Sobolev Inequalities in Disguise

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We present a simple and direct proof of the equivalence of various functional inequalities such as Sobolev or Nash inequalities. This proof applies in the context of Riemannian or sub-elliptic geometry, as well as on graphs and to certain non-local Sobolev norms. It only uses elementary cut-off arguments. This method has interesting consequences concerning Trudinger type inequalities.

1. Introduction. On $\mathbb{R}^n$, the classical Sobolev inequality [27] indicates that, for every smooth enough function f with compact support,

$$\|f\|_{2n/(n-2)} \le C\|\nabla f\|_2 \tag{1.1}$$

where $C > 0$ only depends on $n > 2$. This inequality is of constant use in PDE (e.g., [15]) and is related to a wealth of other inequalities. In particular, it implies the a priori weaker Nash inequality

$$\|f\|_2^{1+2/n} \le C\|\nabla f\|_2 \, \|f\|_1^{2/n}. \tag{1.2}$$

This simply follows from the Hölder inequality $\|f\|_2 \le \|f\|_q^{\vartheta} \, \|f\|_1^{1-\vartheta}$ where $q = 2n/(n-2)$ and $\frac{1}{2} = \vartheta/q + (1-\vartheta)/1$. Inequality (1.2) is one of the main tools used by J. Nash in his celebrated 1958 paper [22] on the Hölder regularity of solutions of divergence form uniformly elliptic equations. In the subsequent work of J. Moser on the subject [20], the inequality

$$\|f\|_{2(1+2/n)}^{1+2/n} \le C\|\nabla f\|_2 \, \|f\|_2^{2/n} \tag{1.3}$$

is used instead. This last inequality also follows from (1.1) and Hölder's inequality because $2 < 2(1+2/n) < 2n/(n-2)$ for $n > 2$.

These inequalities have been used extensively, in more general settings, for the study of heat kernel estimates. See for instance [11, 31] and their references.

Actually, it follows from [29] and [5] that inequalities (1.1), (1.2) and (1.3) are equivalent to a uniform heat kernel upper bound of the order of $t^{-n/2}$. Thereby, (1.1), (1.2) and (1.3) are equivalent when $n > 2$. It is natural to ask whether it is possible to prove this equivalence without the help of heat kernel bounds and, by the same token, to extend the result to inequalities that involve the L^p-norm of the gradient. Such a proof, using capacities, is presented in the book by V. Maz'ja [19]. The main object of this paper is to provide a simple proof of this equivalence which only uses elementary arguments of cutting off functions, and nothing else. We develop this method for interpolation type inequalities generalizing the three examples above. These inequalities are introduced and discussed in details in Section 3. They are stated in terms of a given norm or semi-norm $W(f)$ defined on some class of test functions $\mathcal{F}$, and read

$$(\star) \qquad \|f\|_r \leq (CW(f))^{\vartheta} \|f\|_s^{1-\vartheta}$$

for some $0 < s, r \leq +\infty$ and $\vartheta \in]0,1]$. Inequalities of this type with $W(f) = \|\nabla f\|_p$ were considered, in the late fifties, by E. Gagliardo [14] and by J. Nirenberg [24].

Section 2 describes the simple properties that a Sobolev semi-norm $W(f)$ must satisfy for our method to apply. These properties are easily seen to be satisfied by the semi-norms $W_p(f) = \|\nabla f\|_p$ on Riemannian manifolds. We show in Section 7 that they also hold for certain non-local semi-norms. The main results are stated in Section 3 and proved in Sections 4, 5 and 6. In fact, in order to analyse $(\star)$, three cases must be distinguished depending on the value of the parameter q defined by

$$\frac{1}{r} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s}.$$

Section 4 examines the case where $0 < q < +\infty$ and shows, under some mild hypotheses on W, that $(\star)$ implies the strong Sobolev inequality

$$\|f\|_q \leq BW(f).$$

The case $-\infty < q < 0$ is studied in Section 5. In this case, we show that $(\star)$ implies that a function f such that $W(f) < +\infty$ and $\|f\|_t < +\infty$ for some $0 < t \leq +\infty$ is necessarily bounded. The case $q = +\infty$ is studied in Section 6 where we show that $(\star)$ implies a Trudinger inequality [28] of the type

$$\int [\exp(c|f|^{\gamma}) - 1] \, d\mu \leq C\mu(\operatorname{supp}(f)) \quad \text{for } f \in \mathcal{F} \text{ and } W(f) \leq 1$$

where $\gamma \geq 1$ depends on the basic property that we impose on the norm W. For instance, if $W(f) = \|\nabla f\|_p$ on a Riemannian manifold, we have $\gamma = p/(p-1)$. The arguments leading to the Trudinger inequality in Section 6 depend strongly on the use of Lorentz spaces. Sections 8-10 present various comments, applications and remarks.

To illustrate the method, let us derive directly (1.1) from (1.2) in $\mathbb{R}^n$. Assume $n > 2$ and set $1/q = 1/2 - 1/n$, i.e. $q = 2n/(n-2)$. Let f be a smooth non-negative function with compact support. For every integer k in $\mathbb{Z}$, let $f_k = (f - 2^k)^+ \wedge 2^k$. Applying Nash's inequality (1.2) to f_k, we get

$$\left(\int f_k^2 d\mu\right)^{1+2/n} \leq C^2 \int_{B_k} |\nabla f_k|^2 d\mu \left(\int f_k d\mu\right)^{4/n},$$

where μ is the Lebesgue measure on $\mathbb{R}^n$ and $B_k = \{x : 2^k \leq f(x) < 2^{k+1}\}$, $k \in \mathbb{Z}$. Now, $f_k = f$ on B_k, $f_k = 0$ on the set $\{f < 2^k\}$ and $f_k = 2^k$ on the set $\{f \geq 2^{k+1}\}$. Therefore,

$$\left(2^{2k}\mu(f \geq 2^{k+1})\right)^{1+2/n} \leq C^2 \int_{B_k} |\nabla f|^2 d\mu \left(2^k \mu(f \geq 2^k)\right)^{4/n}.$$

Set $\vartheta = n/(n+2)$ and, for every $k \in \mathbb{Z}$, $a_k = 2^{qk}\mu(f \geq 2^k)$ and $b_k = \int_{B_k} |\nabla f|^2 d\mu$. With this notation, the preceding inequality raised to the power ϑ and multiplied by 2^{q-2} yields

$$a_{k+1} \leq 2^q C^{2\vartheta} b_k^\vartheta a_k^{2(1-\vartheta)}.$$

Now, by Hölder's inequality,

$$\begin{aligned}
\sum_k a_k &= \sum_k a_{k+1} \leq 2^q C^{2\vartheta} \Big(\sum_k b_k\Big)^\vartheta \Big(\sum_k a_k^2\Big)^{1-\vartheta} \\
&\leq 2^q C^{2\vartheta} \Big(\sum_k b_k\Big)^\vartheta \Big(\sum_k a_k\Big)^{2(1-\vartheta)}
\end{aligned}$$

and thus

$$\sum_k a_k \leq \Big[2^q C^{2\vartheta} \Big(\sum_k b_k\Big)^\vartheta\Big]^{1/(2\vartheta-1)}.$$

It is plain that $\sum_k b_k \leq \|\nabla f\|_2^2$ and that $(2^q - 1)\sum_k a_k \geq \|f\|_q^q$. Hence, $\|f\|_q \leq B\|\nabla f\|_2$, which is Sobolev's inequality (1.1) with

$$\begin{aligned}
B &= \left((2^q - 1)(2^q C^{2\vartheta})^{1/(2\vartheta-1)}\right)^{1/q} \\
&= 2^{q-1}(2^q - 1)^{1/q} C.
\end{aligned}$$

Clearly enough, a similar argument would prove that (1.1) is also equivalent to (1.3). Let us observe further that the argument above naturally gives rise to a somewhat better inequality. If we set instead $\vartheta = (n-2)/(n+2)$ and

$a_k = 2^{2k}\mu(f \geq 2^k)^{2/q}$, we get from (1.2) $a_{k+1} \leq 2C^{2\vartheta} b_k^{\vartheta} a_k^{1-\vartheta}$, so that $\sum_k a_k \leq 2^{1/\vartheta} C^2 \sum_k b_k$. This yields

$$\|f\|_{q,2} \leq 2^q C \|\nabla f\|_2,$$

where $\|f\|_{q,2} = \left(2\int_0^\infty (t^q \mu(f \geq t))^{2/q} t^{-1} dt\right)^{1/2}$ is a Lorentz norm. The use of Lorentz norms will have interesting consequences to the limiting case of a Trudinger inequality.

We are not claiming that the type of argument above is new. Indeed, very similar (if not identical) ideas have been used recently by various authors. See [2, 3, 6, 8, 13, 16, 17] among others. In particular, our work was stimulated by the results of G. Carron [6]. After this paper was written we received the preprint [12] which also uses similar argument. However, it seems to us that the extent to which these simple ideas work had not yet been fully recognized. In particular, the treatment of the limit case corresponding to a Trudinger inequality came as a good surprise for us. In the classical setting of $\mathbb{R}^n$, all the inequalities considered in this paper are well known. In more general settings, like manifolds or finitely generated groups, the machinery of this paper produces simple proofs of inequalities that are not in the literature.

Our main motivation for developing in detail the technique presented in this paper comes from the fact that, in practice, Sobolev–type inequalities are often much easier to prove under one of their "weak" forms. Thus, it is important to have simple tools to pass from an apparently weak form to the (apparently) stronger statement. For instance, in $\mathbb{R}^n$, it is not difficult to deduce the weak statement

$$t\,[\mu(|f| \geq t)]^{(n-2)/2n} \leq C\|\nabla f\|_2$$

either from an obvious heat kernel bound of order $t^{-n/2}$ or from a bound of order $|x|^{-n+1}$ on the potential convolution kernel of $\Delta^{-1/2}$ where $\Delta f = -\sum_1^n \partial_i^2 f$. The technique of this paper shows plainly and painlessly that this weak inequality implies the strong version (1.1) (see Section 4). Observe that Sobolev's original approach uses the exact formula $c|x|^{-n+1}$ for the convolution kernel of $\Delta^{-1/2}$ and relies on symmetrization techniques to obtain (1.1). Sections 9 and 10 contain further illustrations of this phenomenon where one first proves a seemingly weak inequality. The statements given there can be used to prove most of the Sobolev inequalities we have ever encountered (except for best constants).

We decided to present the method and all our main results in an abstract setting instead of in the most familiar setting of Riemannian geometry. Our motivation for this is twofold and it seems necessary to give explanations.

(1) Inequality ($\star$) contains three parameters: r, s, and ϑ. Out of these three parameters, one computes a fourth one q, given by $1/r = \vartheta/q + (1-\vartheta)/s$. Three different cases arise depending on whether $0 < q < +\infty$, $-\infty < q <$

0 or $q = +\infty$. In each of these cases distinct useful conclusions can be drawn from inequality ($\star$). The techniques used in each case are similar but different. Now, ($\star$) is stated in terms of a mysterious quantity W. The most classical case is when $W(f) = \|\nabla f\|_p$. Observe that this introduces a new parameter p. Although the four of us have been working with Sobolev inequalities for many years, we were surprised when we realized that the parameter p played only a marginal role in this paper. In fact, the only thing that matters about W is how $W(f)$ controls $W(f_k)$ where $f_k = (f - 2^k)^+ \wedge 2^k$ or, more generally, $f_k = (f - \rho^k)^+ \wedge \rho^k(\rho - 1)$ with $\rho > 1$. Much can be said already if one only knows that $W(f_k) \le AW(f)$. The key property that W may or may not satisfy is

$$(\#) \qquad \Big(\sum_k W(f_k)^\alpha\Big)^{1/\alpha} \le AW(f)$$

for some $0 < \alpha \le +\infty$. Section 2 carefully introduces these inequalities. They will serve as basic hypotheses on W. The reader can easily verify that $W(f) = \|\nabla f\|_p$ satisfies (#) for all $\alpha \ge p$ with $A = 1$. Working with L^p norms of a gradient instead of with an abstract W satisfying (#) brings no simplification whatsoever to the proofs. It makes the notation more cumbersome and the arguments less clear.

(2) The fact that Sobolev inequalities are basic tools in analysis, PDE's and geometry is well known and documented. Our second motivation for working in an abstract setting stems from the less known but nonetheless important applications of Sobolev inequalities to such fields as Markov semigroups or analysis and potential theory on graphs. See, e.g., [1, 2, 5, 10, 17, 29, 30, 31]. For instance, Sobolev inequalities on Cayley graphs of finitely generated groups convey fundamental properties such as whether the group is transient or recurrent. Even in Riemannian geometry, Sobolev inequalities on discrete structures (e.g., the fundamental group of a compact manifold) have proved very useful. It turns out that the natural semi-norms W attached to a graph or to a Markov semigroup satisfy properties like (#). This, however, is not as obvious as in the case of $\|\nabla f\|_p$. It will be proved in Section 7.

2. Sobolev norms and the hypotheses (H_α). This section describes the general framework in which our main results are developed.

Let $(E, \mathcal{E}, \mu)$ be a measurable space with a non-negative σ-finite measure μ. We need to deal with a norm or semi-norm associated with a gradient, or rather the length of a gradient, on some class of functions on E. Let thus $\mathcal{F}$ be a class of functions on E. It will be convenient, although not strictly necessary, to assume that the functions in $\mathcal{F}$ are non-negative. We assume some simple stability properties on $\mathcal{F}$. We mainly need that $f \in \mathcal{F}$ implies $(f - t)^+ \wedge s \in \mathcal{F}$

for all $t, s \geq 0$. In some cases, we will further assume that $\mathcal{F}$ is a cone. In order to avoid any integrability question, one can also suppose that $\mathcal{F}$ is contained in all the L^p-spaces (w.r.t. μ).

Let $W(f)$ be a given norm or semi-norm on $\mathcal{F}$. We say that W satisfies (H_∞^+) if there exists a constant A_∞^+ such that, for any $f \in \mathcal{F}$ and any $s, t \geq 0$,

$$(H_\infty^+) \qquad W\left((f-t)^+ \wedge s\right) \leq A_\infty^+ W(f).$$

For any $\rho > 1$, $k \in \mathbb{Z}$, and any function $f \in \mathcal{F}$, set

$$f_{\rho,k} = (f - \rho^k)^+ \wedge \rho^k(\rho - 1)$$

which, by hypothesis, is also in $\mathcal{F}$. Fix α with $0 < \alpha \leq +\infty$ and $\rho > 1$. We say that W satisfies the condition (H_α^ρ) if there exists a constant $A_\alpha(\rho)$ such that, for any $f \in \mathcal{F}$,

$$(H_\alpha^\rho) \qquad \Big(\sum_{k\in\mathbb{Z}} W(f_{\rho,k})^\alpha\Big)^{1/\alpha} \leq A_\alpha(\rho) W(f),$$

with an obvious modification if $\alpha = +\infty$. If one wishes to extend the results to classes $\mathcal{F}$ containing real or complex functions, one may further assume that $W(|f|) \leq AW(f)$ for all $f \in \mathcal{F}$.

It is clear that (H_α^ρ) implies (H_β^ρ) for all $\alpha \leq \beta \leq +\infty$. Note also that (H_∞^+) is a slightly stronger property than the conjunction of all the (H_∞^ρ) with $\rho > 1$. In most of the examples we have in mind and which are examined below and in Section 7, the semi-norm W will satisfy (H_α^ρ) for some fixed α, uniformly in $\rho \in]1, +\infty[$. We say that W satisfies (H_α) if there exists A_α such that, for all $f \in \mathcal{F}$,

$$(H_\alpha) \qquad \sup_{\rho>1} \Big(\sum_{k\in\mathbb{Z}} W(f_{\rho,k})^\alpha\Big)^{1/\alpha} \leq A_\alpha W(f).$$

This is the only type of properties of W that will be required throughout the paper.

