Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2009, Vol. 45, No. 3, 754–769
DOI: 10.1214/08-AIHP184
© Association des Publications de l'Institut Henri Poincaré, 2009

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

A recursion formula for the moments of the Gaussian orthogonal ensemble

M. Ledoux

Institut de Mathématiques de Toulouse, Université de Toulouse, F-31062 Toulouse, France. E-mail: ledoux@math.univ-toulouse.fr

Received 26 November 2007; accepted 10 June 2008

Abstract. We present an analogue of the Harer–Zagier recursion formula for the moments of the Gaussian Orthogonal Ensemble in the form of a five term recurrence equation. The proof is based on simple Gaussian integration by parts and differential equations on Laplace transforms. A similar recursion formula holds for the Gaussian Symplectic Ensemble. As in the complex case, the result is interpreted as a recursion formula for the number of 1-vertex maps in locally orientable surfaces with a given number of edges and faces. This moment recurrence formula is also applied to a sharp bound on the tail of the largest eigenvalue of the Gaussian Orthogonal Ensemble and, by moment comparison, of families of Wigner matrices.

Résumé. Ce travail présente un analogue de la relation de récurrence de Harer et Zagier pour les moments de l'Ensemble Orthogonal Gaussien sous la forme d'une récurrence à cinq termes. La démonstration s'appuie sur des intégrations par parties gaussiennes et des équations différentielles sur les transformées de Laplace. Une relation similaire est établie pour l'Ensemble Symplectique Gaussien. Comme dans le cas complexe, cette relation s'interprète comme une formule de récurrence pour le nombre de cartes enracinées à nombre de faces et de côtés donné plongées dans des surfaces localement orientées. Cette relation de récurrence sur les moments fournit également une borne sur la loi de la plus grande valeur propre de l'Ensemble Orthogonal Gaussien et, par comparaison de moments, de familles de matrices de Wigner.

MSC: 46L54; 15A52; 33C45; 60E05; 82B31

Keywords: Gaussian Orthogonal Ensemble; Moment recursion formula; Map enumeration; Largest eigenvalue; Small deviation inequality

1. Introduction

Consider a $N \times N$ random matrix $X = X^N$ from the Gaussian Unitary Ensemble (GUE). That is, X is distributed according to the probability distribution

$$\mathbb{P}(\mathrm{d}X) = \frac{1}{Z} \exp\bigl(-\operatorname{Tr}\bigl(X^2\bigr)/2\bigr)\,\mathrm{d}X \tag{1}$$

on the space $\mathcal{H}_N$ of $N \times N$ Hermitian matrices where $\mathrm{d}X$ is Lebesgue measure on $\mathcal{H}_N \cong \mathbb{R}^{N^2}$ and $Z = Z_N$ the normalizing constant. This probability measure is invariant under the action of the unitary group on $\mathcal{H}_N$. Equivalently, $X = X^N = (X_{ij}^N)_{1 \le i,j \le N}$ is a $N \times N$ random Hermitian matrix such that the entries above the diagonal are independent complex (real on the diagonal) Gaussian random variables with mean zero and variance 1 (the real and imaginary parts are independent centered Gaussian variables with variance $\frac{1}{2}$).

The real case is known as the Gaussian Orthogonal Ensemble (GOE), that is the probability distribution on the space $\mathcal{S}_N$ of $N \times N$ symmetric matrices $X = X^N$ given by

$$\mathbb{P}(\mathrm{d}X) = \frac{1}{Z} \exp\bigl(-\operatorname{Tr}\bigl(X^2\bigr)/4\bigr)\,\mathrm{d}X \tag{2}$$

(where now $\mathrm{d}X$ is Lebesgue measure on $\mathcal{S}_N \cong \mathbb{R}^{N(N+1)/2}$). This distribution is invariant by the orthogonal group. Equivalently, $X = X^N = (X_{ij}^N)_{1 \le i,j \le N}$ is a $N \times N$ random real symmetric random matrix such that the entries X_{ij}^N, $1 \le i \le j \le N$, are independent centered real-valued Gaussian random variables with variance 1 (2 on the diagonal).

For such a symmetric or Hermitian random matrix $X = X^N$, denote by $\lambda_1^N \le \cdots \le \lambda_N^N$ the (real) eigenvalues of X^N. It is a classical result due to Wigner [21] that, almost surely,

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^N/\sqrt{N}} \to \mu \tag{3}$$

in distribution as $N \to \infty$, where μ is the semicircle law with density $\frac{1}{2\pi}(4-x^2)^{1/2}$ with respect to Lebesgue measure on $(-2,+2)$. One basic technique towards this result, going back to [21], is the moment method to show the convergence of $\frac{1}{N}\mathbb{E}(\mathrm{Tr}((X^N/\sqrt{N})^p))$ to the p-moment ($p \in \mathbb{N}$) of the semicircle law.

In their algebraic investigation of the genus series of the numbers of ways of obtaining an orientable Riemann surface of given genus by identifying in pairs the sides of a $2p$-gon [7], Harer and Zagier put forward a recursion formula for the (even) moments

$$a_p^N = \mathbb{E}\big(\mathrm{Tr}\big(\big(X^N\big)^{2p}\big)\big), \quad p \in \mathbb{N},$$

of the GUE (the odd moment being zero by symmetry). They namely established with combinatorial tools the following statement.

Theorem 1. *For every integer $p \ge 2$, and every $N \ge 1$,*

$$(p+1)a_p^N = (4p-2)Na_{p-1}^N + (p-1)(2p-1)(2p-3)a_{p-2}^N$$

$(a_0^N = N, a_1^N = N^2)$.

The Harer–Zagier recursion formula was revisited in [12] and [6]. In particular, Haagerup and Thorbjørnsen [6] connected this algebraic formula to the description of the moment generating function of the GUE as a confluent hypergeometric function using classical expansions associated with Hermite polynomials.

The real GOE case is known to be more complicated, and the possibility of an analogue of Theorem 1 in this case is explicitely raised in [6]. In this note, we propose such a recursion formula for the moments of the GOE in the form of a five term recurrence equation. Denote, for every integer p, by

$$b_p^N = \mathbb{E}\big(\mathrm{Tr}\big(\big(X^N\big)^{2p}\big)\big)$$

the even moments of the GOE (the odd moment being zero by symmetry).

Theorem 2. *For every integer $p \ge 4$, and every $N \ge 1$,*

$$\begin{aligned}(p+1)b_p^N &= (4p-1)(2N-1)b_{p-1}^N + (2p-3)\big(10p^2 - 9p - 8N^2 + 8N\big)b_{p-2}^N \\ &\quad - 5(2p-3)(2p-4)(2p-5)(2N-1)b_{p-3}^N \\ &\quad - 2(2p-3)(2p-4)(2p-5)(2p-6)(2p-7)b_{p-4}^N\end{aligned}$$

$(b_0^N = N, b_1^N = N^2 + N, b_2^N = 2N^3 + 5N^2 + 5N, b_3^N = 5N^4 + 22N^3 + 52N^2 + 41N)$.

An alternate recursion formula coupled with the moments of the GUE is somewhat more convenient to generate tables of the moments b_p^N and for a number of applications.

