Perturbations of Functional Inequalities Using Growth Conditions [*]

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Abstract

Perturbations of functional inequalities are studied by using merely growth conditions in terms of a distance-like reference function. As a result, optimal sufficient conditions are obtained for perturbations to reach a class of functional inequalities interpolating between the Poincaré inequality and the logarithmic Sobolev inequality.

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1 Introduction

Let $(E, \mathscr{F}, \mu)$ be a separable complete probability space, and let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a conservative symmetric local Dirichlet form on $L^2(\mu)$ with domain $\mathscr{D}(\mathscr{E})$ in the following sense. Let $\mathscr{A}$ be a dense subspace of $\mathscr{D}(\mathscr{E})$ under the $\mathscr{E}_1^{1/2}$-norm ($\mathscr{E}_1(f) = \|f\|_2^2 + \mathscr{E}(f)$) which is composed of bounded functions, stable under products and composition with Lipschitz functions on $\mathbb{R}$. Let $\Gamma : \mathscr{A} \times \mathscr{A} \to \mathscr{M}_b$ be a bilinear mapping, where $\mathscr{M}_b$ is the set of all bounded measurable functions on E, such that

(1) $\Gamma(f, f) \geq 0$ and $\mathscr{E}(f, g) = \mu(\Gamma(f, g))$ for $f, g \in \mathscr{A}$;

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(2) $\Gamma(\phi \circ f, g) = \phi'(f)\Gamma(f, g)$ for $f, g \in \mathscr{A}$ and $\phi \in C_b^\infty(\mathbb{R})$;
(3) $\Gamma(fg, h) = g\Gamma(f, h) + f\Gamma(g, h)$ for $f, g, h \in \mathscr{A}$ with $fg \in \mathscr{A}$.

It is easy to see that the positivity and the bilinear property imply $\Gamma(f, g)^2 \leq \Gamma(f, f)\Gamma(g, g)$ for all $f, g \in \mathscr{A}$. For simplicity we set below $\Gamma(f, f) = \Gamma(f)$ and $\mathscr{E}(f, f) = \mathscr{E}(f)$.

We shall denote by $\mathscr{A}_{\text{loc}}$ the set of functions f such that for any integer n, the truncated function $f_n = \min(n, \max(f, -n))$ is in $\mathscr{A}$. For such functions, the bilinear map Γ automatically extends and shares the same properties than for functions in $\mathscr{A}$.

We will be concerned in this work with perturbations of the underlying probability measure μ, and thus of the Dirichlet form $\mathscr{E}$. To this task, let ρ be a reference distance-like function. More precisely, let $\rho \in \mathscr{A}_{\text{loc}}$ be a nonnegative function such that $\Gamma(\rho) \leq 1$ and $\rho \in L^1(\mu)$. (The hypothesis that $\rho \in L^1(\mu)$ is not restrictive and will always be satisfied under the functional inequalities assumed on μ below.) Then, for any measurable function V on E such that $\mu(\mathrm{e}^V) < \infty$, let $\mu_V = \mathrm{e}^V\mu/\mu(\mathrm{e}^V)$. If V is ρ-locally bounded (i.e. V is bounded on $\{\rho \leq n\}$ for any $n \geq 1$), then

$$\mathscr{E}_V(f, g) = \mu_V(\Gamma(f, g)), \quad f, g \in \mathscr{A} \cap L^2(\mu_V)$$

is closable in $L^2(\mu_V)$ and the closure $(\mathscr{E}_V, \mathscr{D}(\mathscr{E}_V))$ is a conservative symmetric Dirichlet form (see Proposition 6.1 in the Appendix).

Assume now that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ satisfies a functional inequality, such as for example a Poincaré or a logarithmic Sobolev inequality (see the reference [17] for an exhaustive description of families of functional inequalities in this context). A basic problem addressed in the literature is when does $(\mathscr{E}_V, \mathscr{D}(\mathscr{E}_V))$ satisfy the same inequality (with possibly different constants)? It is classical that all Poincaré-Sobolev type inequalities are stable under bounded perturbations (i.e. V is bounded), see e.g. [4] and references therein. Under some regularity conditions, functional inequalities could also be stable under unbounded perturbations. For instance, Proposition 2.6 in [14] indicates that the family of super Poincaré inequalities are stable under Lipschitz perturbations (i.e. $V \in \mathscr{A}$ and $\Gamma(V)$ is bounded). As another example, the logarithmic Sobolev inequality is stable as soon as $\mu(\mathrm{e}^{\lambda\Gamma(V)}) < \infty$ for some $\lambda > 0$ depending on the logarithmic Sobolev constant [1].

In this paper, we study perturbations of functional inequalities using growth conditions, that is allowing V to be unbounded but with a proper control in its growth. In other words, starting from a given functional inequality for $\mathscr{E}$, we intend to search for the optimal growth condition on V in terms of the reference function ρ such that this inequality is also satisfied by $\mathscr{E}_V$. Unfortunately, we realize (see Theorem 1.2 below for $\alpha_1 = \alpha_2$) that in many cases the optimal growth condition for our purpose is however trivial; that is, without any regularity assumption, V has to be bounded in order to keep the functional inequality. On the basis of this observation, we will try to reach a weaker inequality by making perturbations to a stronger one. In other words, assuming $\mathscr{E}$ satisfies a stronger (e.g. a logarithmic Sobolev) inequality, we aim to search for optimal growth conditions on V such that $\mathscr{E}_V$ satisfies a weaker (e.g. a Poincaré) inequality. As a sample result in this direction, we will prove the following theorem.

Theorem 1.1. *Assume that the logarithmic Sobolev inequality*

$$\mu(f^2 \log f^2) \le C\mathscr{E}(f),\ \ \mu(f^2) = 1, f \in \mathscr{D}(\mathscr{E}), \tag{1.1}$$

holds. Then, for any measurable function V on E such that $|V| \le \log(1+\rho) + K$ for some constant $K > 0$, one has $\mu(\mathrm{e}^V) < \infty$ and the Poincaré inequality

$$\mu_V(f^2) \le C'\mathscr{E}_V(f),\ \ \mu_V(f) = 0, f \in \mathscr{D}(\mathscr{E}_V), \tag{1.2}$$

for $\mathscr{E}_V$ is satisfied for some constant $C' > 0$. Moreover, the growth condition on V is optimal.

We will actually deal with more general inequalities which interpolate between the logarithmic Sobolev inequality and the Poincaré inequality. Such families of interpolating inequalities have been studied quite extensively in the recent literature, and we refer for example to the monograph [17] for an account on the subject. One such interpolation class, which we chose here as a sample example, is the family of inequalities put forward by R. Latała and K. Oleszkiewicz in [7] (see also [17]), namely

$$\sup_{p\in[1,2)} \frac{\mu(f^2) - \mu(|f|^p)^{2/p}}{(2-p)^\alpha} \le C\mathscr{E}(f), \quad f \in \mathscr{D}(\mathscr{E}), \tag{1.3}$$

where $\alpha \in [0,1]$ and $C > 0$ is a constant. This inequality interpolates between the Poincaré and the logarithmic Sobolev inequalities: it reduces to the Poincaré inequality (1.2) for $\alpha = 0$ and to the logarithmic Sobolev inequality (1.1) for $\alpha = 1$. In general, the inequality with larger α is stronger. One specific aspect in the Latała-Oleszkiewicz inequalities is that they precisely describe concentration between exponential and Gaussian [7]. We refer to [4, 16, 17] and the references therein for a more detailed discussion and generalizations. As a particular example of more general perturbation results discussed in the next sections, the next statement is one main conclusion of this work.

Theorem 1.2. *Let $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \le \alpha_2$.*

(**i**) *If the inequality* (1.3) *holds for $\alpha = \alpha_2$, then for any measurable function V on E such that*

$$-\frac{s(\alpha_2-\alpha_1)}{2-\alpha_2}\log(1+\rho) - K \le V \le \frac{(2-s)(\alpha_2-\alpha_1)}{2-\alpha_2}\log(1+\rho) + K \tag{1.4}$$

for some constants $K > 0$ and $s \in [0,2]$, then $\mu(\mathrm{e}^V) < \infty$ and

$$\sup_{p\in[1,2)} \frac{\mu_V(f^2) - \mu_V(|f|^p)^{2/p}}{(2-p)^{\alpha_1}} \le C'\mathscr{E}_V(f), \quad f \in \mathscr{D}(\mathscr{E}_V) \tag{1.5}$$

for some constant $C' > 0$.