Typically, on a Riemannian manifold (M, g), take $\mathcal{F}$ to be the set of non-negative Lipschitz functions with compact support and set

$$W(f) = W_p(f) = \left(\int |\nabla f|^p d\mu\right)^{1/p}$$

for some fixed $0 < p < +\infty$. Then, it is clear that the semi-norm W_p satisfies (H_∞^+) with $A_\infty^+ = 1$ and satisfies (H_α^ρ) for all $\alpha \in [p, +\infty]$ and any $\rho > 1$ with $A_\alpha(\rho) = 1$. Here $d\mu$ can be the Riemannian measure dv or any non-negative σ-finite measure on M and $|\nabla f|^2 = g(\nabla f, \nabla f)$ is the square of the length of the gradient of f.

More generally, we can consider a (measurable) section $\mathcal{A}$ of the bundle of the symmetric endomorphisms (i.e., for each $x \in M$, $\mathcal{A}_x$ is a symmetric endomorphism of the tangent space at x) and set $|\nabla_{\mathcal{A}} f|^2 = g(\mathcal{A}\nabla f, \nabla f)$. Note that $\mathcal{A}$ can be degenerate. This contains the case of the "carré du champ" $\Gamma(f,f)$ of a diffusion operator $\mathcal{L}$ with real coefficients, without constant term, and which is self-adjoint with respect to a measure $d\mu = m dv$. Any such $\mathcal{L}$ determines a section $\mathcal{A}$ of the bundle of the symmetric endomorphisms such that

$$\mathcal{L} = -\frac{1}{m}\mathrm{div}(m\mathcal{A}\nabla f)$$

and $\Gamma(f,f) = |\nabla_{\mathcal{A}} f|^2$. Here, Γ can also be defined intrinsically by $\Gamma(f,f) = \frac{1}{2}|\mathcal{L}f^2 - 2f\mathcal{L}f|$. See [1] for a thorough discussion of Γ. A special case worth mentioning is when we are given a family $X = \{X_i : i = 1, \dots, \ell\}$ of vector fields on M. Then, one can consider $|\nabla_X f|^2 = \sum_1^\ell |X_i f|^2$, which can be viewed as the "carré du champ" of $\mathcal{L}_X = \sum_1^\ell X_i^* X_i$ where X_i^* is the formal adjoint of X_i with respect to $d\mu$. This situation naturally occurs when $M = G$ is a connected Lie group endowed with a Haar measure and X is a set of left invariant vector fields that generates the Lie algebra of G.

Again, in all these situations it is plain that

$$W_p(f) = \left(\int |\nabla f|^p \, d\mu\right)^{1/p}$$

satisfies (H_∞^+) with $A_\infty^+ = 1$ and satisfies (H_α^ρ) for all $\alpha \in [p, +\infty]$ and any $\rho > 1$ with $A_\alpha(\rho) = 1$.

There are several other important cases of semi-norms W satisfying (H_α^ρ) for some α. These include non-local cases and are further discussed in Section 7. For instance, the case where E is vertex set of a locally finite symmetric graph is of special interest (e.g., the Cayley graph of a finitely generated group for a fixed finite set of generators). Write $x \sim y$ if x, y are neighbors. Then,

$$W(f) = W_p(f) = \Big(\sum_{\substack{x,y \\ x\sim y}} |f(x) - f(y)|^p\Big)^{1/p}$$

satisfies (H_∞^+) and (H_α^ρ), $\alpha \in [p, +\infty]$. To see this, however, requires some work and is postponed until Section 7.

It is useful to observe that each L^p-norm (or quasi-norm) satisfies (H_α^ρ) for all $\alpha \in [p, +\infty]$ because it allows us to work with Sobolev norms of the type

$$W_p(f) = (\|\nabla f\|_p^p + c\|f\|_p^p)^{1/p}$$

for some fixed constant c. These norms naturally arise when the total mass of μ is finite. Here again (H_∞^+) clearly holds with $A_\infty^+ = 1$. For finite α, one has the following result:

Lemma 2.1 *If $1 \leq p \leq +\infty$, $\|f\|_p$ satisfies (H_α^ρ) for all $\alpha \in [p, +\infty]$ and all $\rho > 1$ with $A_\alpha(\rho) = (1 - 1/\rho)^{1-1/p} \leq 1$. In particular, it satisfies (H_p).*

If $0 < p < 1$, $\|f\|_p$ satisfies (H_α^ρ) for all $\alpha \in [p, +\infty]$ and all $\rho > 1$ with

$$A_\alpha(\rho) = (1 + (\rho - 1)^p/(\rho^p - 1))^{1/p} \wedge \rho(\rho - 1)/(\rho^p - 1)^{1/p}.$$

Proof. For a fixed $\rho > 1$, let us write

$$\|f_k\|_p^p = p \int_0^{\rho^{k+1} - \rho^k} t^{p-1} \mu(f - \rho^k \geq t)\, dt = p \int_{\rho^k}^{\rho^{k+1}} (s - \rho^k)^{p-1} \mu(f \geq s)\, ds.$$

If $1 \leq p < +\infty$, it follows that

$$\begin{aligned}
\sum_k \|f_k\|_p^p &= p \sum_k \int_{\rho^k}^{\rho^{k+1}} (s - \rho^k)^{p-1} \mu(f \geq s)\, ds \\
&\leq \left[\sup_k \sup_{s \in [\rho^k, \rho^{k+1}]} \left(\frac{s - \rho^k}{s} \right)^{p-1} \right] \left[p \sum_k \int_{\rho^k}^{\rho^{k+1}} s^{p-1} \mu(f \geq s)\, ds \right] \\
&\leq (1 - 1/\rho)^{p-1} \|f\|_p^p.
\end{aligned}$$

The case $0 < p < 1$ is similar and will be omitted. □

3. Sobolev type inequalities. This section describes in detail our main results and the families of functional inequalities that are considered in this paper. We fix a measure space $(E, \mathcal{E}, \mu)$, a set of (non-negative) functions $\mathcal{F}$ stable under contractions (i.e. $(f - t)^+ \wedge s) \in \mathcal{F}$ if $f \in \mathcal{F}$ and $s, t \geq 0$) and a norm or semi-norm W defined on $\mathcal{F}$.

For r, $s \in]0, +\infty]$ and $\vartheta \in]0, 1]$, consider the functional inequality

$$(S_{r,s}^\vartheta) \qquad \|f\|_r \leq (CW(f))^\vartheta \|f\|_s^{1-\vartheta},$$

which may or may not hold for some constant C and all $f \in \mathcal{F}$, and define the associated parameter $q = q(r, s, \vartheta) \in]-\infty\,, 0[\cup]0\,, +\infty[\cup \{\infty\}$ by setting

$$\frac{1}{r} = \frac{\vartheta}{q} + \frac{1 - \vartheta}{s}\,. \tag{3.1}$$

Observe that Hölder's inequality induces a natural partial order among these inequalities. Namely, if $r \geq s$ and q is defined by (3.1), $(S_{r,s}^\vartheta)$ implies $(S_{r',s'}^{\vartheta'})$ for all $r' \geq s'$ such that $r' \leq r$ and $s' \leq s$ with ϑ' given by $1/r' = \vartheta'/q + (1 - \vartheta')/s'$. If $q \leq r \leq s$ where q is again defined by (3.1), $(S_{r,s}^\vartheta)$ implies $(S_{r',s'}^{\vartheta'})$ for all $r' \leq s'$ such that $r \leq r'$ and $s \leq s'$ with ϑ' given by $1/r' = \vartheta'/q + (1 - \vartheta')/s'$ (this second case can only appear when $0 < q < +\infty$).

The aim of this work is to show that whenever one such inequality holds for some fixed r_0, s_0, ϑ_0 then there is a full range of values of r, s and ϑ such that $(S^\vartheta_{r,s})$ holds. Roughly speaking, the range of admissible values can be described by saying that the parameter q stays fixed when r, s and ϑ vary. In other words, we will show that any of the inequalities $(S^\vartheta_{r,s})$ that can be deduced from a given one by Hölder's inequality is, in fact, equivalent to the starting one.

We will prove the following set of results.

Theorem 3.1 (Case: $0 < q < +\infty$***)*** *Assume that $(S^{\vartheta_0}_{r_0,s_0})$ holds for some fixed r_0, $s_0 \in \,]0, +\infty]$ and $\vartheta_0 \in \,]0, 1]$ and that the parameter $q = q(r_0, s_0, \theta_0)$ defined at (3.1) satisfies $0 < q < +\infty$. Assume that W satisfies (H^ρ_q) for some $\rho > 1$. Then, each of the inequalities $(S^\vartheta_{r,s})$ with r, $s \in \,]0, +\infty]$, $\vartheta \in \,]0, 1]$ and $1/r = \vartheta/q + (1 - \vartheta)/s$ is satisfied with a constant C which does not depend on r, s, ϑ. In particular, for all $f \in \mathcal{F}$, we have*

$$\|f\|_q \leq CW(f).$$

Further results involving Lorentz norms will be described in Section 4 under the assumption that W satisfies the hypothesis (H^ρ_α) for some $0 < \alpha \leq +\infty$ and $\rho > 1$.

Theorem 3.2 (Case: $-\infty < q < 0$***)*** *Assume that $(S^{\vartheta_0}_{r_0,s_0})$ holds for some fixed r_0, $s_0 \in \,]0, +\infty]$ and $\vartheta_0 \in \,]0, 1]$ and that the parameter $q = q(r_0, s_0, \vartheta_0)$ defined at (3.1) satisfies $-\infty < q < 0$. Assume that W satisfies (H^+_∞). Then, each of the inequalities $(S^\vartheta_{r,s})$ with $0 < s < r \leq +\infty$, $\vartheta \in \,]0, 1]$ and $1/r = \vartheta/q + (1 - \vartheta)/s$ is satisfied with a constant C depending on r, s, ϑ only through a finite upper bound on s. More precisely, there exists a constant B such that for each $s > 0$ and each $f \in \mathcal{F}$,*

$$\|f\|_\infty \leq BW(f)^{1/(1-s/q)} \|f\|_s^{1/(1-q/s)}.$$

Theorem 3.3 (Case: $q = \infty$***)*** *Assume that $(S^{1-s_0/r_0}_{r_0,s_0})$ holds for some fixed r_0, s_0 with $0 < s_0 < r_0 \leq +\infty$. Assume that W satisfies (H^ρ_∞) for some $\rho > 1$. Then, each of the inequalities $(S^{1-s/r}_{r,s})$ with $0 < s < r < +\infty$ is satisfied with a constant C depending on r, s, ϑ only through a finite upper bound on r.*

This last result can be significantly improved when W satisfies (H_α) for some $0 < \alpha \leq +\infty$. Namely, by an appropriate use of Lorentz norms, the following statement will be proved in Section 6.

Theorem 3.4 *Assume that $(S^{1-s_0/r_0}_{r_0,s_0})$ holds for some fixed r_0, s_0 with $0 < s_0 < r_0 \leq +\infty$, and some class $\mathcal{F}$ of functions. Assume that W satisfies (H_α) for some $0 < \alpha \leq +\infty$. If $0 < \alpha < 1$, the class $\mathcal{F}$ of function s for which $(S^{1-s_0/r_0}_{r_0,s_0})$ is satisfied must be trivial, i.e., $\mathcal{F} = \{0\}$. If $\alpha = 1$, there exists a constant C such* *that, for all $f \in \mathcal{F}$, $\|f\|_\infty \le CW(f)$. Finally, suppose that $1 < \alpha \le +\infty$. Then, there exists a constant C such that, for all $0 < s < r < +\infty$ and all $f \in \mathcal{F}$,*

$$\|f\|_r \le \Big((1 \vee r)^{1-1/\alpha} CW(f)\Big)^{1-s/r} \|f\|_s^{s/r}.$$

It follows that there exist three constants D, C, $c > 0$ such that, for any $s > 0$, any integer $k \ge (\alpha-1)s/\alpha$, and any $f \in \mathcal{F}$,

$$\int_E \exp_k\left([cf/W(f)]^{\alpha/(\alpha-1)}\right) d\mu \le D\left(\|f\|_s/CW(f)\right)^s$$

where $\exp_k(x) = \sum_{\ell=k}^\infty x^\ell/\ell!$.

Each of the three cases $0 < q < +\infty$, $-\infty < q < 0$, and $q = \infty$, will be analyzed in detail in the sequel.

It may be useful to describe here how these cases appear in the Euclidean space $\mathbb{R}^n$ when $W(f) = \|\nabla f\|_p$, $1 \le p < +\infty$. In this classic setting, an easy argument, using dilations, shows that the parameter q must satisfy $1/q = 1/p - 1/n$, i.e. $q = np/(n-p)$. Note also that $W(f) = \|\nabla f\|_p$ satisfies the hypothesis (H_p) and thus (H_α) for any $\alpha \ge p$.

(1) The case $0 < q < +\infty$ corresponds to $1 \le p < n$. For such p, the Sobolev inequality $\|f\|_q \le C\|\nabla f\|_p$ holds and implies, by Hölder's inequality, all the other inequalities of type $(S^\vartheta_{r,s})$ with $r, s \in]0, +\infty]$, $\vartheta \in]0,1]$ and $1/r = \vartheta/q + (1-\vartheta)/s$. A special case worth mentioning is when $s = +\infty$. Then, the inequality reads $\|f\|_r \le (C\|\nabla f\|_p)^{q/r}\|f\|_\infty^{1-q/r}$ where $q < r < +\infty$.

(2) The case $-\infty < q < 0$ corresponds to $n < p \le +\infty$. For these values of p, for each $0 < s < +\infty$, the Gagliardo-Nirenberg inequality $\|f\|_\infty \le (C_s\|\nabla f\|_p)^{1/(1-s/q)}\|f\|_s^{1/(1-q/s)}$ holds. See [14, 24]. The associated inequalities with $0 < s < r < +\infty$ follow by Hölder's inequality.

(3) The third case, $q = \infty$, corresponds to $p = n$. It is known that, in $\mathbb{R}^n$, there exist unbounded functions such that $\|\nabla f\|_n < +\infty$ and $\|f\|_s < +\infty$ for any fixed $1 \le s < +\infty$. The optimal result on local integrability is given by the Trudinger-Moser inequality which shows that $\|\nabla f\|_n \le 1$ implies that $\exp(c_n|f|^{n/(n-1)})$ is locally integrable for some (known) best constant c_n. See [21].

To conclude this illustration, observe that for each $1 \le p \le +\infty$, and any dimension n, the Nash–type inequality

$$\|f\|_p \le (C(n,p)\|\nabla f\|_p)^{\vartheta(n,p)} \|f\|_1^{1-\vartheta(n,p)}, \quad f \in C_0^\infty(\mathbb{R}^n),$$

holds with $\vartheta(n,p) = n(p-1)/(np+p-n)$, and this, whatever the relative positions of p and n might be. Given such an inequality, one can compute q by (3.1) and decide which of the three cases above applies.

Returning to the general setting, consider the (a priori weaker) variant of $(S^{\vartheta}_{r,s})$:

$$(S^{*,\vartheta}_{r,s}) \qquad \sup_{\lambda>0}\left\{\lambda\mu(f\geq\lambda)^{1/r}\right\} \leq (CW(f))^{\vartheta}\left(\|f\|_{\infty}\,[\mu(\mathrm{supp}(f))]^{1/s}\right)^{1-\vartheta}.$$

For $r=+\infty$, the left-hand side is understood as $\|f\|_{\infty}$. The method used in this paper yields immediately a sharper version of the theorems above.

