ANALYTIC TORSION, REGULATORS AND ARITHMETIC HYPERBOLIC MANIFOLDS

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Introduction

This survey paper aims at giving an introduction to the interaction between Riemannian and spectral geometry on one side and number theory on the other, from the point of view of analytic torsion and the Cheeger–Müller theorem. The former is a spectral invariant and the latter relates it to a topological invariant, the Reidemeister torsion. *Regulators* are one of the elements in the relation. The connection to number theory arises when applying this theory to the locally symmetric spaces associated to arithmetic lattices in semisimple Lie groups. The simplest examples where this gives interesting results is when the Lie group is $\mathrm{SL}_2(\mathbb{C})$, and the associated manifolds are then hyperbolic 3–manifolds. The points I want to discuss are more precisely the following:

- Bounding the growth of regulators in sequences of congruence covers of an arithmetic manifold is a step in the program of understanding asymptotic growth of homology in these covers laid in [7] ;
- Regulators themselves seem to have interesting arithmetic properties ;
- Analytic torsion and the Cheeger–Müller theorem only make sense for closed Riemannian manifolds. It is desirable and possible to extend some parts of the theory to manifolds associated to nonuniform arithmetic lattices such as the Bianchi groups $\mathrm{SL}_2(R_D)$, where R_D is a ring of imaginary quadratic integers.

I will concentrate on the first and third points, after giving a short survey of analytic and Reidemeister torsion and hyperbolic manifolds including arithmetic constructions. I will briefly discuss the growth of torsion homology in congruence covers (including some new numerical data); a more complete survey on this topic is given in [5]. The contents of the paper are as follows:

- Section 1 is a short introduction to Reidemeister and analytic torsion, and the Cheeger–Müller theorem relating the two. This is rather old material, from [16] and [30] essentially. This section is mainly here to provide context for the rest of the survey.
- Section 2 concerns hyperbolic 3–manifolds. This section contains a very short introduction to their geometry and topology and then review the relations between regulators and geometric invariants, following [6] and [10].
- Arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ are introduced in Section 3 and a few results about their regulators and cohomology are explained there. The contents of this section are mainly taken from [7], [15] and [6].
- The last section 4 is concerned with the extension to the noncompact case of the results mentioned above, and how to adapt some parts of the previous section to this setting. The contents are mainly from [15] and [35].

As noted above the study of arithmetic groups involves both number theory and differential geometry. I tried to separate as much as possible those results that involve only the latter (and so would be valid in the more general context of locally symmetric spaces) from those which make essential use of the number-theoretical origin of the spaces we consider (and might be false in a more general context).

This survey grew out of a talk I gave about [35] and [37] at the conference "Regulators IV", the contents of which intersected mainly with the last section.

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1. Reidemeister torsion and the Cheeger–Müller equality

1.1. **Reidemeister torsions of CW-complexes.** We will only give a short review of the general theory of Reidemeister torsion, which we will not use much in the sequel. For a complete survey see the book by V. Turaev [41], and some parts in what follows are essentially lifted from [7, Section 2].

If X is a finite CW-complex and $\mathbb{K}$ a commutative ring we let $C_*(X;\mathbb{K}), d_*$ be its chain complex. Suppose that $\mathbb{K}$ is a field and $C_*(X;\mathbb{K})$ is acyclic[1]. Choose graded bases c_*, z_*, b_* for the spaces $C_*, \ker(d_*)$ and $\mathrm{Im}(d_*)$ respectively. Then the *Reidemeister torsion* of X is defined as follows:

$$\tau(X;\mathbb{K}) = \prod_i \det{}_{c_i}\left(d_{i+1}(b_{i+1}) \oplus z_i\right)^{(-1)^i} \tag{1.1}$$

where $\oplus$ denotes concatenation of bases. An elementary verification shows that it does not depend on the various choices, and it is also possible to prove that it is a homeomorphism invariant. If $C_*(X;\mathbb{K})$ is not acyclic then one needs to add a choice of bases h_* for the homology and define the torsion as

$$\tau(X;\mathbb{K}) = \prod_i \det{}_{c_i}\left(d_{i+1}(b_{i+1}) \oplus z_i \oplus \tilde{h}_i\right)^{(-1)^i} \tag{1.2}$$

where $\tilde{h}_*$ is a graded lift of the base h_* of $H_*(X;\mathbb{K})$ to $C_*(X;\mathbb{K})$. The torsion thus defined does depend on the choice of h_*. When $\mathbb{K}$ has characteristic 0 the most natural choice is to take bases coming from the integral structure: since the base change matrices have determinant ± 1 the torsion thus obtained depends on the choice only up to sign. A simple computation shows that in this case we have up to sign:

$$\tau(X;\mathbb{Q}) = \prod_i |H_i(X;\mathbb{Z})_{\mathrm{tors}}|^{(-1)^i} \tag{1.3}$$

where A_{tors} denotes the torsion subgroup of an abelian group A. In general we must add for each i the factor corresponding to the determinant of the chosen base in a $\mathbb{Z}$-base.

When $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ another natural choice for the bases h_* is as follows. There is a natural choice of a basis for C_* given by the cells in each degree. This gives each $C_i(X;\mathbb{K})$ the structure of a euclidean or hermitian space, and then we can identify $H_i(X;\mathbb{K})$ with a subspace of $C_i(X;\mathbb{K})$, namely the orthogonal of $\mathrm{Im}(d_{i+1})$ in $\ker(d_i)$. Then we may take for h_i any

[1] Of course if X is a triangulation of a manifold then $C_*(X;\mathbb{K})$ is never acyclic but taking coefficients in a local system may remedy to this issue.

orthonormal basis of this subspace and the resulting torsion does not depend on this choice up to sign. It can be computed via the formula

$$\tau = \prod_{i=0}^{\dim(X)} \det{}'(d_i)^{(-1)^i} \tag{1.4}$$

where $\det{}'(f)^2$ is the product of all nonzero "singular values" of f, which are the eigenvalues of the associated self-adjoint map $f^* f$. The relation with the torsion in 1.3 is easily worked out to be :

$$\prod_{i=0}^{\dim(X)} \det{}'(d_i)^{(-1)^i} = \prod_i \left(\frac{|H_i(X;\mathbb{Z})_{\mathrm{tors}}|}{R_i(X)} \right)^{(-1)^i} \tag{1.5}$$

where the *combinatorial regulator* $R_i(X)$ is the covolume of the lattice

$$H_i(X;\mathbb{Z})_{\mathrm{free}} = H_i(X;\mathbb{Z})/H_i(X;\mathbb{Z})_{\mathrm{tors}}$$

in the euclidean subspace of $H_i(X;\mathbb{K})$ that it spans.

1.2. **Reidemeister torsions of Riemannian manifolds.** We saw above (cf. 1.4) how to relate the homological torsion of a CW-complex to the "geometry" of its cellular complex via determinants and regulators. The definition given for the latter depended on identifying the free part of the homology to a lattice in a Euclidean space. In this section we will take a CW-complex coming from a triangulation of a smooth manifold and relate its homological torsion to a quantity defined using a Riemannian metric on the manifold. For this purpose D. Ray and I. Singer defined the Reidemeister torsion of a Riemannian manifold : see [38, Definitions 1.1 and 3.6] and also [16, Section 1]. We will define it ex-nihilo in terms of regulators.

Let M be a compact Riemannian manifold with no boundary and of dimension $\dim(M) = d$. We fix a smooth triangulation X of M. For $0 \le k \le d$ let $\mathscr{H}^k(M)$ be the space of harmonic forms on M. By the Hodge-de Rham theorem the linear map

$$\Phi^k : \mathscr{H}^k(M) \to H^k(X;\mathbb{R})$$

given by

$$\Phi^k(\omega)(c) = \int_c \omega$$

(here c is a k-cycle of X, so in particular it is a smooth k-submanifold away from a codimension 1 subset and the integral makes sense) is an isomorphism. On the space $\mathscr{H}^k(M)$ there is the L^2-inner product $\langle\cdot,\cdot\rangle_{L^2}$ defined by integrating the pointwise inner product induced by the Riemannian metric. The kth regulator of M is then defined to be the covolume of $H^k(X;\mathbb{Z})_{\mathrm{free}}$ with respect to the inner product $(\Phi^k)_* \langle\cdot,\cdot\rangle_{L^2}$ on $H^k(X;\mathbb{R})$. We will denote it by $R_k(M)$; note that in general it depends on the choice of the Riemannian metric (but not on the triangulation) but this will not matter for us. Using the explicit form of Φ^k we see that if $c_1, \dots, c_{b_k}$ is a $\mathbb{Z}$-basis of $H_k(X;\mathbb{R})_{\mathrm{free}}$ and $\omega_1, \dots, \omega_{b_k}$ is an orthonormal basis of $\mathscr{H}^k(M)$ then we have

$$R_k(M) = \det\left(\int_{c_i} \omega_j \right)_{1 \le i,j \le b_k}. \tag{1.6}$$

The Reidemeister torsion of M is then defined to be:

$$\tau(M) = \prod_{i=0}^{d} \left(\frac{|H^k(X;\mathbb{Z})_{\mathrm{tors}}|}{R_k(M)} \right)^{(-1)^k}. \tag{1.7}$$

Note that through (1.5) this can be seen as the particular case of (1.2) where one takes integral bases for the $C^k(X;\mathbb{R})$ and L^2-orthonormal bases for the $H^k(X;\mathbb{R})$.

1.3. **Analytic torsion and the Cheeger–Müller theorem.** Now instead of determinants of the differentials of the complex $C^*(X;\mathbb{R})$ we will consider "determinants" of the de Rham differentials. As the spaces $\Omega^k(M)$ are not finite dimensional, and the differentials are not continuous these are not defined as usual. Rather, Ray–Singer used the spectral theory of the Hodge–Laplace operators to define the so-called regularised determinants. We will now describe this without paying attention to the underlying functional analysis.

We will denote by $L^2\Omega^k(M)$ the Hilbert space of square-integrable k-forms. The Hodge–Laplace operator Δ_k on $\Omega^k(M)$ is defined by :

$$\Delta_k = d_k^* d_k + d_{k+1} d_{k+1}^*$$

where d_k^* is the formal adjoint of d_k. The operator Δ_k is unbounded but it has a unique essentially self-adjoint extension in $L^2\Omega^k(M)$. The spectral theorem in Riemannian geometry states that its spectrum is a discrete subset of $[0,+\infty[$. All this means that there exists an Hilbert basis $(\omega_i)_{1\le i}$ of $L^2\Omega^k(M)$ and a nondecreasing sequence sequence $(\lambda_i)_{1\le i}$ such that λ_i goes to infinity and $\Delta_k\omega_i = \lambda_i\omega_i$. Moreover Weyl's law states that

$$|\{i : |\lambda_i| \le \lambda\}| \sim_{\lambda\to+\infty} C(d)\binom{d}{k}\mathrm{vol}(M)\lambda^{d/2}. \tag{1.8}$$

This means that the series

$$\zeta_k(s) = \sum_{i\ge b_k+1} \lambda_i^{-s}$$

converges in the half-plane $\mathrm{Re}(s) > d/2$. Its sum actually extends to a meromorphic function on $\mathbb{C}$, which is often called the Minakshisundaram–Pleijel zeta function when $k=0$. It is regular at $s=0$ and we define the regularised determinant of Δ_k by the following formula:

$$\det{}'(\Delta_k) = \exp(\zeta_k'(0)) \tag{1.9}$$

(note that formally, the value on the right equals the (nonconvergent) product $\prod_{i\ge b_k+1}\lambda_i$).

The *analytic torsion* of M is defined, in analogy with (1.4), by:

$$T(M) = \prod_{k=0}^{d} \left(\det{}'(\Delta_k)\right)^{k(-1)^k}. \tag{1.10}$$

The main motivation for this definition, as well as its use for us, resides in the following result which was conjectured by Ray–Singer and proven shortly afterwards by J. Cheeger and W. Müller independently of each other in [16] and [29] respectively.

