A PROOF OF WEIERSTRASS'S THEOREM[1]

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Weierstrass's theorem with regard to polynomial approximation can be stated as follows:

If $f(x)$ is a given continuous function for $a \leqq x \leqq b$, and if ϵ is an arbitrary positive quantity, it is possible to construct an approximating polynomial $P(x)$ such that

$$|f(x) - P(x)| < \epsilon$$

for $a \leqq x \leqq b$.

This theorem has been proved in a great variety of different ways. No particular proof can be designated once for all as the simplest, because simplicity depends in part on the preparation of the reader to whom the proof is addressed. A demonstration which follows directly from known facts about power series or Fourier series, for example, is not so immediate if a derivation of those facts has to be gone through first. A proof commonly regarded as among the simplest and neatest is the one due to Landau[2] in which an approximating polynomial is given explicitly by means of a certain type of "singular" integral. The purpose of this note is to present a modification or modified formulation of Landau's proof which is believed to possess further advantages of simplicity, at least from some points of view.

Let $f(x)$ be a given continuous function for $a \leqq x \leqq b$. Without essential loss of generality it can be supposed that $0 < a < b < 1$, since any finite interval whatever can be carried over into an interval contained in $(0, 1)$ by a linear transformation, under which any continuous function will go into a continuous function and any polynomial into a polynomial of the same degree. For convenience in the writing of the formulas which enter into the proof, let the function $f(x)$, supposed given in the first instance only for $a \leqq x \leqq b$, be defined outside the interval (a, b) as follows:

$$f(x) = 0 \qquad \text{for} \qquad x \leqq 0,$$

$$f(x) = \frac{x}{a} f(a) \qquad \text{for } 0 < x < a,$$

$$f(x) = \frac{1 - x}{1 - b} f(b) \qquad \text{for } b < x < 1,$$

$$f(x) = 0 \qquad \text{for} \qquad x \geqq 1.$$

Then $f(x)$ is defined and continuous for all values of x. The question at issue is that of approximating it by means of a polynomial for values of x belonging to

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[2] E. Landau, *Über die Approximation einer stetigen Funktion durch eine ganze rationale Funktion*, Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 337–345; see also Courant-Hilbert, *Methoden der Mathematischen Physik*, vol. I, second edition, Berlin, 1931, pp 55–57.

the interval (a, b).

Let J_n denote the constant

$$J_n = \int_{-1}^{1} (1 - u^2)^n du,$$

and let

$$P_n(x) = \frac{1}{J_n}\int_0^1 f(t)[1 - (t - x)^2]^n dt.$$

The integrand in the latter integral is a polynomial of degree $2n$ in x with coefficients which are continuous functions of t, and the integral is for each value of n a polynomial in x of degree $2n$ (at most) with constant coefficients.

If $0 \leqq x \leqq 1$, the value of the integral is not changed if the limits are replaced by $-1+x$ and $1+x$, since $f(t)$ vanishes everywhere outside the interval $(0, 1)$, and vanishes in particular from $-1+x$ to 0 and from 1 to $1+x$:

$$P_n(x) = \frac{1}{J_n}\int_{-1+x}^{1+x} f(t)[1 - (t - x)^2]^n dt.$$

By the substitution $t-x=u$ this becomes

$$P_n(x) = \frac{1}{J_n}\int_{-1}^{1} f(x + u)(1 - u^2)^n du. \tag{1}$$

If the equation

$$1 = \frac{1}{J_n}\int_{-1}^{1} (1 - u^2)^n du$$

is multiplied by $f(x)$, this factor, being independent of u, may be written under the integral sign:

$$f(x) = \frac{1}{J_n}\int_{-1}^{1} f(x)(1 - u^2)^n du. \tag{2}$$

Hence, by subtraction of (2) from (1),

$$P_n(x) - f(x) = \frac{1}{J_n}\int_{-1}^{1} [f(x + u) - f(x)](1 - u^2)^n du. \tag{3}$$

The problem now is to show that the value of this expression approaches zero as n becomes infinite.

Let ϵ be any positive quantity. Since $f(x)$ is (uniformly) continuous there is a $\delta > 0$ (independent of x) such that $|f(x+u)-f(x)| \leqq \epsilon/2$ for $|u| \leqq \delta$. Let M be the maximum of $|f(x)|$. Then $|f(x+u)-f(x)| \leqq 2M$ for all values of u. For $|u| \geqq \delta$, $1 \leqq u^2/\delta^2$, and

$$|f(x + u) - f(x)| \leqq 2Mu^2/\delta^2.$$

For any value of u, one or the other of the quantities $\epsilon/2$, $2Mu^2/\delta^2$ is greater than or equal to $|f(x+u)-f(x)|$, and their sum therefore is certainly greater than or equal to $|f(x+u)-f(x)|$:

$$|f(x+u)-f(x)| \leqq \epsilon/2 + 2Mu^2/\delta^2$$

for all values of u. Consequently, for $0 \leqq x \leqq 1$,

$$\begin{aligned} |P_n(x)-f(x)| &\leqq \frac{1}{J_n}\int_{-1}^{1}(\epsilon/2)(1-u^2)^n du + \frac{1}{J_n}\int_{-1}^{1}\frac{2Mu^2}{\delta^2}(1-u^2)^n du \\ &= \epsilon/2 + \frac{2M}{\delta^2 J_n}\int_{-1}^{1} u^2(1-u^2)^n du. \end{aligned}$$

