HADAMARD GAP THEOREM AND OVERCONVERGENCE FOR FABER-EROKHIN EXPANSIONS.

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Résumé. Principally, we extend the Hadamard-Fabry gap theorem for power series to Faber-Erokhin ones.

1. A SHORT SURVEY ON FABER-EROKHIN BASIS

Let $\Omega \subset \mathbb{C}$ a simply connected domain, $K \subset \Omega$ a compact such that $\Omega \setminus K$ is doubly connected. Under these hypothesis, we know that (up to a rotation) there exists a biholomorphic mapping

$$\Phi : \Omega \setminus K \rightarrow C(0;1,R) = \{ z \in \mathbb{C} : 1 < |z| < R \},$$

where $R > 1$ is the modulus of the condensor $\mathcal{C} = (\Omega, K)$. Let

$$h_{\Omega,K}(z) := \sup \{ u(z) : u \in \text{SH}(\Omega), u \leq 1, u_{/K} \leq 0 \}$$

the relative extremal function and $\Omega_\alpha = \{ z \in \Omega : h_{\alpha,K}(z) < \alpha \}$ its levels sets $(0 < \alpha < 1)$; we have

$$\Omega_\alpha = \Phi^{-1}(D(0,R^\alpha)) = \{ z \in \mathbb{C} : |z| < R^\alpha \}, \quad \forall \alpha \in [0,1].$$

- Let $f \in \mathcal{O}(\Omega)$, then $f \circ \Phi^{-1}$ is holomorphic on the annulus $C(0;1,R)$, we have by the Laurent expansion

$$f \circ \Phi^{-1}(\xi) = \sum_{n=-\infty}^{+\infty} c_n \xi^n, \quad 1 < |\xi| < R$$

where

$$c_n = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f \circ \Phi^{-1}(\zeta)}{\zeta^{n+1}} d\zeta, \quad 1 < \rho < R, \quad n \in \mathbb{Z},$$

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and the series converges normay on compact sets of the annulus. Changing \( \xi \in C(0; 1, R) \) by \( \Phi(z) \in \Omega \setminus K \) the formula (1) is

\[
f(z) = \sum_{n=-\infty}^{+\infty} c_n \Phi(z)^n, \quad z \in \Omega \setminus K
\]

with normal convergence on compact sets of \( \Omega \setminus K \).

But now, unlike \( f \circ \Phi \), the function \( f \) is holomorphic on the whole \( \Omega \) and by Cauchy formula we have forall \( \alpha \in ]0, 1[ \) and \( z \in \Omega_\alpha \)

\[
f(z) = \frac{1}{2i\pi} \int_{\partial \Omega_\alpha} \frac{f(t)}{t-z} dt
\]

So

\[
(3) \quad f(z) = \sum_{n=-\infty}^{+\infty} c_n E_n(z), \quad \forall z \in \Omega
\]

and

\[
(4) \quad E_n(z) = \frac{1}{2i\pi} \int_{\partial \Omega_\alpha} \frac{\Phi(t)^n}{t-z} dt,
\]

where \( \alpha \in ]0, 1[ \) and \( z \in \Omega_\alpha \).

- In the exceptional case where \( \Phi \) extends to a conformal mapping of \( \overline{\mathbb{C}} \setminus K \) with \( \Phi(\infty) = \infty \), then \( E_n = 0, \forall n < 0 \). With (4) it is easy to see that \( E_n, (n \geq 0) \) is a polynomial of degree \( n \), they are the classical Faber polynomials [5]. The Faber polynomial sequence \((E_n)_0^\infty \) is a basis of \( \mathcal{O}(U) \) for all open level set \( U \) of the Green function \( G_K = G(\cdot, \overline{\mathbb{C}} \setminus K, \infty) \) associated to \( K \).