Theorem 3. *For every integer $p \ge 2$, and every $N \ge 1$,*

$$\begin{aligned} b_p^N &= (4N-2)b_{p-1}^N + 4(2p-2)(2p-3)b_{p-2}^N \\ &\quad + a_p^N - (4N-3)a_{p-1}^N - (2p-2)(2p-3)a_{p-2}^N \end{aligned}$$

$(b_0^N = N, b_1^N = N^2 + N)$.

Note that, by induction, $b_p^N \ge a_p^N$ for every p. Here are a few values of the numbers a_p^N (taken from [7]) and b_p^N:

$$\begin{aligned} a_0^N &= N, \\ a_1^N &= N^2, \\ a_2^N &= 2N^3 + N, \\ a_3^N &= 5N^4 + 10N^2, \\ a_4^N &= 14N^5 + 70N^3 + 21N, \\ a_5^N &= 42N^6 + 420N^4 + 483N^2, \end{aligned}$$

$$\begin{aligned} b_0^N &= N, \\ b_1^N &= N^2 + N, \\ b_2^N &= 2N^3 + 5N^2 + 5N, \\ b_3^N &= 5N^4 + 22N^3 + 52N^2 + 41N, \\ b_4^N &= 14N^5 + 93N^4 + 374N^3 + 690N^2 + 509N, \\ b_5^N &= 42N^6 + 386N^5 + 2290N^4 + 7150N^3 + 12143N^2 + 8229N. \end{aligned}$$

In both cases, the factor of highest degree (in N) is given by the Catalan number $\chi_p = \frac{(2p)!}{p!(p+1)!}$, $p \in \mathbb{N}$, which describes the $2p$th moment of the semicircular law in (3).

Denote by $A_p^N = N^{-p-1}a_p^N$ and $B_p^N = N^{-p-1}b_p^N$ the respective moments normalized according to Wigner's law (3). It is then immediate from the recursion equation in Theorem 1 that $A_p^N \to \chi_p$ as $N \to \infty$ for every fixed p. Together with Theorem 3, $B_p^N \to \chi_p$ also. It may in fact even be observed from the recursion formula of Theorem 1 that for every fixed p and every $N \ge 1$,

$$\chi_p \le A_p^N \le \chi_p + \frac{C_p}{N^2},$$

where $C_p > 0$ only depends on p. Combined with the linear recurrence equation for the Catalan numbers $\chi_p = \frac{4p-2}{p+1}\chi_{p-1}$, it actually follows more precisely that the $1/N^2$ correction term α_p given by

$$A_p^N = \chi_p + \frac{\alpha_p}{N^2} + \mathrm{O}\left(\frac{1}{N^4}\right)$$

solves the recurrence equation

$$\alpha_p = \frac{4p-2}{p+1}\alpha_{p-1} + \frac{p(p-1)}{4}\chi_p, \quad p \ge 2$$

($\alpha_0 = \alpha_1 = 0$), yielding thus

$$\alpha_p = \frac{1}{4}\chi_p \sum_{\ell=0}^{p} \ell(\ell-1) = \frac{1}{12}(p-1)p(p+1)\chi_p, \quad p \geq 0. \tag{4}$$

As is classical, the approximation rate is only of the order of $\frac{1}{N}$ in the GOE case, that is

$$\chi_p \leq A_p^N \leq B_p^N \leq \chi_p + \frac{C_p}{N}.$$

Again, we have more precisely from the coupled recursion equation of Theorem 3 that the $1/N$ correction term β_p given by

$$B_p^N = \chi_p + \frac{\beta_p}{N} + \mathrm{O}\left(\frac{1}{N^2}\right)$$

satisfies

$$\beta_p = 4\beta_{p-1} + \chi_{p-1}, \quad p \geq 2$$

($\beta_0 = 0, \beta_1 = 1$), yielding

$$\beta_p = \frac{1}{2}\left[4^p - (p+1)\chi_p\right], \quad p \geq 0, \tag{5}$$

in accordance with the description by Schultz in [16], Theorem 8.1.

The method of proof of Theorems 1–3 we develop here is based on the classical orthogonal polynomial description of the mean spectral measure (cf. [12]). More precisely, we try to reach differential equations on the Laplace transforms (moment generating functions) of the spectral measures that will represent the moment recursion formulas. This is achieved in [6] for the GUE through the explicit representation of this moment generating function as a confluent hypergeometric function. It is not clear whether this is still possible in the GOE case (see, however, the end of Section 3). We instead use simple Gaussian integration by parts arguments to directly reach such a differential equation. Along these lines, the reference [11] develops to this task a general approach based on Markov operators and integration by parts to yield differential equations on Laplace transforms for the classical unitary invariant models. For the matter of completeness and illustration, we reproduce in the next section the argument from [11] in the Hermite case that leads to the GUE recursion formula of Theorem 1. This methodology turns out to be well suited to the real case despite the more complicated form of the mean spectral measure, and Theorem 2 is addressed in this way in Section 3. A first step yields a differential equation for the Laplace transform of the GOE spectral measure coupled with the GUE Laplace transform yielding the recursion formula of Theorem 3. The Gaussian Symplectic Ensemble (GSE) is analyzed similarly in Section 4, with in addition a duality relation between the moments of the GSE of size N and the (formal) moments of the GOE of size $-2N$, in accordance with [13]. The real and symplectic Laguerre and Jacobi ensembles might possibly be analyzed similarly (for the complex case, see [6,11]).

The recursion formula for the moments of the GOE is then applied to the enumeration problem of 1-vertex maps in locally orientable surfaces, the original purpose of the Harer–Zagier recursion formula (in the orientable case). We obtain here the analogous result for unoriented maps, completing in this regard the investigation by Goulden and Jackson [5], where a closed formula for the genus series is put forward.

The last section is devoted to the application of these recursion equations to a sharp moment bound and a small deviation inequality on the largest eigenvalue of the GOE at the Tracy–Widom rate. By simple comparison, the result may be used for classes of Wigner matrices including sign matrices.

The unitary and orthogonal invariance crucially allows for the calculation of the joint law of the eigenvalues $(\lambda_1^N, \ldots, \lambda_N^N)$ of $X = X^N$ of both the GUE and GOE, which in turn may be analyzed by the so-called orthogonal polynomial method (cf. [4,8,12]). Denote by P_ℓ, $\ell \geq 0$, the orthogonal polynomials for the standard Gaussian measure $\mathrm{d}\gamma(x) = \mathrm{e}^{-x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}}$ on $\mathbb{R}$, normalized in $\mathrm{L}^2(\gamma)$, the so-called Hermite polynomials (cf. [18]). Following [12], in the case of the GUE, the mean spectral density is given on every bounded measurable function f on $\mathbb{R}$ by

$$\mathbb{E}\big(\mathrm{Tr}\big(f\big(X^N\big)\big)\big) = \mathbb{E}\left(\sum_{i=1}^N f\big(\lambda_i^N\big)\right) = \int_{\mathbb{R}} f(x) \sum_{\ell=0}^{N-1} P_\ell^2(x)\,\mathrm{d}\gamma(x). \tag{6}$$