Proposition 3.5 *The conclusions of each of the four theorems above still hold when, instead of assuming that $(S^{\vartheta_0}_{r_0,s_0})$ is satisfied, we assume that the weaker inequality $(S^{*,\vartheta_0}_{r_0,s_0})$ is satisfied.*

Corollary 3.6 *Fix $q\in\,]-\infty,0[\,\cup\,]0,+\infty[$. If $q\in\,]0,\infty[$, assume that W satisfies (H^{ρ}_{q}) for some $\rho>1$, whereas, if $q\in\,]-\infty,0[$, assume that W satisfies (H^{+}_{∞}). Then, the inequalities $(S^{\vartheta}_{r,s})$ and $(S^{*,\vartheta}_{r,s})$ with $r,\ s\in\,]0,+\infty]$, $\vartheta\in\,]0,1]$ and $1/r=\vartheta/q+(1-\vartheta)/s$ are all equivalent.*

In the case where $q=\infty$ and if we assume that W satisfies (H^{ρ}_{∞}) for some $\rho>1$, then the inequalities $(S^{1-s/r}_{r,s})$ and $(S^{*,1-s/r}_{r,s})$ whith $0<s<r<+\infty$ are all equivalent.

Let us emphasize that working with $(S^{*,\vartheta}_{r,s})$ instead of $(S^{\vartheta}_{r,s})$ is very valuable in practice since it often happens that the weak inequality $(S^{*,\vartheta}_{r,s})$ is much easier to prove that its strong counterpart.

4. The case $0<q<+\infty$. We start with a simple result which already illustrates the main argument: a given inequality can be improved by applying it to the functions $f_{\rho,k}=(f-\rho^{k})^{+}\wedge\rho^{k}(\rho-1)$, $\rho>1$, $k\in\mathbb{Z}$. For many purposes, taking $\rho=2$ suffices. However, using $\rho>1$ as a parameter gives some extra information. It is essential to do so to obtain sharp results in Section 6 (the case $q=+\infty$).

Theorem 4.1 *Assume that $(S^{*,\vartheta_0}_{r_0,s_0})$ holds for some fixed $r_0,\ s_0\in\,]0,+\infty]$ and $\vartheta_0\in\,]0,1]$ and that the parameter $q=q(r_0,s_0,\vartheta_0)$ defined at (3.1) satisfies $0<q<+\infty$. Suppose that W satisfies (H^{ρ}_{∞}) for some $\rho>1$. Then, for all $f\in\mathcal{F}$, we have the weak Sobolev inequality*

$$\sup_{\lambda>0}\left\{\lambda\mu(f\geq\lambda)^{1/q}\right\}\leq(\rho-1)^{-1}\rho^{1+q/r_0\vartheta_0}C_0A_{\infty}(\rho)W(f).$$

Here C_0 is the constant appearing in $(S^{*,\vartheta_0}_{r_0,s_0})$.

Proof. Fix $f \in \mathcal{F}$ and apply $(S^{*,\vartheta_0}_{r_0,s_0})$ to $f_k = f_{\rho,k} = (f - \rho^k)^+ \wedge \rho^k(\rho - 1)$. Observe that the support of f_k is contained in $\{f \geq \rho^k\}$ and that

$$\{f_k \geq (\rho - 1)\rho^k\} = \{f \geq \rho^{k+1}\}.$$

Since W satisfies (H^ρ_∞), this gives

(4.1)
$$(\rho - 1)\rho^k \mu(f \geq \rho^{k+1})^{1/r_0} \leq (A_\infty(\rho) C_0 W(f))^{\vartheta_0} \left((\rho - 1)\rho^k \mu(f \geq \rho^k)^{1/s_0}\right)^{1-\vartheta_0}.$$

Using the notation

$$\sup_{k \in \mathbb{Z}} \left\{\rho^k \mu(\{f \geq \rho^k\})^{1/q}\right\} = N(f),$$

and the definition (3.1) of q, we get

$$\begin{aligned}\mu(f \geq \rho^{k+1})^{1/r_0} &\leq \rho^{-k(\vartheta_0 + q(1-\vartheta_0)/s_0)} \left(\frac{A_\infty(\rho) C_0}{(\rho - 1)} W(f)\right)^{\vartheta_0} N(f)^{q(1-\vartheta_0)/s_0} \\ &\leq \rho^{-kq/r_0} \left(\frac{A_\infty(\rho) C_0}{(\rho - 1)} W(f)\right)^{\vartheta_0} N(f)^{q(1-\vartheta_0)/s_0}\end{aligned}$$

and thus

$$N(f)^{q/r_0} \leq \rho^{q/r_0} \left((\rho - 1)^{-1} A_\infty(\rho) C_0 W(f)\right)^{\vartheta_0} N(f)^{q(1-\vartheta_0)/s_0}.$$

Simplifying and using (3.1) again, we obtain

$$\sup_{\lambda > 0} \left\{\lambda \mu(f \geq \lambda)^{1/q}\right\} \leq \rho N(f) \leq (\rho - 1)^{-1} \rho^{1 + q/r_0 \vartheta_0} A_\infty(\rho) C W(f)$$

which is the desired inequality. □

If W satisfies (H^+_∞), we can pick $\rho > 1$ as we wish. Setting $\rho = 1 + r_0\vartheta_0/q$ yields:

Corollary 4.2 *Assume that $(S^{*,\vartheta_0}_{r_0,s_0})$ holds for some fixed $r_0, s_0 \in\ ]0, +\infty]$ and $\vartheta_0 \in]0, 1]$, and that the parameter $q = q(r_0, s_0, \vartheta_0)$ defined at (3.1) satisfies $0 < q < +\infty$. Suppose that W satisfies (H^+_∞). Then, for all $f \in \mathcal{F}$, we have the weak Sobolev inequality*

$$\sup_{\lambda > 0} \left\{\lambda \mu(f \geq \lambda)^{1/q}\right\} \leq e(1 + q/r_0\vartheta_0) A^+_\infty C_0 W(f)$$

where C_0 is the constant appearing in $(S^{*,\vartheta_0}_{r_0,s_0})$.

To go further, we need to introduce the full scale of Lorentz "norms"

$$\|f\|_{a,b} = \left(b \int_0^{\infty} (t^a \mu(f \geq t))^{b/a} \frac{dt}{t} \right)^{1/b}$$

with a, $b \in \,]0, +\infty[$. We also set

$$\|f\|_{a,\infty} = \sup_{\lambda > 0} \left\{ \lambda \mu(\{|f| \geq \lambda\})^{1/a} \right\}, \quad \|f\|_{\infty,\infty} = \|f\|_{\infty}.$$

Working with Lorentz norms will play a crucial part in Section 6.

Recall that $\|f\|_{a,a} = \|f\|_a$ for any $0 < a \leq +\infty$ and observe that, for any $\rho > 1$,

$$\begin{aligned} (4.2) \quad \rho^{-b}(\rho^b - 1) \sum_k \rho^{bk} \mu(f \geq \rho^k)^{b/a} \quad &\leq \quad \|f\|_{a,b}^b \\ &\leq \quad (\rho^b - 1) \sum_k \rho^{bk} \mu(f \geq \rho^k)^{b/a}. \end{aligned}$$

For the purpose of this paper, it is useful to introduce the quantities

$$N_{a,b}^{\rho}(f) = \Big(\sum_k \rho^{bk} \mu(f \geq \rho^k)^{b/a} \Big)^{1/b}$$

which clearly satisfy

$$N_{a,b'}^{\rho}(f) \leq N_{a,b}^{\rho}(f) \ \text{ if } \ 0 < b \leq b' \leq +\infty$$

and

$$N_{r,u}^{\rho}(f) \leq \left(N_{s,v}^{\rho}(f)\right)^{\vartheta} \left(N_{t,w}^{\rho}(f)\right)^{1-\vartheta}$$

with

$$\frac{1}{r} = \frac{\vartheta}{s} + \frac{1-\vartheta}{t} \ \text{ and } \ \frac{1}{u} = \frac{\vartheta}{v} + \frac{1-\vartheta}{w}, \quad 0 \leq \vartheta \leq 1.$$

This can be used to show that

$$(4.3) \qquad \|f\|_{a,b'} \leq 2^{2/b} \|f\|_{a,b} \quad \text{for } 0 < b \leq b' \leq +\infty$$

and that these norms satisfy the Hölder–Lorentz inequality

$$\|f\|_{r,u} \leq C(u,v,w) \|f\|_{s,v}^{\vartheta} \ \|f\|_{t,w}^{1-\vartheta}$$

with

$$\frac{1}{r} = \frac{\vartheta}{s} + \frac{1-\vartheta}{t} \ \text{ and } \ \frac{1}{u} = \frac{\vartheta}{v} + \frac{1-\vartheta}{w}, \quad 0 \leq \vartheta \leq 1.$$

Moreover, if $1 \leq u, v, w$, one can take $C(u,v,w) = 4$.

Now, it is natural to extend the definition of $(S^{\vartheta}_{r,s})$ and to consider the family of the inequalities

$$\|f\|_{r,u} \le (CW(f))^{\vartheta} \|f\|^{1-\vartheta}_{s,v} \tag{4.4}$$

with $0 < r, s \le +\infty$, $\vartheta \in]0, 1]$, $0 < u, v \le +\infty$. We can again define the parameter $q = q(r, s, \vartheta) \in]-\infty, 0[\cup]0, +\infty[\cup \{\infty\}$ by setting

$$\frac{1}{r} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s}$$

as in (3.1). Moreover, we can define a second parameter $\gamma = \gamma(u, v, \vartheta)$ by setting

$$\frac{1}{u} = \frac{\vartheta}{\gamma} + \frac{1-\vartheta}{v}.$$

The Hölder-Lorentz inequality stated above induces a natural hierarchy in this family of functional inequalities. Moreover, any transformation using the Hölder-Lorentz inequality preserves the two parameters q and γ. Note that (4.4) easily implies the corresponding $(S^{*,\vartheta}_{r,s})$. More generally, (4.4) is weaker and weaker as u increases and v decreases. We will see below that, if $0 < q \le \infty$, the best parameter γ that can be achieved starting from the weak inequality $(S^{*,\vartheta}_{r,s})$ is equal to the smallest α such that (H^{ρ}_{α}) is satisfied by W for some $\rho > 1$.

Theorem 4.3 *Assume that $(S^{*,\vartheta_0}_{r_0,s_0})$ holds for some fixed r_0, $s_0 \in]0, +\infty]$ and $\vartheta_0 \in]0, 1]$, and that the parameter $q = q(r_0, s_0, \vartheta_0)$ defined at (3.1) satisfies $0 < q < +\infty$. Suppose that W satisfies (H^{ρ}_{α}) for some $\alpha > 0$ and some $\rho > 1$. Then, for all $f \in \mathcal{F}$,*

$$\|f\|_{q,\alpha} \le BW(f)$$

with

$$B = B(\alpha, \rho) = \rho^{q/r_0\vartheta_0}(\rho^{\alpha} - 1)^{1/\alpha}(\rho - 1)^{-1}A_{\alpha}(\rho)C_0$$

where C_0 is the constant appearing in $(S^{*,\vartheta_0}_{r_0,s_0})$. It follows that, for all $f \in \mathcal{F}$, each of the inequalities

$$\|f\|_{r,u} \le D(\vartheta, u, v, \rho)\left(\rho^{q/r_0\vartheta_0}(\rho - 1)^{-1}A_{\alpha}(\rho)C_0W(f)\right)^{\vartheta} \|f\|^{1-\vartheta}_{s,v}$$

is satisfied for all r, $s \in]0, +\infty]$, u, $v \in]0, +\infty]$, and $\vartheta \in]0, 1]$ such that

$$\frac{1}{r} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s}, \quad \frac{1}{u} \le \frac{\vartheta}{\alpha} + \frac{1-\vartheta}{v}.$$

Here, $D(\vartheta, u, v, \rho) = \rho^{1-\vartheta}(\rho^u - 1)^{1/u}(\rho^v - 1)^{-(1-\vartheta)/v}$.

In particular, if W satisfies (H^{ρ}_{q}) for some $\rho > 1$, the Sobolev inequality

$$\|f\|_q \le B(q, \rho)W(f)$$

is satisfied as well as any of the inequalities

$$\|f\|_r \leq (B(q,\rho)W(f))^{\vartheta}\|f\|_s^{1-\vartheta}$$

with $r,\ s \in]0,+\infty]$ *and* $1/r = \vartheta/q + (1-\vartheta)/s$.

Proof. A simple way to prove this theorem is to apply Theorem 4.1, which gives the weak Sobolev inequality

$$\|f\|_{q,\infty} \leq B(\rho)W(f).$$

Then, apply this statement to $f_k = f_{\rho,k}$ again to obtain

$$\rho^k \mu(f \geq \rho^{k+1})^{1/q} \leq B'(\rho)W(f_k).$$

Raise this inequality to the power α. The fact that W satisfies (H^ρ_α) allows us to sum over k and we get

$$\|f\|_{q,\alpha} \leq B''(\rho)W(f).$$

The rest of the theorem is obvious since the Sobolev inequality $\|f\|_{q,\alpha} \leq BW(f)$ implies all the other stated inequalities by a simple application of the Hölder (or Hölder–Lorentz) inequality. □

We write now in detail a slightly more complicated proof which is based on the same ideas and gives better constants. As in the proof of Theorem 4.1, apply $(S^{*,\vartheta_0}_{r_0,s_0})$ to f_k to get

$$\mu(f \geq \rho^{k+1})^{1/r_0} \leq ((\rho-1)^{-1}C_0 W(f_k))^{\vartheta_0}\rho^{-k\vartheta_0}\left(\mu(f \geq \rho^k)^{1/s_0}\right)^{1-\vartheta_0}.$$

Raise this inequality to the power $\alpha r_0/q$ and multiply then both sides by $\rho^{k\alpha}$ to obtain

$$\rho^{k\alpha}\mu(f \geq \rho^{k+1})^{\alpha/q} \leq \left((\rho-1)^{-1}C_0 W(f_k)\right)^{\alpha r_0\vartheta_0/q}\left(\rho^{k\alpha}\mu(f \geq \rho^k)^{\alpha/q}\right)^{(1-\vartheta_0)r_0/s_0}.$$

In this process, we have used the fact that $1/r_0 - \vartheta_0/q = (1-\vartheta_0)/s_0$. This identity can also be written as $1 = r_0\vartheta_0/q + r_0(1-\vartheta_0)/s_0$. Thus, summing over $k \in \mathbb{Z}$, using Hölder's inequality and the hypothesis (H^ρ_α), we get

$$\begin{aligned}
&\sum_k \rho^{\alpha(k+1)}\mu(f \geq \rho^{k+1})^{\alpha/q} \\
\leq\ &\rho^\alpha\left((\rho-1)^{-1}A_\alpha(\rho)C_0 W(f)\right)^{\alpha r_0\vartheta_0/q}\left(\sum_k \rho^{k\alpha}\mu(f \geq \rho^k)^{\alpha/q}\right)^{(1-\vartheta_0)r_0/s_0}.
\end{aligned}$$

After simplifications, this yields

$$N^{\rho}_{q,\alpha}(f) \le \rho^{q/r_0\vartheta_0}(\rho-1)^{-1}A_\alpha(\rho)C_0W(f).$$