- The pioneer work of Erokhin [2], [5] consists to extend the notion of Faber polynomial to a regular condenser \((\Omega, K) \) where \( \Omega \setminus K \) is a doubly connected domain. His work is built on a « fundamental lemma » about the decomposition of a conformal map onto an annulus :

**Erokhin’s Fundamental Lemma 1.** Every conformal map \( \Phi \) from a doubly connected domain \( \Omega \setminus K \) onto an annulus \( C(0, 1, R) = \{ w \in \mathbb{C} : 1 < |w| < R \} \) can be decomposed into \( \Phi = F_2 \circ F_1 \) where \( F_1 \) and \( F_2 \) are conformal maps between simply connected domains, precisely :

1) \( F_1 \) maps conformly the simply connected domain \( \overline{\mathbb{C}} \setminus K \) onto a simply connected domain \( \overline{\mathbb{C}} \setminus L \) where \( L \) is compact in \( \mathbb{C} \). The image
by $F_1$ of the boundary of $\Omega : F_1(\partial \Omega)$ define a simply connected domain $\Omega_1$ which contains $L$.

2) $F_2$ is the biholomorphic map $F_2 : \Omega_1 \to D(0, R)$ such that $F_2(\partial L) = C(0, 1)$.

So we are in the following situation:

\[ \Phi = F_2 \circ F_1 \]

- **The Faber-Erokhin basis**: With this decomposition, the Faber-Erokhin basis is define by analogy with the Faber one by formula (4) with $n \in \mathbb{N}$ only

\[ E_n(z) = \frac{1}{2i\pi} \int_{\partial K} \frac{\Phi(t)^n}{t-z} dt, \quad \forall \alpha \in ]0, 1[ \text{ et } z \in \Omega_\alpha. \]

Erokhin shows that the sequence $(E_n)_{n \geq 0}$ is a common basis for the spaces $\mathcal{O}(\Omega)$, $\mathcal{O}(\Omega_\alpha)$, $(0 < \alpha < 1)$ but generally $E_n \neq 0$ when $n < 0$. The trivial expansion (3) being always transformed in

\[ f(z) = \sum_{0}^{+\infty} a_n E_n(z), \quad z \in \Omega, \]

where the $a_n$ are in general **new coefficients** given by an integral formula usually more complicated than (2). Precisely, we have for all $f \in \mathcal{O}(\Omega_\alpha)$, $0 < \rho < \alpha < 1$:

\[ a_n = \frac{1}{2i\pi} \int_{|\zeta|=\rho} \frac{\varphi_f(\zeta)}{\zeta^{n+1}} d\zeta \]
with for all $|\zeta| < R^\alpha$

$$\varphi_f(\zeta) = \sum_{n=0}^{+\infty} a_n \zeta^n = \frac{1}{2i\pi} \int_{|\tau|=R^\alpha} \frac{f(\Phi^{-1}(\tau))(F_2^{-1})'(\tau)}{F_2^{-1}(\tau) - F_2^{-1}(\zeta)} d\tau.$$  

2. HADAMARD TYPE RESULTS FOR FABER-EROKHIN EXPANSIONS

Let $f$ an holomorphic function on the level set $\Omega_\alpha$ such that $f \not\in \mathcal{O}(\Omega_\gamma)$, $\forall \alpha < \gamma < 1$. Let $f = \sum_{n \geq 0} a_n E_n$ its expansion in the Faber-Erokhin basis, then power series

$$\varphi_f(\zeta) := \sum_{n=0}^{+\infty} a_n \zeta^n$$

has $R^\alpha$ as radius of convergence. Moreover, (5) implies that for all $0 < \beta < \alpha$ and $|\zeta| < R^\beta$:

$$\varphi_f(\zeta) = \sum_{n=0}^{+\infty} a_n \zeta^n = \frac{1}{2i\pi} \int_{|\tau|=R^\beta} \frac{f(\Phi^{-1}(\tau))(F_2^{-1})'(\tau)}{F_2^{-1}(\tau) - F_2^{-1}(\zeta)} d\tau.$$  

**Theorem:** $f$ extends holomorphically across a point $z_0 \in \partial \Omega_\alpha$ iff $\varphi_f$ extends holomorphically across the point $\zeta_0 := \Phi(z_0) \in C(0, R^\alpha)$.

