HOMOLOGICAL TORSION OF ARITHMETIC THREE–MANIFOLDS

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Abstract. In this short preliminary note we study the homological and analytic torsion of arithmetic three–orbifolds defined by division quaternion algebras as the commensurability class varies.

1. Introduction

The aim of this note is to prove the following result (see 2 for the definitions regarding arithmetic subgroups).

Theorem A. *Let Γ_n be a sequence of congruence or maximal torsion-free uniform arithmetic subgroups of $\mathrm{SL}_2(\mathbb{C})$ defined over imaginary quadratic number fields (resp. cubic fields with one complex place). Then there exists a sequence of $\mathbb{Z}$-lattices L_n in $\mathfrak{sl}_2(\mathbb{C})$ (resp. $\mathfrak{sl}_2(\mathbb{C})^3$) preserved by Γ_n and we have the limit:*

$$\lim_{n\to+\infty} \frac{\log|H_p(\Gamma_n; L_n)|}{\operatorname{vol}\Gamma_n\backslash\mathbb{H}^3} = \begin{cases} 0 & \text{if } p = 0, 2; \\ \frac{13}{6\pi} \ \ \textit{(resp.} \frac{13}{2\pi}\textit{)} & \text{if } p = 1. \end{cases}$$

A short argument for the existence statement in the quadratic case is as follows (we prove a more general result in 3 below; see also [6]): let A_n be the quaternion algebra used to define Γ_n, $\mathcal{O}_n$ an order in A_n such that Γ_n normalizes $\mathcal{O}_n$. Let U_n be the subspace of $A_n \otimes \mathbb{C}$ of quaternions having null trace; then $U_n \cap \mathcal{O}_n$ is isomorphic as a Γ_n-module to a lattice in $\mathfrak{sl}_2(\mathbb{C}) \cong U_n \otimes \mathbb{C}$.

The main content of this theorem is the case $p = 1$; the growth rates for degrees 0 and 2 are established using simple algebraic manipulations, but to prove the exponential growth rate of the size of H_1 we use:

- Analytic torsion and the Cheeger–Müller theorem [7];
- The approximation results for L^2-torsion established in [2],[1];
- The convergence results for arithmetic orbifolds from [9].

In fact the proof of Theorem A is little more than a concatenation of these results, and the bulk of this note is dedicated to introduce objects and notation and to prove the statements in degrees 0 and 2.

2. Quaternion algebras and arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$

We record a few results and definitions from [5]; our terminology and notation may differ from those in this reference.

2.1. **Generalities.** A quaternion algebra over a field F is a four-dimensional algebra over F which is simple and whose center is F. Such algebras are constructed as follows: let E be a quadratic extension of F with nontrivial Galois automorphism $x \mapsto \overline{x}$, and fix a nonzero $\theta \in F$. Then the algebra

$$A = E \oplus Ej \tag{2.1}$$

where j is a symbol such that $j^2 = \theta$, with the multiplication law defined by

$$(x' + y'j)(x + yj) = (x'x + \theta y'\overline{y}) + (x'y + y'\overline{x})j$$

is a quaternion algebra over F, and all such are obtined in this manner. It is said to be split if θ is actually a square in E, in which case it is isomorphic to the matrix algebra $M_2(F)$; otherwise it is a division algebra.

The algebra A is endowed with an involution $z = x + yj \mapsto \overline{z} = \overline{x} - yj$ and natural morphisms to F defined as follows:

$$|z|_A = z\overline{z} = |x|_{E/F} + \theta|y|_{E/F}, \quad \mathrm{tr}_A(z) = z + \overline{z} = \mathrm{tr}_{E/F}(x).$$

When $A \cong M_2(F)$ these are identified with the usual determinant and trace. We shall denote by $\mathrm{SL}_1(A)$ the F-algebraic group defined by the kernel of the norm $|\cdot|_A$: it is an almost simple, simply connected group.

If F is a number field or a local non-Archimedean field and $\mathcal{O}_F$ its ring of integers, an order in A is a subring (containing 1) of A which is also a $\mathcal{O}_F$-submodule of rank four (in the matrix case an example is given by $M_2(\mathcal{O}_F)$; we will usually denote such orders by $\mathcal{O}_A$ in the sequel.

