TORSION HOMOLOGY OF THREE–MANIFOLDS

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Abstract. We review various heuristics, questions and conjectures about the torsion part of the homology of compact three–manifolds. We also present a theorem on growth of torsion homology in congruence covers of arithmetic manifolds and give an informal introduction to it's proof.

1. Introduction and overview

Let M be a compact three–manifold; one of the simplest topological invariants of M are it's homology groups $H_p(M;\mathbb{Z})$ which can be computed using any cell structure for M, for example from a Heegard decomposition or from a presentation of M as a Dehn surgery on a link in the three–sphere. The groups $H_p(M;\mathbb{Z})$ are finitely generated abelian groups and as such they decompose as a direct sum

$$H_p(M;\mathbb{Z}) = \mathbb{Z}^{b_p(M)} \oplus T_p(M)$$

where $T_p(M)$ is the torsion subgroup, which is finite. Moreover, at least for closed manifolds and manifolds whose boundary is a disjoint union of tori one can easily see that H_1 determines the others. A basic question which will interest us here is the following: given a compact manifold M, what is the range of $b_1(M')$ and $t_1(M') = |T_1(M')|$ for M' a finite cover of M? Related to this one can ask what the behaviour of these numbers is in specific sequences of finite covers of M. This note is a expanded version of the talk given by the author at the workshop "Growth and Mahler measure in geometry and topology" which was held at the institute Mittag-Leffler from July 1 to 5, 2013. As its title indicates it is mainly focused on the torsion part of the homology; moreover, our main concern will be with hyperbolic manifolds. Our main aim is to provide an informal introduction to the contents of the author's papers [25],[26] and (to a lesser extent, since these contain much more than is talked about here) to the seminal paper of N. Bergeron and A. Venkatesh [4] and to the joint work of the author with M. Abèrt, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov and I. Samet [1].

Let us now describe in some detail what is to be found here. In the first part we will quickly review various results and conjectures about general three–manifolds: first we talk about growth of torsion in cyclic covers, then we explain how to relate homological torsion growth to ℓ^2-invariants in the context of three–manifolds, and finally we present probabilistic results on the homology and volume of random Heegard splittings. All of this motivates the belief that finite–volume hyperbolic manifolds should often have a large torsion subgroup in their first homology, and more precisely that its size should be (in "nice" situations) close to a certain exponent of the volume. The second part is dedicated to put this vague heuristic statement in a more rigorous form, which is achieved through various conjectures and some positive results (including explicit computations of the homology). We conclude the section with a presentation of the special case of congruence covers of arithmetic manifolds, for which we can actually state some proven results. The last section explains informally the analytic methods which are used to prove the results explained in the previous.

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2. Growth of torsion homology in finite covers

Let M be a compact three–manifold; then by trivial considerations, if M' is a finite cover of M of degree d we have $b_1(M) \leq Cd$ where C is the smallest number of 1-simplices in a triangulation of M. There is a similar bound for torsion: by Lemma 5 in [11] there is a constant C' depending on the number of 1-simplices in a triangulation of M and the degree of its 2-simplices such that $t_1(M') \leq C'd$. The rough behaviour of the homology in a sequence of finite covers $M_n \to M$ can thus be studied through the behaviour of the numerical sequences

$$\frac{b_1(M_n)}{[\pi_1(M) : \pi_1(M_n)]} \text{ and } \frac{\log t_1(M_n)}{[\pi_1(M) : \pi_1(M_n)]}.$$

2.1. Cyclic covers.

The only case in which there are complete results on the exponential growth rate of torsion is that of cyclic covers; nontheless this already provides some interesting examples.

2.1.1. *Exponential growth of torsion.*

The setting in this section is as follows: we have a compact three–manifold M with an epimorphism $\pi_1(M) \to \mathbb{Z}$, and we study the sequence of cyclic covers M_n corresponding to the surjections $\pi_1(M) \to \mathbb{Z} \to \mathbb{Z}/n$. Associated to the morphism $\pi_1(M) \to \mathbb{Z}$ (or to the corresponding infinite cyclic cover) is a certain sequence of Laurent polynomials $\Delta_0, \dots, \Delta_k, \dots$, called the Alexander polynomials of the covering, such that Δ_{i+1} divides Δ_i and $\Delta_i = 1$ for large i. The exponential growth rate of the sequence $t_1(M_n)$ is completely understood in terms of the Δ_k; the following result has been proved independantly by T. Le in [15] and by the author in [27] (see also [29], [13], [33] and [4, Section 7]).

