduli of convexity of power type $\max(p, 2)$ (cf. [LiT]). As proved by G. Pisier [P1], every uniformly convex Banach space admits an equivalent norm with a modulus of convexity of power type p for some $p \geq 2$.

Assume now that E is finite dimensional with dimension n and unit ball B with norm $\|\cdot\|$ satisfying (5.1). According to (3.7), Proposition 3.2 and Corollary 4.1 may thus be applied to the measure μ on E with density with respect to Lebesgue measure

$$\frac{d\mu}{dx} = \frac{1}{\Gamma(1 + \frac{n}{p})\mathrm{vol(B)}} \exp(-\|x\|^p). \tag{5.2}$$

For the ℓ_p-norms $\|\cdot\|_p$, $p \geq 2$, (5.1) actually holds with optimal $c = p2^{-p}$ (cf. [LiT]). Therefore, by (3.7), for all $t, s > 0$, $t + s = 1$, and all $x, y \in \mathbb{R}^n$,

$$t\|x\|_p^p + s\|x\|_p^p - \|tx + sy\|_p^p \geq 2^{1-p} \min(t, s)\|x - y\|_p^p. \tag{5.3}$$

As developed by Schmuckenschläger [S], the preceding may be used to recover some aspects of the concentration of measure phenomenon of Gromov and Milman [GM] for uniformly convex bodies. Recall B the unit ball of the uniformly convex finite dimensional Banach space $(E, \|\cdot\|)$ satisfying (5.1). Properly transferring the measure μ of (5.2) to the normalized surface measure $\lambda_{\partial B}$ of B with respect to itself defined by

$$\lambda_{\partial B}(A) = \frac{\mathrm{vol}\left(\bigcup_{0 \leq t \leq 1} tA\right)}{\mathrm{vol}(B)}, \quad A \subset \partial B,$$

yields that, for every Lipschitz function g on ∂B with respect to the metric induced by B, with Lipschitz coefficient less than or equal to 1, and every $t \geq 0$,

$$\lambda_{\partial B}\left(g \geq \int g \, d\lambda_{\partial B} + t \right) \leq 2\, \mathrm{e}^{-nc_p t^p} \tag{5.4}$$

where $c_p > 0$ only depends on c and p. The argument relies on the observation that if g is Lipschitz on ∂B, $G(x) = \|x\| g(x/\|x\|)$ on E is such that $\|G\|_{\mathrm{Lip}} \leq 4\|g\|_{\mathrm{Lip}}$, and the fact that if μ has a density that is invariant under the norm, then the distribution of g under $\lambda_{\partial B}$ coincides with the distribution of $g(x/\|x\|)$ under μ. In addition, $\mathrm{med}_{\lambda_{\partial B}}(g) = \mathrm{med}_{\lambda_\mu}(G)$ (cf. [S]). By standard arguments, (5.4) implies that, for every measurable set A with $\lambda_{\partial B}(A) \geq 1/2$, and every $\varepsilon > 0$,

$$\lambda_{\partial B}(A_\varepsilon) \geq 1 - \mathrm{e}^{-nc'_p \varepsilon^p} \tag{5.5}$$

where A_ε denotes the neighborhood of order $\varepsilon > 0$ in the metric on ∂B induced by the norm $\|\cdot\|$ (and where $c'_p > 0$ is another constant only depending on p). It has been shown by Gromov and Milman [GM] that if $\lambda_{\partial B}$ is the surface measure on the unit sphere ∂B of a uniformly convex n-dimensional normed space E,

$$\lambda(A_\varepsilon) \geq 1 - \mathrm{e}^{-\alpha n \delta'_E(\varepsilon)} \tag{5.6}$$

for some $\alpha > 0$ and every $\varepsilon > 0$ where $\delta'_E(\varepsilon) > 0$ is determined by $\delta_E(\varepsilon) > 0$. A simple proof of this result, also based on the Brunn–Minkowski inequality, was provided recently by J. Arias-de-Reyna, K. Ball and R. Villa [ArBV]. Given two non-empty sets $A, A' \subset B$ at distance $\varepsilon \in (0,1)$, we have by definition of the modulus of convexity that

$$\tfrac{1}{2}(A + A') \subset \big(1 - \delta_E(\varepsilon)\big)B,$$

so that, by the Brunn–Minkowski inequality,

$$\lambda_B^{1/2}(A)\lambda_B^{1/2}(A') \leq \big(1 - \delta_E(\varepsilon)\big)^n$$

where we denote by λ_B uniform measure on B. Taking for A' the complement of A_ε, we get, when $\lambda_B(A) \geq 1/2$,

$$\lambda_B(A_\varepsilon) \geq 1 - 2\big(1 - \delta_E(\varepsilon)\big)^{2n} \geq 1 - 2\,\mathrm{e}^{-2n\delta_E(\varepsilon)}\,.$$

