Markov operators, classical orthogonal polynomial ensembles, and random matrices

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recent study of

random matrix and random growth models

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new asymptotics

common, non central, rate $(\mathrm{mean})^{1/3}$

universal limiting **Tracy-Widom** distribution

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random matrices, longest increasing subsequence,
random growth models, last passage percolation...

P. Forrester, C. Tracy, H. Widom, J. Baik, P. Deift, K. Johansson

invariant random matrix models

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$$P(dX) = \frac{1}{Z} \exp\Big(-\mathrm{Tr}\big(v(X)\big)\Big) dX, \quad v : \mathbb{R} \to \mathbb{R}$$

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$$P^N(dx) = \frac{1}{Z} \big|\Delta_N(x)\big|^\beta \prod_{i=1}^N d\mu(x_i), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

$$\Delta_N(x) = \prod_{i<j} (x_i - x_j), \quad d\mu(x) = e^{-v(x)} dx$$

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$\mu(\{x\}) = (1-q)q^x, \ x \in \mathbb{N}$ geometric distribution law of $(\lambda_1^N, \ldots, \lambda_N^N)$: $\quad P^N(dx) = \frac{1}{Z} \left|\Delta_N(x)\right|^\beta \prod_{i=1}^N d\mu(x_i)$

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large deviation asymptotics (every β)

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minimizer of $2 \int v \, d\nu - \beta \iint \log|x-y| \, d\nu(x) d\nu(y)$

weighted logarithmic potential theory

local regime : individual behavior (spacings, extreme values)

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orthogonal polynomial ensemble

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top edge behavior

largest eigenvalue (particle) fluctuates at the rate $(\mathrm{mean})^{1/3}$

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C. Tracy, H. Widom, P. Deift, X. Zhou, J. Baik, K. Johansson, T. Kriecherbauer, K. McLaughlin, P. Miller, S. Venakides, M. Vanlessen, D. Gioev, P. Bleher, A. Its, A. Kuijlaars, M. Shcherbina, L. Pastur...

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$$\widehat{\mu}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\widehat{\lambda}_i^N} \ \to \ \frac{1}{2\pi} \sqrt{4 - x^2} \, dx \quad \text{on} \quad (-2, +2)$$

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$$F_{\mathrm{TW}}(s) = \exp\bigg(-\int_s^\infty (x-s)u(x)^2 dx\bigg), \quad s \in \mathbb{R}$$

$u'' = 2u^3 + xu$ Painlevé II equation

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K. Johansson (2000)

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- **spectral description** (universal arcsine law)

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- **non asymptotic tail inequalities for largest eigenvalues** (optimal rate)

classical orthogonal polynomial ensembles

Hermite, Laguerre, Jacobi, Charlier, Meixner, Krawtchouk, Hahn

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$P_\ell,\ \ell \in \mathbb{N}$ Hermite polynomials for $\ d\mu(x) = e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$

normalized in $\ L^2(\mu)$

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$-LP_N = N \, P_N$ eigenvectors

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second order differential equation

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Φ Laplace transform of the **arcsine law**

$$\frac{dx}{\pi\sqrt{4 - x^2}} \quad \text{on} \quad (-2, +2)$$

common behavior, with the limiting **arcsine law**

for the classical orthogonal polynomials

of (normalized) measures $P_N^2 \, d\mu$

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compact case : **A. Maté, P. Nevai, V. Totik (1985)**

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U uniform, ξ arcsine, independent

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$\sqrt{U}\,\xi \quad$ semi-circle law on $\quad (-2, +2)$

spectral measure : averaging procedure

GUE $$E\big(\langle f, \widehat{\mu}^N\rangle\big) = \frac{1}{N}\sum_{\ell=0}^{N-1}\int f\bigg(\sqrt{\frac{\ell}{N}}\cdot\frac{x}{\sqrt{\ell}}\bigg)\, P_\ell^2\, d\mu$$

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U uniform, ξ arcsine, independent

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Meixner example (μ geometric) : equilibrium measure example : **last passage percolation** $W_N = \max_\pi \sum_{(i,j)\in\pi} w_{ij}$