Theorem 1.1 (Cheeger, Müller). *Let M be a closed Riemannian manifold. Then*

$$\tau(M) = T(M).$$

1.4. **Local systems.** The work of Ray–Singer as well as that of Cheeger and Müller was done in the slightly more general case where one considers cohomology and differential forms with coefficients in a flat orthogonal vector bundle. For later applications we will need an even larger setting, where the generalisation of the Cheeger–Müller theorem 1.1 was worked out (independently) by J.-M. Bismut–W. Zhang [8] and W. Müller [30].

Let M, X be as above and let ρ be a representation of $\pi_1(M)$ on a finite-dimensional real vector space V, such that $\rho(\pi_1(M)) \subset \mathrm{SL}(V)$. Then we can associate to ρ a *flat unimodular vector bundle* F on M, obtained by quotienting $\widetilde{M}\times V$ by the left-action $\gamma\cdot(x,v) = (\gamma\cdot$

$x, \rho(\gamma) \cdot v)$. There is also an associated cochain complex of finite-dimensional real vector spaces $C^*(X;V) = C^*(\widetilde{X};\mathbb{R}) \otimes_{\mathbb{R}[\pi_1(M)]} V$, with cohomology $H^*(X;V)$. To proceed further one needs a metric on F: in general there is no canonical choice so we will assume that an arbitrary choice has been made. With this metric comes an identification of each $H^k(X;V)$ with a space $\mathscr{H}^k(M;F)$ of harmonic forms on M. It also allows to choose a prefered class of bases of each $C^k(X;V)$ and with these choices we can define a Reidemeister torsion $\tau(M;F)$ as in (1.2). On the analytic side the spectral theory is similar to that described above and the analytic torsion $T(M;F)$ is defined in the same manner. The result of Müller and Bismut–Zhang is then stated as follows.

Theorem 1.2 (Bismut–Zhang, Müller). *Let M be a closed Riemannian manifold and F a flat unimodular bundle over M. Then $\tau(M;F) = T(M;F)$.*

To end this section we will describe a particular case in which the Reidemeister torsion can be expressed with a formula similar to (1.7). Suppose that V contains a lattice L such that $\rho(\pi_1(M))(L) = L$. Then the integral cochain complex $C^*(X;L)$ and its homology $H^*(X;L)$ are well-defined. Thus it is possible to take integral bases to compute the Reidemeister torsion $\tau(M;F)$, and to define the regulators $R_k(M;L)$ as the covolume of $H^k(X;L)_{\mathrm{free}}$ with respect to the L^2-inner product. Then we have the formula:

$$\tau(M;F) = \prod_{k=0}^{d} \left(\frac{|H^k(X;L)_{\mathrm{tors}}|}{R_k(M;L)} \right)^{(-1)^k}. \tag{1.11}$$

2. Regulators of hyperbolic 3–manifolds

2.1. **Hyperbolic manifolds.** An *hyperbolic manifold* is a complete Riemannian manifold all of whose sectional curvatures are equal to -1. Riemann's theorem states that this determines a unique local isometry class, so that there is a unique such manifold which is simply connected. It is referred to as the *d-dimensional hyperbolic space* and usually denoted by $\mathbb{H}^d$. The d-dimensional hyperbolic space admits an algebraic description as the symmetric space associated to the Lie group $\mathrm{SO}(d,1)$, in other words it is isometric to $\mathrm{SO}(d,1)/\mathrm{O}(d)$ with the (suitably normalised) left-$\mathrm{SO}(d,1)$-invariant Riemannian metric. In the sequel we will mostly restrict ourselves to $d = 3$. In this dimension there is a local isomorphism $\mathrm{SO}(3,1) \cong \mathrm{SL}_2(\mathbb{C})$ and we will use the latter group since it is easier to deal with algebraically.

2.1.1. *Thick–thin decomposition.* Recall that the injectivity radius $\mathrm{inj}_x(M)$ of a Riemannian manifold M at a point x is the maximal radius of a ball (in the Riemannian distance) around x which is embedded. For negatively curved manifolds, an alternative description is that $\mathrm{inj}_x(M)$ equals half the minimal length of a closed curve passing through x which is nontrivial in $\pi_1(M)$. For a closed manifold the global injectivity radius $\mathrm{inj}(M)$ is defined to be the minimum of $\mathrm{inj}_x(M)$ over all $x \in M$. We have $\mathrm{inj}(M) > 0$.

The most basic tool to describe the global structure of finite-volume hyperbolic manifolds is the Margulis lemma. We will use one of its corollary which is the *thick-thin decomposition* of hyperbolic manifolds which we will describe in what follows. For an $\varepsilon > 0$ the *ε-thin part* of M is the subset

$$M_{\le\varepsilon} = \{x \in M : \mathrm{inj}_x(M) \le \varepsilon\}$$

and the ε-thick part $M_{>\varepsilon}$is its complement in M. There exists a constant μ (depending on d) called the Margulis constant, such that for all hyperbolic manifolds M of finite volume and $\varepsilon \le \mu$ the ε-thin part of M is diffeomorphic to a finite union of *cusps* and *tubes*. Cusps are the noncompact components, and each is diffeomorphic to $N \times [0, +\infty[$ where N is a closed $(d-1)$-dimensional flat manifold, for example $N = \mathbb{T}^{d-1}$, the flat torus (this is the only possibility in dimension $d = 3$). Tubes are compact, and each of them is obtained as a tubular neighbourhood of a closed geodesic, in particular it is diffeomorphic to $\mathbb{S}^{d-1} \times \mathbb{S}^1$. It then follows from the van Kampen theorem that for $d \geq 3$ the fundamental group of M is generated by that of its subset $M_{\geq\varepsilon}$. If $d \geq 4$ the manifolds M and $M_{\geq\varepsilon}$ have the same fundamental group.

2.1.2. *Triangulations of hyperbolic manifolds.* Suppose that M is a closed Riemannian manifold such that $\mathrm{inj}(M) \geq \varepsilon > 0$. Taking an $\varepsilon/6$-net $x_1, \dots, x_m \in M$ we can look at the pattern of intersections between the balls $B(x_i, \varepsilon/4)$ to define a simplicial complex X, namely X has (x_i) as its 0-skelethon and vertices $x_{i_0}, \dots, x_{i_k}$ define a k-simplex of X if and only if the intersection $B(x_{i_0}, \varepsilon/4) \cap \dots \cap B(x_{i_k}, \varepsilon/4)$ is nonempty. For ε small enough (depending only on the local geometry of the manifold) this triangulation will be homotopy equivalent to M (but not necessarily homeomorphic to it: it can happen that it is not a manifold). Moreover it has at most $C(k, \varepsilon)$ simplices in degree k, and every simplex is adjacent to at most $d(\varepsilon)$ other simplices. In particular $b_k(M) \leq C(k, \varepsilon)$.

In the case where M is an hyperbolic manifold and ε is the Margulis constant, and $\mathrm{inj}(M) < \varepsilon$ it is still possible to construct a triangulation from the thick part $M_{\geq\varepsilon}$ which is homotopy equivalent to it (see [11]). Since in dimension 3 the fundamental group of M is a quotient of that of $M_{\geq\varepsilon}$, and in higher dimension the two manifolds are homotopy equivalent, we see that there exists a constant $C(d)$ such that for every d-dimensional hyperbolic manifold we have :

$$b_k(M) \leq C(d) \cdot \mathrm{vol}(M). \tag{2.1}$$

Less obviously (see for example [40]) this also implies that

$$\log |H_k(M_{\geq\varepsilon}; \mathbb{Z})_{\mathrm{tors}}| \leq C(d) \cdot \mathrm{vol}(M). \tag{2.2}$$

If M has tubes in its thin part this implies that $\log |H_k(M_{\geq\varepsilon}; \mathbb{Z})_{\mathrm{tors}}| \leq C(d) \cdot \mathrm{vol}(M)$ for $k \neq d-2$, but this does not remain true for $k = d-2$: for example in dimension 3 Dehn surgeries on a given manifold can have arbitrarily large torsion in their first homology group while keeping the volume bounded. We do not know whether there is such a uniform bound for $H_{d-2}(M; \mathbb{Z})$ for $d \geq 4$.

2.1.3. *Reflection groups.* Classical constructions of low-dimensional hyperbolic manifolds proceed by using a Coxeter polyhedron and Poincaré's theorem. For example there exists a compact tetrahedron T_2 (we take the notation from [27, Figure 13.1]) in $\mathbb{H}^3$ with Coxeter symbol

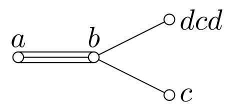
Let Γ_{T_2} be the subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ generated by reflections on the faces of T_2 (we'll denote the reflection in a face by the same letter). Then Γ_{T_2} is a cocompact discrete subgroup in $\mathrm{Isom}(\mathbb{H}^3)$. It has an index-2 subgroup $\Gamma_{T_2}^+$ of orientation-preserving isometries. The latter is generated by $\alpha = ab, \beta = ac$ and $\gamma = ad$ and has the presentation

$$\Gamma_{T_2}^+ = \langle \alpha, \beta, \gamma | \alpha^5, \beta^2, \gamma^3, (\beta\alpha)^3, (\gamma\alpha)^2, (\beta\gamma)^4 \rangle.$$

It is possible to compute an explicit matrix representation of $\Gamma_{T_2}^+$ to $\mathrm{PSL}_2(\mathbb{C})$. More generally formulae for the representations of 3-dimensional hyperbolic tetrahedral groups are given in [22].

Since $\Gamma_{T_2}^+$ is not torsion-free the quotient $\Gamma_{T_2}^+ \backslash \mathbb{H}^3$ is not a manifold. There are two ways to find a finite-index subgroup in $\Gamma_{T_2}^+$ which is torsion-free. We will see the algebraic one later, here we describe a beautiful geometric construction due (in greater generality) to A. Vesnin. The subgroup of Γ_{T_2} fixing the vertex $x_0 = a \cap b \cap c$ is isomorphic to S_5. Thus we get a morphism $\pi : \Gamma_{T_2} \to S_5$ by sending the generator d to the identity. The polyhedron in $\mathbb{H}^3$ corresponding to the subgroup $\ker(\pi)$ is a right-angled dodecahedron D with center x_0 and the group $\Gamma_D = \ker(\pi)$ is generated by the reflections in the sides of D. These sides correspond to the vertices of an icosahedron. Choose a colouring of the icosahedral graph with 4 colours and define a morphism $\rho : \Gamma_D \to (\mathbb{Z}/2\mathbb{Z})^3$ by sending a generator to $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ or $(1,1,1)$ according to its color. Then $\Gamma = \ker(\rho)$ is a torsion-free group and the manifold $\Gamma\backslash\mathbb{H}^3$ is obtained by gluing eight copies of D along their faces according to a certain pattern. Its volume is about $.03588 \times 120 \times 8 \cong 34.4448$.

For later use we note that doubling T_2 along the face d yields another Coxeter tetrahedron T_4, whose Coxeter symbol is



A presentation for its reflection group is

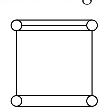
$$\Gamma^+_{T_4} = \left\langle \alpha, \beta, \gamma | \alpha^5, \beta^2, \gamma^2, (\beta\alpha)^3, (\gamma\alpha)^3, (\beta\gamma)^2 \right\rangle . \tag{2.3}$$

Moreover, the following matrices[2] generate a discret cocompact subgroup of $\mathrm{PSL}_2(\mathbb{C})$ isomorphic to $\Gamma^+_{T_4}$:

$$\begin{aligned} \tilde{\alpha} = \frac{1}{2}\begin{pmatrix} a^2 & -e \\ e & a^2 \end{pmatrix}, \ \tilde{\beta} &= \frac{1}{5}\begin{pmatrix} 0 & \left(\left(a^2+2\right)i - 2a^2 + 1\right)e \\ \left(\left(a^2+2\right)i + 2a^2 - 1\right)e & 0 \end{pmatrix}, \\ \tilde{\gamma} &= \frac{1}{5}\begin{pmatrix} 5\left(-a^3+a\right)i & \left(\left(a^2-3\right)i - 2a^2 + 1\right)e \\ \left(\left(a^2-3\right)i + 2a^2 - 1\right)e & 5\left(a^3-a\right)i \end{pmatrix}. \end{aligned} \tag{2.4}$$

Here a is an imaginary root of $X^4 - X^2 - 1$ and $e = \sqrt{4 - a^4}$, so the coefficients belong to an algebraic number field of degree 16.