In case of the GOE,

$$\mathbb{E}\big(\mathrm{Tr}\big(f\big(X^N\big)\big)\big) = \mathbb{E}\left(\sum_{i=1}^N f\big(\lambda_i^N\big)\right) = \int_{\mathbb{R}} f(x)\mu_{\mathrm{GOE}}^N(x)\,\mathrm{d}\gamma(x), \tag{7}$$

where

$$\mu_{\mathrm{GOE}}^N = \sum_{\ell=0}^{N-1} P_\ell^2 + \sqrt{\frac{\pi N}{8}}\varphi\psi P_{N-1} + \frac{\varphi P_{N-1}}{\int \varphi P_{N-1}\,\mathrm{d}\gamma}\mathbf{1}_{N \text{ odd}}$$

with

$$\varphi(x) = \mathrm{e}^{x^2/4} \quad \text{and} \quad \psi(x) = \psi_N(x) = \int_{\mathbb{R}} \mathrm{sgn}(x-y)\varphi(y)P_N(y)\,\mathrm{d}\gamma(y), \quad x \in \mathbb{R}.$$

In particular therefore, $a_p^N = \int x^{2p} \sum_{\ell=0}^{N-1} P_\ell^2\,\mathrm{d}\gamma$ and $b_p^N = \int x^{2p}\mu_{\mathrm{GOE}}^N\,\mathrm{d}\gamma$, $p \in \mathbb{N}$.

Formulas (6) and (7) will be used towards the proofs of Theorems 1–3 together with simple integration by parts arguments with respect to the standard Gaussian measure γ and the Hermite polynomials. Recall in particular that, for every smooth function f on $\mathbb{R}$,

$$\int xf\,\mathrm{d}\gamma = \int f'\,\mathrm{d}\gamma. \tag{8}$$

For each integer N, the Hermite polynomial P_N is an eigenvector with eigenvalue $-N$ of the Ornstein–Uhlenbeck operator L acting on smooth functions f as $\mathrm{L}f = f'' - xf'$. For smooth functions f and g, the operator L satisfies the integration by parts formula with respect to γ,

$$\int f(-\mathrm{L}g)\,\mathrm{d}\gamma = \int f'g'\,\mathrm{d}\gamma. \tag{9}$$

In particular therefore,

$$N\int fP_N\,\mathrm{d}\gamma = \int f(-\mathrm{L}P_N)\,\mathrm{d}\gamma = \int f'P_N'\,\mathrm{d}\gamma. \tag{10}$$

Formula (8) is a particular case of (10) with $P_1 \equiv x$. Note also that P_N is an even (resp. odd) function if N is even (resp. odd) and that, with the normalization chosen here, $P_N' = \sqrt{N}P_{N-1}$.

2. GUE moment recursion formula

As announced, we reproduce here the integration by parts argument developed in [11] to prove Theorem 1 (see also [6,12]). With respect to the more general investigation of [11], we outline, for the matter of clarity, a somewhat more direct exposition.

According to the spectral distribution of the GUE, we investigate the Laplace transform, or moment generating function, u of the mean spectral density, that is

$$u(t) = u_N(t) = \mathbb{E}\big(\mathrm{Tr}\big(\mathrm{e}^{tX^N}\big)\big) = \mathbb{E}\left(\sum_{i=1}^N \mathrm{e}^{t\lambda_i^N}\right), \quad t \in \mathbb{R}.$$

By the orthogonal polynomial representation (6),

$$u = u(t) = \int \mathrm{e}^{tx} \sum_{\ell=0}^{N-1} P_\ell^2 \, \mathrm{d}\gamma.$$

The idea is to show that u satisfies a second-order differential equation which then immediately leads to the three term recurrence equation of Theorem 1.

To this task, first note that for any smooth function f on $\mathbb{R}$, and any $N \geq 1$,

$$\int f' \sum_{\ell=0}^{N-1} P_\ell^2 \, \mathrm{d}\gamma = \sqrt{N} \int f P_{N-1} P_N \, \mathrm{d}\gamma. \tag{11}$$

One argument goes as follows. The recurrence relation for the Hermite polynomials P_N, $N \in \mathbb{N}$, is given by (cf. [18])

$$x P_N = \sqrt{N+1} P_{N+1} + \sqrt{N} P_{N-1}.$$

(This may be seen as a consequence of the fact that P_N is eigenvector of L with eigenvalue $-N$ and that $P_N' = \sqrt{N} P_{N-1}$.) As a consequence, for every ℓ,

$$\left(P_\ell^2\right)' - x P_\ell^2 = P_\ell \left[2 P_\ell' - x P_\ell\right] = \sqrt{\ell} P_{\ell-1} P_\ell - \sqrt{\ell+1} P_\ell P_{\ell+1}.$$

Summing over $\ell = 0, \ldots, N-1$,

$$\left(\sum_{\ell=0}^{N-1} P_\ell^2\right)' - x \left(\sum_{\ell=0}^{N-1} P_\ell^2\right) = -\sqrt{N} P_{N-1} P_N.$$

Integrating against a smooth function f with respect to γ and making use of (8) yields the conclusion (11).

As a consequence of (11), to investigate u we rather analyze

$$\rho = \rho(t) = \int \mathrm{e}^{tx} P_{N-1} P_N \, \mathrm{d}\gamma, \quad t \in \mathbb{R},$$

since $tu = \sqrt{N} \rho$. Actually, it will be convenient to investigate first

$$\sigma = \sigma(t) = \int \mathrm{e}^{tx} P_N^2 \, \mathrm{d}\gamma, \quad t \in \mathbb{R}.$$

Our aim is thus to show that σ satisfies a second-order differential equation. To this task, apply first (8) to get

$$\sigma' = \int x \mathrm{e}^{tx} P_N^2 \, \mathrm{d}\gamma = t\sigma + 2 \int \mathrm{e}^{tx} P_N' P_N \, \mathrm{d}\gamma. \tag{12}$$

Taking the derivative again,

$$\sigma'' - 2t\sigma' + \left(t^2 - 1\right)\sigma = 2 \int \mathrm{e}^{tx} {P_N'}^2 \, \mathrm{d}\gamma + 2 \int \mathrm{e}^{tx} P_N'' P_N \, \mathrm{d}\gamma. \tag{13}$$

Now, by (10) and (12),

$$\begin{aligned} N\sigma &= \int \mathrm{e}^{tx} P_N (-\mathrm{L} P_n) \, \mathrm{d}\gamma = t \int \mathrm{e}^{tx} P_N' P_N \, \mathrm{d}\gamma + \int \mathrm{e}^{tx} {P_N'}^2 \, \mathrm{d}\gamma \\ &= \frac{t}{2}\left(\sigma' - t\sigma\right) + \int \mathrm{e}^{tx} {P_N'}^2 \, \mathrm{d}\gamma \end{aligned}$$

so that

$$2\int \mathrm{e}^{tx} {P_N'}^2 \,\mathrm{d}\gamma = -t\sigma' + \left(t^2 + 2N\right)\sigma. \tag{14}$$