(**ii**) *For any $s \in [0,2], u \in [0,1]$ and positive function $\varphi(r) \uparrow \infty$ as $r \uparrow \infty$, there exist examples such that* (1.3) *holds for $\alpha = \alpha_2$ and $V \in C(E)$ with*

$$-\frac{s(\alpha_2-\alpha_1)}{2-\alpha_2}\log(1+\rho) - (1-u)\varphi(\rho) \le V \le \frac{(2-s)(\alpha_2-\alpha_1)}{2-\alpha_2}\log(1+\rho) + u\varphi(\rho)$$

but for any $C' > 0$, (1.5) *does not hold.*

For pedagogical reasons, we first present in Section 2 a direct proof of Theorem 1.1 relying only on the classical entropic inequality and simple cut-off arguments. Interestingly enough, the proof splits into two steps, the first one, based indeed on the entropic inequality, that produces the appropriate behaviors of norms (from the $L^2 \log L$ for the initial measure to the L^2 norm for the perturbed one), and the second step that takes care (with unfortunately somewhat tedious details) of the tightness property (the functional inequality should imply that when the Dirichlet form of a given function f is 0, then f is constant).

This two-step argument is actually at the basis of the more general investigation developed next. Indeed, to reach perturbations results for interpolating inequalities between logarithmic Sobolev and Poincaré inequalities (such as the preceding Latała-Oleszkiewicz inequalities), we proceed along a similar scheme which decomposes the study of the norm behaviors and the study of the tightness property. The investigation will rely on suitable family of functional inequalities on which the perturbations may be performed efficiently. We namely work with the conjunction of the so-called super and weak Poincaré inequalities extensively discussed in [17]. A super Poincaré inequality for $\mathscr{E}$ is of the form

$$\mu(f^2) \leq r\mathscr{E}(f) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in \mathscr{D}(\mathscr{E}), \tag{1.6}$$

where $\beta : (0,\infty) \to (0,\infty)$ is a decreasing function. This functional inequality typically takes in account the growth of functions f for which $\mathscr{E}(f) < \infty$, and according to the behavior of the function β covers wide families of inequalities. For example, as demonstrated in [16], the Latała-Oleszkiewicz inequality (1.3) (with $\alpha > 0$) is equivalent to the Poincaré inequality (1.2) and the super Poincaré inequality (1.6) with $\beta(r) = \exp[c(1 + r^{-1/\alpha})]$ for some $c > 0$. The proof of Theorem 1.2 then follows from the corresponding perturbation result for super Poincaré inequalities developed in Section 3 for which suitable cut-off arguments may be developed. Now, in order to complete the picture, it is also necessary to establish the Poincaré inequality for the perturbed Dirichlet form $\mathscr{E}_V$. This corresponds to the second step of the argument, which will be accomplished at the level this time of weak Poincaré inequalities of the type

$$\mu(f^2) \leq \alpha(r)\mathscr{E}(f) + r\|f\|_\infty^2, \quad r > 0, f \in \mathscr{D}(\mathscr{E}), \mu(f) = 0, \tag{1.7}$$

where $\alpha : (0,\infty) \to (0,\infty)$ is some decreasing function. Weak Poincaré inequalities appear as the (minimal) technical step to suitably tight functional inequalities. For example, by Proposition 1.3 of [11], the super Poincaré inequality (1.6), actually holding just for one value of $r > 0$ (known then as a defective Poincaré inequality) together with the weak Poincaré inequality (1.7) implies a true Poincaré inequality. We thus develop in Section 4 general perturbation results for weak Poincaré inequalities. These results actually hold under much milder growth conditions on the perturbation potential V than for super Poincaré inequalities.

This investigation plainly justifies the interest in the families of super and weak Poincaré inequalities, and demonstrates their power in this context. It is clear in particular that the perturbation results of Sections 3 and 4 yield more general conclusions than only Theorems 1.1 and 1.2. The super and weak Poincaré inequalities have been introduced in respectively [14] and [11] to describe the essential spectrum and general convergence rate of Markov semigroups.

After the direct proof of Theorem 1.1, the main perturbation arguments for super and weak Poincaré inequalities are developed in Sections 3 and 4. The proof of Theorem 1.2 is then addressed in Section 5. In the Appendix, we make clear the closability of the pre-Dirichlet form $\mathscr{E}_V$ and present the adequate Hardy criterion on the line necessary to construct the examples of Theorem 1.2.

2 A direct proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Start thus with the logarithmic Sobolev inequality for the Dirichlet form $\mathscr{E}$,

$$\mu(f^2\log f^2)\le C\mathscr{E}(f),\ \ \mu(f^2)=1, f\in\mathscr{D}(\mathscr{E}). \tag{2.1}$$

By density, it is enough to deal below with bounded functions f in $\mathscr{A}$. Since $\rho\in L^1(\mu)$, the growth condition on V implies $\mu(\mathrm{e}^V)\le\mathrm{e}^K\mu(1+\rho)<\infty$. Changing thus if necessary V into $V-\log\mu(\mathrm{e}^V)$, we may and do assume that $\mu(\mathrm{e}^V)=1$. To prove the Poincaré inequality (1.2), we assume furthermore that $\mu_V(f)=0$. Let $r,s>0$ and set $g=[(\rho-r)^+\wedge s]/s$ so that $g=0$ on $\{\rho\le r\}$, $g=1$ on $\{\rho>r+s\}$, $0\le g\le 1$ and $\Gamma(g)\le s^{-2}$. Then $g\ge 1_{\{\rho>r+s\}}$ and hence, for every $f\in\mathscr{A}$,

$$\mu_V(f^2)\le\mu_V((fg)^2)+\mu_V(f^2 1_{\{\rho\le r+s\}}). \tag{2.2}$$

In a first step, we treat $\mu_V((fg)^2)$ and prove that for large enough $r>0$ there exists a constant $C(r,\mu,\rho)>0$, only depending on r and μ (to be specified below), such that for every f and every $s>0$,

$$\mu_V((fg)^2)\le C(r,\mu,\rho)\big(\mathscr{E}_V(f)+s^{-2}\mu_V(f^2)\big). \tag{2.3}$$

As announced in the introduction, this step corresponds to the norm control for the perturbed Dirichlet form. By the classical entropic inequality, for every $u>0$, setting for simplicity $\varphi=fg/[D(1+\rho)]^{1/2}$ with $D=\mathrm{e}^K$,

$$\begin{aligned}\mu_V((fg)^2)&=\mu\big(\varphi^2\cdot D(1+\rho)\,\mathrm{e}^V 1_{\{\rho>r\}}\big)\\ &\le\frac{1}{u}\,\mathrm{Ent}_\mu(\varphi^2)+\frac{1}{u}\,\mu(\varphi^2)\log\mu\big(\mathrm{e}^{uD(1+\rho)\mathrm{e}^V 1_{\{\rho>r\}}}\big)\end{aligned} \tag{2.4}$$

where $\mathrm{Ent}_\mu(\varphi^2)=\mu(\varphi^2\log\varphi^2)-\mu(\varphi^2)\log\mu(\varphi^2)$. Now, since $\mathrm{e}^V\le\mathrm{e}^K(1+\rho)=D(1+\rho)$,

$$\begin{aligned}\mu\big(\mathrm{e}^{uD(1+\rho)\mathrm{e}^V\mathbf{1}_{\{\rho>r\}}}\big)&\le\mu(\rho\le r)+\mu\big(\mathrm{e}^{uD^2(1+\rho)^2}1_{\{\rho>r\}}\big)\\ &\le 1+\mu(\rho>r)^{1/2}\mu\big(\mathrm{e}^{2uD^2(1+\rho)^2}\big)^{1/2}\end{aligned}$$

by the Cauchy-Schwarz inequality. Since μ satisfies a logarithmic Sobolev inequality, it is classical that distance-like functions have Gaussian tails (cf. [2, 8, 17]). Therefore, there exists $u_0>0$ small enough so that

$$\mu\big(\mathrm{e}^{2u_0D^2(1+\rho)^2}\big)<\infty.$$

Hence, by the preceding we can find r large enough (only depending on μ) so that

$$\frac{1}{u_0}\,\log\mu\Big(\mathrm{e}^{2u_0D(1+\rho)\mathrm{e}^V\mathbf{1}_{\{\rho>r\}}}\Big)\le\frac{1}{2}\,.$$