Finally, we will also refer to the tetrahedron T_8 which occurs in [13]. Its Coxeter symbol is



and a presentation for its reflection group is

$$\Gamma^+_{T_8} = \left\langle \alpha, \beta, \gamma | \alpha^3, \beta^2, \gamma^5, (\gamma\beta)^3, (\alpha\gamma^{-1})^2, (\alpha\beta)^4 \right\rangle .$$

2.2. **Regulators and geometry.** Note that by Poincaré duality, for a closed Riemannian 3–manifold M it follows from formula (1.6) that :

$$R_2(M) = R_1(M)^{-1}.$$

So to prove upper and lower bound for either R_1 and R_2 it suffices to prove upper bounds for both. We will describe how to do so in both cases, following [6]: in degree 1 the result is good enough for applications, but degree 2 is more complicated.

[2]They are different from those in [22], we simply solved the trace equations given by the presentation (2.3) to find them.

2.2.1. *Bounding R_1.* In degree 1 it is rather easy to estimate regulators. The following is a slightly modified version of [6, Proposition 3.1].

Proposition 2.1 (Bergeron–Şengün–Venkatesh). *There exists a constant $C > 0$ such that for every closed hyperbolic 3–manifold M we have*

$$\log(R_1(M)) \leq Cb_1(M)\log(\mathrm{vol}(M)). \tag{2.5}$$

To get geometric estimates for the regulators here and later one needs to be able to estimate the integral of a harmonic form on a submanifold. In full generality this follows from the Sobolev inequalities, but for hyperbolic manifolds Brock and Dunfield prove the following more precise result (see the proof of [10, Theorem 4.1]).

Theorem 2.2 (Brock–Dunfield). *There exists a constant $C_1 > 0$ such that if M is an hyperbolic manifold and α a harmonic form on M then we have :*

$$\forall x \in M : |\alpha_x| \leq \frac{C}{\sqrt{\mathrm{inj}_x(M)}}\|\alpha\|_{L^2}.$$

Now we can explain the proof of Proposition 2.1. Let μ be the Margulis constant for $\mathbb{H}^3$ and $M_{\geq\mu}$ the μ-thick part of M. Then as we saw it follows from the thick-thin decomposition that $\pi_1(M)$ is generated by $\pi_1(M_{\geq\mu})$. On the other hand one can find a family of closed curves $\gamma_1, \dots, \gamma_m$ which are generators for $\pi_1(M_{\leq\mu})$ and each has length at most $C_2 \cdot \mathrm{diam}(M_{\geq\mu})$. If ω is a harmonic 1-form on M then for $i = 1, \dots, m$ we have

$$\left|\int_{\gamma_i} \omega\right| \leq C\|\omega\|_{L^2}. \tag{2.6}$$

Now there is a subfamily, say $\gamma_1, \dots, \gamma_b$, such that the singular homology classes $[\gamma_1], \dots, [\gamma_b]$ generate $H_1(M;\mathbb{Z})_{\mathrm{free}}$. If $\omega_1, \dots, \omega_b$ is an orthonormal basis of $H^1(M;\mathbb{R})$ for the L^2 norm we get

$$\begin{aligned} R_1(M) &= \det\left(\int_{\gamma_i}\omega_j\right)_{1\leq i,j\leq b} \\ &\leq \prod_{i=1}^{b}\sqrt{\sum_{j=1}^{b}\left|\int_{\gamma_i}\omega_j\right|^2} \\ &\leq (\sqrt{b})^b C^b. \end{aligned}$$

The result follows since we have $b = b_1(M) \ll \mathrm{vol}(M)$.

2.2.2. *Cycle complexity and R_2.* Let S be a 2-cycle in M. The Poincaré dual of the class $[S] \in H_2(M;\mathbb{Z})$ is by definition the unique harmonic 1-form $\omega_S \in H^1(M;\mathbb{R})$ which satisfies

$$\forall \omega \in H^2(M;\mathbb{R}) : \int_S \omega = \int_M \omega_S \wedge \omega.$$

It follows immediately (by the Cauchy–Schwarz inequality) that

$$\left|\int_S \omega\right| \leq \|\omega_S\|_{L^2} \cdot \|\omega\|_{L^2}.$$

So, to estimate the regulator R_2 in a manner similar to the case of R_1 we need bounds on the L^2-norm $\|\omega_S\|_{L^2}$. Getting these bounds is much more intricate than for 1-cycles: in fact there is no equivalent to (2.5) in degree 2. Brock–Dunfield [10, Theorem 1.8] construct a sequence M_n of closed hyperbolic 3–manifolds where $R_2(M_n) \geq c^{\mathrm{vol}(M_n)}$ for some explicit

$c > 1$. However, restricting to specific classes of hyperbolic manifolds (namely arithmetic congruence ones) there is a hope that regulators can be estimated by the volume. We will explain below (see 3.2) some results in this direction due to Bergeron–Şengün–Venkatesh.

In this section we will explain the first step in their argument. For this we need the notion of cycle complexity introduced by them. Let X be a triangulation of M. If $c \in C_2(X;\mathbb{Z})$ then we can represent the homology class $[c]$ by a singular homology class $[S]$ where S is a smooth embedded surface (not necessarily connected). Then a natural notion of complexity for $[c]$ would be the Euler characteristic of S. Note that in a closed negatively-curved manifold any embedded sphere or torus represents the null class in singular homology. The result is then a relation between the cycle complexity and the L^2-norm. The following sharpened version of [6, Proposition 4.1] was proven by Brock–Dunfield [10, Theorem 1.3].

Theorem 2.3 (Bergeron–Şengün–Venkatesh, Brock–Dunfield)**.** *There exists a constant C with the following property. If M is a closed hyperbolic 3–manifold and S is a smooth surface embedded in M then*

$$\|\omega_S\|_{L^2} \leq \frac{C}{\sqrt{\mathrm{inj}(M)}}|\chi(S)|.$$

The proof given by Bergeron–Şengün–Venkatesh in uses the relation between cycle complexity and the Gromov–Thurston norm (Brock and Dunfield use a more direct differential-geometric argument). This is a result due to D. Gabai, which states that $[S]$ can be represented by a singular chain $\sum_i t_i\sigma_i$ where σ_i are triangles and $\sum_i |t_i| = -\chi(S)$. One can then replace each σ_i by a totally geodesic triangle without changing the homology class of the cycle. Since hyperbolic triangles have area at most π it follows by applying Theorem 2.2 that

$$\|\omega_S\|_{L^2}^2 = \int_S \omega_S \leq C_1\|\omega_S\|_{L^2}\pi\sum_k |t_k| = C\chi(S)\|\omega_S\|_{L^2}$$

which proves the inequality (note that this uses only the "easy" inequality in Gabai's result, as was pointed out to me by N. Dunfield).

2.2.3. *Remarks.*

(1) There is also a lower bound for cycle complexity in terms of the L^2-norm: Brock–Dunfield prove that $\|\omega_S\| \geq \pi\,\mathrm{vol}(M)^{-1/2}|\chi(S)|$ is S has maximal Euler characteristic in its singular homology class, and Bergeron–Şengün–Venkatesh prove a weaker version of this. The proof is more complicated.
(2) Brock–Dunfield also give examples showing that it is not possible to remove the dependency on the injectivity radius in Theorem 2.3 (nor that on the volume in the other inequality).

2.3. **Small eigenvalues.** We end this section by discussing a recent work of M. Lipnowski and M. Stern [24], which is not immediately related to regulators but has a similar flavour and which we will have the occasion of mentioning again later in this survey. They study the problem of giving a bound for the lowset eigenvalue of the Laplace operator on 1-forms on a manifold depending only on its geometry. They work only in the setting where M is a finite cover of a compact orbifold M_0. The result they obtain is a bound of the form

$$\lambda_1(M)^{-\frac{1}{2}} \leq C(M)\cdot\left(1 + \sup_\gamma \frac{\mathrm{sArea}(\gamma)}{\ell(\gamma)}\right).$$

Here $C(M)$ is a constant which (in the setting above) is bounded by a polynomial in the volume of M. The *stable area* $\mathrm{sArea}(\gamma)$ is the more interesting part of the bound: the supremum is taken over all closed geodesics γ which are null-homologous, and for such the stable area is the infimum of a bounding surface:

$$\mathrm{sArea}(\gamma) = \inf_{m>0} \inf_{\partial S = m[\gamma^{-1}\gamma']} \left(\frac{\mathrm{vol}(S)}{m} \right).$$

The problem of bounding $\mathrm{sArea}(\gamma)$ is to some extent similar to that of bounding the regulator: instead of low-complexity 1-cycles one looks for low-complexity 2-chains bounding a given curve.

3. Regulators and homology of compact arithmetic manifolds

3.1. **Arithmetic manifolds.** In this section we will recall the construction of the closed hyperbolic arithmetic 3–manifolds. We will work with the convention that an arithmetic manifold is $\Gamma\backslash\mathbb{H}^3$ where $\Gamma \subset SL_2(\mathbb{C})$ is a torsion-free congruence lattice[3]. In turn, such a lattice is described by the following data :

(1) A number field k which has exactly one complex place;
(2) A quaternion algebra A over k which ramifies at all real places of k;
(3) There is then a simply connected algebraic k-group G given by the kernel of the reduced norm of A. To finish the description we need a compact-open subgroup K_f in $\mathrm{G}(\mathbb{A}_f)$ (where $\mathbb{A}_f$ are the finite adèles of k).

To this data is associated the group $\mathrm{G}(k) \cap K_f$ which we will denote by Γ_K. It is naturally a subgroup in $\mathrm{G}(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C})$ if we consider k as a subfield of $\mathbb{C}$. It is always discrete in $\mathrm{SL}_2(\mathbb{C})$ and it is cocompact there if and only if A is a division algebra, which is equivalent to it not being isomorphic to the matrix algebra $M_2(k)$. The condition that Γ_K be torsion-free does not translate nicely in terms of the data above, but a sufficient condition for this is that K_v be a normal pro-p subgroup at some finite place dividing a prime $p > 2$ and at which A is not ramified.

The description above is not very useful to do explicit computations. Rather than using the adélic setting we can describe some arithmetic manifolds in the commensurability class defined by A using orders. An *order* $\mathcal{O}$ in A is a finitely generated subring which generates A as a k-vector space. The group $\mathcal{O}^1$ of norm one elements in A then fits in the family described above. To see this note that if v is a finite place and R_v the ring of v-adic integers then $\mathcal{O}_v = \mathcal{O} \otimes R_v$ is a compact-open subset of $A \otimes k_v$, and the closure of $\mathcal{O}^1$ there is a compact-open subgroup in $\mathrm{G}(k_v)$. Let K_f be the product of these over all finite places, then it is clear that $\mathcal{O}^1 = \mathrm{G}(k) \cap K_f$. If K_f' is contained in a group K_f constructed in this way we say that $\Gamma_{K'}$ is *derived from a quaternion algebra*.