One needs however to transfer this result to the surface measure which can be performed in the same spirit as the arguments developed in [S].

While the preceding arguments yield concentration results for both the uniform and surface measures on a uniformly convex body B, we would like to follow here a similar route but at the level of logarithmic Sobolev inequalities. Following section 3, we prove here some logarithmic Sobolev inequality for the uniform distribution λ_B on B by pushing in an appropriate way the measure (5.2) to λ_B. We first illustrate the procedure for the uniform distribution on the Euclidean unit ball in $\mathbb{R}^n$ for which we determine in particular the order in n of its logarithmic Sobolev constant.

Recall (1.4) the logarithmic Sobolev inequality for the canonical Gaussian measure γ on $\mathbb{R}^n$,

$$\mathrm{Ent}_\mu(f^2) \leq 2\int |\nabla f|^2 d\gamma$$

holding for all smooth functions f. Applying this inequality to functions of the form $f(U)$ with a map $U : \mathbb{R}^n \to B$ that pushes forward γ to λ_B, we get

$$\mathrm{Ent}_{\lambda_B}(f^2) \leq 2\|U\|_{\mathrm{Lip}}^2 \int |\nabla f|^2 d\lambda_B\,, \tag{5.7}$$

where $\|U\|_{\mathrm{Lip}}$ is Lipschitz seminorm of U. Therefore, our task will be to find a map U with Lipschitz constant as small as possible. Since both γ

and λ_B are spherically invariant it seems natural to try to choose U in the class of spherically invariant maps, that is, of the form,

$$U(x) = u(|x|)\tfrac{x}{|x|}\,, \quad x \in \mathbb{R}^n \setminus \{0\}$$

where u is function on $(0,\infty)$. For such maps, we have:

LEMMA 5.1. *$\|U\|_{\mathrm{Lip}} < \infty$ if and only if $\|u\|_{\mathrm{Lip}} < \infty$ and $u(0^+) = 0$. In this case,*

$$\|U\|_{\mathrm{Lip}} = \|u\|_{\mathrm{Lip}}\,.$$

Proof. Write arbitrary points $x_1, x_2 \in \mathbb{R}^n \setminus \{0\}$ as $x_1 = \lambda_1 e_1$, $x_2 = \lambda_1 e_2$ with $\lambda_1 > 0$, $\lambda_2 > 0$, $|e_1| = |e_2| = 1$, so that

$$U(x_1) = u(\lambda_1)e_1\,, \quad U(x_2) = u(\lambda_2)e_2\,.$$

Set $t = \langle e_1, e_2\rangle \in [-1,1]$. Since

$$|x_1 - x_2|^2 = \lambda_1^2 + \lambda_2^2 - 2\,\lambda_1\lambda_2\,t$$

and

$$\big|U(x_1) - U(x_2)\big|^2 = u(\lambda_1)^2 + u(\lambda_2)^2 - 2u(\lambda_1)u(\lambda_2)t\,,$$

the inequality $|U(x_1) - U(x_2)| \le C\,|x_1 - x_2|$ for some $C \ge 0$ is equivalent to

$$u(\lambda_1)^2 + u(\lambda_2)^2 - 2\,u(\lambda_1)u(\lambda_2)\,t \le C^2\,(\lambda_1^2 + \lambda_2^2 - 2\,\lambda_1\lambda_2\,t)\,.$$