By (4.2), the theorem follows. $\square$

Corollary 4.4 *Assume that* $(S^{*,\vartheta_0}_{r_0,s_0})$ *holds for some fixed* r_0*,* $s_0 \in]0,+\infty]$ *and* $\vartheta_0 \in]0,1]$*, and that the parameter* $q = q(r_0,s_0,\vartheta_0)$ *defined at (3.1) satisfies* $0 < q < +\infty$*. Suppose that* W *satisfies* (H_α) *for some* $\alpha > 0$*. Then, if* $0 < \alpha < 1$*, the set* $\mathcal{F}$ *of functions for which* $(S^{*,\vartheta_0}_{r_0,s_0})$ *is satisfied must be trivial (i.e.,* $\mathcal{F} = \{0\}$*). If* $\alpha = 1$*, for all* $f \in \mathcal{F}$*, we have*

$$\|f\|_{q,1} \le AC_0W(f)$$

where $A = \lim_{\rho\to 1} A_1(\rho)$*. If* $\alpha > 1$*, for all* $f \in \mathcal{F}$*, we have*

$$\|f\|_{q,\alpha} \le BA_\alpha C_0W(f)$$

with $B = e\left((1+q/r_0\vartheta_0) \wedge \alpha^{1/\alpha}(1+q/r_0\vartheta_0)^{1-1/\alpha}\right)$*. Here* C_0 *is the constant appearing in* $(S^{*,\vartheta_0}_{r_0,s_0})$*.*

Proof. When $0 < \alpha \le 1$, letting ρ tend to 1 in Theorem 4.2 gives the desired result. When $\alpha > 1$, one cannot let ρ tend to 1, but we choose $\rho = 1 + r_0\vartheta_0/q$. $\square$

Remark 4.5 In the Euclidean space $\mathbb{R}^n$, the Sobolev inequality with Lorentz norm

$$\|f\|_{q,p} \le C\|\nabla f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n)$$

with $1 \le p < n$ and $q = np/(n-p)$ can be found in [23] or [25]. We claim that the Lorentz exponent p in the left hand side of this inequality is optimal. To see this, let $\gamma > 0$ be specified later, and set $r_0 = 0$, $r_k = 2^{qk/n}k^{-\gamma}$ for $k \ge 1$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $f(x) = g(|x|)$ where $g : \mathbb{R} \to \mathbb{R}$ is continuous, decreasing, piecewise linear, and such that

$$g(0) = 1, \quad g' = -2^{-k-1}(r_{k+1}-r_k)^{-1} \ \text{ on } \ [r_k,\ r_{k+1}[, \quad k \ge 0.$$

Hence, $g(r_k) = 2^{-k}$ and the set $\{f \ge 2^{-k}\}$ is just the Euclidean ball with center the origin and radius r_k. Clearly,

$$\|\nabla f\|_p^p \simeq \sum_{k\ge 1} k^{-\gamma(n-p)}$$

and, for every $b > 0$,

$$\|f\|_{q,b}^b \simeq \sum_{k\ge 1} k^{-\gamma bn/q}.$$

If $b < p$, in other words, if $bn/q < n-p$, there exists $\gamma > 0$ such that $\gamma(n-p) > 1$ but $\gamma bn/q \le 1$. The claim follows. This example works as well on more general metric spaces where the balls of radius r have volume of order r^n.

5. The case $-\infty < q < 0$. This section contains the proof of Theorem 3.2. It is based on an idea we learned from Carron [6]. More precisely, we will prove:

Theorem 5.1 *Assume that $(S^{*,\vartheta_0}_{r_0,s_0})$ holds for some fixed $r_0, s_0 \in]0,+\infty]$ and $\vartheta_0 \in]0,1]$, and that the parameter $q = q(r_0, s_0, \vartheta_0)$ defined at (3.1) satisfies $-\infty < q < 0$. Assume that W satisfies (H^+_∞). Then, for each $0 < s < +\infty$ and all $f \in \mathcal{F}$,*

$$\|f\|_\infty \le B\left(A^+_\infty C_0 W(f)\right)^{1/(1-s/q)} \|f\|_s^{1/(1-q/s)}.$$

where $B = e(2 - q(1-\vartheta_0)/s_0\vartheta_0)$ and C_0 is the constant appearing in $(S^{*,\vartheta_0}_{r_0,s_0})$. It follows that, for all $0 < s < r < +\infty$ and all $f \in \mathcal{F}$,

$$\|f\|_r \le (B^{1-s/q} A^+_\infty C_0 W(f))^\vartheta \|f\|_s^{1-\vartheta}$$

where $1/r = \vartheta/q + (1-\vartheta)/s$.

Proof. Fix $f \in \mathcal{F}$ with $\|f\|_\infty \ne 0$. Fix also $\varepsilon > 0$ small enough and $\rho > 1$. Consider the functions

$$f'_{\rho,k} = f'_k = (f - (\|f\|_\infty - \varepsilon - \rho^k))^+ \wedge \rho^{k-1}(\rho - 1)$$

for all $k \le k(f)$ where $k(f)$ is the largest integer k such that $\rho^k < \|f\|_\infty$. Note that $f'_k \in \mathcal{F}$ if $0 < \varepsilon < \|f\|_\infty - \rho^{k(f)}$ and that (H^+_∞) implies $W(f'_k) \le A^+_\infty W(f)$ for all $k \le k(f)$. Set

$$\lambda_k = \lambda_k(\varepsilon) = \|f\|_\infty - \varepsilon - \rho^k.$$

Observe that f'_k is supported on $\{f \ge \lambda_k\}$ and that $\{f'_k \ge \rho^{k-1}(\rho-1)\} = \{f \ge \lambda_{k-1}\}$. If we apply $(S^{*,\vartheta_0}_{r_0,s_0})$ to f'_k, we get

$$\begin{aligned}\rho^{k-1}\mu(f \ge \lambda_{k-1})^{1/r_0} \quad &\le \quad \left((\rho-1)^{-1} A^+_\infty C_0 W(f)\right)^{\vartheta_0} \\ &\qquad \cdot \rho^{(k-1)(1-\vartheta_0)} \mu(f \ge \lambda_k)^{(1-\vartheta_0)/s_0}.\end{aligned}$$

Multiply this inequality by $\rho^{\delta(k-1)}$ and write

$$\begin{aligned}\left(\rho^{r_0(k-1)(1+\delta)}\mu(f \ge \lambda_{k-1})\right)^{1/r_0} \quad &\le \quad \left((\rho-1)^{-1} A^+_\infty C_0 W(f)\right)^{\vartheta_0} \\ &\qquad \cdot \rho^{s_0(k-1)(1+\delta/(1-\vartheta_0))} \mu(f \ge \lambda_k)\big)^{(1-\vartheta_0)/s_0}.\end{aligned}$$

Now, choose δ so that

$$r_0(1+\delta) = s_0(1+\delta/(1-\vartheta_0)).$$

After some algebra, this condition implies $r_0(1+\delta) = q < 0$. Setting $a_k = \rho^{qk}\mu(f \geq \lambda_k)$ yields

$$a_{k-1}^{1/r_0} \leq \rho^{-q(1-\vartheta_0)/s_0} \left((\rho-1)^{-1}A_\infty^+ C_0 W(f)\right)^{\vartheta_0} a_k^{(1-\vartheta_0)/s_0} \tag{5.1}$$

for all $k \leq k(f)$. Observe that $a_k > 0$ for all $k \leq k(f)$ and that $\lim_{k\to-\infty} a_k = +\infty$ because $\mu(f \geq \|f\|_\infty - \varepsilon) > 0$ (this is the reason for the introduction of the parameter ε). Thus, the quantity

$$a = \inf_{k \leq k(f)} a_k,$$

is positive and (5.1) yields

$$a^{1/r_0} \leq \rho^{-q(1-\vartheta_0)/s_0} \left((\rho-1)^{-1}A_\infty^+ C_0 W(f)\right)^{\vartheta_0} a^{(1-\vartheta_0)/s_0}$$

whence

$$a = \inf_{k \leq k(f)} \{\rho^{qk}\mu(f \geq \lambda_k)\} \geq \rho^{-q^2(1-\vartheta_0)/s_0\vartheta_0} \left((\rho-1)^{-1}A_\infty^+ C_0 W(f)\right)^q.$$

Using the inequality $\lambda^s \mu(f \geq \lambda) \leq \left\|f\right\|_s^s$, we get, for all $k \leq k(f)$,

$$\lambda_k^{-s}\rho^{qk}\left\|f\right\|_s^s \geq (\rho-1)^{-q}\rho^{-q^2(1-\vartheta_0)/s_0\vartheta_0}(A_\infty^+ C_0 W(f))^q.$$

Recall that $\lambda_k = \lambda_k(\varepsilon)$ is a function of the small parameter $\varepsilon > 0$. Since the inequality above holds true for all $\varepsilon > 0$ small enough, we can let ε tend to zero. Choosing $k = k(f) - 1$, and observing that $\lambda_k(0) \geq \rho^k(\rho-1) \geq \rho^{-2}(\rho-1)\|f\|_\infty$, we obtain

$$\left\|f\right\|_\infty^{-s+q}\left\|f\right\|_s^s \geq (\rho-1)^{s-q}\rho^{-2s+2q-q^2(1-\vartheta_0)/s_0\vartheta_0}(A_\infty^+ C_0 W(f))^q$$

and thus

$$\|f\|_\infty \leq \rho^2(\rho-1)^{-1}\left(\rho^{-q(1-\vartheta_0)/s_0\vartheta_0} A_\infty^+ C_0 W(f)\right)^{1/(1-s/q)}\left\|f\right\|_s^{1/(1-q/s)}.$$

Since W satisfies (H_∞^+), pick

$$\rho = 1 + \left[2 + \frac{-q(1-\vartheta_0)}{s_0\vartheta_0(1-s/q)}\right]^{-1}.$$

This choice yields the desired inequality. $\square$

Remark 5.2 The proof above shows that one can use $\|f\|_{s,\infty}$ instead of $\|f\|_s$ in deriving the main inequality. Thus, under the hypothesis of Theorem 5.1, one can conclude that

$$\|f\|_{r,u} \leq D(A_\infty^+ C_0 W(f))^\vartheta \|f\|_{s,v}^{1-\vartheta}$$

where $\vartheta \in]0,1]$ is defined by $1/r = \vartheta/q + (1-\vartheta)/s$ and where $0 < v \leq u \leq +\infty$ satisfy $1/u \leq (1-\vartheta)/v$. Here D depends on all the parameters.

6. The case $q = \infty$**.** This section treats the case corresponding, in Euclidean space, to the Trudinger inequality. It relies on the technique already used in the two preceding sections. However, two technical aspects play important parts here. First, using Lorentz norms instead of merely Lebesgue norms is crucial. Second, using all $\rho > 1$ and following the constants carefully is also crucial. The referee pointed out to us that Lorentz spaces are also used in [4] to prove Trudinger–type inequalities in $\mathbb{R}^n$.

Theorem 6.1 *Assume that* $(S_{r_0,s_0}^{*,1-s_0/r_0})$ *holds for some fixed* r_0, s_0 *such that* $0 < s_0 < r_0 \leq +\infty$. *Assume that* W *satisfies* (H_α^ρ) *for some* $\rho > 1$ *and some* $0 < \alpha \leq +\infty$. *Then, for all* $0 < s < r < +\infty$ *and all* $f \in \mathcal{F}$,

$$\|f\|_{r,u} \leq \frac{\rho^{s/r}(\rho^u - 1)^{1/u}}{(\rho^v - 1)^{s/rv}} \left(\rho^{(r_0+r)/(r_0-s_0)}(\rho-1)^{-1}A_\alpha(\rho)C_0 W(f)\right)^{1-s/r} \|f\|_{s,v}^{s/r}$$

for all $0 < v, u \leq +\infty$ *such that* $1/u \leq (1-s/r)/\alpha + (s/r)/v$. *Here* C_0 *is the constant appearing in* $(S_{r_0,s_0}^{*,\vartheta_0})$.

Proof. Fix $f \in \mathcal{F}$ and consider the functions $f_k = (f - \rho^k)^+ \wedge \rho^k(\rho - 1)$ where $\rho > 1$ is fixed. Applying $(S_{r_0,s_0}^{*,1-s_0/r_0})$ to each f_k, we get

$$(\rho-1)\rho^k \mu(f \geq \rho^{k+1})^{1/r_0} \leq (C_0 W(f_k))^{1-s_0/r_0} \left((\rho-1)\rho^k \mu(f \geq \rho^k)^{1/s_0}\right)^{s_0/r_0}$$

and thus

$$(\rho-1)^{r_0}\rho^{kr_0}\mu(f \geq \rho^{k+1}) \leq (C_0 W(f_k))^{r_0-s_0}(\rho-1)^{s_0}\rho^{ks_0}\mu(f \geq \rho^k).$$

Multiply both sides of this inequality by ρ^{tk} for some fixed $t \geq 0$ and raise the resulting inequality to the power $1/(r_0+t)$. Setting $r_t = r_0 + t$, $s_t = s_0 + t$, we get

$$\begin{aligned} &\rho^{k+1}\mu(f \geq \rho^{k+1})^{1/r_t} \\ &\quad\leq \left(\rho^{r_t/(r_0-s_0)}(\rho-1)^{-1}C_0 W(f_k)\right)^{1-s_t/r_t} \left(\rho^k \mu(f \geq \rho^k)^{1/s_t}\right)^{s_t/r_t}. \end{aligned} \tag{6.1}$$

Consider any (v_t, u_t) such that $0 < v_t, u_t \leq +\infty$ and

$$\frac{1}{u_t} = \frac{1 - s_t/r_t}{\alpha} + \frac{s_t/r_t}{v_t}. \tag{6.2}$$

If we raise (6.1) to the power u_t, sum over $k \in \mathbb{Z}$, and use Hölder's inequality and the hypothesis (H_α^ρ), we obtain

$$\begin{aligned} N^\rho_{r_t,u_t}(f) \quad &\leq \quad \left(\rho^{r_t/(r_0 - s_0)}(\rho - 1)^{-1} A_\alpha(\rho) C_0 W(f)\right)^{1 - s_t/r_t} \\ &\qquad \cdot \left(N^\rho_{s_t,v_t}(f)\right)^{s_t/r_t}. \end{aligned} \tag{6.3}$$

Now, fix r and s satisfying $0 < s < r < +\infty$. Fix also $0 < u, v < +\infty$ such that

$$\frac{1}{u} = \frac{1 - s/r}{\alpha} + \frac{s/r}{v}. \tag{6.4}$$

Choose $t = \max\{0\ ,\ r - r_0\ ,\ s - s_0\}$. Thus, $0 < s \leq s_t < r_t$, $0 < s < r \leq r_t < r_0 + r$ and there exist $0 \leq \varepsilon, \eta \leq 1$ such that

$$\frac{1}{r} = \frac{\varepsilon}{s} + \frac{1 - \varepsilon}{r_t} \quad , \quad \frac{1}{s_t} = \frac{\eta}{s} + \frac{1 - \eta}{r_t}.$$

Then, for appropriate choices of u_t, v_t (depending on u, v) satisfying (6.2), Hölder's inequality and (6.3) yield

$$N^\rho_{r,u}(f) \leq \left(\rho^{(r_0 + r)/(r_0 - s_0)}(\rho - 1)^{-1} A_\alpha(\rho) C_0 W(f)\right)^{1 - s/r} \left(N^\rho_{s,v}(f)\right)^{s/r}$$

and, by (4.2),

$$\|f\|_{r,u} \leq \frac{\rho^{s/r}(\rho^u - 1)^{1/u}}{(\rho^v - 1)^{s/rv}} \left(\rho^{(r_0 + r)/(r_0 - s_0)}(\rho - 1)^{-1} A_\alpha(\rho) C_0 W(f)\right)^{1 - s/r} \|f\|^{s/r}_{s,v}.$$

This is the desired inequality. $\square$

To go further observe that, for any $x > 1$,

$$\begin{aligned} \gamma(x-1)x^{\gamma-1} \leq x^\gamma - 1 \leq \gamma(x-1) \quad &\text{if } 0 < \gamma \leq 1 \\ \gamma(x-1) \leq x^\gamma - 1 \leq \gamma(x-1)x^{\gamma-1} \quad &\text{if } 1 \leq \gamma. \end{aligned}$$

A case by case analysis depending on the positions of u and v with respect to 1 shows that

$$\frac{(\rho^u - 1)^{1/u}}{(\rho^v - 1)^{s/rv}} \leq \frac{u^{1/u}}{v^{s/rv}} \rho^{(1 \vee 1/u)} (\rho - 1)^{1/u - s/rv}.$$

Using (6.4), we obtain

$$\begin{aligned} \|f\|_{r,u} \quad &\leq \quad \frac{u^{1/u}}{v^{s/rv}} \rho^{1 + (1 \vee 1/u)} \\ &\qquad \cdot \left(\rho^{(r_0 + r)/(r_0 - s_0)}(\rho - 1)^{-1 + 1/\alpha} A_\alpha(\rho) C_0 W(f)\right)^{1 - s/r} \|f\|^{s/r}_{s,v}. \end{aligned}$$

At this point the reader might wonder why we are carrying out such precise and dull computations involving Lorentz norms and the parameter ρ, but here is the reward. When the hypothesis (H_α^ρ) is satisfied uniformly in ρ, we can either let ρ tend to 1 (when $0 < \alpha \leq 1$) or pick $\rho = 1 + 1/[1 \vee r]$ (when $1 < \alpha \leq +\infty$), and the inequality above yields the following result:

Corollary 6.2 *Assume that* $(S_{r_0,s_0}^{*,1-s_0/r_0})$ *holds for some fixed* r_0, s_0 *such that* $0 < s_0 < r_0 < +\infty$, *and some class* $\mathcal{F}$ *of functions. Assume that* W *satisfies* (H_α) *for some* $0 < \alpha \leq +\infty$.