There are criteria to see whether a group generated by given matrices in $\mathrm{PSL}_2(\mathbb{C})$ is arithmetic or not (assuming one known it is of finite covolume). The simplest test is to look at its *invariant trace field*, which is the field generated by the traces of its elements in the adjoint representation. This is always a number field, and if the lattice is arithmetic then this field equals the field k used in the definition above. Moreover, it is then derived from a quaternion algebra if and only if its trace field (the field generated by the traces of its preimage in $\mathrm{SL}_2(\mathbb{C})$) equals the invariant trace field. This is a necessary but not sufficient condition for a lattice in $\mathrm{PSL}_2(\mathbb{C})$ to be arithmetic; it is also possible to recover the quaternion algebra A in a similar

[3]Non-congruence arithmetic lattices exist in $\mathrm{SL}_2(\mathbb{C})$ and we give an explicit example below—they are in fact rather more aboundant than congruence ones—but for the purposes of this text we will not consider them.

manner, which gives a complete characterisation. We will work out the example of $\Gamma_{T_4}^+$ below, for a more complete description of the contents of this paragraph we refer to [27, 3.5, 3.6, 8.3]

3.1.1. *Asymptotic properties of arithmetic manifolds.* These arithmetic congruence manifolds have very nice geometric and topological properties, for example :

- Their Cheeger constants (or first eigenvalue of the Laplace operator Δ_0) are uniformly bounded away from zero.
- For any $R > 0$ the volume of their R-thin part is a (uniform) $o(\mathrm{vol})$ (this is known as Benjamini–Schramm convergence, see [19]).
- As a consequence of either of the properties above there are only finitely many of them with a given Heegard genus.

3.1.2. *Hecke operators.* Arithmetic manifolds also have a distinguished family of operators acting on differential forms, the Hecke operators. We will give a short description since they occur in the arguments of [6]. By strong approximation, there is a natural bijection between the manifold $\Gamma_K\backslash\mathbb{H}^3$ and the quotient $\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A})/K$ where $K = K_\infty K_f$, $K_\infty = \mathrm{SU}(2)$. For each finite place v of k the space $\mathrm{G}(k_v)/K_v$ is a finite union of trees (one for each coset of K_v in a maximal subgroup) and as such is has an averaging operator δ_v which commutes with the left-$\mathrm{G}(k_v)$-action. Thus these operators act on $C^0(\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A})/K)$, and they actually extend to bounded operators on $L^2(\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A})/K)$. The action of δ_v on $L^2(M)$ is denoted by T_v. Similar constructions can be made for differential forms.

It is immediate that the Hecke operators T_v commute with each other and with the Laplacians. From this it follows that each eigenspace $\ker(\Delta_k - \lambda_i)$ has a decomposition into summands which are eigenspaces for all T_v simultaneously. This applies in particular to the spaces $\mathscr{H}^k(M) = \ker(\Delta_k)$. In general it is not possible to exactly compute eigenspaces of Δ_k (for example there is no exactly known Maass cusp form for $\mathrm{SL}_2(\mathbb{Z})$) but the cohomology can be computed from a triangulation X of M, which can be obtained for example from a fundamental domain, see [32]. With further refinements it is also possible to compute the action of the Hecke operators on $H^*(X;\mathbb{Z})$, see [21]. (We note that the explicit computations use a combinatorial definition for Hecke operators on cohomology, which is obviously equivalent to the analytic definition given just above but does not generalise to other automorphic forms).

3.1.3. *Examples.* The simplest example for k as above is to take an imaginary quadratic field, for example $\mathbb{Q}(\sqrt{-1})$. Then any quaternion algebra A satisfies the conditions. In the next section we will discuss the algebraically simplest example, when $A = M_2(k)$ is split. However, for the compact case many example do not come from quadratic fields.

For example, the cocompact lattice $\Gamma_{T_2}^+$ from 2.1.3 turns out to be arithmetic. In fact its index 2 subgroup $\Gamma_{T_4}^+$ is derived from a maximal order in the quaternion algebra A over the quartic field

$$k = \mathbb{Q}(a),\ a^4 - a^2 - 1 = 0$$

ramified only at real places. This is recorded in [27, 13.1], we will shortly explain how to prove it. (Before getting to the computation let us remark that the commensurable subgroup $\Gamma_{T_2}^+$ is of necessity arithmetic, but because of the maximality of $\Gamma_{T_4}^+$ it cannot be derived from a quaternion algebra. The Vesnin construction gives an example of a non-congruence subgroup.)

We start by computing the traces of all possible products of tuples of pairwise distinct generators of $\Gamma_{T_4}^+$: according to the presentation (2.4) all generators save α and all products of pair of generators have trace 0 or ± 1, and the remaining traces are:

$$\mathrm{tr}(\tilde{\alpha}) = a^2,\ \mathrm{tr}(\tilde{\alpha}\tilde{\beta}\tilde{\gamma}) = -a.$$

It then follows by [27, Lemma 3.5.2] that the invariant trace field is the field k generated by $\mathrm{tr}(\alpha\beta\gamma)$ and also that all traces of elements of $\Gamma_{T_4}^+$ are integral. We see immediately that k has exactly one complex place and two real ones. Now let A be the quaternion algebra generated by $\Gamma_{T_4}^+$. Since $\langle a, b, c\rangle$ generate a group isomorphic to S_5 it is not possible for A to split at a real place (otherwise we would get an embedding of S_5 into $\mathrm{GL}_2(\mathbb{R})$), so it must ramify at all real places. We can finally put all this together and apply Theorem 8.3.2, loc. cit to conclude that Γ_{T4}^+ is arithmetic, and in fact contained with finite index in the unit group of an order in a quaternion algebra. It remains to see that $\Gamma_{T_4}^+$ is the unit group of a maximal order, which can be done by computing the volume $\mathrm{vol}(T_4)$ and comparing it to the covolume of a maximal order given by the volume formula Theorem 11.1.3, loc. cit. We note that we did not check the finite ramification of A, but using a scheme similar to that in section 4.7 of loc. cit. it is possible to get a Hilbert symbol for A and check by hand that it splits at all of them.

There also exists nonarithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ (in fact "most" of them are nonarithmetic). An example is the relection group $\Gamma_{T_8}^+$ introduced in 2.1.3: indeed, its (invariant) trace field is of degree 8 with two complex places (see [27, 13.1]) and thus cannot be the trace field of an arithmetic manifold.

Finally we describe some congruence subgroups of $\Gamma_{T_4}^+$. Let p be a rational prime and v a place of k dividing p. By weak approximation we have a dense embedding of $\Gamma_{T_4}^+$ in $\mathrm{PSL}_2(R_v)$. Let f_v be the residue field of R_v, then we get a surjective map $\Gamma_{T_4}^+ \to \mathrm{PSL}_2(f_v)$. The kernel $\Gamma(\mathfrak{p}_v)$ of this map is a congruence subgroup ("principal congruence subgroup of level $\mathfrak{p}_v$"), as is the preimage $\Gamma_0(\mathfrak{p}_v)$ of the subgroup of upper triangular matrices ("Hecke congruence subgroup of level $\mathfrak{p}_v$"). The former is torsion-free if $p > 2$ but the latter is not.

Let us be more explicit for some primes. Assume that p is a rational prime such that the polynomial $X^4 - X^2 - 1$ has a root a in $\mathbb{F}_p$. Suppose in addition that $4 - a^4$ and -1 are quadratic residues modulo p (the latter actually implies that $X^4 - X^2 - 1$ splits in $\mathbb{F}_p$). Then the matrices in (2.4) make sense in $\mathbb{F}_p$ and "reduction modulo $\mathfrak{p}$" (for $\mathfrak{p}$ one of the prime ideals of k above p) defines a map $\Gamma_{T_4}^+ \to \mathrm{PSL}_2(\mathbb{F}_p)$. Then the subgroup $\Gamma_0(\mathfrak{p})$ is equal to those matrices whose reduction modulo $\mathfrak{p}$ is upper triangular.

Note that in general, for there to be a surjective map $\Gamma_{T_4}^+ \to \mathrm{PSL}_2(\mathbb{F}_p)$ is equivalent to $X^4 - X^2 - 1$ having a root in $\mathbb{F}_p$ (but not necessarily being split). To define the "reduction modulo p" map is in general not obvious using the matrices given in (2.4). However it is easy to solve the equations corresponding to the relations in (2.3) to define a morphism $\Gamma_{T_4}^+ \to \mathrm{PSL}_2(\mathbb{F}_p)$. It is then automatically conjugated to a "reduction modulo $\mathfrak{p}$" morphism.[4]

3.2. **Regulators of closed arithmetic manifolds : estimation by the volume.**

The following theorem is a slight modification of [6, Theorem 6.1] (in the statement we conflate between a closed submanifold and the cohomology class it represents).

Theorem 3.1 (Bergeron–Şengün–Venkatesh). *Let M be a congruence arithmetic manifold defined over an imaginary quadratic field k. Let $H_2^{\mathrm{bc}}(M;\mathbb{Q})$ be the subspace of $H_2(M;\mathbb{Q})$*

[4]This is because $A_5 = \langle \alpha, \beta\rangle$ admits only two representations in $\mathrm{PSL}_2(\mathbb{F}_p)$, which are conjugated to each other, and there are then exactly two nontrivial choices for the remaining generator γ which are equivalent.

spanned by closed imbedded totally geodesic surfaces in M. Then there exist such surfaces $S_1, \dots, S_b$ which span $H_2^{\mathrm{bc}}(M;\mathbb{Q})$ and such that

$$\forall 1 \le i \le b : |\chi(S_i)| \le \mathrm{vol}(M)^C$$

where the constant C depends only on the field k.

The proof of this is rather involved and we will not describe it but we will introduce its main ingredients.

3.2.1. *Totally geodesic surfaces in arithmetic manifolds.* Let k, A and K describe an arithmetic manifold or orbifold as explained above. Then the orbifold $M_K = \Gamma_K \backslash \mathbb{H}^3$ contains totally geodesic 2-dimensional suborbifolds if and only if :

- k is a quadratic extension of a totally real field l ;
- $A = B \otimes_l k$ where B is a quaternion algebra over l which splits at exactly one real place.

Note that there are infinitely many choices for the quaternion algebra B (indeed, we can always add ramification at a finite place which is inert in k/l). However there is a "minimal" choice for B, which ramifies exactly at the places of l which lie below pairs of conjugated places of k. In the sequel we will always assume that B is chosen thus.

As in [6] we will now only consider the case where $l = \mathbb{Q}$, so that B is an anisotropic quaternion algebra over $\mathbb{Q}$ which splits over the reals. Let H be the unit group of B. It is a $\mathbb{Q}$-subgroup of (the Weil restriction to $\mathbb{Q}$ of) G. With respect to $\mathbb{R}$-points this gives an embedding $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{C})$, which in turn induces a totally geodesic embedding $\mathbb{H}^2 \subset \mathbb{H}^3$. In addition $\Lambda_K = \mathrm{H}(k) \cap K$ is a cocompact lattice in $\mathrm{H}(\mathbb{R})$ and the image $S_{g,K}$ of the imbedding $\Lambda_K \backslash \mathbb{H}^2 \to \Gamma_K \backslash \mathbb{H}^3$ is a totally geodesic surface in M.

Note that since we fixed B not all totally geodesic surfaces are obtained by the construction above, as there are infinitely many commensurability classes of totally geodesic surfaces in M [27, Theorem 9.5.6]. But at the cohomological level only we will see below that everything is obtained from the single algebra B (that is, every class in $H_2^{\mathrm{bc}}(M_K;\mathbb{Q})$ has a representative of the form $S_{g,K}$ for some g.