Theorem 2.1. *Notations as above, let r be the smallest index such that $\Delta_r \neq 0$. Then we have*

$$\lim_{n\to+\infty} \frac{\log t_1(M)}{n} = m(\Delta_r) := \int_0^1 \log|\Delta_r(e^{2i\pi\theta})|d\theta.$$

It is a well-known result of L. Kronecker that the right-hand side above (which is the logarithmic Mahler measure of Δ_r) is zero if and only if Δ_r is a cyclotomic polynomial; in general it equals the sum of the $\log|\alpha|$ over the nonzero roots α of Δ_r. We will present various examples where Δ_r is explicitely computed in the sequel; for now let us indicate how to define the Alexander polynomials.

Let $R = \mathbb{Z}[t^{\pm 1}]$ and V be a finitely generated R-module; let $A \in M_{l,m}(R)$ be a presentation matrix for V, then the ith Alexander polynomial $\Delta_i(V)$ of V is a greatest common divisor for the $(l-i)$-minors of A (it does not depend on A). If $\widehat{M} \to M$ is an infinite cyclic covering then the homology group $H_1(\widehat{M}; \mathbb{Z})$ is a R-module (where t is a generator for the covering group) and we put $\Delta_i = \Delta_i(H_1(\widehat{M}; \mathbb{Z}))$. It is defined up to multiplication by a unit $\pm t^k, k \in \mathbb{Z}$, and we see that $r = \mathrm{rk}_R(H_1(M; \mathbb{Z}))$.

2.1.2. *Fibered manifolds.*

We suppose here that M fibers over the circle, i.e. there is a surface S and an homeomorphism ϕ of S such that

$$M \cong S \times [0,1]/\sim \text{ where } (x,0) \sim (\phi(x),1).$$

Then $\pi_1(S)$ is a normal subgroup of $\pi_1(M)$ with quotient $\pi_1(M)/\pi_1(S) = \mathbb{Z}$, the corresponding infinite cyclic covering has $\Delta_r = \Delta_0 = \det(1 - t\phi_*)$.

This example allows to exhibit fibered manifolds which have cyclic covers with exponential growth of homology. For example let M be the fibered Sol-manifold given by the above construction with $S = \mathbb{T}^2$ and $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\Delta_0 = t^2 - 3t + 1$ and $m(\Delta_0) = \log\left(\frac{3+\sqrt{5}}{2}\right)$. One can also give hyperbolic examples with $m(\Delta_0) > 0$, although they are more complicated. Some were computed by T. Koberda, and he made the following conjecture (which if true would imply that every finite–volume hyperbolic manifold has a sequence of covers where t_1 grows exponentially with the volume).

Conjecture 2.2. *Let S be an hyperbolic surface and ϕ a pseudo-Anosov homeomorphism of S. There exists a finite cover S' of S and a lift ϕ' of ϕ to S' such that* $\det(1 - t\phi_*)$ *is not cyclotomic.*

2.1.3. *Knot complements.* Let k be a knot in $\mathbb{S}^3$ and $M = \mathbb{S}^3 - k$. Then $H_1(M) = \mathbb{Z}$ and thus there is a unique cyclic cover M_n of degree n of M. The Alexander polynomial Δ_0 of the infinite cyclic cover of M can be computed from a diagram of the knot (this is actually how it was originally introduced), and it is always nonzero. In this setting Theorem 2.1 is due to D. Silver and S. Williams. For the simplest knots we have:

- if k is the trefoil then $\Delta_0 = t^2 + t - 1$ is cyclotomic;
- if k is the figure-eight then $\Delta_0 = t^2 - 3t + 1$.

Since the complement of the figure-eight is hyperbolic this gives examples of a sequence of covers of a hyperbolic manifold with exponential growth of torsion.

2.1.4. *More general results.* Theorem 2.1 admits a generalization to non-cyclic abelian covers ([15], see also [32] and [27]) which is less precise (although [15] gives the best possible result conditionnally to a number–theoretical conjecture). There is no other, more general type of covers where such a result is known to hold.

2.2. **Relations with ℓ^2-invariants.** Here we state without justification the more or less conjectured relations between homology of covers and the so–called ℓ^2–invariants in the special case of three–manifolds; we will recast them in analytic context in Section 4 below. The reader is refered to [18] for the definition of ℓ^2-invariants; the original paper on approximation is [16].