2.2. **Local results.** If F is a number field, v a finite place of F and F_v the completion of F at v then there is a unique unramified quadratic extension E_v of F_v (obtained by adjoining the square root of a unit). We let moreover $\mathfrak{P}_v$ be the prime ideal of $\mathcal{O}_F$ corresponding to v and π_v a ggenerator for $\mathfrak{P}_v\mathcal{O}_{F_v}$; then there is a unique division quaternion algebra over F_v, obtained by setting $E = E_v$ and $\theta = \pi_v$ in the definition (2.1).

Suppose that A_v is equal to this division algebra; then there is a unique maximal order $\mathcal{O}_{A_v}$ in A_v, which is given by:

$$\mathcal{O}_{A_v} = \mathcal{O}_{E_v} \oplus \mathcal{O}_{E_v} j.$$

The subgroup $K_v = \mathcal{O}_{A_v}^1$ of the F_v-points of the algebraic group $\mathrm{SL}_1(A_v)$ is its unique maximal compact subgroup. If K_v' is a compact-open subgroup it is thus contained in K_v and there is a $m \geq 1$ such that it contains $K_v(m) = 1 + \pi_v^m \mathcal{O}_{A_v}$. If $K_v' \subsetneq K_v$ and m_0 is the maximal such integer we say that K_v' has level $\mathfrak{P}_v^{m_0}$ at v; we define the level of K_v to be $\mathcal{O}_{F_v}$.

In the case where $A_v = M_2(F_v)$ there are infinitely many maximal orders, but they are all conjugated by an inner automorphism of A_v to $\mathcal{O}_{A_v} = M_2(\mathcal{O}_{F_v})$. Likewise, every maximal compact subgroup is conjugated by an inner automorphism to $K_v = \mathrm{SL}_2(\mathcal{O}_{F_v})$, and a basis of compact-open neighbourhoods of the identity is given by the subgroups $1 + \pi_v^m \mathcal{O}_{A_v}, m \geq 1$ of the latter. We define the level of an open subgroup of K_v as above, and extend the definition to all compact open subgroups in the natural way.

2.3. **Global orders and arithmetic lattices.** Let F be a number field of degree r with r_2 complex places and r_1 real ones, and let A a quaternion algebra over F. We shall denote by G the F-algebraic group $\mathrm{SL}_1(A)$ and by G_∞ be the real Lie group

$$G_\infty = \prod_{v \in V_\infty} \mathrm{G}(F_v) \cong \mathrm{SL}_2(\mathbb{C})^{r_2} \times \mathrm{SL}_2(\mathbb{R})^{r_1 - a} \times \mathrm{SU}(2)^a$$

where a is the number of real places of F where A is ramified. We also choose for each v a maximal compact subgroup K_v in $\mathrm{G}(F_v)$ and let G_f be the locally compact group defined by taking the restricted product of all $\mathrm{G}(F_v)$ with respect to those subgroups.

The choice of the quaternion algebra A defines a commensurability class of arithmetic lattices in G_∞. A congruence group in $\mathrm{G}(F)$ is by definition a subgroup $\Gamma \subset \mathrm{G}(F)$ such that there exists a compact-open subgroup K_f of G_f with the property that $\Gamma = \mathrm{G}(F) \cap K_f$: in other words the closure of Γ in G_f is equal to Γ itself. Any such congruence group is a lattice in G_∞ under the diagonal embedding $\mathrm{G}(F)$ in the latter, and the subgroups in G_∞ which are commensurable to one (or any) such lattice form a commensurability class of lattices in G_∞. We will say that such arithmetic subgroups of G_∞ are defined over F, or that their field of definition is F.

If Γ' belongs to this commensurability class then $\Gamma = \Gamma' \cap \mathrm{G}(F)$ does as well, and for each finite place v the closure K_v' of Γ in $\mathrm{G}(F_v)$ is a compact-open subgroup, which therefore has a well-defined level $\mathfrak{P}_v^{m_v}$ for some $m_v \geq 1$. If moreover Γ is a congruence group in $\mathrm{G}(F)$, then we define the level of Γ' to be the ideal $\prod_{v \in V_f} \mathfrak{P}_v^{m_v}$. In particular, the level is defined for congruence or maximal lattices in the commensurability class.