Since $[D(1+\rho)]^{-1}\le\mathrm{e}^{-|V|}\le\mathrm{e}^V$, we have $\mu(\varphi^2)\le\mu_V((fg)^2)$. It thus follows from (2.4) for $u=u_0$ that

$$\begin{aligned}\mu_V((fg)^2)&\le\frac{1}{u_0}\,\mathrm{Ent}_\mu(\varphi^2)+\frac{1}{u_0}\,\mu_V((fg)^2)\log\mu\Big(\mathrm{e}^{u_0D(1+\rho)\mathrm{e}^V1_{\{\rho>r\}}}\Big)\\&\le\frac{1}{u_0}\,\mathrm{Ent}_\mu(\varphi^2)+\frac{1}{2}\,\mu_V((fg)^2).\end{aligned}$$

Therefore

$$\mu_V((fg)^2)\le\frac{2}{u_0}\,\mathrm{Ent}_\mu(\varphi^2).$$

Apply now the logarithmic Sobolev inequality (2.1) to φ to get

$$\mu_V((fg)^2)\le\frac{2}{u_0}\,\mathrm{Ent}_\mu(\varphi^2)\le\frac{2C}{u_0}\,\mathscr{E}(\varphi).$$

Now, since $f\in\mathscr{A}$ and $\rho\in\mathscr{A}_{\mathrm{loc}}$, we may compute $\Gamma(\varphi)$ and get

$$\begin{aligned}\mathscr{E}(\varphi)=\mu(\Gamma(\varphi))&=\mu\big(\Gamma(fg/[D(1+\rho)]^{1/2})\big)\\&\le3\mu\big(g^2\Gamma(f)/[D(1+\rho)]\big)+3\mu\big(f^2\Gamma(g)/[D(1+\rho)]\big)\\&\quad+\frac{3}{4}\,\mu\big((fg)^2/[D(1+\rho)]^3\big)\end{aligned}$$

where we used that $\Gamma(\rho)\le1$. Using successively that $[D(1+\rho)]^{-1}\le\mathrm{e}^V$, $\Gamma(g)\le s^{-2}$ and $g\le1_{\{\rho>r\}}$, it follows that

$$\begin{aligned}\mathscr{E}(\varphi)&\le3\mu_V(\Gamma(f))+3\mu_V(f^2\Gamma(g))+\frac{3}{4}\,\mu_V\Big(\frac{(fg)^2}{(1+\rho)^2}\Big)\\&\le3\mathscr{E}_V(f)+3s^{-2}\mu_V(f^2)+\frac{3}{4(1+r)^2}\,\mu_V((fg)^2).\end{aligned}$$

Summarizing the previous steps,

$$\mu_V((fg)^2)\le\frac{6C}{u_0}\bigg(\mathscr{E}_V(f)+s^{-2}\mu_V(f^2)+\frac{1}{4(1+r)^2}\,\mu_V((fg)^2))\bigg).$$

Hence, provided that r is also large enough so that $(1+r)^2\ge3C/u_0$, we get the claim (2.3) with $C(r,\mu,\rho)=12C/u_0$.

In the second step of the proof, we take care of $\mu_V(f^21_{\{\rho\le r+s\}})$ in (2.2). The argument critically relies on the mean zero property of f and describes the tightness property required to reach the full Poincaré inequality from the defective one (expressed by the first step (2.3)).

Let $t > 0$ and set now $h = [(r+s+t-\rho)^+ \wedge t]/t$ so that $h = 1$ on $\{\rho \le r+s\}$, $h = 0$ on $\{\rho > r+s+t\}$, $0 \le h \le 1$ and $\Gamma(h) \le t^{-2}$. Since $\mu_V(f) = 0$,

$$\begin{aligned} \mu_V(f^2 1_{\{\rho \le r+s\}}) &= \mathrm{Var}_{\mu_V}(f 1_{\{\rho \le r+s\}}) + \mu_V(f 1_{\{\rho > r+s\}})^2 \\ &\le \mathrm{Var}_{\mu_V}(f 1_{\{\rho \le r+s\}}) + \mu_V(f^2)\mu_V(\rho > r+s) \end{aligned} \tag{2.5}$$

where we used the Cauchy-Schwarz inequality in the last step. We first control the term $\mathrm{Var}_{\mu_V}(f 1_{\{\rho \le r+s\}})$. Setting $M = \mu(fh)$,

$$\begin{aligned} \mathrm{Var}_{\mu_V}(f 1_{\{\rho \le r+s\}}) &\le \mu_V\big((f 1_{\{\rho \le r+s\}} - M)^2\big) \\ &\le \mu_V\big((f - M)^2 1_{\{\rho \le r+s\}}\big) + M^2 \mu_V(\rho > r+s). \end{aligned}$$

Replacing μ_V by μ,

$$\begin{aligned} \mu_V\big((f - M)^2 1_{\{\rho \le r+s\}}\big) &= \mu_V\big((fh - M)^2 1_{\{\rho \le r+s\}}\big) \\ &\le \xi(r+s)\mu\big((fh - M)^2 1_{\{\rho \le r+s\}}\big) \\ &\le \xi(r+s)\mathrm{Var}_\mu(fh) \end{aligned}$$

where we set $\xi(u) = \mathrm{e}^K(1+u)$, $u > 0$. On the other hand, since $h \le 1_{\{\rho \le r+s+t\}}$ and $|V| \le \log(1+\rho) + K$, by the Cauchy-Schwarz inequality,

$$M^2 = \big(\mu(fh)\big)^2 \le \xi(r+s+t)^2 \mu_V(f^2).$$

Therefore

$$\mathrm{Var}_{\mu_V}(f 1_{\{\rho \le r+s\}}) \le \xi(r+s)\mathrm{Var}_\mu(fh) + \xi(r+s+t)^2 \mu_V(f^2)\mu_V(\rho > r+s),$$

and together with (2.5),

$$\mu_V(f^2 1_{\{\rho \le r+s\}}) \le \xi(r+s)\mathrm{Var}_\mu(fh) + \big[\xi(r+s+t)^2 + 1\big]\mu_V(f^2)\mu_V(\rho > r+s). \tag{2.6}$$

Now, μ satisfying the logarithmic Sobolev inequality (2.1) also satisfies a Poincaré inequality (with constant $C/2$),

$$\mathrm{Var}_\mu(fh) \le \frac{C}{2}\,\mathscr{E}(fh).$$

Since again $h \le 1_{\{\rho \le r+s+t\}}$, and since $\Gamma(h) \le t^{-2}$,

$$\begin{aligned} \mathscr{E}(fh) = \mu(\Gamma(fh)) &\le 2\mu\big(\Gamma(f) 1_{\{\rho \le r+s+t\}}\big) + 2\mu\big(f^2 \Gamma(h) 1_{\{\rho \le r+s+t\}}\big) \\ &\le 2\xi(r+s+t)\big(\mathscr{E}_V(f) + t^{-2}\mu_V(f^2)\big). \end{aligned}$$

Inserting into (2.6),

$$\begin{aligned} \mu_V(f^2 1_{\{\rho \le r+s\}}) \le\ & \xi(r+s)\xi(r+s+t) C \mathscr{E}_V(f) \\ & + \Big[\xi(r+s)\xi(r+s+t) C t^{-2} \\ & \quad + \big(\xi(r+s+t)^2 + 1\big)\mu_V(\rho > r+s)\Big]\mu_V(f^2). \end{aligned} \tag{2.7}$$

We now complete the argument. Putting together (2.2), (2.3) and (2.7), we get that for every $f \in \mathscr{A}$ with $\mu_V(f) = 0$, for some $r > 0$ large enough only depending on μ, and every $s, t > 0$,

$$\begin{aligned}\mu_V(f^2) \leq & \big[C(r,\mu,\rho) + \xi(r+s)\xi(r+s+t)C\big]\mathscr{E}_V(f) \\ & + \Big[C(r,\mu,\rho)s^{-2} + \xi(r+s)\xi(r+s+t)Ct^{-2} \\ & \quad + \big(\xi(r+s+t)^2 + 1\big)\mu_V(\rho > r+s)\Big]\mu_V(f^2).\end{aligned}$$

By the concentration results (cf. [2, 8, 17]) under logarithmic Sobolev inequalities, we have $\mu(\rho > u) \leq \mathrm{e}^{-cu^2}$ for some $c > 0$ and all u large enough. By the growth assumption on V, it is clear that a similar result holds for μ_V. Then, together with the fact that $\xi(u) = \mathrm{e}^K(1+u)$, $u > 0$, it is immediate to take for example $s = \sqrt{t}$ large enough in the preceding bound so that the factor in front of $\mu_V(f^2)$ is less than 1/2. The Poincaré inequality for μ_V is thus established. Optimality follows from the examples developed in the context of Theorem 1.2. The proof of Theorem 1.1 is thus completed in this way.