Let the last integral be denoted by J'_n. By integration by parts,

$$\begin{aligned} J'_n &= \int_{-1}^{1} u\cdot u(1-u^2)^n du = \left[-u\cdot\frac{(1-u^2)^{n+1}}{2(n+1)}\right]_{-1}^{1} \\ &+ \int_{-1}^{1}\frac{(1-u^2)^{n+1}}{2(n+1)}du = \frac{J_{n+1}}{2(n+1)}. \end{aligned}$$

But $J_{n+1} < J_n$, since $1-u^2<1$ throughout the interior of the interval of integration and hence $(1-u^2)^{n+1}=(1-u^2)(1-u^2)^n<(1-u^2)^n$. So

$$J'_n < \frac{J_n}{2(n+1)}, \qquad \frac{J'_n}{J_n} < \frac{1}{2(n+1)}.$$

It follows that as soon as n is sufficiently large

$$\frac{2MJ'_n}{\delta^2 J_n} < \epsilon/2$$

and consequently

$$|P_n(x)-f(x)| < \epsilon,$$

for $0 \leqq x \leqq 1$ and in particular for $a \leqq x \leqq b$. This is the substance of the conclusion to be proved.

The above proof has something in common with that of S. Bernstein,[3] the fundamental difference being that Landau's integral is used here instead of the algebraic formula for a binomial frequency distribution. It was in fact suggested to the writer, not by consideration of Bernstein's proof as such, but by a conversation with Professor W. L. Hart on the subject of Bernoulli's theorem. A noteworthy characteristic of Bernstein's proof is that it makes Weierstrass's theorem in effect a corollary of that of Bernoulli.

An alternative organization of the present method of proof is as follows. Let $f(x)$ at first be not merely continuous, but subject to the Lipschitz condition

[3] See e.g. Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. I, Berlin, 1925, pp. 66, 230.

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|. \tag{4}$$

If this condition is satisfied by $f(x)$ as originally defined for $a \leqq x \leqq b$, it will be satisfied, possibly with a different value of λ, when the definition is extended to cover all values of x. Then, in (3), $|f(x+u) - f(x)| \leqq \lambda |u|$, and

$$|P_n(x) - f(x)| \leqq \frac{\lambda}{J_n} \int_{-1}^{1} |u| (1 - u^2)^n du = \frac{2\lambda}{J_n} \int_0^1 u(1 - u^2)^n du.$$

Let $\delta = 1/n^{1/2}$, and let

$$I_1 = \int_0^\delta u(1 - u^2)^n du, \qquad I_2 = \int_\delta^1 u(1 - u^2)^n du,$$

so that $|P_n(x) - f(x)| \leqq 2\lambda(I_1 + I_2)/J_n$. In I_1, $u \leqq 1/n^{1/2}$, and

$$I_1 \leqq \int_0^\delta n^{-1/2}(1 - u^2)^n du \leqq \int_0^1 n^{-1/2}(1 - u^2)^n du = \tfrac{1}{2} n^{-1/2} J_n, \quad 2I_1/J_n \leqq 1/n^{1/2}.$$

In I_2, $1/u \leqq n^{1/2}$ and $u = u^2/u \leqq n^{1/2}u^2$, so that by application of the previous reckoning with J_n'

$$I_2 \leqq n^{1/2} \int_\delta^1 u^2(1 - u^2)^n du \leqq n^{1/2} \int_0^1 u^2(1 - u^2)^n du = \tfrac{1}{2} n^{1/2} J_n' \leqq \tfrac{1}{4} n^{1/2} J_n/(n + 1),$$

$$2I_2/J_n \leqq \tfrac{1}{2} n^{1/2}/(n + 1) < 1/n^{1/2}.$$

So it appears not merely that $|P_n(x) - f(x)|$ is less than a quantity independent of x which approaches zero as n becomes infinite, but also, more specifically,[4] that it is less than a quantity of the order of $1/n^{1/2}$. This is proved, to be sure, only for a function satisfying (4). But (4) is satisfied in (a, b) by any continuous function whose graph is a broken line made up of a finite number of straight line segments with finite slope, and as any function whatever that is continuous for $a \leqq x \leqq b$ can be approximated with any desired accuracy by a function of this special type, the general conclusion of Weierstrass's theorem follows immediately.

The method can be applied equally well to the proof of Weierstrass's theorem on the trigonometric approximation of a periodic continuous function, by the use of de la Vallée Poussin's integral[5]

$$\frac{1}{H_n} \int_{-\pi}^{\pi} f(t) \cos^{2n}\left(\frac{t - x}{2}\right) dt, \qquad H_n = \int_{-\pi}^{\pi} \cos^{2n} \frac{u}{2} \, du,$$

and to the proof of corresponding theorems on the approximate representation of continuous functions of more than one variable.

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[4] Cf. C. de la Vallée Poussin, *Sur l' approximation des fonctions d' une variable réelle et de leurs dérivées par des polynômes et des suites limitées de Fourier*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, 1908, pp. 193–254; pp. 221–224.

[5] See de la Vallée Poussin, loc. cit., pp. 228–230.