2.2.1. *Approximation and homology growth.* Suppose that M is a three–manifold with a CW-structure; there is then defined an invariant (not so much as it depends on the chosen CW-structure) called Reidemeister torsion and denoted $\tau(M)$. If $M_\infty \to M$ is an infinite normal cover then there is also defined an ℓ^2-Reidemeister torsion which we will denote $\tau^{(2)}(M_\infty)$. The problem of approximation is then the following: let $\Gamma = \pi_1(M)$, $\Lambda = \pi_1(M_\infty)$; if Γ/Λ is residually finite we can choose a nested sequence of finite-index normal subgroups $\Gamma = \Gamma_0 \supset \ldots \supset \Gamma_n \supset \ldots$ such that $\bigcup_{n \geq 0} \Gamma_n = \Lambda$; letting M_n be the cover of M corresponding to Γ_n with the lifted CW-struture, do we have

$$\lim_{n \to +\infty} \tau(M_n)^{\frac{1}{[\Gamma:\Gamma_n]}} = \tau^{(2)}(M_\infty)? \tag{2.1}$$

This problem is linked to homology growth in the torsion as follows: we have $\tau(M) = t_1(M)^{-1} \times R(M)$ where $R(M)$ comes from the characteristic 0 homology. In some sequences as above it is hoped that $R(M_n)$ has subexponential growth rate with the index, so that if valid (2.1) would yield the exponential growth rate of torsion in the sequence $t_1(M_n)$. This is a way to prove the result about homology growth in cyclic covers (Theorem 2.1).

2.2.2. *Geometrized manifolds.* When $M_\infty = \widetilde{M}$ is the universal cover of M the ℓ^2-torsion is computable in terms of the geometric JSJ-decomposition of M: if $X_1, \ldots, X_m$ are the hyperbolic pieces then

$$\log \tau^{(2)}(\widetilde{M}) = \frac{1}{6\pi} \sum_{i=1}^{m} \operatorname{vol} X_i. \tag{2.2}$$

This does not depend on the choice of a CW-structure on M.

Finally, let us note that the conjectural picture above has been clarified by W. Lück in the case of covers of Seifert fibered manifolds; we will cite the following result from [17] (see Corollary 1.13 there).

Theorem 2.3. *Let M be a compact Seifert fibered three–manifold, and M_n a tower of finite normal covers of M with $\bigcap_n \pi_1(M_n)$ trivial. Then*

$$\lim_{n\to+\infty} \frac{\log t_1(M_n)}{[\pi_1(M):\pi_1(M_n)]} = 0.$$

2.3. **Random manifolds.** Let S be a closed surface, $\mathrm{Mod}(S)$ the mapping class group of S. Let g_k be the kth step of a uniform random walk on $\mathrm{Mod}(S)$ (i.e. g_{k+1} is given by $g_k g$ where g is chosen in a finite symmetric generating set for $\mathrm{Mod}(S)$ with respect to the uniform distribution). Then the manifold M_k obtained by gluing two handlebodies according to X_k (a "random Heegard splitting") is called a random Dunfield–Thurston manifold (after the paper [10] where they were studied first), and the study of the statistical properties of the homology of M_k as $k \to +\infty$ may provide insight. The following theorem is a compilation of results by J. Maher [19], J. Brock–J. Souto (mostly unwritten) and E. Kowalski ([14], which is the main source for this subsection).

Theorem 2.4. *Suppose that the genus of S is at least two, then:*

(i) *M_k is hyperbolic with asymptotic probability 1;*
(ii) *there are $0 < c_1 < c_2$ such that $c_1 k \leq \mathrm{vol}\, M_k \leq c_2 k$ with asymptotic probability 1;*
(iii) *for any sequence u_k such that $u_k \to +\infty$ we have $t_1(M_k) > e^{\frac{k}{u_k}}$ with asymptotic probability 1; moreover, there are $C, \alpha > 0$ such that the expectation of $t_1(M_k)$ is $\geq Ce^{\alpha k}$ for large enough k.*
(iv) *$b_1(M_k) = 0$ with asymptotic probability 1.*

Combining (ii) and (iii) above we see that for this model of random manifold, the torsion homology is almost always proportional to the exponent of the volume; this indicates the the relation between $\log t_1$ and vol for hyperbolic three–manifolds deserves further study.

3. Hyperbolic manifolds

In this section M will always be a finite–volume hyperbolic manifold (which may vary from one occurence to the other).

3.1. **A conjecture.** The behaviour of random manifolds makes it clear that the torsion homology of hyperbolic three–manifolds is an interesting object of study. It exhibits a wide variety of behaviours in finite covers: as we saw in 2.1.3 and 2.1.2 there are sequences where $t_1(M)$ grows as fast as possible, namely exponentially. On the other hand it has been proven by M. Baker, M. Boileau and S. Wang [2] that there exists finite–volume hyperbolic manifolds which have towers of finite covers which are homology spheres. Nevertheless, in view of the expected links between approximation of ℓ^2–invariants and homology growth (2.1),(2.2) we can ask the following question which predicts some kind of uniform behaviour for all hyperbolic manifolds.