3 Perturbations for super Poincaré inequalities

According to the discussion in the introduction, we analyze in this section perturbation results for super Poincaré inequalities that will be used in the proof of the main result. Consider the super Poincaré inequality

$$\mu(f^2) \leq r\mathscr{E}(f) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in \mathscr{D}(\mathscr{E}), \tag{3.1}$$

where $\beta : (0,\infty) \to (0,\infty)$ is a decreasing function. We refer to [17] for a detailed discussion on this family of inequalities which is shown there to cover large families of functional inequalities. We study perturbations of this inequality using cut-off arguments, and to this task define

$$\delta_n(V) = \sup_{\rho \leq n+1} V - \inf_{\rho \leq n+1} V, \quad \sigma_n(V) = \sup_{\rho \leq n+1} V - 2 \inf_{\rho \leq n+1} V, \quad n \geq 1.$$

Let moreover $\beta^{-1}(s) = \inf\{r > 0 : \beta(r) \leq s\}$ for $s > 0$ and $\inf \emptyset = \infty$ by convention. Set

$$\varepsilon_n(V) = \sup_{m \geq n} \beta^{-1}\big(1/2\mu(\rho > m-1)\big)\mathrm{e}^{\delta_{m+1}(V)}, \quad n \geq 1.$$

The following result addresses perturbations for super Poincaré inequalities under growth conditions.

Theorem 3.1. *Assume the super Poincaré inequality* (3.1) *and that* V *is* ρ*-locally bounded.*

(i) *If* $\varepsilon_n(V) \to 0$ *as* $n \to \infty$*, then the super Poincaré inequality*

$$\mu_V(f^2) \leq r\mathscr{E}_V(f) + \tilde{\beta}(r)\mu_V(|f|)^2, \quad r > 0, f \in \mathscr{D}(\mathscr{E}_V) \tag{3.2}$$

holds with

$$\tilde{\beta}(r) = \inf\left\{2\mathrm{e}^{\sigma_n(V)}\beta(s) : 24\varepsilon_n(V) + 4s\mathrm{e}^{\delta_n(V)} \le r\wedge 1\right\} < \infty, \quad r > 0.$$

(**ii**) *If $\varepsilon_n(V) < \infty$ for some $n \ge 1$, then there exist $C_1, C_2 > 0$ such that the following defective Poincaré inequality holds*

$$\mu_V(f^2) \le C_1\mathscr{E}_V(f) + C_2\mu_V(|f|)^2, \quad f \in \mathscr{D}(\mathscr{E}_V). \tag{3.3}$$

Proof. Without loss of generality, we may and do assume that $f \in \mathscr{A}$ and f is bounded. For simplicity, we moreover write δ_n, σ_n and ε_n for respectively $\delta_n(V)$, $\sigma_n(V)$ and $\varepsilon_n(V)$.

(**i**) By (3.1) and the Schwarz inequality, if $f|_{\{\rho\le n-1\}} = 0$ then

$$\mu(f^2) \le r\mathscr{E}(f) + \beta(r)\mu(|f|)^2 \le r\mathscr{E}(f) + \beta(r)\mu(f^2)\mu(\rho > n-1)$$

for all $r > 0$. Taking $r = \beta^{-1}(1/2\mu(\rho > n-1))$, it follows that

$$\mu(f^2) \le 2\beta^{-1}(1/2\mu(\rho > n-1))\mathscr{E}(f), \quad f|_{\{\rho\le n-1\}} = 0. \tag{3.4}$$

To derive inequalities from (3.4) for $\mathscr{E}_V$, we make use of a cut-off argument. Let $h_n = ((\rho - n + 1)^+ \wedge 1)((n+2-\rho)^+\wedge 1)$. We have $h_n = 1$ on $\{n \le \rho \le n+1\}$, $h = 0$ on $\{\rho \le n-1\} \cup \{\rho > n+2\}$, $0 \le h \le 1$ and $\Gamma(h_n) \le 1$. Applying (3.4) to fh_n and recalling that $\delta_n = \sup_{\rho\le n+1} V - \inf_{\rho\le n+1} V$, we get for every $n \ge 1$,

$$\begin{aligned}\mu_V(f^2h_n^2) &\le 2\beta^{-1}(1/2\mu(\rho > n-1))\mathrm{e}^{\delta_{n+1}}\mu_V(\Gamma(fh_n))\\ &\le 4\beta^{-1}(1/2\mu(\rho > n-1))\mathrm{e}^{\delta_{n+1}}\mu_V(\Gamma(f)1_{\{n-1\le\rho\le n+2\}})\\ &\quad + 4\beta^{-1}(1/2\mu(\rho > n-1))\mathrm{e}^{\delta_{n+1}}\mu_V(f^2 1_{\{n-1\le\rho\le n+2\}}).\end{aligned}$$

Taking summations on both sides over n from some fixed integer, we arrive at

$$\mu_V(f^2 1_{\{\rho\ge n\}}) \le 12\varepsilon_n\mathscr{E}_V(f) + 12\varepsilon_n\mu_V(f^2), \quad n \ge 1. \tag{3.5}$$

On the other hand, let $g_n = (n+1-\rho)^+\wedge 1$. By the super Poincaré inequality (3.1) again, and the definition of σ_n, we have for every $s > 0$,

$$\begin{aligned}\mu_V(f^2 1_{\{\rho\le n\}}) \le \mu_V(f^2g_n^2) &\le \mu(f^2g_n^2)\mathrm{e}^{\sup_{\rho\le n+1} V}\\ &\le \mathrm{e}^{\sup_{\rho\le n+1} V}(s\mu(\Gamma(fg_n)) + \beta(s)\mu(|fg_n|)^2)\\ &\le 2s\mathrm{e}^{\delta_n}(\mathscr{E}_V(f) + \mu_V(f^2)) + \beta(s)\mathrm{e}^{\sigma_n}\mu_V(|f|)^2.\end{aligned} \tag{3.6}$$

Combining this estimate with (3.5) we conclude that for every $n \ge 1$ and $s > 0$,

$$\begin{aligned}\mu_V(f^2) &\le (12\varepsilon_n + 2s\mathrm{e}^{\delta_n})\mathscr{E}_V(f)\\ &\quad + (12\varepsilon_n + 2s\mathrm{e}^{\delta_n})\mu_V(f^2) + \beta(s)\mathrm{e}^{\sigma_n}\mu_V(|f|)^2.\end{aligned}$$

Therefore, for any $n \ge 1$ and $s > 0$ such that $24\varepsilon_n + 4s\mathrm{e}^{\delta_n} \le r\wedge 1$, we have

$$\begin{aligned}\mu_V(f^2) &\le 2(12\varepsilon_n + 2s\mathrm{e}^{\delta_n})\mathscr{E}_V(f) + 2\beta(s)\mathrm{e}^{\sigma_n}\mu_V(|f|)^2\\ &\le r\mathscr{E}_V(f) + 2\beta(s)\mathrm{e}^{\sigma_n}\mu_V(|f|)^2.\end{aligned}$$

This completes the proof of **(i)**.

