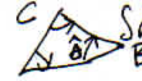

Soit un triangle, montrer que :

$$\Delta = \begin{vmatrix} \tan(\hat{A}/2) & \cos(\hat{A}) & 1 \\ \tan(\hat{B}/2) & \cos(\hat{B}) & 1 \\ \tan(\hat{C}/2) & \cos(\hat{C}) & 1 \end{vmatrix} = \begin{vmatrix} \tan(\hat{A}/2) & \cos\hat{A} & 1 \\ \tan(\hat{B}/2)-\tan(\hat{A}/2) & \cos(\hat{B})-\cos(\hat{A}) & 0 \\ \tan(\hat{C}/2)-\tan(\hat{A}/2) & \cos(\hat{C})-\cos(\hat{A}) & 0 \end{vmatrix}$$

remarquer qu'il est bien défini car $\hat{A}+\hat{B}+\hat{C}=\pi \Rightarrow 0<\frac{\hat{A}}{2},\frac{\hat{B}}{2},\frac{\hat{C}}{2}<\pi/2$ si $=\frac{\pi}{2}$ triangle dégénéré..

$$= \begin{vmatrix} \tan(\hat{B}/2)-\tan(\hat{A}/2) & \cos(\hat{B})-\cos(\hat{A}) \\ \tan(\hat{C}/2)-\tan(\hat{A}/2) & \cos(\hat{C})-\cos(\hat{A}) \end{vmatrix}$$

$\tan(U-V)$ si $\tan U$ et $\tan V$ ont un sens (*)

or $\cos(U)-\cos(V) = -2\sin\left(\frac{U-V}{2}\right)\sin\left(\frac{U+V}{2}\right)$

$\tan(U)-\tan(V) = \tan(U-V)\,[1+\tan(U)\tan(V)]$

donc

$$\Delta = \begin{vmatrix} \tan(\hat{B}-\hat{A}/2)\{1+\tan(\hat{A}/2)\tan(\hat{B}/2)\} & -2\sin\left(\frac{\hat{B}-\hat{A}}{2}\right)\sin\left(\frac{\hat{B}+\hat{A}}{2}\right) \\ \tan\left(\frac{\hat{C}-\hat{A}}{2}\right)\left(1+\tan\left(\frac{\hat{C}}{2}\right)\tan\left(\frac{\hat{A}}{2}\right)\right) & -2\sin\left(\frac{\hat{C}-\hat{A}}{2}\right)\sin\left(\frac{\hat{C}+\hat{A}}{2}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\sin(\hat{B}-\hat{A}/2)}{\cos(\hat{B}-\hat{A}/2)}\{1+\tan\hat{A}/2\tan\hat{B}/2\} & -2\sin(\hat{B}-\hat{A}/2)\sin(\hat{B}+\hat{A}/2) \\ \frac{\sin(\hat{C}-\hat{A}/2)}{\cos(\hat{C}-\hat{A}/2)}\{1+\tan(\hat{C}/2)\tan(\hat{A}/2)\} & -2\sin(\hat{C}-\hat{A}/2)\sin(\hat{C}+\hat{A}/2) \end{vmatrix}$$

$$= -2\sin\left(\frac{\hat{B}-\hat{A}}{2}\right)\sin\left(\frac{\hat{C}-\hat{A}}{2}\right)\begin{vmatrix} \frac{1+\tan\hat{A}/2\tan\hat{B}/2}{\cos(\hat{B}-\hat{A}/2)} & \sin\left(\frac{\hat{B}+\hat{A}}{2}\right) \\ \frac{1+\tan\hat{A}/2\tan\hat{C}/2}{\cos(\hat{C}-\hat{A}/2)} & \sin\left(\frac{\hat{C}+\hat{A}}{2}\right) \end{vmatrix}$$

or $\hat{A}+\hat{B}+\hat{C}=\pi$

donc $\sin(\hat{B}+\hat{A}/2) = \sin\left(\frac{\pi}{2}-\frac{\hat{C}}{2}\right) = \cos(\hat{C}/2)$:

$$\Delta = -2\sin\left(\frac{\hat{B}-\hat{A}}{2}\right)\sin\left(\frac{\hat{C}-\hat{A}}{2}\right)\underbrace{\begin{vmatrix} \frac{1+\tan\hat{A}/2\tan\hat{B}/2}{\cos(\hat{B}-\hat{A}/2)} & \cos\hat{C}/2 \\ \frac{1+\tan\hat{A}/2\tan\hat{C}/2}{\cos\hat{C}-\hat{A}/2} & \cos(\hat{B}/2) \end{vmatrix}}_{\Delta'}$$

Calculons Δ' :

$$\Delta' = \frac{1+\tan\hat{A}/2\tan\hat{B}/2}{\cos(\hat{A}-\hat{B}/2)}\cos\hat{B}/2 - \cos\hat{C}/2\,\frac{1+\tan\hat{A}/2\tan\hat{C}/2}{\cos(\hat{C}-\hat{A}/2)}$$

$$= \frac{\cos\hat{A}/2\cos\hat{B}/2+\sin\hat{A}/2\sin\hat{B}/2}{\cos\hat{A}/2\,\underbrace{\cos(\hat{A}-\hat{B}/2)}_{\cos(\hat{A}-\hat{B}/2)}} - \frac{\cos\hat{C}/2\cos\hat{A}/2+\sin\hat{C}/2\sin\hat{A}/2}{\cos\hat{A}/2\,\underbrace{\cos(\hat{C}-\hat{A}/2)}_{\cos(\hat{C}-\hat{A}/2)}}$$

$$= \frac{\cos(\hat{C}-\hat{A}/2)(\cos\hat{A}/2\cos\hat{B}/2+\sin\hat{A}/2\sin\hat{B}/2) - \cos(\hat{A}-\hat{B}/2)(\cos\hat{C}/2\cos\hat{A}/2+\sin\hat{A}/2\sin\hat{C}/2)}{\cos\hat{A}/2\,\cos(\hat{A}-\hat{B}/2)\,\cos(\hat{C}-\hat{A}/2)}$$

$= 0$! Le déterminant est donc nul.