Conjecture 3.1. *For any finite-volume hyperbolic three–manifold M there is a tower of finite covers $\ldots \to M_n \to \ldots \to M_0 = M$ with $\bigcap_n \pi_1(M_n) = \{1\}$ and*

$$\lim_{n\to+\infty} \frac{\log t_1(M_n)}{\mathrm{vol}\, M_n} = \frac{1}{6\pi}.$$

It is not expected (at least not by everybody) that for *any* tower satisfying the assumptions of the question the growth of torsion homology will be as indicated; indeed there are numerical computations suggesting that for certain sequences this might not be the case [5, Figure 4.5], [31] . On the other hand, the same computations suggest that if a tower as in the conjecture satisfies the additional assumption that $H_1(M_n;\mathbb{Q}) = 0$ for all n then the conclusion holds for that sequence.

3.2. **Convergence of hyperbolic manifolds and homology.** We will put the sequence of covers for which the conjecture was made above in a more general geometric context. Note that if M is a compact manifold, then the condition in Question 3.1 is equivalent to $\mathrm{inj}(M_n) \to +\infty$ for a tower. It is thus natural to consider sequences of compact hyperbolic manifolds with $\mathrm{inj}(M_n) \to +\infty$; we will be interested in sequence satisfying the following weaker condition, which was introduced in [1, Definition 1.1] under the name of "Benjamini–Schramm"[1] convergence to $\mathbb{H}^3$[2]:

$$\forall R > 0, \lim_{n\to+\infty} \frac{\mathrm{vol}\{x \in M_n :\ \mathrm{inj}_x(M_n) \le R\}}{\mathrm{vol}\, M_n} \xrightarrow[n\to+\infty]{} 0. \tag{3.1}$$

That this is a good setting in which to study the asymptotics of homology is illustrated by the following result [1, Theorem 1.8], [25, Proposition C]:

Theorem 3.2. *Let M_n be a sequence of finite-volume hyperbolic three–manifolds which BS-converges to $\mathbb{H}^3$; then we have*

$$\lim_{n\to+\infty} \frac{b_1(M_n)}{\mathrm{vol}\, M_n} = 0.$$

Note that even for sequences of covers the convergence to 0 can be arbitrarily slow [12]. For torsion the picture is less clear; it is known by work of J. Brock and N. Dunfield that BS-convergence to $\mathbb{H}^3$ is far from implying exponential growth of torsion (in [5] they construct a sequence M_n which satisfies (3.1) but has $H_1(M_n;\mathbb{Z}) = 0$ for all n). The following conjecture can nevertheless be made.

Conjecture 3.3. *Let M_n be BS-convergent to $\mathbb{H}^3$ and such that the Cheeger constants of the M_n are bounded away from 0 and $H_1(M_n;\mathbb{Q}) = 0$ for all n; then*

$$\lim_{n\to+\infty} \frac{\log t_1(M_n)}{\mathrm{vol}\, M_n} = \frac{1}{6\pi}.$$

3.3. **Congruence manifolds.**

3.3.1. *Congruence groups.* Let G be a $\mathbb{Q}$–form of $\mathrm{PSL}_2(\mathbb{C}) \times G'$ where G' is a compact group. Let $\mathbb{A}_f$ be the ring of finite adèles of $\mathbb{Q}$; then $\mathrm{G}(\mathbb{Q})$ is dense in $\mathrm{G}(\mathbb{A}_f)$, and a subgroup $\Gamma \subset \mathrm{G}(\mathbb{Q})$ is said to be a congruence group in $\mathrm{G}(\mathbb{Q})$ if the closure K_f of Γ in $\mathrm{G}(\mathbb{A}_f)$ is compact and open and moreover $\Gamma = \mathrm{G}(\mathbb{Q}) \cap K_f$. The following theorem is a slight generalization of [1, Theorem 1.12] and [26, Theorem B].

Theorem 3.4. *Let G as above and Γ_n be the images in $\mathrm{SL}_2(\mathbb{C})$ of a sequence of pairwise distinct congruence groups in $\mathrm{G}(\mathbb{Q})$; then the sequence of orbifolds $M_n = \Gamma_n\backslash\mathbb{H}^3$ is BS-convergent to $\mathbb{H}^3$.*

We will call a hyperbolic three–manifold a congruence manifold if its fundamental group is conjugated in $\mathrm{PSL}_2(\mathbb{C})$ to the image of a congruence group in some $\mathrm{G}(\mathbb{Q})$. The Cheeger constant of congruence three–manifolds is known to be bounded below by a uniform constant [9]; the rational homology of a congruence manifold can be nonzero, so that Conjecture 3.3 does not necessarily apply to a sequence of congruence manifolds. The following conjecture is nevertheless believed to be true.

Conjecture 3.5. *Let $\mathrm{G}, \Gamma_n, M_n$ be as in Theorem 3.4. Then we have*

$$\lim_{n\to+\infty} \frac{\log t_1(M_n)}{\mathrm{vol}\, M_n} = \frac{1}{6\pi}.$$

[1] After the work [3] of these authors on a similar notion for regular graphs.