(ii) The proof is a slight variation on the preceding. Observe that if there exists $n \ge 1$ such that $24\varepsilon_n < 1$, then $\tilde\beta(1)$ defined in (1) is finite. Thus, the above proof implies (3.3) for $C_1 = 1$ and $C_2 = \tilde\beta(1)$. In general, for any $N > 0$ let

$$V_N = (V - N)1_{\{V \ge N\}} + (V + N)1_{\{V \le -N\}}.$$

Then $|V_N| = (|V| - N)^+$ and

$$\begin{aligned}24\varepsilon_n(V_N) &\le 24 \sup_{m \ge n} \beta^{-1}\big(1/2\mu(\rho > m-1)\big)\mathrm{e}^{(\delta_m(V)-2N)^+}\\ &\le 24\mathrm{e}^{-2N}\varepsilon_n + 24\beta^{-1}\big(1/2\mu(\rho > n-1)\big).\end{aligned}$$

Since ε_n is decreasing in n and $\beta^{-1}\big(1/2\mu(\rho > n-1)\big) \to 0$ as $n \to \infty$, we conclude that $24\varepsilon_n(V_N) < 1$ for sufficiently large n and N. Therefore, there exists $N > 1$ such that (3.3) holds for $\mathscr{E}_{V_N}$ in place of $\mathscr{E}_V$. But (3.3) is stable under bounded perturbations (up to constants) and $|V - V_N| \le N$, thus (3.3) also holds for $\mathscr{E}_V$. The proof of **(ii)**, and thus of Theorem 3.1, is complete. □

We next illustrate some examples of super Poincaré inequalities for which the conclusions of Theorem 3.1 are relevant.

Proposition 3.2. *Let $\theta > 1/2$. Assume that the super Poincaré inequality* (3.1) (*for $\mathscr{E}$*) *holds for $\beta(r) = \exp[c(1 + r^{-\theta})]$ for some $c > 0$. Let $\varepsilon \in [0, (2\theta - 1)^{-1})$.*

(i) *If $-s\varepsilon\log(1+\rho) - K \le V \le (2-s)\varepsilon\log(1+\rho) + K$ for some $K > 0$ and $s \in [0, 2]$, then the super Poincaré inequality* (3.2) *holds for*

$$\tilde\beta(r) = \exp\left[c'\big(1 + r^{-\theta/(1-\varepsilon(2\theta-1))}\big)\right]$$

for some constant $c' > 0$.

(ii) *If $|V| \le (2\theta - 1)^{-1}\log(1+\rho) + K$ for some $K > 0$, then the defective Poincaré inequality* (3.3) *holds.*

Proof. In the preceding notation, obviously

$$\delta_n = \delta_n(V) \le 2\varepsilon\log(n+3) + 2K, \quad \sigma_n = \sigma_n(V) \le 4\varepsilon\log(n+2) + 3K.$$

Furthermore, the super Poincaré inequality (3.1) implies $\mu(\rho > n-1) \le \exp[-\lambda n^{2\theta/(2\theta-1)}]$ for some $\lambda > 0$ and large enough n (cf. [15, Corollary 5.1] or [17, Corollary 3.3.22]). Then

$$\beta^{-1}(1/2\mu(\rho > n-1)) \le c_1 n^{-2/(2\theta-1)} \tag{3.7}$$

for some constant $c_1 > 0$ and all n large enough. Therefore, taking $s = n^{-2/(2\theta-1)}$, we get

$$24\varepsilon_n + 4s\mathrm{e}^{\delta_n} \le c_2 n^{2\varepsilon - 2/(2\theta-1)}, \quad n \ge n_0,$$

for some constant $c_2 > 0$ and some $n_0 \ge 1$. Hence, if $\varepsilon < (2\theta - 1)^{-1}$, for any $r \in (0, 1]$,

$$\begin{aligned}\tilde{\beta}(r) &= \inf\left\{2\mathrm{e}^{\sigma_n}\beta(s) : 24\varepsilon_n + 4s\mathrm{e}^{\delta_n} \le r\right\}\\ &\le \inf\left\{2\mathrm{e}^{\sigma_n}\beta(n^{-2/(2\theta-1)}) : n \ge n_0, c_2 n^{2\varepsilon-2/(2\theta-1)} \le r\right\}\\ &\le \exp[c'(1 + r^{-\theta/(1-\varepsilon(2\theta-1))})]\end{aligned}$$

for some constant $c' > 0$. The first part of the statement is thus established. Finally, if $\varepsilon = (2\theta - 1)^{-1}$ then $\varepsilon_n = \varepsilon_n(V) < \infty$ for large n. Hence, the proof is completed by Theorem 3.1. □

Remark. Since according to [15, Corollary 5.1] or [17, Corollary 3.3.22] ρ has to be bounded if (3.1) holds for $\beta(r) = \exp[c(1 + r^{-\theta})]$ for some $c > 0$ and $\theta < 1/2$, Proposition 3.2 contains a reasonable class of super Poincaré inequalities for our study of unbounded perturbations using growth conditions in terms of ρ. Next, as shown in Proposition 5.1 below, growth conditions presented in Proposition 3.2 are sharp.

4 Perturbations for weak Poincaré inequalities

As discussed in the introduction, in order to reach the more classical logarithmic Sobolev or Latała-Oleszkiewicz inequalities from the family of super Poincaré inequalities, and more precisely their tightness property, one needs to complement them with suitable weak Poincaré inequalities. In this section, we thus consider perturbation results for weak Poincaré inequalities. These actually hold under milder assumptions than for super Poincaré inequalities. In particular, only a suitable control on the growth of $\sigma_n(V)$ with respect to the tail of μ is necessary. However, the centerings induce several technical issues in the proofs.

The following weak Poincaré inequality

$$\mu(f^2) \le \alpha(r)\mathscr{E}(f) + r\|f\|_\infty^2, \quad r > 0, \mu(f) = 0, f \in \mathscr{D}(\mathscr{E}), \tag{4.1}$$

where $\alpha : (0,\infty) \to (0,\infty)$ is a decreasing function has been related to the convergence rate of the associated Markov semigroup in [11]. Note that (4.1) with a constant function α is just the standard Poincaré inequality. We take again the notation of the preceding section.

We only need below perturbation results of the weak Poincaré inequality (4.1) with constant function α (that is the classical Poincaré inequality). It is however of interest to state a general result in this regard.

Proposition 4.1. *Assume that the weak Poincaré inequality* (4.1) *holds and let* $\gamma(s) = \alpha(s)/s$.

(i) *If* $\mu(\rho > n) > 0$ *for all* $n > 0$ *and* $\mathrm{e}^{\sigma_n(V)}\gamma^{-1}(1/\mu(\rho > n)) \to 0$ *as* $n \to \infty$, *then*

$$\mu_V(f^2) \le \tilde{\alpha}(r)\mathscr{E}_V(f) + r\|f\|_\infty^2, \quad r > 0, \mu_V(f) = 0, f \in \mathscr{D}(\mathscr{E}_V), \tag{4.2}$$

holds for

$$\tilde\alpha(r) = \inf\left\{2\mathrm{e}^{2\sigma_n(V)}\mu(\rho > n)^{-1}\gamma^{-1}(1/\mu(\rho > n))\right\} < \infty,\ \ r > 0,$$

where the infimum is running over all n's such that

$$12\mathrm{e}^{\sigma_n(V)}\gamma^{-1}(1/\mu(\rho > n)) + 3\mu_V(\rho > n) \le r.$$

(ii) *If $\mu(\rho > n) = 0$ for some $n > 0$, then (4.2) holds for $\tilde\alpha(r) = \mathrm{e}^{2\sigma_n(V)}\alpha(r/4\mathrm{e}^{\sigma_n(V)})$.*

Proof. Let $f \in \mathscr{A}$ be bounded such that $\mu_V(f) = 0$ and let $f_n = (n+1-\rho)^+ \wedge 1$. Set again $\sigma_n = \sigma_n(V)$ for simplicity. We have $\mu_V(ff_n)^2 = \mu_V(f(1-f_n))^2 \le \|f\|_\infty^2\mu_V(\rho > n)^2$ so that

$$\mu_V(f^2) \le \mu_V(f^2f_n^2) + \|f\|_\infty^2\mu_V(\rho > n) \le \mathrm{Var}_{\mu_V}(ff_n) + 2\|f\|_\infty^2\mu_V(\rho > n).$$

With $M = \mu(ff_n)$,

$$\begin{aligned}\mathrm{Var}_{\mu_V}(ff_n) &\le \mu_V\big((ff_n - M)^2)\big)\\ &\le \mu_V\big((ff_n - M)^2 1_{\{\rho\le n+1\}}\big) + M^2\mu_V(\rho > n+1)\\ &\le \mathrm{e}^{\sigma_n}\mathrm{Var}_\mu(ff_n) + \|f\|_\infty^2\mu_V(\rho > n).\end{aligned}$$