**Proof:**

- **Necessary condition:** Suppose that there exists a neighbourhood $V_{z_0} \subset \Omega \setminus K$ of $z_0$ such that $f$ extends holomorphically on $\Omega_\alpha \cup V_{z_0}$. Let $r > 0$ such that

$$D(\zeta_0, r) \subset \Phi(V_{z_0}) \subset C(0, 1, R),$$

and choose $0 < \beta < \alpha$ enough close to $\alpha$ so that

$$D(\zeta_0, r) \cap D(0, R^\beta) \neq \emptyset,$$

now, consider the oriented path $\gamma_{z_0}$ bellow
Then the function defined by the formula
\[ (7) \quad \psi(\zeta) = \frac{1}{2i\pi} \int_{\gamma_0} \frac{f(\Phi^{-1}(\tau))(F_2^{-1}(\tau))'}{F_2^{-1}(\tau) - F_2^{-1}(\zeta)} d\tau, \quad \zeta \in D(\zeta_0, r) \cup D(0, R^3). \]
is clearly holomorphic on \( D(\zeta_0, r) \cup D(0, R^3) \).

On the other hand with Cauchy
\[ (8) \quad \frac{1}{2i\pi} \int_{\gamma_0} \frac{f(\Phi^{-1}(\tau))(F_2^{-1}(\tau))'}{F_2^{-1}(\tau) - F_2^{-1}(\zeta)} d\tau = 0, \quad \forall \zeta \in D(\zeta_0, r). \]
Formula (8) combined with (6) and (7) assure that
\[ \psi = \varphi_f \quad \text{on} \quad D(0, R^3) \cap D(\zeta_0, r) \neq \emptyset, \]
so we succeed to extend holomorphically \( \varphi_f \) across \( \zeta_0 \).

- **Sufficient condition:** The proof is the same, it is built on the dual formula of (5)
\[ (5') \quad f(z) = \sum_{0}^{+\infty} a_n E_n(z) = \frac{1}{2i\pi} \int_{\partial\Omega} \frac{\varphi_f(\Phi(t))}{t - z} dt, \quad \forall z \in \Omega_\beta. \]

2.1. **Applications:** By contradiction, we have the following property: \( f \in \mathcal{O}(\Omega_\alpha) \) has \( \Omega_\alpha \) as domain of holomorphy, if and only if, \( \varphi_f \) has the disc \( D(0, R^\alpha) \) as domain of holomorphy.

So we are able to extend for expansions following the Faber-Erokhin basis some theorems on the boundary behaviour of an powers series. For example, we have

- **(Hadamard):** Let \( f(z) = \sum_{0}^{+\infty} a_n E_n(z) \in \mathcal{O}(D_\alpha) \) such that \( f \not\in \mathcal{O}(D_\beta) \), \( \forall \beta > \alpha \). If there exists a constant \( c > 0 \) such that \( n_{k+1} - n_k > c \cdot n_k \), \( \forall k \in \mathbb{N} \), then \( D_\alpha \) is the domain of holomorphy of \( f \).

Or stronger

- **(Fabry-Pólya):** Let \( f(z) = \sum_{0}^{+\infty} n_k E_{n_k}(z) \in \mathcal{O}(D_\alpha) \) such that \( f \not\in \mathcal{O}(D_\beta) \), \( \forall \beta > \alpha \). If \( \lim_{n_k} \frac{n_k}{\phi} = \infty \) then \( \Omega_\alpha \) is the domain of holomorphy of \( f \). Conversely (Pólya), every increasing sequence of integers \( n_0 < n_1 < \ldots \) such that every series \( \sum_{0}^{+\infty} a_{n_k} E_{n_k} \) has \( \Omega_\alpha \) as domain of holomorphy, satisfies \( \lim_{n_k} \frac{n_k}{\phi} = \infty \).

For example, the functions \( f(z) = \sum_{0}^{+\infty} R^{-2n^\alpha} E^{2n}(z) \) (Hadamard) or \( g(z) = \sum_{0}^{+\infty} R^{-n^\alpha} E_{n^2}(z) \) (Fabry) admits \( \Omega_\alpha \) as domain of holomorphy.
but this is not the cases for \( h(z) = \sum_{0}^{+\infty} R^{-na} E_n(z) \) who presents an unique singular (which of course is \( \Phi^{-1}(1) \ldots \) ) on the boundarys \( \partial \Omega_\alpha \).