[2] It fits into a more general notion of convergence for finite–volume manifolds, which also incorporates the convergence towards an infinite Galois cover; see[1, Definition 3.1 and Lemma 3.5]

This is supported by the computations of M.H. Şengun [30] and of Brock–Dunfield [5, Figure 4.4]. It is also motivated by arithmetic considerations, see [4].

3.3.2. *Example: the Bianchi groups.* The simplest (algebraically) exemple of congruence groups is given by the Bianchi groups. Let $F = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and $\mathcal{O}_F$ its ring of integers. We get a $\mathbb{Q}$-form of $\mathrm{PSL}_2(\mathbb{C})$ by taking the Weil restriction G of PSL_2/F to $\mathbb{Q}$. At finite places we get:

- (i) $\mathrm{G}(\mathbb{Q}_p) = \mathrm{PSL}_2(\mathbb{Q}_p) \times \mathrm{PSL}_2(\mathbb{Q}_p)$ if p is split in $F/\mathbb{Q}$;
- (ii) $\mathrm{G}(\mathbb{Q}_p) = \mathrm{PSL}_2(F_p)$ where $F_p = \mathbb{Q}_p(\sqrt{-d})$ is a quadratic extension of $\mathbb{Q}_p$ if p is inert or ramified in $F/\mathbb{Q}$.

Let $\Gamma = \mathrm{PSL}_2(\mathcal{O}_F)$; then the closure of Γ in $\mathrm{PSL}_2(\mathbb{Q}_p)$ is equal to $\mathrm{PSL}_2(\mathbb{Z}_p) \times \mathrm{PSL}_2(\mathbb{Z}_p)$ in case (i) and to $\mathrm{PSL}_2(\mathcal{O}_p)$ in case (ii) (where $\mathcal{O}_p$ is the closure of $\mathcal{O}_F$ in F_p), and we see that Γ is a congruence group in $\mathrm{G}(\mathbb{Q})$. We can define many other congruence subgroups inside Γ: for an ideal $\mathfrak{I}$ in $\mathcal{O}_F$ let $\Gamma(\mathfrak{I})$ be the kernel of the reduction morphism $\mathrm{PSL}_2(\mathcal{O}_F) \to \mathrm{PSL}_2(\mathcal{O}_F/\mathfrak{I})$, then $\Gamma(\mathfrak{I})$ is easily seen to be a congruence group, and it follows that the preimage in Γ of any subgroup in $\mathrm{PSL}_2(\mathcal{O}_F/\mathfrak{I})$ is also a congruence groups; it is usual to denote by $\Gamma_0(\mathfrak{I})$ the preimage of the upper triangular matrices and by $\Gamma_1(\mathfrak{I})$ the preimage of the subgroup of unipotent matrices.

3.3.3. *Local coefficients.* Conjecture 3.5 is wide open at present; however there is a scheme of proof which we will describe in the next section and allows to get a better understanding of what is involved in the conjecture; moreover it actually succeeds in proving results similar to the conjecture, by replacing the trivial local system $\mathbb{Z}$ by other $\pi_1(M)$-modules. We will describe here a generalized version of Conjecture 3.5 and state the results obtained in [4], [1] and [26] in this direction.

We will consider here lattices in $\mathrm{SL}_2(\mathbb{C})$ rather than in $\mathrm{PSL}_2(\mathbb{C})$ for reasons that will soon be apparent; in any case, torsion free lattices in $\mathrm{PSL}_2(\mathbb{C})$ at least lift to isomorphic lattices in $\mathrm{SL}_2(\mathbb{C})$ so there is no loss of generality when considering manifolds. We let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a lattice and ρ a representation of Γ in $\mathrm{SL}(L)$ for some free, finitely generated $\mathbb{Z}$-module L. There is then a ℓ^2-torsion associated to the chain complex $C_*(M; \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} L)$ and a torsion homology $H_1(M; L)_{\mathrm{tors}}$ and we can ask how the growth of the latter in covers of M relates to the former, in the spirit of ?ref?. We will formulate a conjecture in the case where Γ is arithmetic and ρ comes from a representation of $\mathrm{SL}_2(\mathbb{C})$. The real representations of $\mathrm{SL}_2(\mathbb{C})$ are on the spaces

$$V_{m,q} = \mathrm{Sym}^m(\mathbb{C}^2) \otimes \mathrm{Sym}^q(\overline{\mathbb{C}}^2)$$

where $\mathrm{SL}_2(\mathbb{C})$ acts on $\overline{\mathbb{C}}^2$ by conjugate matrices. For any lattice Γ the ℓ^2-torsion of $C_*(M; \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} V)$ is given for $V = V_{m,q}$ by

$$\log \tau^{(2)}(\widetilde{M}; V) = \mathrm{vol}(M) t^{(2)}(V),$$

where

$$t^{(2)}(V) = \frac{-1}{48\pi}\left((m+q+2)^3 - |m-q|^3 + 3|m-q|(m+q+2)(m+q+2-|m-q|)\right) \tag{3.2}$$

has been computed in [4, 5.9.3, Example (3)].