(1) *If* $0 < \alpha < 1$, *the class* $\mathcal{F}$ *must be trivial (i.e.,* $\mathcal{F} = \{0\}$*).*
(2) *If* $\alpha = 1$, *for all* $0 < s < r < +\infty$, *all* $0 < v, u \leq +\infty$ *such that* $1/u = (1 - s/r) + s/rv$, *and all* $f \in \mathcal{F}$,

$$\|f\|_{r,u} \leq \frac{u^{1/u}}{v^{s/rv}} \left(AC_0 W(f)\right)^{1-s/r} \|f\|_{s,v}^{s/r},$$

where $A = \lim_{\rho \to 1} A_1(\rho)$.
(3) *Finally, if* $\alpha > 1$, *for all* $0 < s < r < +\infty$, *all* $0 < v, u \leq +\infty$ *such that* $1/u = (1 - s/r)/\alpha + s/rv$, *and all* $f \in \mathcal{F}$,

$$\|f\|_{r,u} \leq \frac{u^{1/u}}{v^{s/rv}} 2^{1+(1\vee 1/u)} \left(e^{(1+r_0)/(r_0-s_0)} [1\vee r]^{1-1/\alpha} A_\alpha C_0 W(f)\right)^{1-s/r} \|f\|_{s,v}^{s/r}.$$

From this, we deduce the following:

Corollary 6.3 *Assume that* $(S_{r_0,s_0}^{*,1-s_0/r_0})$ *holds for some fixed* r_0, s_0 *such that* $0 < s_0 < r_0 < +\infty$, *and some class* $\mathcal{F}$ *of functions. Assume that* W *satisfies* (H_α) *for some* $1 \leq \alpha \leq +\infty$.

(1) *If* $\alpha = 1$, *we have*

$$\|f\|_\infty \leq AC_0 W(f)$$

for all $f \in \mathcal{F}$ *where* $A = \lim_{\rho \to 1} A_1(\rho)$.
(2) *If* $1 < \alpha \leq +\infty$, *then, for any* $0 < s < +\infty$ *and any integer* $k \geq (\alpha - 1)s/\alpha$,

$$\int_E \exp_k \left([b|f|/CW(f)]^{\alpha/(\alpha-1)} \right) d\mu \leq B \left[\|f\|_s / CW(f) \right]^s$$

for all $f \in \mathcal{F}$. *Here,* $\exp_k(x) = \sum_{\ell=k}^{+\infty} x^\ell/\ell!$, $C = DA_\alpha C_0$ *where* D *is a numerical constant, and* B, $b > 0$ *depend only on* α.

Proof. To obtain the first statement, fix s, $v > 0$ and let r tend to infinity in Part 2 of Theorem 6.2.

In order to obtain the second statement, first observe that, for $\alpha \geq 1$, $1 \leq s < 2s \leq r$, $v = s$ and $u \leq r$ given by $1/u = (1 - s/r)/\alpha + 1/r \leq 2$, (4.3) and Corollary 6.2 yield

$$\|f\|_r \leq 4^{1/u}\|f\|_{r,u} \leq \left(r^{1-1/\alpha} D A_\alpha C_0 W(f)\right)^{1-s/r} \|f\|_s^{s/r}$$

where D is a numerical constant that may change from line to line. Using Hölder's inequality as indicated at the begining of Section 3 yields

$$\|f\|_r \leq \left((1 \vee r)^{1-1/\alpha} D A_\alpha C_0 W(f)\right)^{1-s/r} \|f\|_s^{s/r}$$

for all $0 < s \leq r < +\infty$. Using the notation $C = DA_\alpha C_0$ introduced in Corollary 6.3, rewrite this inequality as

$$\left[\frac{\|f\|_r}{CW(f)}\right]^r \leq (1 \vee r)^{(1-1/\alpha)(r-s)} \left[\frac{\|f\|_s}{CW(f)}\right]^s . \tag{6.5}$$

Now, fix $s > 0$ and let $k = \lceil (\alpha - 1)s/\alpha \rceil$. For $\ell \geq k \geq 1$, set $r = r(\ell) = \ell\alpha/(\alpha - 1)$ and observe that $r \geq s$ and $r \geq 1$. Thus, for each $\ell \geq k$, inequality (6.5) gives

$$\int \left[\frac{|f|}{CW(f)}\right]^{\ell\alpha/(\alpha-1)} d\mu \leq \left(\frac{\alpha}{\alpha - 1}\right)^{\ell} \ell^{\ell} \left[\frac{\|f\|_s}{CW(f)}\right]^s .$$

By the definition of $\exp_k$, this clearly yields the desired bound if $b = b(\alpha)$ is chosen small enough. □

For functions in $\mathcal{F}$ that are supported in a given set Ω with $\mu(\Omega) < +\infty$, letting s tend to 0 in the last statement of Corollary 6.3 yields the following result:

Corollary 6.4 *Assume that* $(S^{*,1-s_0/r_0}_{r_0,s_0})$ *holds for some fixed* r_0, s_0 *such that* $0 < s_0 < r_0 < +\infty$. *Assume that* W *satisfies* (H_α) *for some* $1 < \alpha \leq +\infty$. *Then, for all sets* Ω *of finite measure and all functions* $f \in \mathcal{F}$ *supported in* Ω,

$$\int_\Omega \exp\left([bf/CW(f)]^{\alpha/(\alpha-1)}\right) d\mu \leq B\mu(\Omega)$$

where B, b, *and* C *are as in Corollary 6.3.*

7. Non local gradients satisfy (H^ρ_α)**.** Section 2 introduced the basic assumptions (H^ρ_α) that were used in Sections 3, 4, 5, and 6. As already noticed, these are readily satisfied by semi-norm W derived from a gradient on a manifold. This section presents further examples. In particular, it shows that certain non-local semi-norms W satisfy (H^ρ_α).

First, observe that, without a differentiable structure, one can still define the quantity $\nabla f(x)$ for any Lipschitz function on a metric space (E,d) by setting

$$\nabla f(x) = \limsup_{y\to x} \frac{|f(x)-f(y)|}{d(x,y)}, \quad x \in E.$$

In all these cases,

$$W(f) = W_p(f) = \left(\int |\nabla f|^p \, d\mu\right)^{1/p}$$

satisfies (H_∞^+) with $A_\infty^+ = 1$ as well as (H_α^ρ) for every $\alpha \in [p,+\infty]$ and any $\rho > 1$ with $A_\alpha(\rho) = 1$.

An interesting setting where the above does not directly apply is when E is the vertex-set of a connected, locally finite non-oriented graph. Let $m(x)$ be the number of neighbors of x and set $K(x,y) = 1/m(x)$ if x and y are neighbors and $K(x,y) = 0$ otherwise. For each $0 < p \le +\infty$, one can define (the length of) a gradient by setting

$$\nabla_p f(x) = \Big(\sum_y |f(x)-f(y)|^p K(x,y)\Big)^{1/p}, \quad x \in E.$$

Of course, when the graph is uniformly locally finite (i.e. $\sup_x m(x) < +\infty$), the parameter p does not play any significant role in this definition. More generally, consider a non-negative kernel K (not necessarily a Markov kernel) on a σ-finite measure space $(E,\mathcal{E},\mu)$. In other words, for each $x \in E$, $K(x,\cdot\,)$ is a non-negative σ-finite measure on $(E,\mathcal{E})$ and $K(\,\cdot\,,A)$ is a measurable function on E for every $A \in \mathcal{E}$. In most cases of interest, $d\mu(x)K(x,dy)$ will be a symmetric measure on $E\times E$, but it is not necessary to assume so. Fix $p > 0$ and set

$$\nabla_p f(x) = \left(\int_E |f(x)-f(y)|^p K(x,dy)\right)^{1/p}, \quad x \in E.$$

Here, the set $\mathcal{F}$ must be chosen so that the integral above exists for all $f \in \mathcal{F}$. Then, we have the following result:

Lemma 7.1 *Let K be a non-negative kernel on $(E,\mathcal{E},\mu)$ and fix $0 < p \le a$. Set*

$$W_{p,a}(f) = \|\nabla_p f\|_a = \left(\int_E \left(\int_E |f(x)-f(y)|^p K(x,dy)\right)^{a/p} d\mu(x)\right)^{1/a}.$$

If $p \ge 1$, $W_{p,a}$ satisfies (H_∞^+) with $A_\infty^+ = 1$. It also satisfies (H_α) for every $\alpha \in [a,+\infty]$ with $A_\alpha = (1+2^{a/p-1}(1+(2p)^{a/p}))^{1/a}$.

When $0 < p < 1$, $W_{p,a}$ satisfies (H_∞^+) with $A_\infty^+ = 1$ and satisfies (H_α^ρ) with $A_\alpha(\rho) = (1+2^{a/p-1}(1+(2/p)^{a/p}(1-1/\rho)^{a(1-1/p)}))^{1/a}$ for all $\alpha \in [a,+\infty[$ and all $\rho > 1$.

Proof. Fix some appropriate class $\mathcal{F}$ of non-negative functions, and fix $\rho > 1$. For every $k \in \mathbb{Z}$, recall that $f_k = f_{\rho,k} = (f - \rho^k)^+ \wedge \rho^k(\rho - 1)$ and set $B_k = \{\rho^k < f \le \rho^{k+1}\}$. Since the map $t \to (t - \rho^k)^+ \wedge \rho^k(\rho - 1)$ is a contraction, for every x, $y \in E$ and every $k \in \mathbb{Z}$,

$$|f_k(x) - f_k(y)| \le |f(x) - f(y)| .$$

It follows that $\nabla_p f_k \le \nabla_p f$ and thus

$$\sum_k \int_{B_k} (\nabla_p f_k)^a \, d\mu \le \sum_k \int_{B_k} (\nabla_p f)^a \, d\mu = \int_E (\nabla_p f)^a \, d\mu. \tag{7.1}$$

Let us now consider $\int_{B_k^c} (\nabla_p f_k)^a \, d\mu$. For every k,

$$\begin{aligned}
\int_{B_k^c} (\nabla_p f_k)^a \, d\mu \;&\le\; 2^{a/p-1} \Bigg\{ \int_{B_k^c} \left(\int_{B_k} |f_k(x) - f_k(y)|^p K(x, dy) \right)^{a/p} d\mu(x) \\
&\qquad + \int_{B_k^c} \left(\int_{B_k^c} |f_k(x) - f_k(y)|^p K(x, dy) \right)^{a/p} d\mu(x) \Bigg\} \\
&= \; 2^{a/p-1} (J_1(k) + J_2(k)).
\end{aligned}$$

For the first term we use the fact that $a/p \ge 1$ to obtain

$$\begin{aligned}
\sum_k J_1(k) \;&\le\; \int_E \left(\sum_k \int_{B_k} |f(x) - f(y)|^p K(x, dy) \right)^{a/p} d\mu(x) \\
&= \; \int (\nabla_p f)^a \, d\mu.
\end{aligned} \tag{7.2}$$

To deal with the second term, we observe that, in the region $B_k^c \times B_k^c$, $|f_k(x) - f_k(y)| = 0$ unless $(x, y) \in Z_k \cup Z_k^*$ where

$$Z_k = \{(x, y) : f(y) \le \rho^k < f(x)/\rho\}$$

and Z_k^* is the symmetric of Z_k. Moreover, on Z_k or Z_k^*, we have $|f_k(x) - f_k(y)| = \rho^k(\rho - 1)$. Therefore,

$$\sum_k J_2(k) = \int_E \left(\int_E \left(\sum_k (\rho - 1)^p \rho^{pk} (1_{Z_k}(x, y) + 1_{Z_k^*}(x, y)) \right) K(x, dy) \right)^{a/p} d\mu(x).$$

For every $(x, y) \in Z_k$, let $k_1 \ge k_2$ be the integers such that

$$\rho^{k_1} < f(x)/\rho \le \rho^{k_1+1} \quad \text{and} \quad \rho^{k_2-1} < f(y) \le \rho^{k_2}.$$

Then,

$$\sum_k (\rho-1)^p \rho^{pk} 1_{Z_k}(x,y) = (\rho-1)^p \sum_{k=k_2}^{k_1} \rho^{pk} = \frac{(\rho-1)^p}{\rho^p-1}(\rho^{p(k_1+1)} - \rho^{pk_2})$$

and $f(x) - f(y) > \rho^{k_1+1} - \rho^{k_2}$. Consider first the case where $p \geq 1$. Since $k_1 \geq k_2$, we have $\rho^{k_1+1} - \rho^{k_2} \geq \rho^{k_1+1}(1-\rho^{-1})$ and

$$\begin{aligned} \rho^{p(k_1+1)} - \rho^{pk_2} &\leq p\,(\rho^{k_1+1} - \rho^{k_2})\rho^{(p-1)(k_1+1)} \\ &\leq p\,\frac{\rho^{p-1}}{(\rho-1)^{p-1}}|f(x) - f(y)|^p. \end{aligned}$$

It follows that

$$\sum_k (\rho-1)^p \rho^{pk} 1_{Z_k}(x,y) \leq p|f(x) - f(y)|^p.$$

Reversing the roles of x and y shows that we also have

$$\sum_k \rho^{pk} 1_{Z_k^*}(x,y) \leq p|f(x) - f(y)|^p.$$

Therefore,

$$\sum_k J_1(k) \leq (2p)^{a/p} \int_E (\nabla_p f)^p \, d\mu.$$

Together with (7.1) and (7.2), this proves Lemma 7.1 when $p \geq 1$. Now, if $0 < p < 1$, we have $\rho^{p(k_1+1)} - \rho^{pk_2} \leq (\rho^{k_1+1} - \rho^{k_2})^p$ and thus

$$\sum_k (\rho-1)^p \rho^{pk} 1_{Z_k}(x,y) \leq \frac{(\rho-1)^p}{\rho^p - 1}|f(x) - f(y)|^p.$$

In this case, using the inequality $\rho^p - 1 \geq p\,(\rho-1)\rho^{p-1}$, we get

$$\sum_k J_1(k) \leq (2/p)^{a/p}(1-1/\rho)^{a(1-1/p)} \int_E (\nabla_p f)^p \, d\mu$$

and the desired result follows as before. □

As an example, consider the case of a metric space (E, d) endowed with a σ-finite Borel measure μ. Set $V(x,y) = \mu(B(x, d(x,y)))$ and, for fixed $1 \leq p \leq +\infty$ and $0 < \gamma \leq 1$, define

$$K(x, dy) = d(x,y)^{-\gamma p} V(x,y)^{-1} \, d\mu(y).$$

Then,

$$W^{\gamma}_{p,p}(f)=\left(\int_E\int_E\left(\frac{|f(x)-f(y)|}{d(x,y)^{\gamma}V(x,y)^{1/p}}\right)^p d\mu(y)\,d\mu(x)\right)^{1/p}$$

satisfies (H_α) for all $\alpha\in[p,+\infty]$. When the metric space is the n-dimensional Euclidean space and μ is the Lebesgue measure, we get the classical Besov semi-norm

$$W^{\gamma}_{p,p}(f)=\left(c\int_{\mathbb{R}^n}\left(\frac{\|f(\,\cdot\,)-f(\,\cdot\,+h)\|_p}{|h|^{\gamma}}\right)^p\frac{dh}{|h|^n}\right)^{1/p}.$$

We now state a variant of Lemma 7.1 whose proof is similar and will be omitted.