According to the local picture described above, if $\Gamma \subset \mathrm{G}(F)$ is an arithmetic group in the commensurability class defined by A then at each finite place v there is an order $\mathcal{O}_{A_v}$ in $A_v = A \otimes F_v$ which is normalized by Γ; thus the global order $\mathcal{O}_A$ of A defined by $\mathcal{O}_A = A \cap \prod_v \mathcal{O}_{A_v}$ is normalized by Γ. Since any arithmetic lattice in the commensurability class normalizes a Γ as above it is not necessary to suppose that $\Gamma' \subset \mathrm{G}(F)$ for Γ' to normalize an order.

In the sequel we will restrict (though until 5 we do not need to) to the case where $G_\infty = \mathrm{SL}_2(\mathbb{C}) \times K_\infty'$ where K_∞' is compact (in the notation above, this means that $r_2 = 1$ and $r_1 = a$). A. Borel's formula for the volume of congruence lattices [5, Theorem 11.1.3] together with the upper bounds for the index $[\Gamma : \Gamma \cap \mathrm{G}(F)]$ when Γ is a maximal lattice, yield the following lower bound for the covolume: let Γ be a congruence or maximal lattice of level $\mathfrak{I}$, then:

$$\mathrm{vol}(\Gamma\backslash\mathbb{H}^3) \gg 2^{-s} \cdot |\mathfrak{I}| \cdot D \tag{2.2}$$

where s is the number of primes dividing $\mathfrak{I}$ and $D = \prod_{v\in S}(q_v+1)$ where S is the set of finite places at which A ramifies.

3. Local systems

As was explained briefly in the notes to Theorem A, arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ defined over quadratic fields always preserve a lattice in the adjoint representation. More generally, if we let V_{2m}, ρ_{2m} denote the representation of $\mathrm{SL}_2(\mathbb{C})$ on the complex vector space $\mathrm{Sym}^m(\mathfrak{sl}_2(\mathbb{C}))$ then it remains true that the image by ρ_{2m} of such an arithmetic group preserves a lattice in V_{2m} (see also [6, Proposition 3.3] for a more conceptual proof). On the other hand, if Γ is an arithmetic lattice defined over a non-quadratic field there may not be any $\rho_{2m}(\Gamma)$-invariant lattice in V_{2m}. One has to take multiples of the representation to ensure their existence.

Proposition 3.1. *If Γ is an arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$ defined over a number field F of degree r over $\mathbb{Q}$ then for all $m \geq 1$ there is a Γ-invariant lattice L in the representation $(k_{m,r}\rho_{2m}, V_{2m}^{k_{m,r}})$, where*

$$k_{m,r} = \binom{3(r-2)+m-1}{m}$$

is the dimension of the mth symmetric power of $\mathbb{R}^{3(r-2)}$).

Proof. We note for future use that we shall prove below a more explicit (albeit also more cumbersomely stated) result, which goes as follows: let A be the F-quaternion algebra defining the commensurability class of Γ and let $\mathcal{O}_A$ be an $\mathcal{O}_F$-order in A normalized by Γ; then as a Γ-module the lattice L is isomorphic to the mth symmetric power of the elements of trace 0 in $\mathcal{O}_A$.

First we prove this for $m=1$; moreover, since the representations ρ_{2m} factor through $\{\pm 1\}$ we shall deal with lattices in the adjoint group $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$. Notation as above, let W denote the F-vector space of elements of trace 0 in A, and let $\overline{\mathrm{G}}$ be the Weil restriction from F to $\mathbb{Q}$ of the group $\mathrm{PGL}_1(A)$. The latter acts on W by conjugation, and the subgroup $\overline{\Gamma}$ of $\overline{\mathrm{G}}(F)$ preserves the submodule $L = (\mathcal{O} \cap W)$. Now on the complex vector space $W_\infty = W \otimes_{\mathbb{Q}} \mathbb{R}$ there is a representation ρ_∞ of $G_\infty = \overline{\mathrm{G}}(\mathbb{R})$, and $\rho_\infty(\Gamma)$ preserves the lattice $L \subset W_\infty$. Let σ_1 be a complex embedding of F, and let