3.2.2. *Base change.* We will now explain brielfy the analytic construction of the cohomology classes dual to totally geodesic surfaces. It uses the *base-change* construction of R. Langlands. The latter associates (in our special case) an automorphic representation of SL_2/k to an automorphic representation of SL_2/l. To use it in the case of a cocompact lattice one needs to use in addition the *Jacquet–Langlands correspondence* which is a bijection between automorphic representations of an anisotropic group G as above and SL_2/l. We may sum up these constructions via the diagram:

$$\begin{array}{ccc} \mathscr{A}(\mathrm{SL}_2(\mathbb{A}_l)) & \xrightarrow{\text{base change}} & \mathscr{A}(\mathrm{SL}_2(\mathbb{A}_k)) \\ {\scriptstyle\text{Jacquet–Langlands}}\uparrow & & \uparrow{\scriptstyle\text{Jacquet–Langlands}} \\ \mathscr{A}(\mathrm{H}(\mathbb{A}_l)) & & \mathscr{A}(\mathrm{G}(\mathbb{A}_k)) \end{array}$$

where $\mathscr{A}$ denotes the space of automorphic forms on a group. The Jacquet–Langlands maps are not bijections but the one on the right is when restricted to the image by base-change of $\mathscr{A}(\mathrm{H}(\mathbb{A}_l))$ and we get a map $\mathscr{A}(\mathrm{H}(\mathbb{A}_l)) \to \mathscr{A}(\mathrm{G}(\mathbb{A}_k))$ which completes the diagram above. We will call it the base change map associated to $B \subset A$. It respects Laplace eigenvalues and it is Hecke-equivariant, and it follows in particular that they define a Hecke-invariant subspace $H^2_{\mathrm{bc}}(M;\mathbb{C})$. We note that this subspace is defined over $\mathbb{Q}$. The following lemma is a less precise version of [6, Proposition 6.9]

Lemma 3.2. *The spaces $H^2_{\mathrm{bc}}(M;\mathbb{C})$ and $H_2^{\mathrm{bc}}(M;\mathbb{C})$ are dual to each other.*

This proves the claim above: the image of the base change map between $\mathscr{A}(\mathrm{H}(\mathbb{A}_l))$ and $\mathscr{A}(\mathrm{G}(\mathbb{A}_k))$ depends on the choice of B but the image of the map for our "minimal" choice above contains all the others, hence by the lemma any totally geodesic surface is cobordant to a class $S_{g,K}$.

Now the volume of such a surface has an upper bound in terms of the reduced discriminant of B and the volume of K and the denominator d_g of g by the volume formula. Since $\mathrm{disc}(B)^2 = \mathrm{disc}(A)$ we see that $\log\mathrm{disc}(B) \ll \log\mathrm{vol}(M)$ and we get a rough estimate:

$$\chi(S_{g,K}) = 2\pi\,\mathrm{vol}(S_{g,K}) \le (d_g\,\mathrm{vol}(M))^C \tag{3.1}$$

for some absolute C.

3.2.3. The Hecke operators T_v act on $H^2(M;\mathbb{Q})$ and by duality on $H_2(M;\mathbb{Q})$ as well. As we said above this action preserves $H^2_{\mathrm{bc}}(M;\mathbb{Q})$ and by the lemma its dual action preserves the subspace spanned by the $S_{g,K}$. In fact it follows from the combinatorial definition of the Hecke operators acting on cohomology that $T_v S_{g,K}$ is a linear combination of $S_{g_i,K}$ where $d_{g_i} \le C d_g q_v$. What Bergeron–Şengün–Venkatesh prove, which implies the statement of Theorem 3.1 in view of (3.1), is the following:

There exists $v_1, \dots, v_l$ and $g_1, \dots, g_l$ such that $d_{g_i}, q_{v_i} \le \mathrm{vol}(M)^C$ and the classes $T_{v_i} S_{g_i,K}$ span $H_2^{\mathrm{bc}}(M;\mathbb{Q})$.

We will not discuss the proof of this result and instead refer the reader to [6, Section 6.10] where a detailed outline is provided.

3.3. **Growth of torsion homology.** We saw in (2.2) that for M a closed hyperbolic manifold the torsion subgroup of $H_1(M;\mathbb{Z})$ has its size bounded above by $C^{\mathrm{vol}(M)}$ for some C depending only on the injectivity radius of M. It was recently proven by M. Frączyk [19] that the bound $|H_1(M;\mathbb{Z})_{\mathrm{tors}}| \le C^{\mathrm{vol}(M)}$ holds with a uniform C for the class of all hyperbolic manifolds (conditionally on a positive answer to Lehmer's question the injectivity radius of all arithmetic manifolds is bounded away from zero but Frączyk's proof holds unconditionally). A much simpler observation is that if we restrict attention to the finite covers of a given manifold then the injectivity radius is bounded below and so the bound applies with an absolute C. A natural question in either context is whether it is sharp. A recent preprint of Y. Liu [25, Theorem 1.2] together with known results (for example [7, Theorem 7.3]) imply that the exponential aspect is. However the covers produced are not congruence and the constant C is not specified independently of M. Closer to our center of interest Bergeron and Venkatesh made the following more precise conjecture.

Conjecture 3.3. *Let Γ be an arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$ and Γ_n a sequence of pairwise distinct congruence subgroups of Γ. Then*

$$\lim_{n\to+\infty}\left(|H_1(\Gamma_n;\mathbb{Z})_{\mathrm{tors}}|^{\frac{1}{\mathrm{vol}(\Gamma_n\backslash\mathbb{H}^3)}}\right) = e^{\frac{1}{6\pi}}.$$

As a corollary of Theorem 3.1, the Cheeger–Müller theorem 1.1, and the "limit multiplicity for analytic torsion" from [7] we get the following theorem (see [6, Theorem 1.2], note that the assumption the the cohomology is purely base change implies the hypothesis on the Betti numbers there).

Theorem 3.4 (Bergeron–Şengün–Venkatesh)**.** *Let M_n be a sequence of closed congruence arithmetic hyperbolic manifolds. Assume that the M_n have "few small eigenvalues" (this is an asymptotic condition precised in (i) of* [6, Theorem 1.2]*) and that the characteristic zero cohomology of each M_n consists only of totally geodesic classes. Then Conjecture 3.3 holds for M_n.*

Unfortunately there is no known example of a sequence satifying the "few small eigenvalues conditions" (the results of Lipnowski–Stern [24] mentioned in 2.3 are still far from providing even a sufficiently good lower bound on the first nonzero eigenvalue). F. Calegari and N. Dunfield [14] construct examples of sequences M_n with $b_1 = 0$ (so the condition on cohomology is vacuously true) but there is no example where there are nontrivial classes but they all are totally geodesic.

Regarding the upper limit in Conjecture 3.3, a much more general result due to T. Lê holds.

Theorem 3.5 (Lê)**.** *Let M be a closed hyperbolic 3–manifold and M_n a sequence of finite covers which converges in the Benjamini–Schramm sense to $\mathbb{H}^3$. Then*

$$\limsup_{n\to+\infty} \frac{\log |H_1(M_n; \mathbb{Z})_{\mathrm{tors}}|}{\mathrm{vol}(M_n)} \leq \frac{1}{6\pi}.$$

We note that the proof of this last theorem is mostly topological; see [26] for a survey of its setting.

3.3.1. *Numerical computations.* The first computational evidence for Conjecture 3.3 was given by Şengün in [12]. He computed the abelianisation of $\Gamma_0(\mathfrak{p})$ for Γ a Bianchi group and a set of prime ideals $\mathfrak{p}$ with norm close to 20000 (note that $\Gamma_0(\mathfrak{p})$ is not torsion-free) and found the size to be in accordance with Bergeron and Venkatesh's prediction. For cocompact lattices N. Dunfield made computations for $\Gamma_0(\mathfrak{n})$ where $\mathfrak{n}$ is prime or a prime power and Γ is an arithmetic orbifold coming from a simple topological construction. The results of these computations are recorded in [9] and they also support the conjecture. An interesting phenomenon in this numerical data is that prime levels exhibit a much faster convergence towards the $1/6\pi$ growth rate.

There are also computations for congruence covers of nonarithmetic manifolds, by Şengün [13] (for example for the tetrahedral group Γ_{T_8}) and Dunfield [9]. In both cases the data seems to indicate that when $b_1(M_n) > 0$ the size $|H_1(M_n; \mathbb{Z})_{\mathrm{tors}}|^{1/\mathrm{vol}(M_n)}$ is much smaller than $e^{1/6\pi}$.

We also performed some new computations for subgroups $\Gamma_0(\mathfrak{p})$ in $\Gamma_{T_4}^+$. Note that the tetrahedral group $\Gamma_{T_4}^+$ is not commensurable to the examples in Şengün's computations, but it is commensurable to the fundamental group of the orbifold $T(2,5)$ appearing in Dunfield's computations in [14] and [9].

We performed the claculations using GAP [20], the raw data is available at [36]. The range of computation was $1000 < |\mathfrak{p}| < 26000$. We only recorded data from subgroups $\Gamma_0(\mathfrak{p})$ for ideals $\mathfrak{p}$ of degree 1, since in any case very few ideals of degree 2 and none of degree 4 have norm in this range. Figure 1 is a graphic representation of the ratios of $\log |H_1(M_p; \mathbb{Z})_{\mathrm{tors}}|$ by $\mathrm{vol}(M_p)/(6\pi)$ where p is a rational prime which totally splits in the trace field k and M_p is one of the congruence covers $\Gamma_0(\mathfrak{p})\backslash\mathbb{H}^3$ for $\mathfrak{p}$ a prime factor of p. Blue dots correspond to subgroups with $b_1 = 0$, and red ones to the others. We removed the "small primes" part of the homology, i.e. all p-parts where $p = 2, 3, 5$ since those can come from the torsion of elements of $\Gamma_0(\mathfrak{p})$ itself (note that this contribution is however known to vanish in the limit, by Benjamini–Schramm convergence).

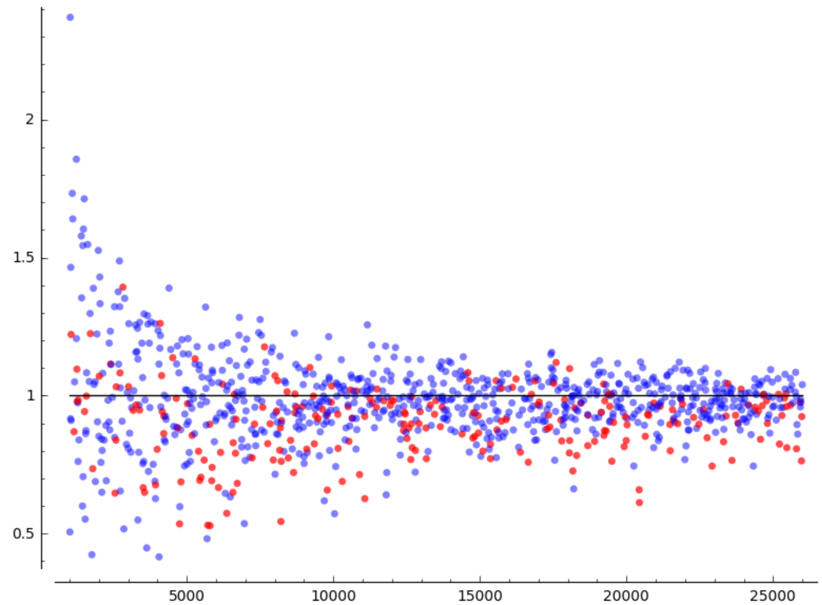


FIGURE 1. Torsion ratio for $\Gamma_0(\mathfrak{p})$, $1000 < |\mathfrak{p}| < 26000$

3.4. **Homology with coefficients.**

3.4.1. *Strongly acyclic local systems.* Conjecture 3.3 can be formulated in greater generality: instead of looking only at the homology group $H_1(\Gamma_n;\mathbb{Z})$ one can study the groups $H_1(\Gamma_n;L)$ where L is an *arithmetic* Γ*-module*, meaning it is of the type discussed before (1.11): there exists a finite-dimensional representation $\rho : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}(V)$ such that L is a lattice in V preserved by $\rho(\Gamma)$. The lattice Γ being arithmetic means that such modules do exist (their description depends on the algebraic group G—we will give specific ones below). Perhaps counter-intuitively, a huge simplification can occur in this more general setting: as proven in [7], certain ρ are *strongly acyclic*, meaning that the Hodge–Laplace operators on the spaces of square-integrable *forms* with coefficients in the flat bundle F associated to ρ have a uniform—not depending on the base manifold—spectral gap.[5] In particular there are no regulator terms in (1.11) and thus Theorem 1.2 establishes a direct relation between the order of H_1 and the analytic torsion. Using the uniformity of the spectral gap it is relatively easy to establish the asymptotic behaviour of the analytic torsion (see [7, Section 4]), leading to the following result.