Now, it follows from (4.1) that

$$\begin{aligned}\mathrm{Var}_\mu(ff_n) &\le \alpha(s)\mu(\Gamma(ff_n)) + 4s\|f\|_\infty^2\\ &\le 2\alpha(s)\mu(\Gamma(f)1_{\{\rho\le n+1\}}) + (2\alpha(s)\mu(\rho > n) + 4s)\|f\|_\infty^2\\ &\le 2\alpha(s)\mathrm{e}^{\sigma_n}\mathscr{E}_V(f) + (2\alpha(s)\mu(\rho > n) + 4s)\|f\|_\infty^2.\end{aligned}$$

Therefore,

$$\mu_V(f^2) \le 2\alpha(s)\mathrm{e}^{2\sigma_n}\mathscr{E}_V(f) + \big(\mathrm{e}^{\sigma_n}(2\alpha(s)\mu(\rho > n) + 4s) + 3\mu_V(\rho > n)\big)\|f\|_\infty^2.$$

Taking $s = \gamma^{-1}(1/\mu(\rho > n))$ if $\mu(\rho > n) > 0$, it follows that

$$\begin{aligned}\mu_V(f^2) \le{}& 2\,\mathrm{e}^{2\sigma_n}\mu(\rho > n)^{-1}\gamma^{-1}(1/\mu(\rho > n))\mathscr{E}_V(f)\\ &+ \big(12\gamma^{-1}(1/\mu(\rho > n))\mathrm{e}^{\sigma_n} + 3\mu_V(\rho > n)\big)\|f\|_\infty^2.\end{aligned}$$

This completes the proof of **(i)**.

If $\mu(\rho > n) = 0$, then (4.1) implies for every $s > 0$,

$$\begin{aligned}\mu_V(f^2) = \mathrm{Var}_{\mu_V}(f) &\le \mu_V((f - \mu(f))^2)\\ &\le \mathrm{e}^{\sigma_n}\mu((f - \mu(f))^2)\\ &\le \alpha(s)\mathrm{e}^{\sigma_n}\mu(\Gamma(f)) + 4s\mathrm{e}^{\sigma_n}\|f\|_\infty^2\\ &\le \alpha(s)\mathrm{e}^{2\sigma_n}\mathscr{E}_V(f) + 4s\mathrm{e}^{\sigma_n}\|f\|_\infty^2.\end{aligned}$$

Hence **(ii)** follows, and Proposition 4.1 is established. □

To illustrate this result, assume for example that a Poincaré inequality for $\mathscr{E}$ holds, so that the weak Poincaré inequality (4.1) with α a constant function is satisfied. Then, by the concentration results under Poincaré inequalities $\mu(\rho > n) \leq \mathrm{e}^{-cn}$ for some $c > 0$ and all large n's (cf. [8, 17]). In particular, only mild growth conditions on the perturbation V are enough in order that a weak Poincaré inequality for $\mathscr{E}_V$ holds. The following corollary is one such example.

Corollary 4.2. *Assume that the Poincaré inequality*

$$\mu(f^2) \leq C\mathscr{E}(f), \ \mu(f) = 0, f \in \mathscr{D}(\mathscr{E}),$$

for $\mathscr{E}$ holds for some $C > 0$, and that $|V| \leq \psi(\rho)$ where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and such that $\limsup_{n\to\infty} \psi(n)/n < C^{-1/2}$. Then the weak Poincaré inequality (4.2) for $\mathscr{E}_V$ holds for some function $\tilde{\alpha}$.

Proof. Simply note that the Poincaré inequality implies $\gamma^{-1}(s) = C/s$ and (cf. [8, 17]) $\mu(\mathrm{e}^{C^{-1/2}\rho}) < \infty$. $\square$

5 Proof of Theorem 1.2

With the material of the preceding sections, we now address the proof of the main Theorem 1.2 which will follow from the suitable combination of the results of Sections 3 and 4. We start with the first part (**i**). By the classical bounded perturbation results, we may assume that $\alpha_2 > \alpha_1$. As discussed in the introduction, the observation justifying the introduction of super Poincaré inequalities is that the Latała-Oleszkiewicz inequality (1.3) may be described equivalently by such a super Poincaré inequality. More precisely, according to [16, Corollary 1.2], the Latała-Oleszkiewicz inequality (1.3) with $\alpha = \alpha_2 \in [0, 1]$ is equivalent to a Poincaré inequality and to a super Poincaré inequality (3.1) with $\beta(r) = \exp[c(1 + r^{-1/\alpha_2})]$ for some $c > 0$, where we regard (3.1) with $\beta(r) = \exp[c(1+r^{-1/0})]$ as the defective Poincaré inequality (3.3) since in this case $\beta(r) = \mathrm{e}^c$ makes sense for $r > 1$. Of course, a similar claim holds for the perturbed inequality (1.5) (with $\alpha_1 \in [0, 1]$). By the results of [11] (cf. Proposition 1.3 there), the Poincaré inequality may actually be replaced by a weak Poincaré inequality. The conclusion then immediately follows from the conjunction of Proposition 3.2 and Corollary 4.2. More precisely, if $\alpha_1 > 0$, Proposition 3.2 applied to $\theta = \alpha_2^{-1} \geq 1$ and $\varepsilon = \frac{\alpha_2 - \alpha_1}{2-\alpha_2} < (2\theta - 1)^{-1}$ (so that $\frac{\theta}{1-\varepsilon(2\theta-1)} = \frac{1}{\alpha_1}$), implies (3.2) with $\tilde{\beta}(r) = \exp[c'(1 + r^{-1/\alpha_1})]$ for some $c' > 0$, and the result follows from Corollary 4.2. When $\alpha_1 = 0$ (that is the Latała-Oleszkiewicz inequality (1.5) amounts to the Poincaré inequality for $\mathscr{E}_V$) the second part of Proposition 3.2 ensures that the defective Poincaré inequality (3.3) for $\mathscr{E}_V$ holds. We then conclude together with Corollary 4.2 as in the previous case. Part **(i)** is thus established.

We are left with the construction of the examples **(ii)**. To this task, it is enough again to work at the level of the super Poincaré inequalities as in Proposition 3.2 for $\theta \geq 1$. Indeed, we have the following more general examples which also include the case $\theta \in (\frac{1}{2}, 1)$.

Proposition 5.1. *Consider $\mu(\mathrm{d}x) = Z^{-1}\mathrm{e}^{-\rho^{2\theta/(2\theta-1)}}\mathrm{d}x$ on $\mathbb{R}$, where $\rho = |\cdot|$, $\theta \in (1/2, \infty]$ and Z is the normalization. Let $\Gamma(f, g) = f'g'$ for $\mathscr{A}$ the set of all Lipschitz functions. Then*

(3.1) *holds for* $\beta(r) = \mathrm{e}^{c(1+r^{-\theta})}$ *for some constant* $c > 0$, *where for* $\theta = \infty$ *we mean* (3.3) *holds. Moreover:*

(i) *Let* $\theta < \infty$, $\varepsilon \in (0, (2\theta - 1)^{-1})$ *and* $s \in [0, 2]$. *For any* $u \in [0, 1]$ *and positive function* $\varphi(r) \uparrow \infty$ *as* $r \uparrow \infty$, *there exists* $V \in C(\mathbb{R})$ *such that*

$$-s\varepsilon \log(1 + \rho) - (1 - u)\varphi(\rho) \le V \le (2 - s)\varepsilon \log(1 + \rho) + u\varphi(\rho) \tag{5.1}$$

but for any $c' > 0$, (3.2) *with*

$$\tilde{\beta}(r) = \exp\left[c'\big(1 + r^{-\theta/(1-\varepsilon(2\theta-1))}\big)\right]$$

does not hold.