Lemma 7.2 *Let K be a non-negative kernel on $(E,\mathcal{E},\mu)$. Fix $p>1$. Set*

$$\mathcal{D}_p(f)=\left(\int_{E\times E}(f^{p-1}(x)-f^{p-1}(y))(f(x)-f(y))K(x,dy)\,d\mu(x)\right)^{1/p}$$

where f belongs to some appropriate class $\mathcal{F}$ of non-negative functions. Then, $\mathcal{D}_p(f)$ satisfies (H_α) for every $\alpha\in[p,+\infty[$ for some finite constant A_α and (H^+_∞) with $A^+_\infty=1$.

Corollary 7.3 *Let P_t be a symmetric Markov semigroup acting on the spaces $L^p(E,\mu)$. Denote by $-\mathcal{L}$ the infinitesimal generator of P_t. Then, on its domain, the Dirichlet norm $\mathcal{D}(f)=\langle f,\mathcal{L}f\rangle^{1/2}$ satisfies (H_α) for all $\alpha\in[2,+\infty[$ with $A_\alpha=\sqrt{6}$ and satisfies (H^+_∞) with $A^+_\infty=1$. More generally, for $1<p<+\infty$, $\mathcal{D}_p(f)=\langle f^{p-1},\mathcal{L}f\rangle^{1/p}$ satisfies (H_α) for all $\alpha\in[p,+\infty[$ and satisfies (H^+_∞) with $A^+_\infty=1$.*

Proof. The quantity $\mathcal{D}_p(f)^p=\langle f^{p-1},\mathcal{L}f\rangle$ satisfies

$$\mathcal{D}_p(f)^p=\lim_{t\to 0}\frac{1}{2t}\int_{E\times E}(f^{p-1}(x)-f^{p-1}(y))(f(x)-f(y))P_t(x,dy)\,d\mu(x)$$

where $P_t(x,dy)$ is the Markov kernel of the semigroup P_t. More precisely, take $\mathcal{F}=\mathcal{F}_p$ to be the set of non-negative functions in $L^p(E,\mu)$ such that the limit above exists. The stated result then follows from Lemma 7.2. When P_t is a diffusion semigroup and $f\in\mathcal{F}$, we have $\mathcal{D}_p(f)^p=(p-1)\int_E\Gamma(f^{p/2},f^{p/2})\,d\mu$ where Γ is the "carré du champ". See [1]. □

8. Moving from one case to another. In practice, it often happens that we are dealing not only with one semi-norm W but with a family of semi-norms W_p, $1 \le p \le +\infty$, such that, for each p, W_p satisfies (H_∞^+) and (H_p). For instance, this is the case if

$$W_p(f) = \left(\int_M |\nabla f|^p \, d\mu \right)^{1/p}$$

for any local gradient ∇ on a manifold, or if

$$W_p(f) = \left(\int_M \int_M |f(x) - f(y)|^p K(x,y) \, d\mu(y) \, d\mu(x) \right)^{1/p}$$

for some fixed non-negative kernel K such that

$$\int_E K(x,y) \, d\mu(y) + \int_E K(x,y) \, d\mu(x) \le C < +\infty.$$

In these two (most important) cases, the following property is satisfied:

$$\forall\, p \in [1, +\infty], \;\; \forall\, \gamma \ge 1, \;\; W_p(f^\gamma) \le C(p,\gamma) \|f\|_{\gamma p}^{\gamma - 1} W_{\gamma p}(f). \tag{8.1}$$

In fact, for a local gradient ∇,

$$|\nabla (f^\gamma)| \le \gamma |f|^{\gamma-1} |\nabla f|$$

which, by Hölder's inequality, yields (8.1) with $C(p,\gamma) = \gamma^{1/p}$. In the non local case, use the estimate

$$|u^\gamma - v^\gamma| \le \gamma |u - v| (u^{\gamma-1} + v^{\gamma-1})$$

to see that (8.1) is again satisfied. Note that (8.1) is only a special case of a more general inequality that can be obtained by using the Hölder inequality with different parameters. It has been chosen for its simplicity and because it suffices for our purpose.

Theorem 8.1 *Assume that we are given a family $\mathcal{W}$ of semi-norms W_p, $1 \le p \le +\infty$, defined on a set $\mathcal{F}$ of non-negative functions such that, for each p, W_p satisfies (H_∞^+) and (H_p). Assume further that $f \in \mathcal{F}$ implies $f^\gamma \in \mathcal{F}$ for all $\gamma \ge 1$ and that $\mathcal{W}$ satisfies* (8.1). *Finally, assume that there exist $p_0 \in [1, +\infty[$, $s_0, r_0 \in \,]0, +\infty]$ and $\vartheta_0 \in \,]0, 1]$ such that W_{p_0} satisfies $(S_{r_0,s_0}^{*,\vartheta_0})$. Let $q(p_0)$ be defined by*

$$1/r_0 = \vartheta_0 / q(p_0) + (1 - \vartheta_0)/s_0$$

and assume that $1/q(p_0) \le 1/p_0$ (recall that $q(p_0)$ may be negative). Define $n \in [0, +\infty]$ by setting

$$1/q(p_0) = 1/p_0 - 1/n.$$

Then, for any $p_0 \le p \le +\infty$, W_p *satisfies* $(S^{\vartheta}_{r,s})$ *for all* $\vartheta \in]0,1]$ *and all* $r, s \in]0,+\infty]$ *such that*

$$1/r = \vartheta/q(p) + (1-\vartheta)/s$$

where $q(p)$ *is defined by* $1/q(p) = 1/p - 1/n$.

Proof. Fix $p > p_0$ and set $\gamma = p/p_0$. Using the results stated in Section 3, we can assume without loss of generality that W_{p_0} satisfies $(S^{\vartheta_1}_{p_0,s_1})$ where

$$\frac{1}{p_0} = \frac{\vartheta_1}{q(p_0)} + \frac{1-\vartheta_1}{s_1}$$

(here, in the case where $0 < q(p_0) < +\infty$, we use the hypothesis that $1/q(p_0) \le 1/p_0$). Then, (8.1) implies that W_p satisfies $(S^{\vartheta_2}_{p,s_2})$ with $s_2 = \gamma s_1$ and $\vartheta_2 = \vartheta_1/(\vartheta_1 + \gamma(1-\vartheta_1))$. It follows that $(1-\vartheta_1)/\vartheta_1 = (1-\vartheta_2)/\gamma\vartheta_2$ and

$$\begin{aligned} \frac{1}{n} &= \frac{1}{p_0} - \frac{1}{q(p_0)} = \frac{1-\vartheta_1}{\vartheta_1}\left(\frac{1}{s_1} - \frac{1}{p_0}\right) = \frac{1-\vartheta_2}{\gamma\vartheta_2}\left(\frac{1}{s_1} - \frac{1}{p_0}\right) \\ &= \frac{1-\vartheta_2}{\vartheta_2}\left(\frac{1}{s_2} - \frac{1}{p}\right) = \frac{1}{p} - \frac{1}{q(p)}. \end{aligned}$$

This gives $1/p = \vartheta_2/q(p) + (1-\vartheta_2)/s_2$. Now, we can again use the results of Section 3 to obtain the full statement of Theorem 8.1. □

Theorem 8.1 formalizes in our setting the well-known principle that Sobolev inequalities are decreasing in strength with respect to the parameter p. A classical application of this principle shows that, on a Riemannian manifold, an inequality of the type $\|f\|_q \le C\|\nabla f\|_1$ implies bounds on the heat kernel via the L^2 Sobolev inequality $\|f\|_{q'} \le C'\|\nabla f\|_2$.

Among the many corollaries of Theorem 8.1, we will only state two.

Corollary 8.2 *Assume that we are given a family* $\mathcal{W}$ *of semi-norms* W_p, $1 \le p \le +\infty$, *defined on* $\mathcal{F}$ *and such that, for each* p, W_p *satisfies* (H^+_∞) *and* (H_p). *Assume that* $f \in \mathcal{F}$ *implies* $f^\gamma \in \mathcal{F}$ *for all* $\gamma \ge 1$ *and that* $\mathcal{W}$ *satisfies* (8.1).

Assume that there exist $p_0 \in [1,+\infty[$, s_0, $r_0 \in]0,+\infty]$ *and* $\vartheta_0 \in]0,1]$ *such that* W_{p_0} *satisfies* $(S^{*,\vartheta_0}_{r_0,s_0})$. *Let* $q(p_0)$ *be defined by* $1/r_0 = \vartheta_0/q(p_0) + (1-\vartheta_0)/s_0$. *Assume that* $p_0 < q(p_0) < +\infty$ *and define* n *by* $1/n = 1/p_0 - 1/q(p_0)$. *Then, there exist two constants* $0 < b, B < +\infty$ *such that, for any function* $f \in \mathcal{F}$ *with* $W_n(f) \le 1$,

$$\int_E \left(e^{b|f|^{n/(n-1)}} - 1\right) d\mu \le B\|f\|_1.$$

Moreover, if f is supported in a set Ω of finite measure and satisfies $W_n(f) \leq 1$,

$$\int_\Omega e^{b|f|^{n/(n-1)}}\, d\mu \leq B\mu(\Omega).$$

Let us give two more specific examples:

(1) If (M, g) is a Riemannian manifold which satisfies the isoperimetric inequality

$$v_n(\Omega)^{(n-1)/n} \leq C v_{n-1}(\partial\Omega)$$

for all domains Ω with smooth enough boundary $\partial\Omega$, we deduce from the co-area formula and Corollary 8.2 that (M, g) satisfies the Trudinger inequality

$$\int_\Omega e^{b|f|^{n/(n-1)}}\, d\mu \leq B\mu(\Omega).$$

for all f supported in Ω such that $\int |\nabla f|^n dv \leq 1$.

(2) Let G be a finitely generated group, and fix a symmetric generating set $S \subset G$ containing the neutral element. Set $K(x, y) = 1_S(y^{-1}x)$ and $W_p(f) = \left(\sum_{x,y\in G} |f(x) - f(y)|^p K(x, y)\right)^{1/p}$. Consider the Cayley graph of G associated with S and let $V(\ell)$ be the number of elements at distance less than or equal to ℓ from the neutral element in this Cayley graph. If we assume that $V(\ell) \geq c\ell^n$ for all $\ell = 1, 2, \ldots$, it follows from Varopoulos isoperimetric inequality that

$$\|f\|_{n/(n-1)} \leq CW_1(f)$$

for all f with finite support. See [30, 31, 10] for details. Thus, Corollary 8.2 implies that we also have

$$\sum_{x\in\Omega} e^{b|f(x)|^{n/(n-1)}} \leq B|\Omega|$$

for all functions f supported in Ω and such that $W_n(f) \leq 1$. Here, $|\Omega|$ is the cardinality of Ω.

Corollary 8.3 *Assume that our underlying space E is a metric space associated with a distance function ρ and that $\mathcal{F}$ is the space of the non-negative Lipschitz functions with compact support. Let $\mathcal{W} = \{W_p : 1 \leq p \leq +\infty\}$ be a family of semi-norms defined on $\mathcal{F}$ which satisfies* (8.1) *and such that, for each p, W_p satisfies (H_∞^+) and (H_p). Set $\phi_{x,r}(y) = \max\{0\ , r - \rho(x, y)\}$. Assume that, for all $x \in E$, all $r > 0$, and all $1 \leq p \leq +\infty$,*

$$W_p(\phi_{x,r}) \leq CV(x, r)^{1/p} \tag{8.2}$$

where $V(x,r) = \mu(\{y : \rho(x,y) \le r\})$.

If there exist $p_0 \in [1,+\infty[$, s_0, $r_0 \in]0,+\infty]$ *and* $\vartheta_0 \in]0,1]$ *such that* W_{p_0} *satisfies* $(S^{*,\vartheta_0}_{r_0,s_0})$ *with* $1/r_0 = \vartheta_0/q(p_0) + (1-\vartheta_0)/s_0$ *and* $1/q(p_0) \le 1/p_0$*, then*

$$\forall\ r > 0,\ \forall\ x \in E, \quad V(x,r) \ge cr^n$$

where $n \in [0,+\infty]$ *is defined by* $1/q(p_0) = 1/p_0 - 1/n$.

Proof. Using Theorem 8.1, we can assume that $q(p_0) < 0$. In that case, we have

$$\|f\|_\infty \le (CW_{p_0}(f))^{-q(p_0)/(1-q(p_0))}\ \|f\|_1^{1/(1-q(p_0))}$$

for all $f \in \mathcal{F}$. In particular, for $f = \phi_{x,r}$, we get

$$r \le (CV(x,r))^{-q(p_0)/p_0(1-q(p_0))}(rV(x,r))^{1/(1-q(p_0))}$$

which yields

$$r \le C'V(x,r)^{1/p_0-1/q(p_0)} = C'V(x,r)^{1/n}.$$

□

The abstract parameter n introduced in Theorem 8.1 can be interpreted as a "dimension". In the classical case of $\mathbb{R}^n$, it indeed coincides with the topological dimension. Moreover, in the setting of Theorem 8.1, the three cases $0 < q < +\infty$, $-\infty < q < 0$ and $q = \infty$ correspond to $1 \le p < n$, $n < p < +\infty$ and $p = n$. Corollary 8.3 shows that this abstract notion of dimension is related to the volume growth of balls in many cases. Indeed, the hypothesis (8.2) is obviously satisfied if the distance function has a "bounded gradient".