Theorem 3.6 (Bergeron–Venkatesh)**.** *Let* Γ *be a uniform congruence arithmetic lattice in* $\mathrm{SL}_2(\mathbb{C})$*. Let* ρ *be a real linear representation of* $G = \mathrm{PSL}_2(\mathbb{C})$ *on a vector space* V *which has no irreducible component fixed by the Cartan involution of* G*. Suppose that* L *is a lattice in* V *which is preserved by* $\rho(\Gamma)$*. Then there is a constant* $c_\rho > 0$ *(effectively computable from the decomposition of* ρ *into irreducible factors) such that for any sequence* M_n *of torsion-free*

[5]Note that this is never the case for trivial coefficients: in addition to the 0 eigenvalue on functions, even when there is a uniform spectral gap on functions the spectral gap on 1-forms always tends to 0 in a sequence of manifolds with volume going to infinity.

congruence subgroups of M we have

$$\lim_{n\to+\infty} \frac{\log |H_1(\Gamma_n; L)|}{[\Gamma : \Gamma_n]} = c_\rho \operatorname{vol}(\Gamma\backslash\mathbb{H}^3).$$

An example of such a representation is obtained by the following construction (see [7, Section 8.2]). Let A be a quaternion algebra (satisfying the conditions in 3.1) and $\mathcal{O}$ and order in A. Let Γ be the group of norm 1 elements in $\mathcal{O}$ and L the sub-$\mathbb{Z}$-module of elements of trace 0 in $\mathcal{O}$. Then L is a Γ-stable lattice in $A \otimes_{\mathbb{Q}} \mathbb{R}$ on which G acts by conjugation. As a representation of $\mathrm{SL}_2(\mathbb{C})$ it is isomorphic to $3^{[k:\mathbb{Q}]-2}$ copies of the adjoint representation and hence satisfies the conditions of the theorem. The constant c_ρ in this case has been computed in [7] to be equal to $13/6\pi$.

3.4.2. *Remarks.*

1. It is possible to prove a similar statement for a sequence of pairwise noncommensurable arithmetic lattices, using the Benjamini–Schramm convergence of arithmetic lattices proven in [19] and elementary arguments. Note that in this case the lattices L are not fixed, and if the degree of the field of definition goes to infinity then their rank has to go to infinity.
2. Bergeron–Venkatesh prove a result which is valid for all locally symmetric spaces. In its general form it does not state that torsion homology in a certain degree witnesses exponential growth but only that at least one does (its parity is the only thing that is determined). They conjecture a precise statement for the general case. Some numerical data for case beyond $\mathrm{SL}_2(\mathbb{C})$ is collected in [3].

3.4.3. *Changing coefficients.* Another type of result on torsion growth, which has the same scheme of proof as outlined above, consists in fixing the lattice Γ and changing the coefficients modules. We will only quote the following result from [28].

Theorem 3.7 (Marshall–Müller)**.** *Let Γ be a uniform torsion-free congruence arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$, defined over an imaginary quadratic field. Let* hol *its holonomy representation, $L_2 \subset \mathfrak{sl}_2(\mathbb{C})$ a lattice preserved by $\mathrm{Ad}(\mathrm{hol}\,\Gamma)$ and $L_{2m} = \mathrm{Sym}^m L_2$. Then*

$$\lim_{m\to+\infty} \frac{\log H_1(\Gamma; L_{2m})}{m^2} = \frac{2\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi}.$$

3.5. **Rationality properties of regulators of closed arithmetic manifolds.** We will say a few short words about some results concerning the fine arithmetic properties of regulators of arithmetic manifolds proven in [15] and [4].

3.5.1. *Regulators and L-values.* In [15, Theorem 5.2.3] Calegari and Venkatesh relate $R_1(M_K)$ rationally to a product of (essentially) L-values associated to the cohomological representations occuring in the space of K-invariant automorphic forms. The precise statement of their result needs too much notation for us to quote here. While interesting from a number-theoretical point of view this result is useless if we want to estimate the size of R_2 : to do this we would need to "pin down $R_2(M)$ integrally rather than rationally" to paraphrase [15]. We will see below (cf. 4.3) how to do this in a similar but simpler context; in general we are not aware of any results on this for closed manifolds. There are some hints of this (from a number-theoretical point of view) in [15, 5.2.5].

3.5.2. *Spectrally related manifolds.* Suppose that Γ is a cocompact lattice in $\mathrm{PSL}_2(\mathbb{C})$ and there is a surjection $\pi : \Gamma \to Q$ where Q is a finite group which contain two subgroups H_1, H_2 which are not conjugated by an element of $\mathrm{Aut}(G)$ but such that $\mathbb{C}[G/H_i]$ are isomorphic as G-spaces (H_1 and H_2 are then said to be *almost conjugated*). There are numerous examples of such Γ, G. Assuming further that $\Gamma_i := \pi^{-1}(H_i)$ are torsion-free it was discovered by T. Sunada that the manifolds $M_i = \Gamma_i \backslash \mathbb{H}^3$ are isospectral to each other (together they are called a *Sunada pair*). By the Cheeger–Müller theorem 1.1 and (1.7) it follows that

$$\frac{R_1(M_1) \cdot R_2(M_2)}{R_2(M_1) \cdot R_1(M_2)} \in \mathbb{Q}^\times. \tag{3.2}$$

A. Bartel and A. Page give in [4, Theorem 1.7] a more precise description of the quotient. It relies only on the representation theory of the finite group G, and so in particular it is independent of the Cheeger–Müller theorem.

The other well-known construction of isospectral hyperbolic 3–manifolds is by M. F. Vignéras, who uses nonconjugated maximal orders in a quaternion algebra. A different but somewhat similar construction is that of *Jacquet–Langlands pairs*. These are not isospectral but the spectra are still related. The construction proceeds as follows: take two quaternion algebras A_1, A_2 and let Γ_i be the lattice associated to a maximal order in A_i. Let S_1, S_2 be the sets of finite places where A_1, A_2 respectively ramify, and assume $S_1 \neq S_2$. Let $S \supset S_1 \cup 2_2$ and let $\Gamma_{0,i}(N_i)$ be the Hecke subgroups of level N_i in each $M_i = \Gamma_i$, where $N_i = \prod_{v \in S \setminus S_i} \mathfrak{p}_v$. Jacquet and Langlands proved that there exists a relation between the spaces of automorphic forms on $\Gamma_i \backslash \mathrm{PSL}_2(\mathbb{C})$ and in [15] this is exploited to relate regulators on M_1 and M_2. The relation is more complicated than for Sunada pairs and they only prove partial results. The statements are similar in form to (3.2) but unfortunately they are too involved to be explained here.

4. Regulators of finite-volume arithmetic manifolds

4.1. **Non-compact hyperbolic manifolds.**

In this preliminary subsection we will detail a bit more the structure of noncompact hyperbolic manifolds of finite volume, and the arithmetic examples of such.

4.1.1. *Cusps and height functions.* For a noncompact manifold of finite volume the injectivity radius is always 0 (as the manifold must scrunch at infinity to have finite volume). Another metric invariant in this case is the *systole* $\mathrm{sys}(M)$, which by definition is the smallest length of a closed geodesic on M (in the case where M is closed it equals twice the injectivity radius). In this section we will work under the ssumption that $\mathrm{sys}(M) > \varepsilon > 0$ for a fixed $\varepsilon > 0$. Thus, according to the thick-thin decomposition, the ε-thin part $M_{\leq \varepsilon}$ consists only of cusps. The metric description of a cusp is as follows: let T be a 2-dimensional torus with holomorphic coordinate z and flat Riemannian metric $|dz|^2$. Then the warped product

$$C_T = T \times [1, +\infty[, \ \frac{|dz|^2 + dy^2}{y^2}$$

is locally hyperbolic and of finite volume. For ε smaller than the Margulis constant of $\mathbb{H}^3$ the thin part of M is then made of a finite disjoint union of cusps $C_{T_1}, \ldots, C_{T_h}$. We see that this decomposition specifies a *height function* on M, which in a cusp C_{T_i} is given by the y-coordinate and which we take to equal 1 on the ε-thick part. We note that this function is well-defined independently of the choice of ε (or more generally of a parametrisation of each cusp) only up to multiplication by a constant in each cusp but this will not affect the objects we will define later in a way that we need to worry about.

In the sequel we will also use the Borel–Serre compactification $\overline{M}$ of a manifold M. This is the disjoint union $M \cup \partial\overline{M}$ where

$$\partial\overline{M} = T_1 \sqcup \cdots \sqcup T_h$$

and the topology in $C_{T_i} \cup T_i$ is defined by $(y_n, x_n) \to x$ if $y_n \to +\infty$ and $x_n \to x$. It is a smooth manifold with boundary, diffeomorphic to the level sets of a height function, and the inclusion $M \subset \overline{M}$ is a homotopy equivalence.

4.1.2. *Reflection group examples.* It is much easier to come with examples of noncompact manifolds than compact ones. For example if T_r is a regular ideal tetrahedron in $\mathbb{H}^3$ then its dihedral angles are equal to $\pi/3$ and hence the reflection group Γ_{T_r} is a nonuniform lattice in $\mathrm{PSL}_2(\mathbb{C})$. It is also possible to glue the faces of T_r to obtain a nonorientable manifold (the Gieseking manifold), which is the noncompact hyperbolic manifold of smallest volume [1]. Its orientation cover is the figure eight-knot complement. These examples are commensurable to each other and arithmetic (more on this below). Another arithmetic example (which is not commensurable to the previous ones) is the reflection group of the regular ideal octahedron.

There are also nonarithmetic reflection examples. A list is given in [27], the simplest example there is U_1 whose Coxeter symbol is:

4.1.3. *Arithmetic examples.* As for the geometric constructions, the arithmetic construction of lattices in $\mathrm{SL}_2(\mathbb{C})$ is much simpler. By general theory an arithmetic lattice is noncompact if and only if its algebraic group is not anisotropic. In the construction given in 3.1 above this amounts to the quaternion algebra A is not a division algebra, which in turn means that A is isomorphic to the matrix algebra $M_2(k)$ for some imaginary quadratic field k. If R is the ring of integers in A then $M_2(R)$ is an ordre in A and the associated group of units is $\mathrm{SL}_2(R)$. These groups are called Bianchi groups, and as a consequence of this paragraph every nonuniform arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$ is commensurable (up to conjugation) to one of these. As we mentioned above the figure-eight complement is arithmetic; the image of its holonomy map is in fact contained in $\mathrm{PSL}_2(\mathbb{Z}[e^{2i\pi/3}])$[6].

We also note that it is much easier to describe congruence subgroups for the Bianchi groups than for uniform lattices. For example if $\Gamma = \mathrm{SL}_2(R)$ and $\mathfrak{n}$ is an ideal in R the Hecke congruence subgroup of level $\mathfrak{n}$ is simply

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R) : \; c \in \mathfrak{n} \right\}.$$

4.2. **Definitions of the regulators.** In this section we will use $H^*(M;\mathbb{K})$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) to denote de Rham cohomology, which is when needed identified with the cohomology of a triangulation of M. We will denote the class of a differential form ω in de Rham cohomology by $[\omega]$. When A is a subgring of $\mathbb{C}$ we write $H^*(M;A)$ to denote the A-submodule of those de Rham classes whose integral against integral cycles (i.e. submanifolds) lie in A.