μ geometric, K_N Meixner kernel, $\widehat{\lambda}_i^N = \lambda_i^N / N$

spectral measure

$$\widehat{\mu}^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\widehat{\lambda}_i^N} \ \to \ \text{equilibrium measure on } (a,b)$$

largest "particle" $\lambda_N^N \sim W_N/N \to b$

$$N^{-1/3}\big[W_N - bN\big] \ \to \ F_{\mathrm{TW}}$$ **Tracy-Widom** distribution

K. Johansson (2000)

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$$\frac{1}{1-q}\Big(\sqrt{qU(1+U)}\,\xi + \big[U + q(1+U)\big]\Big)$$

moment recursion formula

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$$E\left(e^{t\,\mathrm{Tr}(X^N)}\right) = \psi(t) = \int e^{tx} \sum_{\ell=0}^{N-1} P_\ell^2 \, d\mu, \quad t \in \mathbb{R}$$

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recurrence equation on moments

$$E\Big(\mathrm{Tr}\left((X^N)^p\right)\Big) = \int x^p \sum_{\ell=0}^{N-1} P_\ell^2 \, d\mu, \quad p \in \mathbb{N}$$

$$a_p^N = E\Big(\mathrm{Tr}\,\big((X^N)^{2p}\big)\Big), \quad X^N \text{ GUE}$$

three term recurrence equation

$$(p+1)a_p^N = (4p-2)Na_{p-1}^N + (p-1)(2p-1)(2p-3)a_{p-2}^N$$

J. Harer, D. Zagier (1986)

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$\varepsilon_0(p)$ Catalan numbers

Markov operator technology

similar recursion formulas

for the **classical** orthogonal polynomial ensembles

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continuous variable : Hermite, Laguerre, Jacobi

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continuous variable : Hermite, Laguerre, Jacobi

discrete variables : Charlier, Meixner, Krawtchouk, Hahn

explicit expressions for the (factorial) moments

Gaussian Orthogonal Ensemble $\beta = 1$

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$$P^N(dx) = \frac{1}{Z}\,|\Delta_N(x)|\prod_{i=1}^{N} d\mu(x_i), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

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limiting spectral distribution : **Wigner** semi-circle law

Tracy-Widom edge asymptotics

moment recursion formula ?

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$$\mu^N_{\mathrm{GOE}} = \sum_{\ell=0}^{N-1} P_\ell^2 + \frac{e^{x^2/4} P_{N-1}}{\int e^{x^2/4} P_{N-1} d\mu} \mathbf{1}_{N \text{ odd}}$$

$$+ \sqrt{\frac{\pi N}{8}}\, e^{x^2/4} P_{N-1} \int \mathrm{sgn}\,(x-y)\, e^{x^2/4}(y) P_N(y) d\mu(y)$$

Markov operator tools : differential equation

on the Laplace transform of the spectral measure

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effective values

GUE

$$
\begin{aligned}
a_0^N &= N \\
a_1^N &= N^2 \\
a_2^N &= 2N^3 + N \\
a_3^N &= 5N^4 + 10N^2 \\
a_4^N &= 14N^5 + 70N^3 + 21N
\end{aligned}
$$

GOE

$$
\begin{aligned}
b_0^N &= N \\
b_1^N &= N^2 + N \\
b_2^N &= 2N^3 + 5N^2 + 5N \\
b_3^N &= 5N^4 + 22N^3 + 52N^2 + 41N \\
b_4^N &= 14N^5 + 93N^4 + 374N^3 + 690N^2 + 509N
\end{aligned}
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duality with Symplectic Ensemble $(\beta = 4)$

recursion formulas lead to

sharp moment bounds

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Gaussian Unitary Ensemble (GUE)

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similar bounds for

classical orthogonal polynomial ensembles $(\beta = 2)$

Gaussian Orthogonal Ensemble (GOE) $(\beta = 1)$

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deviation inequality on last passage percolation $\quad W_N$

K. Johansson (2000)

moment comparison

Wigner matrices (independent entries)

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