In the same way, starting from (12),

$$\begin{aligned} N\left(\sigma' - t\sigma\right) &= 2\int \mathrm{e}^{tx} P_N'(-\mathrm{L}P_N)\,\mathrm{d}\gamma \\ &= 2t\int \mathrm{e}^{tx} {P_N'}^2\,\mathrm{d}\gamma + 2\int \mathrm{e}^{tx} P_N' P_N''\,\mathrm{d}\gamma, \end{aligned}$$

so that, together with (14),

$$2\int \mathrm{e}^{tx} P_N' P_N''\,\mathrm{d}\gamma = \left(t^2 + N\right)\sigma' - t\left(t^2 + 3N\right)\sigma. \tag{15}$$

Now, $P_N' = \sqrt{N} P_{N-1}$ is an eigenfunction of L with eigenvalue $-(N-1)$. Thus by (10) again for $N-1$, and (12),

$$\begin{aligned} (N-1)\left(\sigma' - t\sigma\right) &= 2\int \mathrm{e}^{tx} P_N\left(-\mathrm{L}P_N'\right)\mathrm{d}\gamma \\ &= 2t\int \mathrm{e}^{tx} P_N P_N''\,\mathrm{d}\gamma + 2\int \mathrm{e}^{tx} P_N' P_N''\,\mathrm{d}\gamma. \end{aligned}$$

Together with (15),

$$2t\int \mathrm{e}^{tx} P_N P_N''\,\mathrm{d}\gamma = -\left(t^2+1\right)\sigma' + t\left(t^2 + 2N + 1\right)\sigma. \tag{16}$$

It remains to insert (16) and (14) into (13) to get that σ solves the second-order differential equation in t,

$$t\sigma'' + \sigma' - t\left(t^2 + 4N + 2\right)\sigma = 0 \tag{17}$$

which is the desired claim.

From (12) and $P_N' = \sqrt{N} P_{N-1}$, $\sigma' - t\sigma = 2\sqrt{N}\rho$. From this equation and (17), it is not difficult to see that ρ solves the differential equation

$$t^2\rho'' + t\rho' - \left[t^2\left(t^2 + 4N\right) + 1\right]\rho = 0.$$

Since, by (11), $\sqrt{N}\rho = tu$, this differential equation shows in turn that u satisfies

$$tu'' + 3u' - t\left(t^2 + 4N\right)u = 0. \tag{18}$$

The solution of this equation is a confluent hypergeometric function (cf. [6]). As announced, this differential equation may immediately be translated into a three term recurrence equation on the (even) moments $a_p^N = \int x^{2p} \sum_{\ell=1}^{N-1} P_\ell^2 \,\mathrm{d}\gamma$ since, as an entire function,

$$u = u(t) = \sum_{p=0}^{\infty} \frac{t^{2p}}{(2p)!} a_p^N.$$

Theorem 1 is established in this way.

3. GOE moment recursion formula

We follow here the same strategy as in the previous section, but have to work with the more delicate spectral distribution (7).

Recall $\varphi(x) = \mathrm{e}^{x^2/4}$ and $\psi(x) = \int \mathrm{sgn}(x-y)\varphi(y) P_N(y)\,\mathrm{d}\gamma(y)$, $x \in \mathbb{R}$. Note that $\varphi\psi' = \sqrt{\frac{2}{\pi}} P_N$.

Set first

$$\tau_1 = \tau_1(t) = \int \mathrm{e}^{tx} \varphi\psi P_{N-1}\,\mathrm{d}\gamma, \quad t \in \mathbb{R}.$$

To start with,

$$\tau_1' = 2t\tau_1 + 2\sqrt{\frac{2}{\pi}}\rho + 2\int \mathrm{e}^{tx}\varphi\psi P_{N-1}'\,\mathrm{d}\gamma, \tag{19}$$

where we recall that $\rho = \int \mathrm{e}^{tx} P_{N-1} P_N\,\mathrm{d}\gamma$. Taking again the derivative,

$$\tau_1'' = 2\tau_1 + 2t\tau_1' + 2\sqrt{\frac{2}{\pi}}\rho' + 2\int x\mathrm{e}^{tx}\varphi\psi P_{N-1}'\,\mathrm{d}\gamma. \tag{20}$$

Now, by (10),

$$\begin{aligned}(N-1)\tau_1 &= \int \mathrm{e}^{tx}\varphi\psi(-\mathrm{L}P_{N-1})\,\mathrm{d}\gamma \\ &= t\int \mathrm{e}^{tx}\varphi\psi P_{N-1}'\,\mathrm{d}\gamma + \frac{1}{2}\int x\mathrm{e}^{tx}\varphi\psi P_{N-1}'\,\mathrm{d}\gamma + \sqrt{\frac{2}{\pi}}\int \mathrm{e}^{tx} P_N P_{N-1}'\,\mathrm{d}\gamma.\end{aligned}$$

Therefore, together with (19),

$$\begin{aligned}4(N-1)\tau_1 &= 2t\left[\tau_1' - 2t\tau_1 - 2\sqrt{\frac{2}{\pi}}\rho\right] \\ &\quad + 2\int x\mathrm{e}^{tx}\varphi\psi P_{N-1}'\,\mathrm{d}\gamma + 4\sqrt{\frac{2}{\pi}}\int \mathrm{e}^{tx} P_N P_{N-1}'\,\mathrm{d}\gamma.\end{aligned}$$

By difference with (20),

$$\tau_1'' - \left(4t^2 + 4N - 2\right)\tau_1 = \sqrt{\frac{8}{\pi}}\left(\rho' + 2t\rho - 2\int \mathrm{e}^{tx} P_N P_{N-1}'\,\mathrm{d}\gamma\right). \tag{21}$$

Now, by (8),

$$\rho' = t\rho + \int \mathrm{e}^{tx} P_{N-1}' P_N\,\mathrm{d}\gamma + \int \mathrm{e}^{tx} P_{N-1} P_N'\,\mathrm{d}\gamma. \tag{22}$$

On the other hand, by (10) applied for N and $N-1$,

$$N\rho = t\int \mathrm{e}^{tx} P_{N-1} P_N'\,\mathrm{d}\gamma + \int \mathrm{e}^{tx} P_{N-1}' P_N'\,\mathrm{d}\gamma$$

and

$$(N-1)\rho = t\int \mathrm{e}^{tx} P_N P_{N-1}'\,\mathrm{d}\gamma + \int \mathrm{e}^{tx} P_N' P_{N-1}'\,\mathrm{d}\gamma.$$

By difference,

$$\rho = t\int \mathrm{e}^{tx} P_{N-1} P_N' \,\mathrm{d}\gamma - t\int \mathrm{e}^{tx} P_N P_{N-1}' \,\mathrm{d}\gamma.$$

Together with (22), for $t \neq 0$,

$$\begin{aligned}
&\rho' + 2t\rho - 2\int \mathrm{e}^{tx} P_N P_{N-1}' \,\mathrm{d}\gamma \\
&\quad = 3t\rho + \int \mathrm{e}^{tx} P_{N-1} P_N' \,\mathrm{d}\gamma - \int \mathrm{e}^{tx} P_N P_{N-1}' \,\mathrm{d}\gamma \\
&\quad = \left(3t + \frac{1}{t}\right)\rho.
\end{aligned}$$

Recall that $\sqrt{N}\rho = tu$. Therefore, Eq. (21) shows finally that τ_1 solves the second-order differential equation in t,

$$\tau_1'' - \left(4t^2 + 4N - 2\right)\tau_1 = \sqrt{\frac{8}{\pi N}}\left(3t^2 + 1\right)u. \tag{23}$$