9. How to prove a Sobolev inequality. This section may help motivate the results developed in this paper. Roughly speaking, this work elaborates the observation that certain "weak" inequalities imply their "strong" counterparts (here weak and strong are understood in a heuristic sense). This could only be a formal remark if it did not turn out that the "weak" versions are really easier to obtain that their "strong" counterparts. To illustrate this point, we now describe a rather general approach that yields Sobolev type inequalities.

Theorem 9.1 *Assume that* W *is a semi-norm defined on* $\mathcal{F}$. *Fix* $R \in]0,+\infty]$ *and* $p \in [1,+\infty[$. *Assume that there exists a family* $\{M_r,\ r \in]0,R[\}$ *of operators such that, for all* $f \in \mathcal{F}$,

$$\forall\ r \in]0,R[\,,\ \|M_r f\|_\infty \le C_1 r^{-n}\|f\|_1 \tag{9.1}$$

for some $n > 0$, *and*

$$\forall\ r \in]0,R[\,,\ \|f - M_r f\|_p \le C_2 r\, W(f). \tag{9.2}$$

Then, setting $\tau = 1 + 1/n$, the inequality

$$\sup_{\lambda>0}\left\{\lambda\mu(f \geq \lambda)^{1/p\tau}\right\} \leq C\left(W(f)^p + R^{-p}\|f\|_p^p\right)^{1/p\tau}\|f\|_1^{1-1/\tau}$$

is satisfied for all $f \in \mathcal{F}$.

If we set $1/q = 1/p - 1/n$, and if we assume further that W satisfies (H_p) and (H_∞^+), this implies that each of the inequalities $(S_{r,s}^\vartheta)$ with $\vartheta \in]0,1]$, r, $s \in]0,+\infty]$ and $1/r = \vartheta/q + (1-\vartheta)/s$ is satisfied by $W'(f) = (W(f)^p + R^{-p}\|f\|_p^p)^{1/p}$.

Proof. Let us fix $\lambda > 0$. We always have

$$\lambda^p\mu(f \geq \lambda) \leq \|f\|_p^p.$$

It follows that, if λ satisfies $\lambda \leq 3C_1R^{-n}\|f\|_1$, we have

$$\lambda^{p\tau}\mu(f \geq \lambda) \leq (3C_1)^{p/n}R^{-p}\|f\|_p^p\|f\|_1^{p/n}.$$

We can thus assume that $\lambda > 3C_1R^{-n}\|f\|_1$. Now, if we choose $0 < r < R$ so that $3C_1r^{-n}\|f\|_1 = \lambda$ and use the hypothesis (9.1),

$$\begin{aligned} \mu(f \geq \lambda) &\leq \mu(|f - M_rf| \geq \lambda/2) + \mu(|M_rf| \geq \lambda/2) \\ &\leq \mu(|f - M_rf| \geq \lambda/2). \end{aligned}$$

Then, (9.2) implies

$$\begin{aligned} \mu(f \geq \lambda) &\leq (2/\lambda)^p\|f - M_rf\|_p^p \leq (2C_2\,r/\lambda)^pW(f)^p \\ &\leq (2C_2)^p(3C_1)^{p/n}\lambda^{-p(1+1/n)}W(f)^p\|f\|_1^{p/n}. \end{aligned}$$

This proves the theorem. □

Let us mention a few typical applications.

First, assume that $H_t = e^{-tL}$ is a Markov self-adjoint semigroup on $L^2(E,\mu)$. Then, $\|f - H_tf\|_2^2 \leq 2t\mathcal{E}(f,f)$ where $\mathcal{E}$ is the associated Dirichlet form (i.e., $\mathcal{E}(f,f) = \langle Lf, f\rangle$). By Corollary 7.3, $\mathcal{E}$ satisfies H_∞^+ and H_α for all $\alpha \geq 2$. Assuming that $\|H_tf\|_\infty \leq Ct^{-n/2}\|f\|_1$ for $0 < t < T$, Theorem 9.1 shows that $W(f) = \left(\mathcal{E}(f,f) + T^{-1}\|f\|_2^2\right)^{1/2}$ satisfies $(S_{r,s}^\vartheta)$ for all r, $s \in]0,+\infty]$, $\vartheta \in]0,1]$ such that $1/r = \vartheta/q + (1-\vartheta)/s$ where $q = 1/2 - 1/n$. Of course, this is nothing but one half of the equivalence between Sobolev-Nash inequalities and heat kernel decay that was mentioned in the introduction. See [29, 31, 11, 5].

As a second example, consider the case where E is a metric space with balls $B(x,r)$ of volume $V(x,r)$ and set $M_rf(x) = (1/V(x,r))\int_{B(x,r)} f\,d\mu$. Even when the distance ρ is naturally associated with a local gradient ∇, it is not always true that $\|f - M_r f\|_p \leq rC\|\nabla f\|_p$ for all $0 < r < +\infty$. But there are a number of important cases where these inequalities are satisfied. In particular, they are satisfied by the natural metrics on Lie groups associated with invariant vector fields. See [26, 31, 10]. D. Robinson [26] was the first to use this remark to prove a Nash type inequality.

We will describe our last example in more detail. Let again E be a metric space. Note $\rho(x,y)$ the distance from x to y, and set $V(x,r) = \mu(B(x,r))$ and $V(x,y) = V(x,\rho(x,y))$. Fix $0 < \gamma \leq 1$ and consider the Besov semi-norm

$$W_p(f) = \left(\int_E \int_E \frac{|f(x) - f(y)|^p}{\rho(x,y)^{p\gamma}} \frac{d\mu(x)\, d\mu(y)}{V(x,y)}\right)^{1/p}$$

defined on a set $\mathcal{F}$ of sufficiently regular non-negative functions with compact support. By Lemma 7.1, W_p satisfies H_∞^+ and H_α for all $\alpha \geq p$. Setting $M_r f(x) = (1/V(x,r)) \int_{B(x,r)} f\, d\mu$, it is easy to check that

$$|f(x) - M_r f(x)|^p \leq r^{p\gamma} \int_E \frac{|f(x) - f(y)|^p}{\rho(x,y)^{p\gamma}} \frac{d\mu(y)}{V(x,y)}.$$

It follows that, for all $0 < r < +\infty$,

$$\|f - M_r f\|_p \leq r^\gamma W_p(f).$$

Assuming that

$$\forall\, 0 < r < R, \quad \inf_{x \in E} V(x,r) \geq cr^n,$$

Theorem 9.1 (after substituting $r^{1/\gamma}$ to r in the definition of M_r) shows that, for each $1 \leq p < +\infty$,

$$W'_p(f) = (W_p^p(f) + R^{-p\gamma}\|f\|_p^p)^{1/p}$$

satisfies all of the inequalities $(S_{r,s}^\vartheta)$ with $\vartheta \in\,]0,1]$, r, $s \in\,]0,+\infty]$, and $1/r = \vartheta/q(p) + (1-\vartheta)/s$, where $q(p)$ is defined by $1/q(p) = 1/p - \gamma/n$. In particular, if $1 \leq p < n/\gamma$,

$$\|f\|_{q(p)} \leq CW'_p(f)$$

whereas, if $p = n/\gamma$ and $W'_{n/\gamma}(f) \leq 1$,

$$\int \left(e^{b|f|^{n/(n-\gamma)}} - 1\right) d\mu \leq C\mu(\mathrm{supp}(f)).$$

10. Further remarks.

10.1 A. W, p*–**Capacities.*** We already mentioned in the introduction that V. Maz'ja develops in [19] the type of results described in Section 3 and that his approach is based on capacities and the use of non-trivial co-area formulas. The method of this paper allows us, conversely, to obtain some results concerning capacities. These results are similar to some of the conclusions in [17]. In our abstract setting, define the (W, p)-capacity of a set $S \subset E$ by

$$\mathrm{Cap}_{W,p}(S) = \inf\{W(f)^p : f \in \mathcal{F},\ \inf_S f \geq 1\}.$$

Consider now the "isoperimetric" inequality

$$\forall\ S \subset E,\ \mu(S)^{p/q} \leq C\,\mathrm{Cap}_{W,p}(S) \tag{10.1}$$

with $q > 0$. Clearly, the Sobolev inequality $\|f\|_{q,\infty} \leq CW(f)$ implies (10.1), for any p. Conversely, assume that W satisfies (H^2_∞) and that (10.1) holds true. Let $f \in \mathcal{F}$ and set $S_k = \{f \geq 2^{k+1}\}$, $k \in \mathbb{Z}$. If we assume that $\mathcal{F}$ is a cone, for every $k \in \mathbb{Z}$, the function $2^{-k} f_k = 2^{-k}[(f - 2^k)^+ \wedge 2^k]$ belongs to $\mathcal{F}$ and is larger than or equal to 1 on S_k. Hence, by (10.1) and (H^2_∞),

$$2^{pk}\mu(f \geq 2^{k+1})^{p/q} \leq CW(f)^p$$

for all $k \in \mathbb{Z}$. This shows that $\|f\|_{q,\infty} \leq C'W(f)$. Thus, if $\mathcal{F}$ is a cone, the capacity inequality (10.1) with $q > 0$, is equivalent to the weak Sobolev inequality

$$\|f\|_{q,\infty} \leq C'W(f).$$

If we further assume that W satisfies (H_p), we obtain that (10.1) is equivalent to

$$\|f\|_{q,p} \leq C'W(f).$$

Note that the case $q = +\infty$ and $q < 0$ have to be excluded in the discussion above. It is not clear what the interpretation of these cases in terms of capacity should be.

10.2 B. Faber-Krahn and logarithmic Sobolev inequalities. We now discuss two other types of inequalities that appear in the literature and are limit cases of $(S^\vartheta_{r,s})$.

Consider $(S^\vartheta_{r,s})$ for some fixed $r > s > 0$. On one hand, by Jensen inequality, one has

$$\|f\|_s \leq (\mu(\mathrm{supp}(f)))^{1/s\,-1/r}\,\|f\|_r.$$

Thus, we see that $(S^\vartheta_{r,s})$ implies

$$\|f\|_r \leq CW(f)\,(\mu(\mathrm{supp}(f)))^\sigma \tag{FK^σ_r}$$

with $\sigma = (1 - \vartheta)(r - s)/\vartheta rs = 1/r - 1/q > 0$. On the other hand, for all r,

$$\|f\|_r \leq \|f\|_\infty\,(\mu(\mathrm{supp}(f)))^{1/r},$$

whence it follows that (FK_r^σ) implies $(S_{r,s}^{*,\vartheta})$ with $1/s = 1/r + \sigma\vartheta/(1-\vartheta)$. Thus, we have the following result:

Theorem 10.1 *Fix $q \in]-\infty, 0[\cup]0, +\infty[$. If $q \in]0, \infty[$, assume that W satisfies (H_q^ρ) for some $\rho > 1$. If $q \in]-\infty, 0[$, assume that W satisfies (H_∞^+). Then, the inequalities (FK_r^σ) with $0 \le \sigma < +\infty$, $0 < r \le +\infty$, and $\sigma = 1/r - 1/q$ are all equivalent and they are also equivalent to the inequalities $(S_{r,s}^\vartheta)$ where $1/r = \vartheta/q + (1-\vartheta)/s$, $s, r \in]0, +\infty]$ and $\vartheta \in]0, 1]$.*

If W satisfies (H_∞^ρ), for some $\rho > 1$, the inequalities $(FK_r^{1/r})$ with $0 < r < +\infty$, are all equivalent and they are also equivalent to the inequalities $(S_{r,s}^{1-s/r})$ where $0 < s < r < +\infty$. This corresponds to the case where $q = +\infty$.

Here, FK stands for Faber-Krahn inequality because, when $W(f) = \|\nabla f\|_2$ and $r = 2$, (FK_r^σ) is equivalent to the following statement: for all $\Omega \subset E$, the first eigenvalue

$$\lambda(\Omega) = \inf \left\{ \left\|\nabla f\right\|_2^2 : \operatorname{supp}(f) \subset \Omega,\ \left\|f\right\|_2^2 = 1 \right\}$$

satisfies

$$\lambda(\Omega) \ge C^{-2}\mu(\Omega)^{-2\sigma}. \tag{10.2}$$

A. Grigor'yan relates this type of inequality to heat kernel bounds in [16]. G. Carron [6] proves directly the equivalence of (10.2) with the classical Sobolev inequality. Faber-Krahn inequalities corresponding to $W(f) = \|\nabla f\|_p$, $1 \le p \le +\infty$, appear in [8] where Carron's result is extended. Note that the inequality (FK_r^σ) with $\sigma = 1/r - 1/q$ can be seen as the limit case of $(S_{r,s}^{*,\vartheta})$ when s tends to 0, ϑ tends to 1, and $(1-\vartheta)/\vartheta s$ tends to $1/r - 1/q$.

Another limit case of interest is related to logarithmic Sobolev inequalities. See [1, 11]. Assume that the family of inequalities $(S_{r,s}^\vartheta)$ is satisfied for some fixed q, some fixed $0 < r < +\infty$ and all $s < r$, $\vartheta \in]0, 1]$, satisfying $1/r = \vartheta/q + (1-\vartheta)/s$, with a constant C independent of s (the theorems stated above assert that this is the case as soon as $(S_{r,s}^\vartheta)$ is satisfied for one value of $s < r$). Since $1/s - 1/r = \vartheta(1/s - 1/q)$, we can write $(S_{r,s}^\vartheta)$ as

$$\left(\frac{\|f\|_r}{\|f\|_s}\right)^{1/(1/s-1/r)} \le \left(\frac{CW(f)}{\|f\|_s}\right)^{1/(1/s-1/q)}, \quad f \in \mathcal{F}.$$

Taking logarithms, we get

$$\frac{\log\|f\|_r - \log\|f\|_s}{1/s - 1/r} \le (1/s - 1/q)^{-1} \log(CW(f)/\|f\|_s), \qquad f \in \mathcal{F}. \tag{10.3}$$

Now, letting $s < r$ tend to r, we deduce from (10.3) that

$$(LS_r^q) \qquad \int \left[f^r \log\left(\frac{f}{\|f\|_r}\right)^r \right] d\mu \le (1/r - 1/q)^{-1} \|f\|_r^r \log\left(\frac{CW(f)}{\|f\|_r}\right), \qquad f \in \mathcal{F}.$$

Here we have used the fact that $u \to \phi(u) = \log \|f\|_{1/u}$ satisfies

$$-\|f\|_r^r \phi'(1/r) = \int [f^r \log(f/\|f\|_r)^r] \, d\mu$$

for $r \in]0, +\infty[$. This shows that (LS_r^q) can be interpreted as a possible definition of $(S_{r,r}^0)$.

In fact, the Hölder inequality shows that the function ϕ is convex on $[0, +\infty]$. Thus, the slope

$$\psi(u) = [\phi(u) - \phi(1/r)]/(u - 1/r)$$

is an increasing function of u which can be taken to be equal to $\phi'(1/r)$ at $u = 1/r$. If we now assume that (LS_r^q) holds for some $0 < r < +\infty$ and $1/r > 1/q$ (recall that q may be negative) then, for any $0 < s < r$, we obtain

$$-\psi(1/s) \le -\psi(1/r) = -\phi'(1/r) \le (1/r - 1/q)^{-1} \log\left(\frac{CW(f)}{\|f\|_r}\right)$$

and thus

$$\|f\|_r \le (CW(f))^\tau \|f\|_r^{-\tau} \, \|f\|_s$$

with $(1/s - 1/r) = \tau(1/r - 1/q)$. This is exactly $(S_{r,s}^\vartheta)$ with $\vartheta = \tau/(1+\tau)$, i.e. $1/r = \vartheta/q + (1-\vartheta)/s$. We can state:

Theorem 10.2 *Fix $q \in]-\infty, 0[\cup]0, +\infty[$. If $q \in]0, \infty[$ assume that W satisfies (H_q^ρ) for some $\rho > 1$ whereas, if $q \in]-\infty, 0[$, assume that W satisfies (H_∞^+). Then, the inequalities (LS_r^q) with $0 < r < +\infty$ and $1/q < 1/r$, are all equivalent and they are also equivalent to the inequalities $(S_{r,s}^\vartheta)$ where $1/r = \vartheta/q + (1-\vartheta)/s$, $s, r \in]0, +\infty]$ and $\vartheta \in]0, 1]$.*

If W satisfies (H_∞^ρ) for some $\rho > 1$, the inequalities (LS_r^∞) with $0 < r < +\infty$, are all equivalent and they are also equivalent to the inequalities $(S_{r,s}^{1-s/r})$ where $0 < s < r < +\infty$.