This inequality is linear in t. Hence, it holds for all $t \in [-1,1]$ if and only if it holds for the end points $t = 1$ and $t = -1$. In the case $t = 1$, i.e. when $e_1 = e_2$, it reads

$$\big|u(\lambda_1) - u(\lambda_2)\big| \le C|\lambda_1 - \lambda_2|$$

where the best constant is $C = \|u\|_{\mathrm{Lip}}$. In the case $t = -1$, i.e., when $e_1 = -e_2$, it reads

$$\big|u(\lambda_1) + u(\lambda_2)\big| \le C|\lambda_1 + \lambda_2|\,.$$

Clearly, it holds for all $\lambda_1, \lambda_2 > 0$ if and only if it holds for $\lambda_1 = \lambda_2 = \lambda > 0$, in which case it becomes $|u(\lambda)| \le C\lambda$. It is necessary for C to be finite that $u(0^+) = 0$. But in this case, the above inequality holds with $C = \|u\|_{\mathrm{Lip}}$ and the lemma is proved in this way. □

In order to determine a sharp constant in the logarithmic Sobolev inequality (5.7) for the uniform measure λ_B on the Euclidean unit ball B, we thus reduced the problem to find $u : [0,\infty) \to [0,1)$ with $u(0) = 0$ and smallest Lipschitz norm. If we restrict ourselves to increasing non-negative functions, the function u such that U pushes forward γ into λ_B is uniquely determined by the relation

$$\gamma\big(x \in \mathbb{R}^n; |U(x)| < s\big) = \lambda_B\big(y \in B; |y| < s\big)\,, \quad s \in (0,1)\,,$$

which is just

$$\gamma\big(x \in \mathbb{R}^n; u(|x|) < s\big) = s^n\,, \quad s \in (0,1)\,.$$

Lemma 5.2. *$\|u\|_{\mathrm{Lip}} = \omega_n^{1/n}/\sqrt{2\pi}$ where $\omega_n = \pi^{n/2}/\Gamma(\frac{n}{2}+1)$ is the volume of B.*

Proof. Setting $s = u(r)$, $r > 0$, the definition of u becomes

$$u(r)^n = \gamma\big(x \in \mathbb{R}^n; |x| < r\big) = \frac{0}{(2\pi)^{n/2}} \int_{\{|x|<r\}} \mathrm{e}^{-|y|^2/2} dy$$
$$= \frac{n\omega_n}{(2\pi)^{n/2}} \int_0^r t^{n-1} \mathrm{e}^{-t^2/2} dt \,.$$

Clearly, u is strictly increasing, positive and of class $C^\infty(0,\infty)$. We will show that u is concave so that

$$\|u\|_{\mathrm{Lip}} = u'(0^+) \,. \tag{5.8}$$

Indeed, set $g(r) = \int_0^r t^{n-1}\mathrm{e}^{-t^2/2}dt$ so that

$$u(r) = \frac{(n\omega_n)^{1/n}}{\sqrt{2\pi}} g(r)^{1/n} \,.$$

Thus, u is concave if and only if $g^{1/n}$ is concave. For a positive function g, we have in general

$$(g^{1/n})' = \tfrac{1}{n} g^{(1/n)-1} g'$$

and

$$(g^{1/n})'' = \tfrac{1}{n}\left(\tfrac{1}{n} - 1\right) g^{(1/n)-2} g'^2 + \tfrac{1}{n} g^{(1/n)-1} g''$$

so that $(g^{1/n})'' \leq 0$ if and only if $gg'' \leq \big(1 - \frac{1}{n}\big) g'^2$. In our case

$$g'(r) = r^{n-1}\mathrm{e}^{-r^2/2} \quad \text{and} \quad g''(r) = \big((n-1) - r^2\big) r^{n-2} \mathrm{e}^{-r^2/2} \,.$$

Therefore the inequality $gg'' \leq \big(1 - \frac{1}{n}\big) g'^2$ amounts to

$$\big((n-1) - r^2\big) g(r) \leq \big(1 - \tfrac{1}{n}\big) r^n \mathrm{e}^{-r^2/2} \,.$$