If Γ is a discrete, torsion-free subgroup of $\mathrm{PSL}_2(\mathbb{C})$ then there is a well-defined maximal essentially self adjoint extension of the Hodge–Laplace operators from smooth compactly supported forms to square-integrable ones. On the other hand if M is the hyperbolic manifold $\Gamma\backslash\mathbb{H}^3$ it is not true that $H^k(M;\mathbb{R}) \cong \mathscr{H}^k(M)$ when M is noncompact. For related reasons it is not possible to define the analytic torsion using the same definition as in the compact case. In the case where M is of finite volume it is possible to extend both Hodge theory and the Cheeger–Müller theorem using the theory of Eisenstein series, which gives an explicit

[6] This is actually how its hyperbolic structure was first discovered by R. Riley, see [39].

description of the orthogonal complement to the subspace where the Laplace has a discrete spectrum.

4.2.1. *Eisenstein cohomology.* Let $M = \Gamma\backslash\mathbb{H}^3$ be a non-compact hyperbolic manifold of finite volume. We will first describe the cohomology group $H^1(M;\mathbb{R})$ by analytic means. The first important fact is that the "cuspidal" and "L^2" cohomologies coincide in our setting.

Lemma 4.1. *Let $H^1_{\mathrm{cusp}}(M;\mathbb{R})$ be the kernel of the restriction map $i_1^* : H^1(M;\mathbb{R}) \to H^1(\partial\overline{M},\mathbb{R})$ and $\mathscr{H}^1_{L^2}(M) = \ker(\Delta_1)$. Then the Hodge–de Rham map $\mathscr{H}^1_{L^2}(M) \to H^1_{\mathrm{cusp}}(M;\mathbb{R})$ is an isomorphism.*

This gives a decomposition

$$H^1(M;\mathbb{R}) \cong \mathscr{H}^1_{L^2}(M) \oplus \mathrm{Im}(i_1^*).$$

Duality and the long exact sequence imply that the image has dimension half that of $H^1(\partial\overline{M};\mathbb{R})$, which is equal to the number h of cusps of M. To proceed further we need to give a description of the second summand by automorphic forms on M. The theory of Eisenstein cohomology has been introduced for precisely this purpose, we will give a short account here.

The cohomology $H^1(\partial\overline{M},\mathbb{C})$ has a Hodge decomposition: if $T_1,\dots,T_h$ are the boundary components of $\overline{M}$ then each of them has a holomorphic coordinate z_i. The forms dz_i and $d\overline{z}_i$ are both harmonic and the cohomology is then decomposed as

$$H^1(\partial\overline{M};\mathbb{C}) = H^{1,0}(\partial\overline{M};\mathbb{C}) \oplus H^{0,1}(\partial\overline{M};\mathbb{C})$$

where

$$H^{1,0}(\partial\overline{M};\mathbb{C}) = \bigoplus_{i=1}^{h} \mathbb{C}[dz_i] \text{ and } H^{0,1}(\partial\overline{M};\mathbb{C}) = \bigoplus_{i=1}^{h} \mathbb{C}[d\overline{z}_i].$$

There is then a distinguished injective map

$$E^1 : H^{1,0}(\partial\overline{M};\mathbb{C}) \to H^1(M;\mathbb{C})$$

such that $H^1(M;\mathbb{C}) = \mathscr{H}^1_{L^2}(M) \oplus \mathrm{Im}(E^1)$. To define it formally Eisenstein series are needed, which we won't introduce. We will assume that everything is defined using the normalisations from [15, Chapter 6]. If $\omega \in H^{1,0}(\partial\overline{M};\mathbb{C})$ the Eisenstein series $E(0,\omega)$ is a smooth harmonic 1-form on M which is not square-integrable. It is closed and the class $E^1(\omega)$ is defined by $E^1([\omega]) = [E(0,\omega)]$. We will not care about the exact definition of Eisenstein series but we will record the following analytic properties :

(1) There exists a linear isomorphism $\Phi(0)$ ("intertwining operator") from $H^{1,0}(\partial\overline{M};\mathbb{C}) \to H^1(M;\mathbb{C})$ to $H^{0,1}(\partial\overline{M};\mathbb{C}) \to H^1(M;\mathbb{C})$ such that

$$E(0,\omega)_P = \omega + \Phi(0)(\omega). \tag{4.1}$$

(2) We have the "Maass–Selberg formula":

$$\int_{M^Y} \langle E(0,\omega), E(0,\omega')\rangle = \log(Y)\langle\omega,\omega'\rangle. \tag{4.2}$$

In (4.1) we used f_P to denote the *constant term* of a differential form on M. This is defined as the collection of zeroth Fourier coefficient for restriction of f to the cross-sections of the cusps ; note that in general it depends on the height at which the cross section is taken but not for $s = 0$.

In degree 2 the construction of Eisenstein classes is slightly different. As above we define $\mathscr{H}^2_{L^2}(M) = \ker(\Delta_2)$ and $H^2_{\mathrm{cusp}}(M;\mathbb{R}) = \ker(i_2^*)$. The same proof as that of Lemma 4.1 yields that $\mathscr{H}^2_{L^2}(M) \cong H^2_{\rm cusp}(M;\mathbb{R})$. Then we look at the long exact sequence of $\overline{M}, \partial\overline{M}$ to see that $\dim(\mathrm{Im}(i_2^*)) = h-1$, and let

$$Z = \mathrm{Im}(i_2^*) = \left\{ \sum_i [a_i dz_i \wedge d\overline{z}_i] : \sum_i a_i = 0 \right\}.$$

The Eisenstein map is a map $E^2 : Z \to H^2(M;\mathbb{C})$ such that

$$H^2(M;\mathbb{C}) = \mathrm{Im}(E^2) \oplus H^2_{\rm cusp}(M;\mathbb{C}).$$

We define it following [15, 6.3.1]: if $\eta \in H^2(\partial\overline{M};\mathbb{C})$ we take its Poincaré dual $f = *\eta$ which is just a collection $(a_1,\ldots,a_h)$. Eisenstein series for the constant function have a pole at $s=1$ but if $\sum_i a_i = 0$ we may still construct the Eisenstein series $E(1,f)$. It is then a harmonic function (not square-integrable) and thus the 2-form $*dE(1,f)$ is closed. Finally, we put $E^2([\eta]) = [*dE(1,f)]$.

4.2.2. *Reidemeister torsion.* According to the above paragraph we can define a Hermitian inner product on $H^1(M;\mathbb{C})$ by putting :

$$\begin{aligned} \langle [f_1],[f_2]\rangle &= \langle f_1,f_2\rangle_{L^2} && \text{if } f_1,f_2 \in \mathscr{H}^2_{L^2}(M);\\ \langle [f], E^1([\omega])\rangle &= 0 && \text{if } f \in \mathscr{H}_{L^2}(M), \omega \in H^1(\partial\overline{M};\mathbb{C});\\ \langle E^1([\omega_1]), E^1([\omega_2])\rangle &= \langle \omega_1,\omega_2\rangle && \text{if } \omega_1,\omega_2 \in H^1(\partial\overline{M};\mathbb{C}). \end{aligned}$$

and similarly for $H^2(M;\mathbb{C})$. We can then define the regulators $R_i(M)$ for $i = 1,2$ using the formula (1.6) (0- and 3-cohomology groups are represented by square-integrable function, see [15]), and the Reidemeister torsion by the formula (1.7). It is also possible to define an analytic torsion $T_R(M)$ using the Selberg trace formula (see [15], [31]).

There should be a “Cheeger–Müller” relation between $\tau(M)$ and $T_R(M)$, possibly with additional terms coming from the cusps. A proof for this is almost given in the course of proving Theorem 6.8.3 in [15], which however misses a crucial ingredient (see the remarks there after the statement of the theorem).

4.3. **Estimating regulators.** If V is a $\mathbb{C}$-vector space which contains a spanning $\mathbb{Q}$-subspace $V_\mathbb{Q}$ of the same dimension we say that a subspace $W \subset V$ is *defined over* $\mathbb{Q}$ if $W_\mathbb{Q} := W \cap V_\mathbb{Q}$ spans W. The subspace $H^1_{\rm cusp}(M;\mathbb{C})$ is obviously defined over $\mathbb{Q}$ (with respect to $H^1(M;\mathbb{Q})$). It is also true and not hard to see that the subspaces $H^{1,0}$ and $H^{0,1}(\partial\overline{M};\mathbb{C})$ are defined over $\mathbb{Q}$. The following result then implies that the subspace $\mathrm{Im}(E^1)$ spanned by Eisenstein series is defined over $\mathbb{Q}$ as well.

Proposition 4.2. *The map* $\Phi(0)$ *is defined over* $\mathbb{Q}$*.*

In what follows we will thus denote $H^1_{\rm Eis}(M;\mathbb{Q})$, etc. the subspaces $\mathrm{Im}(E^1) \cap H^1(M;\mathbb{Q})$, etc. Over $\mathbb{Q}$ we thus have the decomposition

$$H^1(M;\mathbb{Q}) = H^1_{\rm cusp}(M;\mathbb{Q}) \oplus H^1_{\rm Eis}(M;\mathbb{Q}).$$

Note that this decomposition is in general not true over $\mathbb{Z}$, that is $H^1_{\rm cusp}(M;\mathbb{Z}) \oplus H^1_{\rm Eis}(M;\mathbb{Z})$ is a finite index subgroup in $H^1(M;\mathbb{Z})$ but they are not equal up to modding out torsion subgroups. It follows that we cannot estimate directly $R_1(M)$ by estimating factors coming from cuspidal and Eisenstein homology. In the rest of this subsection we will mainly explain how one should deal with this problem. Estimating the Eisenstein factor is easy as we will mention below. The cuspidal factor in degree 1 can be dealt with as in the compact case, and in degree 2 it is likely to be much harder to estimate, we present a partial result due to Bergeron–Şengün–Venkatesh about this in the last subsection.

4.3.1. *Denominators and estimate.* Let us now define formally the different parts of the regulator that we will be dealing with.

- The "cuspidal regulator" $R_{\mathrm{cusp},1}$ is defined to be the covolume of $H^1_{\mathrm{cusp}}(M;\mathbb{Z})$ in $\mathscr{H}_{L^2}(M;\mathbb{R})$;
- The "Eisenstein regulator" $R_{\mathrm{Eis},1}$ is the covolume of $H^1(\partial\overline{\mathbb{Z}}) \cap \mathrm{im}(i_1^*)$ in its $\mathbb{R}$-span in $\mathscr{H}^1(\partial\overline{M})$.

It is easy to see that $\log R_{\mathrm{Eis},1}(M) \ll \log \mathrm{vol}(M)$ (see [37, Lemma 6.4]. To relate the product $R_{\mathrm{cusp},1}(M)R_{\mathrm{Eis},1}(M)$ to $R_1(M)$ we need to introduce the *denominator* of the map E^1: by definition this is the smallest integer $a_{E^1} \geq 1$ such that

$$\forall[\omega] \in H^1(\partial\overline{M};\mathbb{Z}) : aE^1([\omega]) \in H^1(M;\mathbb{Z}).$$

It is then easy to see that

$$a_{E^1} \cdot H^1(M;\mathbb{Z}) \subset H^1_{\mathrm{cusp}}(M;\mathbb{Z}) \oplus H^1_{\mathrm{Eis}}(M;\mathbb{Z})$$

up to torsion, and this implies that

$$R_1(M) \leq a_{E^1}^{b_1(M)} \cdot R_{\mathrm{cusp},1}(M) \cdot R_{\mathrm{Eis},1}(M). \tag{4.3}$$

4.3.2. *Arithmetic structure of the intertwining map.* In view of (4.3), to estimate the regulator it is necessary to estimate the denominator a_{E^1} of E^1. In order to do this we need first to do it "on the boundary": we let a_Φ be the smallest integer such that $a_\Phi\Phi(0)([\omega]) \in H^{0,1}(\partial\overline{M};\mathbb{Z})$ for all $[\omega] \in H^{0,1}(\partial\overline{M};\mathbb{Z})$.