Conjecture 3.6. *Let Γ be a congruence groups and Γ_n a sequence of pairwise distinct congruence groups contained in Γ; let ρ, V be a representation of $\mathrm{SL}_2(\mathbb{C})$ and suppose*[3] *that there is a lattice L in V which is preserved by $\rho(\Gamma)$. Then we have*

$$\lim_{n \to +\infty} \frac{\log t_1(M_n; L)}{\mathrm{vol}\, M_n} = -t^{(2)}(V).$$

[3]This is always the case is Γ is defined over a quadratic imaginary field.

As we promised, here are results proving this conjecture for a large portion of these coefficients systems; note that there were previous result on exponential growth of homology in this context (due independantly to J. Pfaff [24] and the author [28, Section 6.5], both relying in some way on [20]) but they are far less precise than the following statement (and their proof cannot be expected to gice such a result).

Theorem 3.7. *Let Γ be an arithmetic lattice, $V = V_{m,q}$ and suppose there is a lattice L in V preserved by Γ. Let Γ_n be a sequence of pairwise distinct, torsion–free congruence subgroups of Γ. Suppose furthermore that $m \neq q$.*

[1] *If Γ is cocompact then*

$$\lim_{n\to+\infty} \frac{\log t_1(M_n; L)}{\operatorname{vol} M_n} = -t^{(2)}(V).$$

[26] *If Γ is a Bianchi group and the sequence Γ_n is moreover cusp-uniform then the same conclusion holds.*

Being cusp-uniform means that the conformal structures on the boundary components do not degenerate. We will give the proof of the first result in the next section (not that in this form it is not explicitely stated in [1]) and explain how to deal with the second case, when Γ is not cocompact.

4. Trace formula and analytic torsion

4.1. **Analytic torsion and an attempt to prove Conjecture 3.3 in the compact case.** We suppose here that M is a compact hyperbolic manifold and let E be a flat bundle with a Euclidean metric on M; the Hodge-Laplace operators $\Delta^p[M]$ on p-forms on M then have a self-adjoint extension to L^2-forms and the space $L^2\Omega^p(M;E)$ of these forms decomposes as a Hilbert sum of finite-dimensional eigenspaces. Let $0 < \lambda_1 \leq \ldots \leq \lambda_k \leq \ldots$ be the positive eigenvalues of $\Delta^p[M]$; then the Minakshisundaram–Pleijel expansion for the trace of the heat kernel allows to prove that the series

$$\zeta_p(s) = \sum_{k\geq 1} \lambda_k^{-s}$$

defines a holomorphic function in the half-plane $\mathrm{Re}(s) > 3/2$[4] which extends to a meromorphic function on $\mathbb{C}$ regular at 0. One then defines the Ray–Singer analytic torsion of M with coefficients in E by the expression

$$T(M;E) = \prod_{p=1}^{3} \exp(\zeta_p'(0)). \tag{4.1}$$

This is useful to study homological torsion in view of the Cheeger–Müller Theorem[5], which amazingly relates this spectrally defines invariant to a topological one, in the spirit of Hodge–de Rham theory. We will state it first for $E = \mathbb{R}$ the trivial line bundle; it then amounts to an equality

$$T(M;\mathbb{R}) = \frac{R^1(M)}{\mathrm{vol}(M)\cdot|H_1(M;\mathbb{Z})_{\mathrm{tors}}|} \tag{4.2}$$

where $R^1(M)$ is defined to be the covolume of the lattice of non-torsion integral cohomology classes inside the space of harmonic forms. Now for another case of interest to us: suppose $E = E_\rho$ as

[4] Weyl's law for the asymptotics of λ_k are actually enough to deduce this.

[5] The original result by Cheeger [8] and Müller is proven for orthogonal bundles; Cheeger's proof has been extended by Müller [21] to cover also unimodular bundles, and an even more general result has been proven by J.M. Bismut and W. Zhang.

above, Γ is arithmetic and L is a lattice in V which is preserved by $\rho(\Gamma)$. Suppose in addition that $H^*(M; E_\rho) = 0$; then

$$T(M; E_\rho) = \frac{|H_2(M; L)| \cdot |H_0(M; L)|}{|H_1(M; L)|}. \tag{4.3}$$

Thus the cocompact part of Theorem 3.7 follows from the following results:

Theorem 4.1. *Notations as in Theorem 3.7, if Γ is cocompact we have*

$$\lim_{n \to +\infty} \frac{\log T(M; E_\rho)}{\operatorname{vol} M_n} = t^{(2)}(V).$$

Lemma 4.2. *If Γ_n is a sequence of congruence subgroups of Γ then for $p = 0, 2$*

$$\log |H_p(M; L)_{\text{tors}}| = o(\operatorname{vol} M_n).$$

Lemma 4.2 is proven in [4, Section 8.6] in a more general context; a very short proof for subgroups of Bianchi groups is given in [26, Lemma 6.5].