10.3 C. Non-polynomial families of inequalities. The inequality $\|f\|_r \le (CW(f))^\vartheta \|f\|_s^{1-\vartheta}$ can be written $\|f\|_r \le C\,(\|f\|_s/\|f\|_r)^{(1-\vartheta)/\vartheta} W(f)$. Here, recall that q is defined by $1/r = \vartheta/q + (1-\vartheta)/s$. This gives

$$\frac{1-\vartheta}{\vartheta} = \left(\frac{1}{r} - \frac{1}{q}\right) \frac{rs}{r-s}.$$

Assuming that $s < r$, it is natural to replace the power function $u \to Cu^{1/r-1/q}$ by a more general non-decreasing function $\phi_r : [0, +\infty[\to [0, +\infty[$, and to consider inequalities of the type

$$\|f\|_r \le \phi_r\left((\|f\|_s/\|f\|_r)^{rs/(r-s)}\right) W(f).$$

For $s < r$, the inequality above implies

$$\|f\|_r \le \phi_r(\mu(\mathrm{supp}(f)))W(f)$$

which generalizes the Faber-Krahn inequality (FK_r^σ) of Section 3. It would be interesting to elucidate systematically the various relationships between these inequalities in the spirit of what has been done in this paper for power functions. Some results in this direction can be found in [9]. Indeed, such inequalities occurs on groups and on certain manifolds when the volume growth is superpolynomial. See [30, 10, 16]. In [16], A. Grigor'yan relates an inequality of this type to the decay of the heat kernel. The results below must also be compared with those obtained by G. Carron in [7].

Proposition 10.3 *Assume that W satisfies (H_∞^+). Fix $0 < s < r < +\infty$ and a non-decreasing function $\phi : [0, +\infty[\to [0, +\infty[$. The inequalities*

$$\forall\, f \in \mathcal{F}, \quad \|f\|_r \le C_1\phi\big(C_2(\|f\|_s/\|f\|_r)^{rs/(r-s)}\big)W(f)$$

and

$$\forall\, f \in \mathcal{F}, \quad \|f\|_r \le C_3\phi(C_4\mu(\mathrm{supp}(f)))W(f)$$

are equivalent.

Proof. We already noticed that one direction is obvious. To show that the second inequality implies the first, we write

$$\|f\|_r^r \le 2^r\|(f-t)^+\|_r^r + \int_{\{f\le 2t\}} f^r\, d\mu \le 2^r\|(f-t)^+\|_r^r + (2t)^{r-s}\big\|f\big\|_s^s$$

and we observe that, by hypothesis,

$$\|(f-t)^+\|_r^r \le C_3^r\phi(C_4\mu(\{f\ge t\}))^r W(f)^r$$

since $\mathrm{supp}((f-t)^+) = \{f \ge t\}$. Using $\mu(f\ge t) \le t^{-s}\big\|f\big\|_s^s$, we get

$$\|f\|_r^r \le 2^r C_3^r \phi(C_4 t^{-s}\big\|f\big\|_s^s)^r W(f)^r + (2t)^{r-s}\big\|f\big\|_s^s.$$

We now choose $2t = [\|f\|_r^r/(2\big\|f\big\|_s^s)]^{1/(r-s)}$. This choice yields

$$\|f\|_r \le 2^{1+1/r}C_3\phi\left(2^{s(1+1/(r-s))}C_4(\|f\|_s/\|f\|_r)^{rs/(r-s)}\right)W(f),$$

the desired inequality with $C_1 = 2^{1+1/r}C_3$ and $C_2 = 2^{s(1+1/(r-s))}C_4$. $\square$

Theorem 3.1 of [17] generalizes to the present setting and reads as follows.

Theorem 10.4 *Fix $0 < p < +\infty$ and a non-decreasing function $\phi : [0,+\infty[\to [0,+\infty[$. Assume that W satisfies (H_p^ρ), for some $\rho > 1$. The inequality*

$$\forall\ f \in \mathcal{F}, \quad \mu(S) \le \phi(\mu(S))\mathrm{Cap}_{W,p}(S)$$

implies

$$\forall\ f \in \mathcal{F}, \quad \|f^p\|_\Phi^{1/p} \le CW(f)$$

where Φ is is a Young function such that

$$\phi(t) \le t\Phi^{-1}(1/t),$$

and $\|f\|_\Phi$ is the Orlicz norm associated with Φ.

Here, the Orlicz norm is defined by duality: if Ψ denote the Young conjugate of Φ defined by $\Psi(y) = \sup_{x>0}\{xy - \Phi(y)\}$, the norm $\|f\|_\Phi$ is given by

$$\|f\|_\Phi = \sup\left\{\int fg\,d\mu : \int \Psi(g)\,d\mu \le 1\right\}.$$

To see that our statement is the same as in Kaimanovich [17], recall that any two Young conjugate functions Φ, Ψ satisfy

$$\forall\ s > 0, \quad s \le \Phi^{-1}(s)\Psi^{-1}(s) \le 2s.$$

We will need a few more facts about Orlicz norms which we borrow to [17]. We also refer the reader to [18] for details on Orlicz norms. For any set E and for any function $f \ge 0$, we have

$$\|1_E\|_\Phi = \mu(E)\Psi^{-1}(1/\mu(E)), \quad \|f^p\|_\Phi^p = \int_0^\infty \|1_{\{f\ge t\}}\|_\Phi\,dt^p.$$

It follows that, for any fixed $\rho > 1$,

$$\|f^p\|_\Phi^p \le 2(\rho^p - 1)\sum_k \frac{\rho^{kp}}{\Phi^{-1}\left[1/\mu\left(f \ge \rho^k\right)\right]}\,. \tag{10.4}$$

The following result is similar to Theorem 10.4 but more subtle. We need to introduce the condition

$$\exists\ B > 1,\ \forall\ s > 0, \quad \Phi(2s) \le B\Phi(s) \tag{Δ_2}$$

that may or may not be satisfied by a Young function Φ. Note that, under the condition (Δ_2), the Orlicz norm $\|f\|_\Phi$ is comparable to the quantity

$$\inf\left\{s > 0 : \int \Phi(f/s)\,d\mu \le 1\right\}.$$

Theorem 10.5 *Fix* $0 < r, p < +\infty$ *and a non-decreasing function* $\phi : [0, +\infty[\to [0, +\infty[$. *Assume that* W *satisfies* (H_p^ρ), *for some* $\rho > 1$, *and that the inequality*

$$\forall\ f \in \mathcal{F}, \quad \|f\|_r \le \phi(\mu(\mathrm{supp}(f)))W(f) \tag{10.5}$$

holds. Assume further that there exists a Young function Φ *satisfying the condition* (Δ_2) *and such that*

$$\phi^p(t) \le t^{p/r}\Phi^{-1}(1/t).$$

Then,

$$\forall\ f \in \mathcal{F}, \quad \|f^p\|_\Phi^{1/p} \le CW(f)\,.$$

Proof. For simplicity, we will write the proof for $\rho = 2$. The proof is two steps. Set

$$N = \sup_k \frac{2^{kp}}{\Phi^{-1}\left(1/\mu\left(f \ge 2^k\right)\right)}.$$

The first step is to show

$$N \le CW(f)\,, \tag{10.6}$$

which is a weak version of the desired inequality (cf. Theorem 4.1).

The second step is then very easy. Apply (10.6) to $f_k = (f - 2^k)^+ \wedge 2^k$, sum over k, and use (10.4) and (H_p^2). The desired inequality follows.

We now proceed to prove (10.6). By the very definition of N, we have

$$\mu\left(f \ge 2^k\right) \le \frac{1}{\Phi\left(2^{kp}/N\right)}.$$

Since ϕ is non-decreasing, inequality (9.3) applied to f_k yields

$$2^k \mu\left(f \ge 2^{k+1}\right)^{1/r} \le A_\infty W(f)\phi\left[\frac{1}{\Phi\left(2^{kp}/N\right)}\right]. \tag{10.7}$$

Recall that ϕ and Φ are related by

$$\phi^p(t) \le t^{p/r}\Phi^{-1}(1/t).$$

Raising (10.7) to the power p, we get

$$2^{kp}\mu\left(f \ge 2^{k+1}\right)^{p/r} \le \left(A_\infty W(f)\right)^p \Phi\left(2^{kp}/N\right)^{-p/r}\frac{2^{kp}}{N}.$$

Hence,

$$\frac{N^{r/p}\Phi\left(2^{kp}/N\right)}{\left(A_\infty W(f)\right)^r} \le \frac{1}{\mu\left(f \ge 2^{k+1}\right)}.$$

Taking Φ^{-1} of both sides yields

$$\Phi^{-1}\left[\frac{N^{r/p}\Phi\left(2^{kp}/N\right)}{(A_\infty W(f))^r}\right] \leq \Phi^{-1}\left[\frac{1}{\mu\left(f \geq 2^{k+1}\right)}\right].$$

and

$$N = \sup_k \frac{2^{kp}}{\Phi^{-1}\left[\frac{1}{\mu(f\geq 2^k)}\right]} \leq \sup_k \frac{2^{p(k+1)}}{\Phi^{-1}\left[\frac{N^{r/p}\Phi(2^{kp}/N)}{(A_\infty W(f))^r}\right]}$$

or

$$N \leq \frac{2^p}{\inf_k \frac{1}{2^{kp}}\Phi^{-1}\left[\frac{N^{r/p}\Phi(2^{kp}/N)}{(A_\infty W(f))^r}\right]}. \tag{10.8}$$

For $y > 0$, define

$$\omega(y) = \inf_{x>0}\left\{\frac{1}{x}\Phi^{-1}\left[y\Phi(x)\right]\right\}.$$

Observe that ω is non-decreasing and that

$$\inf_k \left\{\frac{N}{2^{kp}}\Phi^{-1}\left[\frac{N^{r/p}\Phi\left(2^{kp}/N\right)}{(A_\infty W(f))^r}\right]\right\} \geq \frac{1}{2^p}\omega\left(\frac{N^{r/p}}{(A_\infty W(f))^r}\right).$$

Thus, (10.8) yields

$$\omega\left(\frac{N^{r/p}}{(A_\infty W(f))^r}\right) \leq 4^p.$$

To finish up the proof, we just need to check that $\omega(s) \leq 4^p$ implies $s \leq C$ for some constant C. But, the condition (Δ_2), satisfied by Φ, implies that $\lim_{s\to\infty}\omega(s) = +\infty$. This ends the proof of Theorem 10.5. □

10.4 Remarks. 1. It is obvious that one can replace the norm $\|f\|_r$ by the norm $\|f\|_{r,\infty}$ in (10.5) without changing the conclusion of Theorem 10.5. Moreover, (10.5) can also be replaced by

$$\forall\, f \in \mathcal{F},\ \ \forall\, \lambda > 0, \quad \lambda\mu(f \geq \lambda)^{1/r} \leq \phi(\|f\|_1/\lambda)W(f). \tag{10.9}$$

Indeed, applying this inequality to f_k with $\lambda = 2^k$, or applying (10.5) to f_k yields, basically, the same inequality.

2. The quantity $\|f^p\|_\Phi^{1/p}$ is comparable to the Orlicz norm $\|f\|_{\Phi_p}$ where $\Phi_p(s) = \Phi(s^p)$.

3. If we assume that the inequality $\|f^p\|_\Phi^{1/p} \le CW(f)$ holds for all $f \in \mathcal{F}$ and if the function $s \to \left[\Phi^{-1}(s)\right]^{r/p}$ is concave, then,

$$\forall\ f \in \mathcal{F}, \quad \|f\|_r \le \phi(\mu(\mathrm{supp}(f)))W(f)$$

where $\phi(t) = Ct^{p/r}\Phi^{-1}(1/t)$. This offers a converse of Theorem 10.5.

It is important to see how Theorem 10.5 can be used in practice. For this purpose, there is a non-polynomial version of Theorem 9.1.

Proposition 10.6 *Fix $r \in]0,+\infty]$. Assume that W is a semi-norm defined on $\mathcal{F}$. Assume that there exists a family of operators M_s, $s > 0$, defined on $\mathcal{F}$ such that,*

$$\forall\ f \in \mathcal{F},\ \forall\ s > 0, \quad \|f - M_s f\|_r \le C\, s\, W(f), \text{ and } \|M_s f\|_\infty \le V(s)^{-1}\|f\|_1$$

for some non-decreasing function V satisfying $\lim_{+\infty} V = +\infty$. Then, W satisfies

$$\forall\ f \in \mathcal{F},\ \forall\ \lambda > 0, \quad \lambda\mu(f \ge \lambda)^{1/r} \le 2C\phi\left(3\|f\|_1/\lambda\right) W(f)$$

where $\phi(t) = \inf\{s > 0 : V(s) > t\}$.

Proof. Write

$$\mu(f \ge \lambda) \le \mu(|f - M_s f| \ge \lambda/2) + \mu(|M_s f| \ge \lambda/2)$$

and

$$\mu(|f - M_s f| \ge \lambda/2) \le \left(2\lambda^{-1}\|f - M_s f\|_r\right)^r \le \left(2C\lambda^{-1}\, s\, W(f)\right)^r.$$

Since $M_s f \le V(s)^{-1}\|f\|_1$, choosing $s = \phi(3\|f\|_1/\lambda)$ yields

$$\mu(f \ge \lambda) \le \left(2C\lambda^{-1}\phi(3\|f\|_1/\lambda)W(f)\right)^r$$

or

$$\lambda\mu(f \ge \lambda)^{1/r} \le 2C\phi(3\|f\|_1/\lambda)W(f).$$

□

Corollary 10.7 *Fix $0 < r, p < +\infty$. Assume that W is a semi-norm defined on $\mathcal{F}$ and satisfies (H_p^ρ) for some $\rho > 1$. Assume that there exists a family of operators defined on $\mathcal{F}$ by $M_s f(x) = \int_E M_s(x,y)f(y)\,d\mu(y)$ with $s > 0$ and such that,*

$$\forall\ f \in \mathcal{F},\ \ \forall\ \ s > 0, \quad \|f - M_s f\|_r \le C\, s\, W(f), \quad \textit{and} \quad \|M_s f\|_\infty \le V(s)^{-1}\|f\|_1$$

where V is non-decreasing and satisfies $\lim_{+\infty} V = +\infty$. *Set* $\phi(t) = \inf\{s > 0 : V(s) > t\}$ *and assume that* Φ *is a Young function satisfying* (Δ_2) *and such that* $\phi(t) \leq t^{p/r}\Phi^{-1}(1/t)$. *Then, we have*

$$\forall\, f \in \mathcal{F}, \quad \|f^p\|_{\Phi}^{1/p} \leq C'W(f).$$

Proof. Apply Proposition 10.6 and Remark 1 following Theorem 10.5. □

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