It is easy to see that, similarly, if we let

$$\tau_2 = \tau_2(t) = \int \mathrm{e}^{tx} \varphi P_{N-1} \,\mathrm{d}\gamma, \quad t \in \mathbb{R},$$

then τ_2 solves the equation

$$\tau_2'' - \left(4t^2 + 4N - 2\right)\tau_2 = 0. \tag{24}$$

Set

$$\alpha_N = \sqrt{\frac{\pi N}{8}} \quad \text{and} \quad \beta_N = \left(\int \varphi P_{N-1} \,\mathrm{d}\gamma\right)^{-1} \mathbf{1}_{N \text{ odd}}.$$

According to the spectral distribution (7) of the GOE, write $\tau = \alpha_N \tau_1 + \beta_N \tau_2$ that, as a consequence of (23) and (24), solves the differential equation

$$\tau'' - \left(4t^2 + 4N - 2\right)\tau = \left(3t^2 + 1\right)u, \tag{25}$$

where we recall that u is the moment generating function of the spectral distribution of the GUE.

Denote below by

$$v(t) = v_N(t) = \mathbb{E}\left(\mathrm{Tr}\left(\mathrm{e}^{tX^N}\right)\right) = \mathbb{E}\left(\sum_{i=1}^N \mathrm{e}^{t\lambda_i^N}\right) = \int \mathrm{e}^{tx} \mu_{\mathrm{GOE}} \,\mathrm{d}\gamma, \quad t \in \mathbb{R},$$

the Laplace transform (moment generating function) of the spectral distribution of the GOE. Since by (7), $v = u + \tau$, we thus conclude to the following statement.

Proposition 4. *Under the preceding notation, v solves the differential equation*

$$v'' - \left(4t^2 + 4N - 2\right)v = u'' - \left(t^2 + 4N - 3\right)u.$$

Since $v(t) = \sum_{p=0}^\infty t^{2p} b_p^N/(2p!)$, Proposition 4 immediately translates into a recurrence equation on the (even) moments b_p^N, $p \in \mathbb{N}$, of the GOE coupled with the moments of the GUE to yield Theorem 3. Note that $b_1^N = N^2 + N$ may be computed directly as $\mathbb{E}(\mathrm{Tr}((X^N)^2))$.

We now make use of Proposition 4 to reach Theorem 2. Set

$$V = V(t) = v'' - \big(4t^2 + 4N - 2\big)v$$

so that Proposition 4 reads $V = u'' - (t^2 + 4N - 3)u$. On the other hand, recall from Section 2 that u solves the equation

$$tu'' + 3u' - t\big(t^2 + 4N\big)u = 0. \tag{26}$$

Therefore $tV = 3(tu - u')$. Taking the derivative and using (26) once more yields

$$tV' + \big(t^2 + 4\big)V = -12(N-1)u.$$

Therefore

$$tV'' + \big(t^2 + 5\big)V' + 2tV = -12(N-1)u',$$

and together with $tV = 3(tu - u')$, it follows that V solves the differential equation

$$tV'' + 5V' - t\big(t^2 + 4N - 2\big)V = 0.$$

Note that this is the generic second-order differential equation for confluent hypergeometric functions. Replacing V by $v'' - (4t^2 + 4N - 2)v$ then shows that v solves the fourth-order differential equation

$$\begin{aligned} &tv^{(4)} + 5v^{(3)} - t\big(5t^2 + 8N - 4\big)v'' - \big(36t^2 + 20N - 10\big)v' \\ &\quad + t\big[4t^4 + (20N - 10)t^2 + 16N^2 - 16N - 44\big]v = 0. \end{aligned} \tag{27}$$

We then conclude to the recursion formula of Theorem 2 as for the GUE. The first terms are determined from Theorem 3.

4. GSE moment recursion formula

In this short section, we briefly outline the corresponding recursion formula for the Gaussian Symplectic Ensemble (GSE). A random Hermitian matrix $X = X^N = (X_{ij}^N)_{1\le i,j\le N}$ with real quaternion elements is said to belong to the GSE if the real quaternions X_{ii}^N, $1 \le i \le N$, are independent normal with mean zero and variance 1 while the quaternion elements X_{ij}^N, $1 \le i < j \le N$, are independent and their four coordinates are independent Gaussian variables with mean zero and variance $\frac{1}{2}$.

According again to [12], the mean spectral measure is given in this case by

$$\mathbb{E}\left(\sum_{i=1}^{N} f\big(\lambda_i^N\big)\right) = \int_{\mathbb{R}} f(x)\mu_{\mathrm{GSE}}^N(x)\,\mathrm{d}\gamma(x),$$

where now

$$\mu_{\mathrm{GSE}}^N = \frac{1}{2}\sum_{\ell=0}^{2N} P_\ell^2 + \frac{1}{2}\sqrt{\frac{\pi(2N+1)}{8}}\varphi\psi_{2N+1}P_{2N}.$$

Recall that here

$$\varphi(x) = \mathrm{e}^{x^2/4} \quad \text{and} \quad \psi_{2N+1}(x) = \int_{\mathbb{R}} \operatorname{sgn}(x-y)\varphi(y)P_{2N+1}(y)\,\mathrm{d}\gamma(y), \qquad x \in \mathbb{R}.$$

Slight modifications of the arguments developed in the preceding section then yield analogous recursion formulas for the even moments c_p^N, $p \in \mathbb{N}$, of the GSE. First, observe that the moment generating function w of the GSE satisfies the differential equation of Proposition 4 with N replaced by $2N+1$ and with $u = u^{2N+1}$ the moment generating function of the GUE of size $2N+1$. As a consequence, w solves the fourth-order differential equation (27) with N replaced by $2N+1$. The sequence of moments c_p^N, $p \in \mathbb{N}$, thus satisfied statements analogous to Theorems 2 and 3.

Of more interest perhaps is a recursion formula coupled with the moments b_p^{2N+1} of the GOE of dimension $2N+1$. Since

$$\mu_{\mathrm{GSE}}^N = \frac{1}{2}\mu_{\mathrm{GOE}}^{2N+1} - \frac{1}{2}\frac{\varphi P_{2N}}{\int \varphi P_{2N}\,\mathrm{d}\gamma},$$

and since the Laplace transform τ_2 of $\varphi P_{2N}/\int \varphi P_{2N}\,\mathrm{d}\gamma$ solves the differential equation $\tau_2'' - (4t^2 + 8N + 2)\tau_2 = 0$, the following conclusion may be deduced.

Theorem 5. *For every integer $p \geq 2$, and every $N \geq 1$,*

$$\begin{aligned} 2c_p^N &= (16N+4)c_{p-1}^N + 8(2p-2)(2p-3)c_{p-2}^N \\ &\quad + b_p^{2N+1} - (8N+2)b_{p-1}^{2N+1} - 4(2p-2)(2p-3)b_{p-2}^{2N+1} \end{aligned}$$

$(c_0^N = N, c_1^N = 2N^2 - N)$.