$$W_{\mathrm{c}} = W^{\sigma_1} \otimes_F \mathbb{C}, \quad W_{\mathrm{r}} = \bigotimes_\sigma W^\sigma \otimes_F \mathbb{R}$$

where the tensor product is over all real embeddings of F. Let ρ be the restriction of ρ_∞ to the subgroup $G_{\mathrm{c}} = \mathrm{PGL}_1(A^{\sigma_1} \otimes_F \mathbb{C}) \cong \mathrm{PGL}_2(\mathbb{C})$. It is isomorphic to $\rho_{\mathrm{c}} \otimes_{\mathbb{R}} 1_{W_{\mathrm{r}}}$ where ρ_{c} is the natural representation of G_{c} on W_{c} and 1_V denotes the trivial representation on a vector space V. Since as a representation of $\mathrm{PGL}_2(\mathbb{C})$, ρ_{c} is isomorphic to the adjoint representation and $\dim_{\mathbb{R}} W_{\mathrm{r}} = 3(r-2)$ this finishes the proof of the proposition for $m=1$.

The case of any $k \geq 1$ follows, since we have a map of $\mathrm{SL}_2(\mathbb{C})$-modules from $\mathrm{Sym}^m(V_2^{k_{2,r}})$ to $V_{2m}^{k_{m,r}}$ which is rational. □

4. Growth of homology in degrees 0 and 2

We will give here upper bounds for the order of the torsion subgroups of H_0, H_2 for arithmetic lattices, in terms of the level and discriminant. They are sufficient to deduce the part of Theorem A dealing with these degrees.

4.1. Bounds for the order of co-invariants.

Proposition 4.1. *Let A be a quaternion algebra over a number field of degree r and $\mathcal{O}_A$ an order of level $\mathfrak{I}$ in A. Let L be the lattice in $k_{2,r}V_2$ preserved by $\rho(\mathcal{O}_A^1)$ given by Proposition 3.1; then there is a $N \in \mathbb{Z}$ depending only on m, r such that:*

$$N\mathfrak{I}D_A L \subset (\mathcal{O}_A^1 - 1) \cdot L.$$

The proof occcupies the rest of the section. Since the algebraic group $\mathrm{SL}_1(A)$ satisfies absolute strong approximation it suffices to prove the corresponding statement at each finite place of F; this is done for ramified places in Lemma 4.2 and for split ones in Lemma 4.3. We will use the notation of 2

Lemma 4.2. *If A ramifies at a finite place v which does not divide 2 we have $\pi_v L_v \subset (\mathcal{O}_{A_v}^1 - 1) \cdot L_v$; in fact it is equal to $\pi_v L_v + \mathcal{O}_{E_v} j$.*

Proof. We first deal with the case $m=1$. Then L_v is isomorphic as a $\mathcal{O}_{A_v}^1$-module to the set of trace 0 elements in $\mathcal{O}_{A_v}$ on which $\mathcal{O}_{A_v}^1$ acts by conjugation. Let $a = x + yj \in \mathcal{O}_{A_v}^1$ and $u = v + wj \in L_v$. A straightforward computation yields:

$$\begin{aligned} aua^{-1} &= (|x|_{E_v/F_v} + |y|_{E_v/F_v}\pi_v)v + \pi_v(\overline{x}y\overline{w} - x\overline{y}w) \\ &\quad + (x^2 w - 2xyv - \pi_v \overline{w})j. \end{aligned}$$

Setting $y=0$ we get that $(\mathcal{O}_{A_v}^1 - 1)L_v$ contains $(x^2-1)\mathcal{O}_{E_v} j$ for any x with $|x|_{E_v/F_v} = 1$. Now since $v \not| 2$ there exists such a x which is not congruent to 1 modulo $\pi_v \mathcal{O}_{E_v}$; hence we end up with $\mathcal{O}_{E_v} j \subset (\mathcal{O}_{A_v}^1 - 1)L_v$.