### 3. The case of an arbitrary common basis.

With the same hypothesis on the pair \((K, \Omega)\) let us consider now an arbitrary common basis \((\varphi_n)_n\) for the spaces \( \mathscr{O}(K), \mathscr{O}(\Omega) \). It extends as a common basis of the intermediate spaces \( \mathscr{O}(\Omega_\alpha), \) (\( 0 < \alpha < 1 \)). This is not difficult to see that the preceeding results are no longer true for any common basis \((\varphi_n)_n\) : consider the simple example where \( K = D(0, 1/2) \subset \Omega = D(0, 2) \). This condensor admits as level sets the discs \( \Omega_\alpha = D(0, 2^{\frac{1}{2}+\frac{1}{2}}) \). Consider the common basis

\[
\varphi_n(z) = z^{\pi(n)}, \quad n \in \mathbb{N}
\]

where \( \pi : \mathbb{N} \to \mathbb{N} \) is a bijection such that \( \pi(2^n) = 2n \). Then the function \( f(z) = \sum_{0}^{+\infty} \varphi_{2n}(z) \) satisfies the Hadamard lacunary condition but

\[
f(z) = \sum_{0}^{+\infty} \varphi_{2n}(z) = \sum_{0}^{+\infty} z^{2n} = \frac{1}{1 - z^2}
\]

holomorphic on \( D(0, 1) = \Omega_{1/3} \) admits \( \mathbb{C} \setminus \{ \pm 1 \} \) as domain of holomorphy.

**Remarks:** In [1], J.A. Adepoju proved the Fabry-type gap theorem for Faber polynomials, his proof followed the classical one for entire series and is rather complicated.

In [4] we extend Fatou-type theorems to all common bases of the pair \((\mathscr{O}(K), \mathscr{O}(\Omega))\) in a more general situation.

### 4. Overconvergence.

In the spirit of the proof of theorem 1, the formulas \((5)\) and \((5')\) lead us to transport overconvergence phenomena to Faber-Erokhin series. Let \( f = \sum_{0}^{+\infty} a_n E_n \in \mathscr{O}(\Omega_\alpha) \), if \( f \) is not holomorphic on larger level sets \( \Omega_\beta, \alpha < \beta \) then we will say that the series \( \sum_{0}^{+\infty} a_n E_n \) is overconvergent if there exists a subsequence \((m_k)_k\) such that the corresponding partial sums

\[
s_{m_k}(f, z) := \sum_{\mu=0}^{m_k} a_\nu E_\nu(z),
\]

converges compactly in a domain that contains properly \( \Omega_\alpha \).
Unicity of coefficients in the Faber-Erokhin expansion and formula (5) gives

\[ s_m^k(\varphi_f, \zeta) := \sum_{\nu=0}^{m_k} a_{\nu} z^\nu = \frac{1}{2i\pi} \int_{|\tau|=R^3} \frac{s_m^k(f, \Phi^{-1}(\tau))(F_2^{-1})'(\tau)}{F_2^{-1}(\tau) - F_2^{-1}(\zeta)} d\tau. \]

Suppose now that the sequence \((s_m^k((f, \cdot))_k\) converges uniformly on a neighborhood \(V_{z_0}\) of a boundary point \(z_0 \in \partial \Omega_\alpha\), then as in theorem 1, we have

\[
\sup_{\zeta \in D(\zeta_0, r)} |s_m^k(\varphi_f, \zeta) - s_m^k(\varphi_f, \zeta)| \leq \sup_{\zeta \in V_{z_0}} |s_m^k(f, z) - s_m^k(f, z)| \times \int_{\gamma_{z_0}} \frac{|F_2^{-1}(\tau)| \cdot |d\tau|}{|F_2^{-1}(\tau) - F_2^{-1}(\zeta)|} \leq C \cdot \sup_{\zeta \in V_{z_0}} |s_m^k(f, z) - s_m^k(f, z)|
\]

where as before \(\zeta_0 = \Phi(z_0), D(z_0, r) \subset \Phi(V_{z_0})\). This implies that \((s_m^k(\varphi_f, \cdot))_k\) is an uniformly convergent Cauchy sequence on the disc \(D(\zeta_0, r)\): the series \(\sum_{k=0}^{+\infty} a_k z^k\) is overconvergent. By duality the overconvergence of \(\sum_{k=0}^{+\infty} a_k E_k\) implies the one for \(\sum_{k=0}^{+\infty} a_k E_k\).

Références