4.1.1. *Strong acyclicity and the proof of Theorem 4.1.* To study analytic torsion it is convenient to use an expression for $\zeta_p'(0)$ in terms of the trace of the heat kernel $e^{-t\Delta^p[M]}$ of M (with coefficients in some bundle E); it is defined to be

$$\operatorname{Tr} e^{-t\Delta^p[M]} := \dim \ker \Delta^p[M] + \sum_{k \geq 1} e^{-t\lambda_k}$$

A formal computation shows that we have

$$\zeta_p(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \operatorname{Tr} e^{-t\Delta^p[M]} t^s \frac{dt}{t} \tag{4.4}$$

where $\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} dt$ is Euler's function. In particular, for any $t_0 > 0$ it follows that:

$$\zeta_p'(0) = \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} \operatorname{Tr} e^{-t\Delta^p[M]} t^s \frac{dt}{t} \right)_{s=0} + \int_{t_0}^{+\infty} \operatorname{Tr} e^{-t\Delta^p[M]} \frac{dt}{t}. \tag{4.5}$$

There is also an heat operator $e^{-t\Delta^p[\mathbb{H}^3]}$ on the hyperbolic space, and it has a "L^2-trace" $\operatorname{Tr} e^{-t\Delta^p[\mathbb{H}^3]}$ which we will define below; one defines the L^2–analytic torsion for $\mathbb{H}^3$ with coefficients in E using (4.5) with the heat operator $e^{-t\Delta^p[M]}$ replaced by $e^{-t\Delta^p[\mathbb{H}^3]}$, and a computation using the Harish-Chandra Plancherel formula yields (3.2) for its logarithm $t^{(2)}(V)$. The first part of the proof of Theorem 4.1 is thus the following result, the proof of which we will sketch in 4.1.3 below.

Lemma 4.3. *Let M_n be a sequence of compact hyperbolic three–manifolds which is BS-convergent to $\mathbb{H}^3$ and such that there exists a $\delta > 0$ for which* $\operatorname{inj}(M_n) \geq \delta$ *for all n. Then for any $t_0 > 0$ we have*

$$\frac{1}{\operatorname{vol} M_n} \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} \left(\operatorname{Tr} e^{-t\Delta^p[M_n]} - (\operatorname{vol} M_n) \operatorname{Tr} e^{-t\Delta^p[\mathbb{H}^3]} \right) t^s \frac{dt}{t} \right)_{s=0} \xrightarrow[n \to +\infty]{} 0.$$

To conclude the proof of Theorem 4.1 one needs to show that

$$\limsup_{t_0 \to +\infty} \left(\sup_n \int_{t_0}^{+\infty} \operatorname{Tr} e^{-t\Delta^p[M_n]} \frac{dt}{t} \right) = 0. \tag{4.6}$$

To prove this one needs to estimate *in a uniform manner in n* the exponential decay as $t \to +\infty$ of $\operatorname{Tr} e^{-t\Delta^p[M_n]}$. Taking for granted that the trace at a given time (say $t = 1$) $\operatorname{Tr} e^{-\Delta^p[M_n]}$ is bounded, this follows from the following result [4, Lemma 4.1].

Lemma 4.4. *If $m \neq q$, then there is a $\lambda_0 > 0$ such that for any cocompact lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ the Laplace operators $\Delta^p[M_n]$ do not have eigenvalues beneath λ_0.*

4.1.2. *A Selberg-type conjecture.* The existence of a uniform spectral gap is not a necessary condition for (4.6) to hold. Indeed, one hopes that it stays true in the case of trivial coefficients, where is is known that there has to appear eigenvalues veruy close to zero on 1-forms. We propose the following conjecture, which if true implies that Theorem 4.1 extends to trivial coefficients. For $\lambda > 0$ let $m(\lambda; M_n) = \dim\ker(\Delta^p[M_n] - \lambda\,\mathrm{Id})$ and let $m_p([0,\delta]; M_n) = (\sum_{\lambda\in[0,\lambda_1]} m_p(\lambda; M_n)$ be the number of eigenvalues of $\Delta^p[M_n]$ below λ_1.