To compute the denominator a_Φ via intertwining integrals we need a finer decomposition of the Eisenstein cohomology. For this we interpret the boundary as

$$\partial M \cong \mathrm{P}(F)\backslash\mathrm{G}(\mathbb{A})/K$$

where P is the parabolic k-subgroup of upper triangular matrices, and $\cong$ signifies homotopy retraction. The group $\mathbb{A}^\times/F^\times$ acts on the right-hand side (by multiplication by diagonal matrices). Thus we have a decomposition of $H^1(\partial\overline{M};\overline{\mathbb{Q}})$ over according to Hecke characters, we will denote by $H^1(\partial\overline{M};\overline{\mathbb{Q}})_\chi$ eigenspace with eigencharacter χ. Note that $H^1(\partial\overline{M};\overline{\mathbb{Q}})_\chi$ is contained in the $(0,1)$ or $(1,0)$ part of the cohomology according to whether $\chi_\infty = z^2/|z|^2$ or $\overline{z}^2/|z|^2$.

Proposition 4.3. *If χ is a Hecke character with infinite part $z^2/|z|^2$ and $\omega \in H^1(\partial\overline{M};\overline{\mathbb{Z}})_\chi$ we have the expression :*

$$\Phi(0)(\omega) = \frac{M}{N} \cdot \frac{L(\chi,0)}{L(\chi,1)}\overline{\omega} \tag{4.4}$$

where $\overline{\omega} \in H^1(\partial\overline{M};\overline{\mathbb{Z}})_{\overline{\chi}}$ and M, N are integers with N polynomially bounded in the level of Γ.

4.3.3. *Modular symbols.* The expression (4.4) deals with a_Φ, in other words it estimates the integrality of classes $E^1([\omega])$ against chains in the image of $H_1(l\overline{M};\mathbb{Z})$ inside $H_1(M;\mathbb{Z})$. We will say a few words about the next steps necessary to estimate a_{E^1}.

For this we need to estimate integrality of $E^1([\omega])$ against lifts of chains in $H_1(\overline{M},\partial\overline{M};\mathbb{Z})$. The latter are represented by *modular symbols*, which are bi-infinite geodesics between two (not necessarily distinct) cusps. We will present a simplified version of the exposition from [15]. For $\alpha,\beta \in \mathbb{P}^1(k)$ let $c_{\alpha,\beta}$ be the image in M of the geodesic line from α to β in $\mathbb{H}^3$. By the exponential decay of cuspidal forms it is possible to integrate any element of $\mathscr{H}^1_{L^2}(M)$ against $c_{\alpha,\beta}$; it is also true that the harmonic forms $E^1(\omega)$ are integrable against the $c_{\alpha,\beta}$, because their constant term is orthogonal to the geodesics going to a cusp. Since the $c_{\alpha,\beta}$ generate $H_1(\overline{M}, \partial\overline{M}; \mathbb{Z})$ we see that the class $[E^1(\omega)]$ is integral if and only if $\int_{c_{\alpha,\beta}} E^1(\omega) \in \mathbb{Z}$ for all α, β (see also [15, 6.7.5]). These integrals are computed in [15, 6.7.6] where an expression similar to (4.4) is obtained :

$$\int_{c_{\alpha,\beta}} E(0,\omega) = \frac{M}{N}\sum_{\zeta} \frac{L(1/2,\chi\zeta)L(1/2,\overline{\chi}\zeta)}{L(1,\chi^2)} \tag{4.5}$$

(the sum runs over all Hecke characters obtained from characters of the class group of k).

4.3.4. *Degree 2.* In degree 2 it is actually much simpler to separte the cuspidal and Eisenstein part in the regulator. This is done in [15, 6.3.3], we will shortly explain the argument. Let i^2_* be the inclusion map $H_2(\partial\overline{M};\mathbb{Z}) \to H_2(M;\mathbb{Z})$, then as $H_2(\overline{M};\partial\overline{M})$ is torsion-free we have

$$H_2(M;\mathbb{Z}) = H_{2,\mathrm{cusp}}(M;\mathbb{Z}) \oplus \mathrm{Im}(i^2_*).$$

In addition, if $f \in \mathscr{H}^2_{L^2}(m)$ then f is a cuspidal form and as such $\int_c f = 0$ for any cycle $c \in \mathrm{Im}(i^*_2)$. Thus the matrix of periods appearing in (1.6) is block-diagonal and it follows that

$$R_2(M) = R_{2,\mathrm{cusp}}(M) \cdot R_{2,\mathrm{Eis}}(M). \tag{4.6}$$

In addition the Eisenstein regulator is immediately computable: per [15] its value is

$$R_{2,\mathrm{Eis}}(M) = \left(\frac{\prod_{i=1}^h \mathrm{vol}(T_i)}{\sum_{i=1}^h \mathrm{vol}(T_i)}\right)^{-1/2}.$$

As in the closed case the cuspidal part remains hard to evaluate because we do not know whether it is possible to generate $H_{2,\mathrm{cusp}}$ with cycles of low complexity (polynomial in the volume). A partial result is given in [6, Theorem 7.2], which we will only informally describe. The authors prove that for the orbifold $M = \Gamma_0(\mathfrak{n})$ (where Γ is a Bianchi group and $\mathfrak{n}$ an ideal in its trace ring), if $\dim(H^1_{\mathrm{cusp}}(M;\mathbb{C}) = 1$, and under additional assumptions related to the number-theoretical side of the Langlands programme, then $H^1_{\mathrm{cusp}}(M;\mathbb{Z})$ can be generated by a harmonic form of L^2-norm at most $\mathrm{vol}(M)^C$ (for some C which a priori depends on the base field). Of course the dual cycle must have small area, hence small complexity.

4.4. **Nontrivial coefficients.** Let k_D be the imaginary quadratic field of discriminant D and R_D its ring of integers. If Γ is a subgroup of the Bianchi group $\mathrm{SL}_2(\mathcal{O}_D)$ there are natural Γ-modules which fit in the setup of 1.4. These are the symmetric powers $\mathrm{Sym}^m(R_D^2)$, which are lattices in the space $\mathrm{Sym}^m(\mathbb{C}^2)$ on which $\mathrm{SL}_2(\mathbb{C})$ acts. There are Eisenstein series with coefficients in the associated fiber bundle F_m and there are also Eisenstein maps

$$E^i : H^i(\partial\overline{M}; F_m) \to H^i(M; F_m)$$

for $i = 1, 2$ which are defined respectively by $[\omega] \mapsto [E(m,\omega)]$ and $[*f] \mapsto [*dE(m+1, f)]$. The major difference with trivial coefficients is that for $m \geq 1$ the cuspidal homology vanishes, hence $H^i(M;\mathbb{C}) = \mathrm{im}(E^i)$. Thus the only difficulty in estimating regulators lies in the denominator a_{E^1} of E^1. Moreover there is an equality between the Reidemeister torsion and an analytic torsion due to Pfaff [34] (see also [2] which states that

$$\tau_{\mathrm{Eis}}(M; L_m) = T_R(M; L_m) + B \tag{4.7}$$

where B depends only on the conformal structure of the boundary $\partial\overline{M}$ and the choices made to define both the Reidemeister and analytic torsions.

4.5. **Application to homology growth.** It is possible to generalise in part Theorem 3.6 to the case of congruence subgroups of the Bianchi groups. The upper limit is proven to hold in [37] (we note that the main result of [23] includes finite volume manifolds, and the proof is actually simpler in this case). The lower bound is given in [33].

Theorem 4.4. *Let Γ be a torsion-free nonuniform lattice in $\mathrm{PSL}_2(\mathbb{C})$ and Γ_n a sequence of congruence subgroups. Fix notation as in the statement of Theorem 3.6. Assume in addition that the sequence Γ_n is "cusp-uniform"[7]. Then*

$$\left(c_\rho - \frac{12}{\pi}\right)\mathrm{vol}(\Gamma\backslash\mathbb{H}^3) \le \liminf_{n\to+\infty} \frac{\log|H_1(\Gamma_n; L)_{\mathrm{tors}}|}{[\Gamma:\Gamma_n]}$$

and

$$\limsup_{n\to+\infty} \frac{\log|H_1(\Gamma_n; L)_{\mathrm{tors}}|}{[\Gamma:\Gamma_n]} \le c_\rho\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3).$$

The proof uses an argument similar to that of Theorem 3.6; the hypothesis on the cusps (which can be slightly weakened) is used to control the additional terms in the trace formula. The reason why we do not get an equality as in Theorem 3.6 is because we lack a good upper bound for the norm of the algebraic part of the L-values appearing in (4.4) and (4.5). The lower bound obtained by Pfaff is based on a trick using duality and the long exact sequence to show that the growth of R_1 must be compensated by an equivalent growth of the torsion in H_1, and which avoids directly dealing with intertwining operators.

We won't give more detail about this argument here, rather we will explain the proof of the following theorem from [35] which uses similar arguments but also an estimate for a_Φ. The result itself is a partial generalisation of Marshall and Müller's theorem 3.7. We note that we give here a slighty less precise statement that the one in loc. cit.

Theorem 4.5. *There exists $C > 0$ such that for any square-free negative integer D, if Γ is a principal congruence subgroup of large enough level in the Bianchi group $\mathrm{SL}_2(R_D)$ and $L(m) = \mathrm{Sym}^m(R_D^2)$ then we have*

$$\frac{1}{C}\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3) \le \liminf_{m\to+\infty} \frac{\log|H_1(\Gamma; L(m))_{\mathrm{tors}}|}{m^2}$$

and

$$\limsup_{m\to+\infty} \frac{\log|H_1(\Gamma; L(m))_{\mathrm{tors}}|}{m^2} \le C\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3).$$

The interesting part is the proof of the lower bound ; the upper bound follows from the generic arguments explained before 2.2. The first step towards proving it is to upper and lower bounds for the growth of the Reidemeister torsions appearing $\tau_{\mathrm{Eis}}(M; \mathcal{L}(m))$ appearing in Theorem [34]. This follows from earlier work of Müller and Pfaff together with (4.7), using a trick comparing growth between two different Γs at once (this explains the indeterminacy of the constant $1/C$ in the lower bound). Then, instead of estimating the regulator directly we prove the following lemma [35, Lemma 4.1].

Lemma 4.6. *We have :*

$$R_1(M; \mathcal{L}(m)) \le |H_1(\Gamma; L(m))_{\mathrm{tors}}| \cdot a_\Phi \cdot R_{\mathrm{Eis},1}(M). \tag{4.8}$$

[7]This means that the subgroups $\Gamma_n \cap P$ for P a parabolic subgroup stay within a compact subset of the moduli space of lattices in $\mathbb{C}$.

The proof is elementary. Assuming that we know that $\log(a_\Phi) = o(m^2)$ (and $R_2(M; \mathcal{L}(m)) = o(m^2)$ which is simpler to prove) the lemma then basically gives

$$\liminf_{m\to+\infty} \frac{\log |H_1(\Gamma; L(m))_{\mathrm{tors}}|}{m^2} \geq \frac{1}{2} \liminf \frac{\log \tau_{\mathrm{Eis}}(M; \mathcal{L}(m))}{m^2}$$

which finishes the proof. To estimate the denominator a_Φ we use the formula (4.4) and two papers of R. Damerell [17, 18]. The first one almost states that there exists $\Omega \in \mathbb{C}^\times$ such that $L(\chi, m)/\Omega$ and $L(\chi, m+1)/\Omega$ are both algebraic numbers. We need to refine that statement to get that in addition the absolute norm of both is bounded above by $(m!)^c$ for some c depending only on k, which is done by examining carefully Damerell's argument. The second paper proves that $L(\chi, m)/\Omega, L(\chi, m+1)/\Omega \in 1/c\overline{\mathbb{Z}}$ where $c \in \mathbb{Z}$ depends only on Γ. These two theorems together immediately imply that $\log(a_\Phi) = O(m \log(m))$, which concludes the proof.

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