(ii) *Similarly, for any* $s \in [0, 2], u \in [0, 1]$ *and positive function* $\varphi(r) \uparrow \infty$ *as* $r \uparrow \infty$, *there exists* $V \in C(\mathbb{R})$ *such that*

$$-\frac{s}{2\theta - 1} \log(1 + \rho) - (1 - u)\varphi(\rho) \le V \le \frac{2 - s}{2\theta - 1} \log(1 + \rho) + u\varphi(\rho)$$

but for any $C_1, C_2 > 0$, (3.3) *does not hold.*

Proof. Since it is well-known that (1.2) holds if $\theta = \infty$ (cf. [13, Corollary 1.4]), the assertion on (3.1) follows from [14, Corollary 2.5] or [15, Corollary 6.1]. To construct examples for **(i)** and **(ii)**, we make use of the Hardy criterion Proposition 6.2 below. To this end, we work with a further equivalent description of the super Poincaré inequalities (generalizing the logarithmic Sobolev inequality). Recall namely that (see e.g. [15, Proposition 1.3]), for any $\theta > 0$, the super Poincaré inequality (3.1) with $\beta(r) = \mathrm{e}^{c(1+r^{-\theta})}$ for some $c > 0$ holds if and only if

$$\mu(f^2 \log^{1/\theta}(1 + f^2)) \le C_1\mathscr{E}(f) + C_2, \quad \mu(f^2) = 1 \tag{5.2}$$

holds for some $C_1, C_2 > 0$.

We turn to the construction of the examples. We may assume that $\varphi(r) \le \log(1 + r)$ by using $\varphi(r) \wedge \log(1 + r)$ in place of $\varphi(r)$. Let $\varepsilon \in [0, (2\theta - 1)^{-1}]$ be fixed. Let

$$K_n = (2 - s)\varepsilon \log(n + 1) + u\varphi(n), \quad \tilde{K}_n = s\varepsilon \log(n + 1) + (1 - u)\varphi(n), \quad n \ge 2,$$

and take

$$V(x) = \begin{cases} (\tilde{K}_n + K_n)(n - x) + K_n, & x \in [n + \mathrm{e}^{-n}, n + 1), n \ge 2, \\ (\tilde{K}_n + K_n + K_n\mathrm{e}^n + \tilde{K}_{n-1}\mathrm{e}^n)(x - n) - \tilde{K}_{n-1}, & x \in [n, n + \mathrm{e}^{-n}), n \ge 2, \\ -\tilde{K}_1, & x < 2. \end{cases}$$

Then $V \in C(\mathbb{R})$ and (5.1) holds. According to Proposition 6.2 below and the correspondence between (3.1) and (5.2) mentioned above, it suffices to show that

$$
\begin{aligned}
(5.3)\qquad &\limsup_{n\to\infty}\Big(\int_{n-1+\mathrm{e}^{1-n}}^{n} \mathrm{e}^{x^{2\theta/(2\theta-1)}-V(x)}\mathrm{d}x\Big)\\
&\qquad\times \mu_V([n+\mathrm{e}^{-n},n+1])\log^{(1-\varepsilon(2\theta-1))/\theta}(1/\mu_V([n+\mathrm{e}^{-n},n+1]))=\infty.
\end{aligned}
$$

Indeed, since the function $r\mapsto r\log^{\alpha} r^{-1}$ is increasing in r for small $r>0$, where $\alpha=(1-\varepsilon(2\theta-1))/\theta$, we have

$$
\begin{aligned}
\delta_{+0}&\geq \limsup_{n\to\infty}\mu([n+\mathrm{e}^{-n},\infty))\log^{\alpha}\mu([n+\mathrm{e}^{-n},\infty))^{-1}\int_0^{n+\mathrm{e}^{-n}}\mathrm{e}^{-C(y)}\mathrm{d}y\\
&\geq \limsup_{n\to\infty}\mu([n+\mathrm{e}^{-n},n+1])\log^{\alpha}\mu([n+\mathrm{e}^{-n},n+1])^{-1}\int_{n-1+\mathrm{e}^{1-n}}^{n}\mathrm{e}^{-C(y)}\mathrm{d}y.
\end{aligned}
$$

Thus, (5.3) implies $\delta_{+0}=\infty$ since in the present case we have $-C(y)=|y|^{2\theta/(2\theta-1)}-V(y)$. Obviously, since $\varphi(n)\leq\log(1+n)$, there exists $\varepsilon_1,\varepsilon_2\in(0,1)$ such that

$$
\begin{aligned}
&\int_{n-1+\mathrm{e}^{1-n}}^{n}\mathrm{e}^{x^{2\theta/(2\theta-1)}-V(x)}\mathrm{d}x=\int_{n-1+\mathrm{e}^{1-n}}^{n}\mathrm{e}^{x^{2\theta/(2\theta-1)}+(K_{n-1}+\tilde{K}_{n-1})(x-n+1)-K_{n-1}}\mathrm{d}x\\
&\geq\frac{\varepsilon_1}{n^{2\theta/(2\theta-1)-1}}\int_{n-1+\mathrm{e}^{1-n}}^{n}\mathrm{e}^{x^{2\theta/(2\theta-1)}+(K_{n-1}+\tilde{K}_{n-1})(x-n+1)-K_{n-1}}\mathrm{d}\{x^{2\theta/(2\theta-1)}+(K_{n-1}+\tilde{K}_{n-1})x\}\\
&\geq\frac{\varepsilon_2}{n^{1/(2\theta-1)}}\,\mathrm{e}^{n^{2\theta/(2\theta-1)}+s\varepsilon\log n+(1-u)\varphi(n-1)},\quad n\geq 2.
\end{aligned}
$$

Similarly, there exists small $\varepsilon_3>0$ such that

$$
\begin{aligned}
\mu_V([n+\mathrm{e}^{-n},n+1])&=Z_V^{-1}\int_{n+\mathrm{e}^{-n}}^{n+1}\mathrm{e}^{-x^{2\theta/(2\theta-1)}+(\tilde{K}_n+K_n)(n-x)+K_n}\mathrm{d}x\\
&\geq\frac{\varepsilon_3}{n^{1/(2\theta-1)}}\mathrm{e}^{-n^{2\theta/(2\theta-1)}+(2-s)\varepsilon\log(n+1)+(1-u)\varphi(n)},
\end{aligned}
$$

where Z_V is the normalization. Noting that

$$
2\varepsilon-\frac{2}{2\theta-1}+\frac{2\theta}{2\theta-1}\cdot\frac{1-\varepsilon(2\theta-1)}{\theta}=0,
$$

we may find out a constant $c'>0$ such that

$$
\begin{aligned}
&\Big(\int_{n-1+\mathrm{e}^{1-n}}^{n}\mathrm{e}^{x^{2\theta/(2\theta-1)}-V(x)}\mathrm{d}x\Big)\\
&\quad\times\mu_V([n+\mathrm{e}^{-n},n+1])\log^{(1-\varepsilon(2\theta-1))/\theta}(1/\mu_V([n+\mathrm{e}^{-n},n+1]))\geq c'\mathrm{e}^{u\varphi(n)+(1-u)\varphi(n-1)}
\end{aligned}
$$

which goes to ∞ as $n\to\infty$. Thus, (5.3) holds. $\square$

6 Appendix

6.1 The closability of $(\mathscr{E}_V, \mathscr{A}\cap L^2(\mu_V))$

Proposition 6.1. *If V is locally ρ-bounded, then $(\mathscr{E}_V, \mathscr{A}\cap L^2(\mu_V))$ is closable in $L^2(\mu_V)$.*

Proof. Since $\mathscr{A}$ is dense in $L^2(\mu)$ and V is ρ-locally bounded, it is easy to check from assumptions on $\mathscr{A}$ that $\mathscr{A}\cap L^2(\mu_V)$ is dense in $L^2(\mu_V)$. Moreover, the contraction property of $\mathscr{E}_V$ follows from that of Γ. So, we need only to verify the closability. Assume that $\{f_n\}\subset\mathscr{A}$ satisfies $\mu_V(f_n^2)\to 0$ and $\mathscr{E}_V(f_n-f_m)\to 0$ as $n,m\to\infty$. We only need to prove that $\mathscr{E}_V(f_n)\to 0$. To this end, we make use of a cut-off argument so that the closability of $\mathscr{E}$ can be applied. For any $N\geq 1$, let $h_N=(N+1-\rho)^+\wedge 1$. Since V is bounded on $\{\rho\leq N+1\}$, we have $\mu(f_n^2h_N^2)\to 0$ and

$$\begin{aligned}\mathscr{E}((f_n-f_m)h_N,(f_n-f_m)h_N) &\leq C_N\mu_V(\Gamma((f_n-f_m)h_N))\\ &\leq 2C_N\mathscr{E}_V(f_n-f_m)+2C_N\mu_V((f_n-f_m)^2)\to 0\end{aligned}$$

as $n,m\to\infty$, where $C_N>0$ is a constant. By the closability of $\mathscr{E}$ we have $\mathscr{E}(f_nh_N)\to 0$ as $n\to\infty$. Hence,

$$\lim_{n\to\infty}\mu_V(\Gamma(f_n)1_{\{\rho\leq N\}})=0. \tag{6.1}$$

On the other hand, since

$$\Gamma(f_n-f_m)=\Gamma(f_n)+\Gamma(f_m)-2\Gamma(f_n,f_m)\geq\frac{1}{2}\Gamma(f_n)-\Gamma(f_m),$$

we have

$$\mu_V(\Gamma(f_n)1_{\{\rho>N\}})\leq 2\mathscr{E}_V(f_n-f_m)+2\mu_V(\Gamma(f_m)1_{\{\rho>N\}}).$$