Since equality holds at $r = 0$, in order to check the preceding inequality it suffices to compare the derivatives of both sides. This yields

$$-2r\, g(r) + \big((n-1) - r^2\big) r^{n-1} \mathrm{e}^{-r^2/2} \leq (n-1)\, r^{n-1} \mathrm{e}^{-r^2/2} - \big(1 - \tfrac{1}{n}\big)\, r^{n+1} \mathrm{e}^{-r^2/2}$$

that reduces to $-2g(r) \leq \frac{1}{n}\, r^n \mathrm{e}^{-r^2/2}$. Since the latter is clearly satisfied, we conclude that $g^{1/n}$ is indeed concave on $(0,\infty)$. Now, from the definition, $g(r) = \frac{r^n}{n} + O(r^{n+1})$ as $r \to 0^+$. Hence $g^{1/n}(r)$ is equivalent to $n^{-1/n} r$ near zero. Therefore,

$$(g^{\frac{1}{n}})'(r) = \tfrac{1}{n} g^{\frac{1}{n}-1}(r) g'(r) = \tfrac{1}{n} g^{\frac{1}{n}-1}(r)\, r^{n-1} \mathrm{e}^{-r^2/2} \to n^{-\frac{1}{n}} \quad \text{as} \quad r \to 0^+.$$

As a result,

$$u'(0^+) = \frac{(n\omega_n)^{1/n}}{\sqrt{2\pi}} \big(g^{1/n}\big)'(0^+) = \frac{\omega_n^{1/n}}{\sqrt{2\pi}} \,.$$

By (5.8), Lemma 5.2 is established. □

Combining Lemmas 5.1 and 5.3 with (5.7), we may conclude to the following result.

Proposition 5.3. *For the uniform measure λ_B on the Euclidean unit ball B,*

$$\mathrm{Ent}_{\lambda_B}(f^2) \le \frac{\omega_n^{2/n}}{\pi} \int |\nabla f|^2 d\lambda_B\,.$$

Note that $\frac{\omega_n^{2/n}}{\pi} = \Gamma\big(\frac{n}{2}+1\big)^{-2/n}$ behaves as $\frac{1}{n}$ for large n. The inequality is, up to a constant, sharp as can be seen by testing it on the functions $x \to \mathrm{e}^{\lambda x_1}$, $\lambda \in \mathbb{R}$. It may be shown that the Poincaré constant is of the same order. Somewhat surprisingly, when restricted to radial functions on B, the logarithmic Sobolev constant is still of the order of $1/n$ while the Poincaré constant behaves as $1/n^2$ (cf. [Bo], [L]).

On the basis of this example, it is possible to extend similarly Proposition 5.3 to uniformly convex bodies. Let $(E, \|\cdot\|)$ be an n-dimensional normed space with unit ball $B = \{x \in E : \|x\| \le 1\}$ equipped with the uniform distribution λ_B on B.

As in the preceding sections, assume B is uniformly convex in the sense that, for all $t, s > 0$ with $t + s = 1$ and all $x, y \in E$ (equivalently, $x, y \in B$),

$$t\|x\|^p + s\|y\|^p - \|tx + sy\|^p \ge \tfrac{c}{p}\big(s + o(s)\big)\|x - y\|^p \tag{5.9}$$

where $p \ge 2$, $c > 0$, and where $o(s)$ is a function of $s \in (0,1)$ such that $o(s)/s \to 0$, as $s \to 0$. Denote by q the conjugate number of p, and by $(E^*, \|\cdot\|_*)$ the dual space.

Proposition 5.4. *For every smooth non-negative function f on B,*

$$\mathrm{Ent}_{\lambda_B}(f^q) \le \frac{4}{\Gamma\big(\frac{n}{p}+1\big)^{2/n}} \Big(\frac{q}{c}\Big)^{q-1} \int \|\nabla f\|_*^q d\lambda_B\,.$$

Proof. As for Proposition 5.3, the proof is based on two steps corresponding to Lemma 5.1 and Lemma 5.2.

Lemma 5.5. *Let $U : E \to E$ be a map defined by*

$$U(x) = u(\|x\|)\frac{x}{\|x\|}\,, \quad x \in E\,,$$

where u is function on $[0,\infty)$ with a finite Lipschitz constant and such that $u(0) = 0$ (by continuity, $U(0) = 0$). Then,

$$\|U\|_{\mathrm{Lip}} \le 3\,\|u\|_{\mathrm{Lip}}\,.$$

If moreover, u is non-decreasing, then $\|U\|_{\mathrm{Lip}} \le 2\,\|u\|_{\mathrm{Lip}}$.