A further connection with the moments of the GOE was suggested to us by Haagerup. Namely, w thus solves the differential equation (27) with N replaced by $2N+1$. Then $w(\mathrm{i}t)$, $t \in \mathbb{R}$, solves the differential equation (27) with N replaced by $-2N$. Checking the initial conditions $c_0^N, c_1^N, c_2^N, c_3^N$, for example with the help of Theorem 5 above, this observation yields a duality identity between the moments c_p^N of the GSE and the formal moments b_p^{-2N} of the GOE of size $-2N$ in the form of the following statement. The duality between the Orthogonal and Symplectic Ensembles was put forward by Mulase and Waldron in their paper [13] in terms of graphical and Feynman diagram expansions of Gaussian integrals. The extension to multi-matrix models is discussed in the recent work [2].

Theorem 6. *For every integer p, and every $N \geq 1$,*

$$2c_p^N = (-1)^{p+1} b_p^{-2N}.$$

For example,

$$\begin{aligned} c_0^N &= N, \\ c_1^N &= 2N^2 - N, \\ c_2^N &= 8N^3 - 10N^2 + 5N, \\ c_3^N &= 40N^4 - 88N^3 + 104N^2 - 41N, \\ c_4^N &= 224N^5 - 744N^4 + 1496N^3 - 1380N^2 + 509N, \\ c_5^N &= 1344N^6 - 6176N^5 + 18320N^4 - 28600N^3 + 24286N^2 - 8229N. \end{aligned}$$

5. Map enumeration in locally orientable surfaces

In their work [7], Harer and Zagier described a map enumeration problem by Gaussian matrix integrals. For every integer p, denote by $\varepsilon_g(p)$ the number of ways of putting the consecutive sides of a $2p$-gon into p pairs, each side belonging to one and only one pair, and such that if the sides in each pair are identified, one obtains an orientable surface of genus g. An alternate equivalent description characterizes $\varepsilon_g(p)$ as the number of oriented 1-vertex maps with p edges and k faces imbedded into a (orientable) surface of genus g ($k = p + 1 - 2g$) (cf. [22] for an accessible introduction). The numbers $\varepsilon_g(p)$ are non-zero only for $p \geq 2g$. By the Wick calculus for integrals of Gaussian polynomials, Harer and Zagier showed that the pth moments a_p^N, $p \in \mathbb{N}$, of the GUE describes the generating series in N,

$$a_p^N = \sum_{g \geq 0} \varepsilon_g(p) N^{p+1-2g} \tag{28}$$

of the numbers $\varepsilon_g(p)$. In particular, the recursion formula of Theorem 1 may be translated equivalently into the following recursion formula for the numbers $\varepsilon_g(p)$, $g \geq 1$, $p \geq 2$,

$$(p+1)\varepsilon_g(p) = (4p-2)\varepsilon_g(p-1) + (p-1)(2p-1)(2p-3)\varepsilon_{g-1}(p-2) \tag{29}$$

(with the boundary conditions $\varepsilon_0(p) = \chi_p$ for every p). Note that, from (4), $\varepsilon_1(p) = \frac{1}{12}(p-1)p(p+1)\chi_p$. A purely geometric proof of this formula seems still to be lacking (cf. [22]). This formula may also be used to give the closed form

$$a_p^N = \frac{(2p)!}{2^p p!} \sum_{r=0}^{p} 2^r \binom{p}{r} \binom{N}{r+1} \tag{30}$$

(see [5,7,12]). A direct combinatorial and self-contained proof of the equality between the right-hand sides of (28) and (30) is provided in [9]. Actually, setting $\tilde{a}_p^N = (2^p p!/(2p)!)a_p^N$, Theorem 1 shows that

$$(p+1)\tilde{a}_p^N = 2N\tilde{a}_{p-1}^N + (p-1)\tilde{a}_{p-2}^N, \quad p \geq 2.$$

This recursion equation is easily solved by the generating series

$$1 + 2\sum_{p=0}^{\infty} \tilde{a}_p^N x^{p+1} = \left(\frac{1+x}{1-x}\right)^N.$$

Expanding the binomial $(1+x)^N(1-x)^{-N}$ then yields (30).

The real analogue has been studied by Goulden and Jackson [5] who described the generating series of the numbers of pairings of the sides of a $2p$-gon leading to unoriented surfaces. It will be more convenient to adopt the language of 1-vertex maps for which the moments b_p^N, $p \in \mathbb{N}$, of the GOE represent the generating series

$$b_p^N = \sum_{k \geq 1} \eta_k(p) N^k,$$

where $\eta_k(p)$ is the number of 1-vertex maps with p edges and k faces in locally orientable surfaces (of genus g such that $k = p + 1 - 2g$ for orientable surfaces and $k = p + 1 - g$ for non-orientable ones). For example, in accordance with [5], $b_2^N = 2N^3 + 5N^2 + 5N$. In particular, $\eta_1(2) = 5$, that is there are 5 maps in locally orientable surfaces with 1 vertex, 2 edges and 1 face. As in the GUE case, $\eta_{p+1}(p) = \chi_p$ for every p, while, from (5), $\eta_p(p) = \frac{1}{2}[4^p - (p+1)\chi_p]$. The numbers $b_p^N - a_p^N$ describe the genus series for 1-vertex maps in non-orientable surfaces with p edges. Using classical expansions associated with the Hermite polynomials, Goulden and Jackson found the following closed form of the moments

$$b_p^N = p! \sum_{r=0}^{p} 2^{2p-r} \sum_{\ell=0}^{p} \binom{p-1/2}{p-\ell} \binom{r+\ell-1}{r} \binom{(N-1)/2}{\ell} + a_p^{N-1}. \tag{31}$$

The strategy in [5] also relies on the spectral density (7), expressing the Hermite polynomials by their coefficients, performing the integrations and then coming back to the Hermite polynomials by reverse formulas. It is not clear whether this approach can also lead to recursion formulas, and conversely whether the recursion formulas can easily yield the closed form (31). Nevertheless, setting as in the complex case $\tilde{b}_p^N = (2^p p!/(2p)!) b_p^N$, Theorem 3 shows that the generating function $\psi(x) = \sum_{p=0}^\infty \tilde{b}_p^N x^{p+1}$ solves the differential equation

$$4x\big(1-4x^2\big)\psi' - 2\big[3+(4N-3)x\big]\psi = 4Nx - 3(1-x)\left[1 - \left(\frac{1+x}{1-x}\right)^N\right]$$

which has an explicit solution. In any case, the recursion formulas of Theorems 2 and 3 are of course more efficient to generate tables of the moments b_p^N. An algorithm for map enumeration is presented in the recent [14].

In the next statement, we draw from Theorem 2 the analogue of the Harer–Zagier recursion formula (29) for the numbers $\eta_k(p)$.