Combining this with (6.1) we obtain

$$\limsup_{n\to\infty}\mathscr{E}_V(f_n)\leq 2\limsup_{n\to\infty}\mathscr{E}_V(f_n-f_m)+2\mu_V(\Gamma(f_m)1_{\{\rho>N\}}).$$

Then the proof is finished by first letting $N\to\infty$ then $m\to\infty$. □

6.2 Hardy's criterion

Let E be either $[0,\infty)$ or $(-\infty,\infty)$, consider $L=a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}+b(x)\frac{\mathrm{d}}{\mathrm{d}x}$, where $a(>0)$ and b are measurable functions. Let $C(x)=\int_0^x\frac{b(r)}{a(r)}\mathrm{d}r$ and assume that $\mu(\mathrm{d}x)=Z^{-1}\frac{\mathrm{e}^{C(x)}}{a(x)}\mathrm{d}x$ is a probability measure, where $Z>0$ is the normalization.

We recall the following Hardy criterion due to [12] (see also [17, Theorem 6.2.4]), which generalizes Bobkov-Götze's corresponding result on logarithmic Sobolev inequalities. The analogous for birth-death processes is also available. We include below a simple proof for readers' reference.

For any $\alpha \geq 0$, let

$$\delta_{+0} := \sup_{x\in[0,\infty)} \mu([x,\infty)) \log^{\alpha} \mu([x,\infty))^{-1} \int_0^x \mathrm{e}^{-C(y)} dy,$$

$$\delta_{-0} := \sup_{x\in(-\infty,0]} \mu((-\infty,x]) \log^{\alpha} \mu((-\infty,x])^{-1} \int_x^0 \mathrm{e}^{-C(y)} dy.$$

We study the generalized logarithmic Sobolev inequality (5.2) by using an argument in [3] due to Hardy's inequality.

Proposition 6.2. *Let $\alpha \geq 0$ be fixed and $\mathscr{E}(f,g) = \mu(af'g')$ for $f \in C_b^1$. By convention, when $\alpha = 0$, (5.2) for $\theta = 1/\alpha$ means the Poincaré inequality.*

(1) *Let $E = [0,\infty)$. (5.2) for $\theta = 1/\alpha$ and some $C_1,\ C_2 > 0$ implies $\delta_{+0} < \infty$. If $\alpha \leq 1$ then they are equivalent.*

(2) *Let $E = \mathbb{R}$. (5.2) for $\theta = 1/\alpha$ and some $C_1, C_2 > 0$ implies $\delta_{-0} + \delta_{+0} < \infty$. If $\alpha \leq 1$ then they are equivalent.*

Proof. By the classical weighted Hardy inequality (cf. [5, 9]), it suffices to consider the case that $\alpha > 0$. We only prove (1) since the proof of (2) is similar. To this end, let us apply the weighted Hardy inequality. Let $\Psi(s) = |s| \log^{\alpha}(1+s)$, which is a Young function, i.e. a nonnegative, continuous, convex and even function satisfying $\Psi(x) = 0$ if and only if $x = 0$, $\lim\limits_{x\to 0} \Psi(x)/x = 0$, $\lim\limits_{x\to\infty} \Psi(x)/x = \infty$. Let

$$\|f\|_{\Psi} = \inf\{\lambda > 0 : \mu(\Psi(f/\lambda)) \leq 1\}.$$

On the other hand, the dual function

$$\Psi^*(y) = \sup\{x|y| - \Psi(x) : x \geq 0\},\ y \in \mathbb{R}$$

is once again a Young function. Let $\mathscr{G} = \{g \geq 0 : \mu(\Psi^*(g)) \leq 1\}$ and $\|f\|_{\mathscr{G}} = \inf\{\mu(|fg|) : g \in \mathscr{G}\}$. By [10, Proposition 3.3.4], we have

$$\|f\|_{\Psi} \leq \|f\|_{\mathscr{G}} \leq 2\|f\|_{\Psi}. \tag{6.2}$$

For any $n \geq 0$, define

$$B_n = \sup_{r>n} \|\mathbf{1}_{[r,\infty)}\|_{\mathscr{G}} \int_n^r e^{-C(x)} dx.$$

Let A_n be the smallest positive constant such that

$$\|f^2\|_{\mathscr{G}} \leq A_n \mu(af'^2), \quad f \in C_0^1([0,\infty)),\ f|_{[0,n]} = 0.$$

Then the weighted Hardy inequality introduced in [9] indicates that (cf. [5, Theorem 1.1] for more refined estimates)

$$B_n \leq A_n \leq 4B_n. \tag{6.3}$$

Since

$$\|\mathbf{1}_{[x,\infty)}\|_\Psi = \frac{1}{\Psi^{-1}\big(\mu([x,\infty))^{-1}\big)}, \quad x \geq 0,$$

and since

$$\frac{t}{c\log^\alpha t} \leq \Psi^{-1}(t) \leq \frac{ct}{\log^\alpha t}$$

holds for some $c > 0$ and all $t > 2$, $\delta_{+0} < \infty$ if and only if $B_n < \infty$ for some (hence all) $n \geq 0$. Let $\lambda = \mu(f^2 \log^\alpha(1+f^2)) + 1$. We have

$$\mu(f^2\lambda^{-1}\log^\alpha(1+f^2\lambda^{-1})) \leq \lambda^{-1}\mu(f^2\log^\alpha(1+f^2)) \leq 1.$$

Then

$$\|f^2\|_\Psi \leq \mu(f^2\log^\alpha(1+f^2)) + 1.$$

So, (5.2) implies

$$\|f^2\|_\Psi \leq C_1\mu(af'^2) + (C_2+1)\mu(f^2).$$

If $f|_{[0,n]} = 0$, then by Hölder's inequality for Orlicz norms and the fact that $\|1_{[n,\infty)}\|_{\Psi^*} \to 0$ as $n \to \infty$,

$$\mu(f^2) \leq c_1\|f^2\|_\Psi\|1_{[n,\infty)}\|_{\Psi^*} \leq \frac{1}{2(C_2+1)}\|f^2\|_\Psi$$

holds for sufficiently large n. Therefore $A_n < \infty$ (hence $B_n < \infty$) for large n, and hence, $\delta_{+0} < \infty$.

Finally, assume $\alpha \in (0,1]$. If $\mu(f^2) = 1$ and $\lambda = \frac{1}{2}\mu(f^2\log(1+f^2)) \geq \mathrm{e}$, then

$$\begin{aligned}\lambda^{-1}\mu(f^2\log^\alpha(1+f^2\lambda^{-1})) &= \lambda^{-1}\mu(f^2[\log(\lambda+f^2) - \log\lambda]^\alpha)\\ &\geq \lambda^{-1}\mu(f^2\log^\alpha(1+f^2)) - \lambda^{-1}\log^\alpha\lambda \geq 2-1 = 1.\end{aligned}$$

Hence,

$$\frac{1}{2}\mu(f^2\log^\alpha(1+f^2)) - \mathrm{e} \leq \|f^2\|_\Psi, \quad \mu(f^2) = 1.$$

Since $\delta_{+0} < \infty$ is equivalent to $A_n < \infty$, (5.2) holds for f with $f|_{[0,n]} = 0$. Then the proof is completed by a standard cut-off argument and the Sobolev inequality on finite intervals. □

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