On the other hand, when $w=0$ the summand of $aua^{-1} - u$ on $\mathcal{O}_{E_v}$ becomes $2(|x|_{E_v/F_v} - 1)v$. Fixing some $x \in 1 + \pi_v \mathcal{O}_{E_v}$, $x \notin 1 + \pi^2 \mathcal{O}_{E_v}$ and letting v range among all possibilities we get that $(\mathcal{O}_{A_v}^1 - 1)L_v$ a submodule of $\pi_v L_v$ whose projection on $\mathcal{O}_{E_v}$ is the same as that of $\pi_v L_v$. As we have proven above that $(\mathcal{O}_{A_v}^1 - 1)L_v$ contains $\mathcal{O}_{E_v} j$ it follows that the former contain $\pi_v L_v$ (and in fact equals $\pi_v L_v + \mathcal{O}_{E_v} j$). □

The proof of the following lemma is essentially contained in [8, Lemma 6.5], and could possibly also be deduced along the lines of the proof of [6, Proposition 4.2]

Lemma 4.3. *There is a set S of finite places of F such that if v is a finite place not in S where A is split, and $\mathcal{O}_A$ has level $\mathfrak{P}_v^a$ then we have $\pi_v^a L_v \subset (\mathcal{O}_{A_v}^1 - 1) \cdot L_v$. Moreover, for all $v \in S$ there is an integer m_v such that if A is split at v and $\mathcal{O}_{A_v}$ has level $\mathfrak{P}_v^a$ then $\pi_v^{a+m_v} L_v \subset (\mathcal{O}_{A_v}^1 - 1) \cdot L_v$*

Proof. We let S be the set of all finite places of F which divide a rational prime smaller than $2m$. In particular, for any finite place $v \notin S$ the binomial coefficient $\binom{k}{l}$ is a unit in $\mathcal{O}_{F_v}$ for all $k = 0, \dots, 2m$, $l = 0, \dots, k$.

Suppose that $v \notin S$, and that the order $\mathcal{O}_A$ has level $\mathfrak{P}_v^a$ at v. Then up to conjugating $\mathcal{O}_{A_v}$ by an element of the adjoint group $\mathrm{PGL}_2(F_v)$ (which does not affect the quotient $L_v/(\mathcal{O}_{A_v}^1 - 1)L_v$) we may suppose that

$$\mathcal{O}_{A_v}^1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_{F_v}) : a, d \in 1 + \pi_v^a \mathcal{O}_{F_v},\, b, c \in \pi_v^a \mathcal{O}_{F_v} \right\}.$$

In particular, $\mathcal{O}_{A_v}^1$ contains the subgroups

$$U = \left\{ u_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \pi_v^a \mathcal{O}_{F_v} \right\} \text{ and } L = \left\{ l_y = \begin{pmatrix} 1 & \\ y & 1 \end{pmatrix} : y \in \pi_v^a \mathcal{O}_{F_v} \right\}.$$

Now we have $L_v = \mathrm{Sym}^{2m} \mathcal{O}_{F_v}$, which has a basis $e_1^k e_2^{2m-k}$, $k = 0, \dots, 2m$ on which the action of L, U is given by:

$$\begin{aligned} u_x \cdot e_1^k e_2^{2m-k} &= \sum_{l=0}^{k} \binom{k}{l} x^l e_1^l e_2^{2m-l} \\ l_y \cdot e_1^k e_2^{2m-k} &= \sum_{l=k}^{2m} \binom{2m-k}{2m-l} y^l e_1^l e_2^{2m-l}. \end{aligned}$$

By using the first formula and the fact that the binomial coefficients are units (since $v \notin S$) we get by an easy inductive argument on k that

$$\forall k = 1, \dots, 2m : \ \pi_v^a e_1^k e_2^{2m-k} \in (U - 1)L_v; \tag{4.1}$$

similarly, we have

$$\forall k = 0, \dots, 2m-1 : \ \pi_v^a e_1^k e_2^{2m-k} \in (L - 1)L_v. \tag{4.2}$$

Since $U, L \subset \mathcal{O}_{A_v}^1$ we can conclude that $\pi_v^a l_v \subset (\mathcal{O}_{A_v}^1 - 1)L_v$.