Corollary 7. *Denote by $\eta_k(p)$ the number of unoriented 1-vertex maps with p edges and k faces. For every $p \geq 4$,*

$$\begin{aligned}(p+1)\eta_k(p) = {} & (8p-2)\eta_{k-1}(p-1) - (4p-1)\eta_k(p-1) \\ & + p(2p-3)(10p-9)\eta_k(p-2) - 8(2p-3)\eta_{k-2}(p-2) \\ & + 8(2p-3)\eta_{k-1}(p-2) - 10(2p-3)(2p-4)(2p-5)\eta_{k-1}(p-3) \\ & + 5(2p-3)(2p-4)(2p-5)\eta_k(p-3) \\ & - 2(2p-3)(2p-4)(2p-5)(2p-6)(2p-7)\eta_k(p-4)\end{aligned}$$

with the convention that $\eta_k(p) = 0$ if $k \leq 0$ or $k > p+1$, and the boundary conditions $\eta_{p+1}(p) = \chi_p$ for every p, $\eta_1(1) = 1$, $\eta_1(2) = \eta_2(2) = 5$, $\eta_1(3) = 41$, $\eta_2(3) = 52$, $\eta_3(3) = 22$.

6. Small deviation inequality on the largest eigenvalue

In this final section, we draw from the recursion equation of Theorem 3 a simple bound on the moments of the GOE that entails the deviation inequality at the Tracy–Widom fluctuation regime.

Theorem 8. *Let X^N belong to the GOE of size $N \geq 1$. For every integer p such that $p^3 \geq N^2$,*

$$\mathbb{E}\big(\mathrm{Tr}\big(\big(X^N\big)^{2p}\big)\big) \leq C(4N)^p \left(1 + \frac{p^2}{N^2}\right)^{2p},$$

where $C > 0$ is numerical.

Proof. First, we recall from [11] the corresponding bound for the GUE. As in the Introduction, set $A_p^N = N^{-p-1} a_p^N$, $p \in \mathbb{N}$, $N \geq 1$. Then the Harer–Zagier formula from Theorem 1 reads, for every $N \geq 1$ and $p \geq 2$,

$$A_p^N = \frac{4p-2}{p+1} A_{p-1}^N + \frac{4p-2}{p+1} \cdot \frac{4p-6}{p} \cdot \frac{p(p-1)}{4N^2} A_{p-2}^N.$$

Recall the Catalan numbers

$$\chi_p = \frac{4p-2}{p+1}\chi_{p-1} = \frac{(2p)!}{p!(p+1)!}, \qquad p \in \mathbb{N}.$$

Thus, by a simple induction, for every p,

$$A_p^N \leq \left(1 + \frac{p^2}{4N^2}\right)^p \chi_p. \tag{32}$$

Furthermore, by Stirling's formula, there is a numerical constant $C > 0$ such that $\chi_p \leq C 4^p p^{-3/2}$ for every $p \geq 1$. In particular, Theorem 8 holds for the GUE.

We proceed along the same lines with the coupled recursion formula of Theorem 3 for the moments of the GOE. Set similarly $B_p^N = N^{-p-1} b_p^N$, $p \in \mathbb{N}$, $N \geq 1$. To this task, it will be convenient to work with the numbers $D_p^N = B_p^N - A_p^N$, $p \in \mathbb{N}$, which, according to Theorem 3, satisfy

$$D_p^N = \left(4 - \frac{2}{N}\right) D_{p-1}^N + \frac{4(2p-2)(2p-3)}{N^2} D_{p-2}^N + \frac{1}{N} A_{p-1}^N + \frac{3(2p-2)(2p-3)}{N^2} A_{p-2}^N$$

for every $p \geq 2$. Note that $D_0^N = 0$ and $D_1^N = \frac{1}{N}$. Define

$$E_p^N = \frac{1}{N} A_{p-1}^N + \frac{16p^2}{N^2} A_{p-2}^N, \quad p \geq 2$$

($E_0^N = E_1^N = 0$), so that, for every $p \geq 2$,

$$D_p^N \leq 4 D_{p-1}^N + \frac{16p^2}{N^2} D_{p-2}^N + E_p^N.$$

By induction, it follows that for every p,

$$D_p^N \leq 4^p \left(D_1^N + \sum_{\ell=0}^{p} 4^{-\ell} E_\ell^N\right)\left(1 + \frac{p^2}{N^2}\right)^p.$$

Now, by (32), it is easily checked that for $p^3 \geq N^2$,

$$\sum_{\ell=0}^{p} 4^{-\ell} E_\ell^N \leq \frac{C}{N}\left(1 + \frac{p^2}{N^2}\right)^p$$

for some numerical constant $C > 0$. Since $B_p^N = D_p^N + A_p^N$, the conclusion follows. □

One major advance in modern random matrix theory is the Tracy–Widom fluctuation result [19] indicating that for a random matrix X^N from the GUE,

$$\mathbb{P}\left(\lambda_N^N \leq 2\sqrt{N} + sN^{-1/6}\right) \to F_2(s), \quad s \in \mathbb{R},$$

where λ_N^N is the largest eigenvalue of X^N and F_2 is the so-called Tracy–Widom distribution (cf. e.g. [8] for an introduction). This law has a somewhat intricate description. For our purpose here, note that it is non-centered and its behavior at $+\infty$ is given by

$$C^{-1} \mathrm{e}^{-Cs^{3/2}} \leq 1 - F_2(s) \leq C \mathrm{e}^{-s^{3/2}/C} \tag{33}$$

for s large and C numerical. A similar result holds for the GOE model [20] with a limiting distribution F_1 of the same nature, satisfying in particular also (33).

These Gaussian results have been extended by Soshnikov [17], using a moment comparison principle, to families of symmetric (or Hermitian) Wigner matrices $Y^N = (Y_{ij}^N)_{1 \leq i,j \leq N}$ with independent entries Y_{ij}^N, $1 \leq i \leq j \leq N$, with $\mathbb{E}((Y_{ij}^N)^2) = 1$ and having a symmetric distribution with subgaussian tail, that is

$$\mathbb{E}\left(\left(Y_{ij}^N\right)^{2p}\right) \leq (Cp)^p$$

for some $C > 0$ and every $p \in \mathbb{N}$. This moment hypothesis has been weakened in [15].

In [1,10,11], it is shown that for the GUE, for every $0 < \varepsilon \le \sqrt{N}$,

$$\mathbb{P}\big(\lambda_N^N \ge 2\sqrt{N} + \varepsilon\big) \le C\mathrm{e}^{-N^{1/4}\varepsilon^{3/2}/C},$$

where $C > 0$ is numerical, in accordance thus with the Tracy–Widom asymptotics and (33) (choose $\varepsilon = sN^{-1/6}$). On the basis of Theorem 8, we show here the analogous bound in case of the GOE. The real case has the advantage to produce similar bounds for classes of Wigner matrices by a simple moment comparison argument. In particular, this result complements Soshnikov's [17] extension of the Tracy–Widom theorem. Such comparisons may be developed in the complex setting as well but take a less appealing form.

Corollary 9. *Let $Y^N = (Y_{ij}^N)_{1\le i,j\le N}$ be a $N \times N$ symmetric matrix such that the entries Y_{ij}^N, $1 \le i \le j \le N$, are independent symmetric random variables such that, for every integer p,*

$$\mathbb{E}\big(\big(Y_{ij}^N\big)^{2p}\big) \le \mathbb{E}\big(\big(X_{ij}^N\big)^{2p}\big),$$

where X^N is taken from the GOE of size N. Denote by $\lambda_N^N(Y)$ the largest eigenvalue of Y^N. Then, for every $0 < \varepsilon \le \sqrt{N}$,

$$\mathbb{P}\big(\lambda_N^N(Y) \ge 2\sqrt{N} + \varepsilon\big) \le C\mathrm{e}^{-N^{1/4}\varepsilon^{3/2}/C},$$

where $C > 0$ is numerical.