Proof. As we have seen from Lemma 5.1, the constants 3 and 2 may be replaced by 1 in the case of the Euclidean norm. In general however, we have to use another argument which deteriorates the original constant. The function $x \to \|x\|$ has a finite Lipschitz constant, so, by Rademacher's theorem, it is differentiable at almost every point $x \neq 0$. For such points, for all $h \in E$, we have Taylor's representation

$$\|x + \varepsilon h\| = \|x\| + \big\langle D(x), h\big\rangle \varepsilon + o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0$$

where $D(x) \in E^*$, and moreover $\|D(x)\|_* \leq 1$. By an approximation argument, we may assume that u is everywhere differentiable, so that

$$u\big(\|x + \varepsilon h\|\big) = u\big(\|x\|\big) + u'\big(\|x\|\big)\big\langle D(x), h\big\rangle \varepsilon + o(\varepsilon)\,.$$

Therefore,

$$\begin{aligned} U(x+\varepsilon h) &= u\big(\|x + \varepsilon h\|\big)\frac{x + \varepsilon h}{\|x + \varepsilon h\|} \\ &= U(x) + \varepsilon\left[\left(-\tfrac{u(\|x\|)}{\|x\|^2} + \tfrac{u'(\|x\|)}{\|x\|}\right)\big\langle D(x), h\big\rangle x + \tfrac{u(\|x\|)}{\|x\|}\,h\right] + o(\varepsilon)\,. \end{aligned}$$

Hence, U is differentiable at x, and its derivative $U'(x) : E \to E$ is given by

$$\begin{aligned} U'(x)h &= \lim_{\varepsilon \to 0} \frac{U(x + \varepsilon h) - U(x)}{\varepsilon} \\ &= \left(-\tfrac{u(\|x\|)}{\|x\|} + u'(\|x\|)\right)\big\langle D(x), h\big\rangle \tfrac{x}{\|x\|} + \tfrac{u(\|x\|)}{\|x\|}\,h\,. \end{aligned}$$

Since u is Lipschitz, $|u(\|x\|)| \leq \|u\|_{\mathrm{Lip}}\|x\|$ and $|u'(\|x\|)| \leq \|u\|_{\mathrm{Lip}}$, while $|\langle D(x), h\rangle| \leq \|h\|$. Consequently,

$$\|U'(x)h\| \leq 3\,\|u\|_{\mathrm{Lip}}\|h\|\,.$$

In case u is non-decreasing, for some $t_0 \in (0, \|x\|)$, $u(\|x\|) = u'(t_0)\|x\|$, so that

$$\left\|\left(-\tfrac{u(\|x\|)}{\|x\|} + u'(\|x\|)\right)\big\langle D(x), h\big\rangle \tfrac{x}{\|x\|}\right\| \leq \big|u'(\|x\|) - u'(t_0)\big\langle D(x), h\big\rangle\big|$$

which is clearly bounded by $\|u\|_{\mathrm{Lip}}\|h\|$. The lemma follows. □

As in the proof of Proposition 5.3, one can uniquely push forward the measure μ of (5.2) to λ_B by a map U of the form

$$U(x) = u\big(\|x\|\big)\tfrac{x}{\|x\|}\,, \quad x \in E\,,$$

where $u : [0, \infty) \to [0, 1)$ is an increasing non-negative function. This function is defined by the relation

$$\mu\big(x \in E; \|U(x)\| < s\big) = \lambda_B\big(y \in B : \|y\| < s\big)\,, \quad s \in (0, 1)\,,$$

that is,

$$\mu\big(x \in E; u(\|x\|) < s\big) = s^n\,, \quad s \in (0, 1)\,.$$

Exactly as in Lemma 5.2, (replacing $\mathrm{e}^{-t^2/2}$ by e^{-t^p}) one can show that

$$\|u\|_{\mathrm{Lip}} = \Gamma\big(\tfrac{n}{p} + 1\big)^{-1/n}\,.$$

It is then easy to conclude the proof of Proposition 5.4 as the proof of Proposition 5.3 by means of the logarithmic Sobolev inequality of Proposition 3.2 applied to the measure μ of (5.2). □

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S.G. Bobkov, School of Mathematics, University of Minnesota, 206 Church St. SE, Minneapolis, MN 55455, USA bobkov@math.umn.edu

M. Ledoux, Département de Mathématiques, Laboratoire de Statistique et Probabilités associé au C.N.R.S., Université Paul-Sabatier, 31062 Toulouse, France
ledoux@cict.fr

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