Now if $v \in S$ and A_v is split we can apply exactly the same argument, taking into account the fact that the binomial coefficients are not units. Letting m_v be the maximal valuation at v of all $\binom{k}{l}$ for $k = 0, \dots, 2m$, $l = 0, \dots, k$ we can establish that (4.1) and (4.2) hold with a replaced by $a + m_v$, from which we conclude that $\pi_v^{a+m_v} L_v \subset (\mathcal{O}_{A_v}^1 - 1)L_v$. $\square$

4.2. **Conclusion.** It follows from Proposition 4.1 that there are an exponent $c \geq 1$ and a rational integer $N > 0$ depending only on m, r such that: if Γ is a congruence lattice in $\mathrm{SL}_2(\mathbb{C})$ defined over a field F of degree r, with discriminant and level $D_A, \mathfrak{I} \subset \mathcal{O}_F$, and L is the lattice in $V_2^{k_2,r}$ preserved by Γ; we have

$$|L/(\Gamma - 1)L| \leq N \cdot (|D_A| \cdot |\mathfrak{I}|)^c.$$

By the lower bound (2.2) for the covolume of congruence lattices, if Γ_n is any sequence of congruence lattices which only satisfy the condition that the degrees of their fields of definition be bounded, we get:

$$\lim_{n \to +\infty} \frac{\log |H_0(\Gamma_n; L_n)|}{\operatorname{vol} M_n} = 0. \tag{4.3}$$

Now suppose that the lattices Γ_n are moreover uniform and torsion free. Then we can use Poincaré duality and [4, Corollary 3.3 in Chapter 3] to deduce that the torsion subgroups of $H_p(\Gamma_n; L_n)$ are isomorphic for $p = 0, 2$, hence it follows from (4.3) that we have also:

$$\lim_{n \to +\infty} \frac{\log |H_2(\Gamma_n; L_n)|}{\operatorname{vol} M_n} = 0. \tag{4.4}$$

Remark Since the degree of the fields of definition of the lattices we consider are bounded, and the homology $H_*(\Gamma; V_2) = 0$ the arguments of [3] can be applied to prove (4.4) also for lattices containing torsion elements.

5. Analytic torsion, Cheeger–Müller theorem and exponential growth for H_1

We conclude here the proof of Theorem A. For $m, k \geq 1$ there is a representation of $\mathrm{SL}_2(\mathbb{C})$ on the space $V_{2m}^k = \mathrm{Sym}^m(\mathfrak{sl}_2(\mathbb{C}))^k$ and we denote by E_ρ the associated vector bundle on any $\Gamma\backslash\mathbb{H}^3$ whose total space is $\Gamma\backslash(\mathbb{H}^3 \times V_{2m}^k)$.

By Theorem A in [9] any sequence of manifolds M_n as in the statement of the theorem is BS-convergent to $\mathbb{H}^3$. It then follows from Theorem 10.9 in [1] for all ρ as above we have the limit

$$\lim_{n \to +\infty} \frac{\log T(M_n; E_\rho)}{\operatorname{vol} M_n} = k t_{2m}^{(2)} \tag{5.1}$$

where (see [2, 5.9.3, Example (3)]):

$$t_{2m}^{(2)} = \frac{-1}{48\pi}\big((2m+2)^3 - 8m^3 + 24m(m+1)\big)$$

and $T(M_n; E_\rho)$ is the Ray–Singer analytic torsion with coefficients in the bundle E_ρ. For the definition of the latter we refer to [7], where it is also proven that

$$T(M_n; E_\rho) = \frac{|H_0(M_n; L_n)| \cdot |H_2(M_n; L_n)|}{|H_1(M_n; L_n)|} \tag{5.2}$$

where L_n is the lattice in V_{2m}^k preserved by Γ_n (we suppose that all Γ_n are defined over fields of the same degree r and that $k = k_{m,r}$). In the case $m = 1$, putting together (5.2),(5.1),(4.3) and (4.4) we finally obtain:

$$\lim_{n\to+\infty} \frac{\log|H_1(\Gamma_n; L_n)}{\operatorname{vol} M_n} = -k_{2,r} t_2^{(2)}.$$

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