The proof of the corollary simply follows from the fact that, under the hypotheses,

$$\mathbb{E}\big(\mathrm{Tr}\big(\big(Y^N\big)^{2p}\big)\big) = \mathbb{E}\bigg(\sum_{1\le i_1,\dots,i_{2p}\le N} Y_{i_1i_2}^N Y_{i_2i_3}^N \cdots Y_{i_{2p}i_1}^N\bigg) \le \mathbb{E}\big(\mathrm{Tr}\big(\big(X^N\big)^{2p}\big)\big)$$

for every integer p. Then, by Markov's inequality and Theorem 8, for every p such that $p^3 \ge N^2$ and $0 < \varepsilon \le \sqrt{N}$,

$$\begin{aligned}
\mathbb{P}\big(\lambda_N^N(Y) \ge 2\sqrt{N} + \varepsilon\big) &\le \big(2\sqrt{N} + \varepsilon\big)^{-2p}\mathbb{E}\Bigg(\sum_{i=1}^N \big(\lambda_i^N(Y)\big)^{2p}\Bigg) \\
&\le \big(2\sqrt{N} + \varepsilon\big)^{-2p}\mathbb{E}\big(\mathrm{Tr}\big(\big(X^N\big)^{2p}\big)\big) \\
&\le C_1\big(2\sqrt{N} + \varepsilon\big)^{-2p}(4N)^p\mathrm{e}^{2p^3/N^2} \\
&\le C_1\mathrm{e}^{-\varepsilon p/2\sqrt{N} + 2p^3/N^2},
\end{aligned}$$

where C_1 is the constant of Theorem 8. Assume first $N^{1/4}\varepsilon^{3/2} \ge 10^3$. Choose p to be the integer part of $\sqrt{\frac{\varepsilon}{12}}N^{3/4}$. Then $p^3 \ge N^2$ and

$$\mathrm{e}^{-\varepsilon p/2\sqrt{N} + 2p^3/N^2} \le \sqrt{\mathrm{e}}\mathrm{e}^{-N^{1/4}\varepsilon^{3/2}/6\sqrt{3}}.$$

Thus the inequality of the corollary follows in this case with $C = \max(\sqrt{\mathrm{e}}C_1, 6\sqrt{3})$. When $N^{1/4}\varepsilon^{3/2} \le 10^3$,

$$\mathbb{P}\big(\lambda_N^N(Y) \ge 2\sqrt{N} + \varepsilon\big) \le 1 \le \mathrm{e}^{1-(N^{1/4}\varepsilon^{3/2}/10^3)}$$

and the result follows here with $C = 10^3$.

Examples of matrices Y^N include matrices such that Y_{ij}^N, $1 \le i \le j \le N$, are independent and symmetric with $|Y_{ij}^N| \le 1$ almost surely, in particular $Y_{ij}^N = \pm 1$ with equal probability. Another example consist of entries Y_{ij}^N with symmetric distributions $\mathrm{e}^{-v_{ij}}\,\mathrm{d}x$ on $\mathbb{R}$ such that $v_{ij} - \frac{x^2}{2}$ is convex. Such distributions are known [3] to be 1-Lipschitz images of the standard Gaussian distribution γ so that the moment comparison of Corollary 9 holds.

Acknowledgment

I am grateful to Professor U. Haagerup for pointing out the connection to the reference [16] through (5) and for suggesting that the methods developed here could lead to Theorem 6.

References

[1] G. Aubrun. An inequality about the largest eigenvalue of a random matrix. In *Séminaire de Probabilités XXXVIII* 320–337. *Lecture Notes in Math*. **1857**. Springer, Berlin, 2005. MR2126983
[2] W. Bryc and V. Pierce. Duality of real and quaternionic random matrices, 2008.
[3] L. Caffarelli. Monotonicity properties of optimal transportation and the FKG and related inequalities. *Comm. Math. Phys.* **214** (2000) 547–563. MR1800860
[4] P. A. Deift. *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach. CIMS Lecture Notes* **3**. Courant Institute of Mathematical Sciences, New York, 1999. MR1677884
[5] I. Goulden and D. Jackson. Maps in locally orientable surfaces and integrals over real symmetric surfaces. *Can. J. Math.* **49** (1997) 865–882. MR1604106
[6] U. Haagerup and S. Thorbjørnsen. Random matrices with complex Gaussian entries. *Expo. Math.* **21** (2003) 293–337. MR2022002
[7] J. Harer and D. Zagier. The Euler characteristic of the moduli space of curves. *Invent. Math.* **85** (1986) 457–485. MR0848681
[8] W. König. Orthogonal polynomial ensembles in probability theory. *Probab. Surv.* **2** (2005) 385–447. MR2203677
[9] B. Lass. Démonstration combinatoire de la formule de Harer–Zagier. *C. R. Acad. Sci. Paris Ser. I Math.* **333** (2001) 155–160. MR1851616
[10] M. Ledoux. A remark on hypercontractivity and tail inequalities for the largest eigenvalues of random matrices. In *Séminaire de Probabilités XXXVII* 360–369. *Lecture Notes in Mathematics* **1832**. Springer, Berlin, 2003. MR2053053
[11] M. Ledoux. Differential operators and spectral distributions of invariant ensembles from the classical orthogonal polynomials. The continuous case. *Electron. J. Probab.* **9** (2004) 177–208. MR2041832
[12] M. L. Mehta. *Random Matrices*. Academic Press, Boston, MA, 1991. MR1083764
[13] M. Mulase and A. Waldron. Duality of orthogonal and symplectic random matrix integrals and quaternionic Feynman graphs. *Comm. Math. Phys.* **240** (2003) 553–586. MR2005857
[14] V. Pierce. An algorithm for map enumeration (2006).
[15] A. Ruzmaikina. Universality of the edge distribution of eigenvalues of Wigner random matrices with polynomially decaying distributions of entries. *Comm. Math. Phys.* **261** (2006) 277–296. MR2191882
[16] H. Schultz. Non-commutative polynomials of independent Gaussian random matrices. The real and symplectic cases. *Probab. Theory Related Fields* **131** (2005) 261–309. MR2117954
[17] A. Soshnikov. Universality at the edge of the spectrum in Wigner random matrices. *Comm. Math. Phys.* **207** (1999) 697–733. MR1727234
[18] G. Szegö. *Orthogonal Polynomials. Colloquium Publications* **XXIII**. Amer. Math. Soc., Providence, RI, 1975. MR0372517
[19] C. Tracy and H. Widom. Level-spacing distribution and the Airy kernel. *Comm. Math. Phys.* **159** (1994) 151–174. MR1257246
[20] C. Tracy and H. Widom. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.* **177** (1996) 727–754. MR1385083
[21] E. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Ann. Math.* **62** (1955) 548–564. MR0077805
[22] A. Zvonkin. Matrix integrals and map enumeration: An accessible introduction. *Math. Comput. Modelling* **26** (1997) 281–304. MR1492512