Question 4.5. *Does there exists $\lambda_0 > 0$ such that for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that for all n and $\lambda_1 \leq \lambda_0$ we have*

$$\frac{\sum_{\lambda\in[0,\lambda_1]} m_1(\lambda; M)}{\mathrm{vol}\, M} \leq C_\varepsilon \delta^{1+\varepsilon}\,\mathrm{vol}\, M \quad ?$$

(Or less precisely, does this hold for some exponent $c > 0$ in place of $1+\varepsilon$ on the right-hand side?)

Of course, the conjecture is that the answer is positive when M_n is a sequence of congruence covers of an arithmetic manifold.

4.1.3. *The trace formula and the proof of the Main Lemma, compact case.* If M is a compact hyperbolic three–manifolds the "trace formula" for M is in its crudest form the equality

$$\mathrm{Tr}\, e^{-t\Delta^p[M]} = \int_M \mathrm{tr}\, K_t^p[M](x,x)dx$$

where $K_t^p[M]$ is a kernel on $M \times M$ with coeficients in the right bundle[6], such that the operator $e^{-t\Delta^p[M]}$ is given by convolution with K_t^p. There is a similar kernel k_t^p on $\mathbb{H}^3$, and we will show unsing the formula above that for any $t_0 > 0$ there is a $C > 0$ such that for all $t \in]0, t_0]$

$$\mathrm{tr}\, K_t^p[M](x,x) - \mathrm{tr}\, k_t^p(x,x) \leq C\,\mathrm{inj}_x(M)^{-3} t^{-\frac{3}{2}} e^{-\frac{\mathrm{inj}_x(M)^2}{10t}} \tag{4.7}$$

for all $x \in M$. Now since the L^2-trace is given by

$$\mathrm{Tr}\, e^{-t\Delta^p[\mathbb{H}^3]} = \mathrm{tr}\, k_t^p(x_0, x_0) \tag{4.8}$$

for any $x_0 \in \mathbb{H}^3$, the proof of Lemma 4.3 follows without difficulties from (4.7).

The proof of (4.7) follows [4, (4.5.1)], [1, Lemma 8.23]; the principle is as follows: there is an expansion

$$K_t^p[M](x,x) = \sum_{\gamma\in\Gamma} k_t^p(x, \gamma x)$$

so we have to show that the sum over nontrivial elements is bounded by the right-hand side of (4.7); this is an immediate consequence of the estimate $|k_t^p(x,y)| \leq Ct^{-\frac{3}{2}} e^{-\frac{d(x,y)^2}{5t}}$ and of the well-known bound

$$|\{\gamma \in \Gamma :\ d(x, \gamma x) \leq r\}| \leq C\,\mathrm{inj}_x(M)^{-3} e^{cr}.$$

4.2. **The trace formula and analytic torsion in the non-compact case.** In the non-compact case the Laplace operator has continuous spectrum, the heat kernel is not trace–class and we cannot define the analytic torsion immediately as in the compact case. What is to be done is to define the trace using the Selberg trace formula and analyse it to get the analytic continuation and regularity at 0 as in the compact case. This is done in [23], [22] and [25], we will follow the approach of the latter paper which is better suited to the problems at hand.

[6]So that $K_t^p[M](x,y)$ is a linear map from $\wedge^p T_y^* M \otimes E_y$ to $\wedge^p T_x^* M \otimes E_x$.

4.2.1. *Selberg trace formula.* Let us try first to explain the Selberg Trace formula in our case (i.e. cusped hyperbolic three–manifolds) in some detail. Let M be a finite-volume, non-compact hyperbolic three–manifold; as in the compact case there is on M a kernel $K_t^p[M]$ the convolution with which is the heat operator on p-forms. The non-compactness of M implies that the function $\operatorname{tr} K_t^p[M]$ is not integrable on M, and the trace formula results from an asymptotic estimate of the integral of it on certain compact subsets which exhaust M.

More precisely, there exists a compact subset $M^1 \subset M$ which is a submanifold with smooth boundary consisting of h flat (with the induced metric) tori $T_1, \dots, T_h$ and such that $M - M^1$ is the disjoint union of $T_j \times [1, +\infty[$ with the metric $\frac{dx^2+dy_j^2}{y_j^2}$ where dx^2 is the flat metric on T_j and $y_j = \log d(\cdot, T_j)$ (d being the hyperbolic distance on M). Now putting $M^Y = \{x \in M \ \max_j y_j(x) \leq Y\}$ for $Y \in [1, +\infty[$, one can express the integral of $\operatorname{tr} K_t^p[M]$ on M^Y in two ways, using either the geometric expansion of the heat kernel or the spectral expansion; this yield two expansions as $Y \to \infty$ which are of the form $A \log Y + B + o(1), A' \log Y + B' + o(1)$ for some A, B, A', B' which we will describe presentely, and the trace formula is the equality $B = B'$.

4.2.2. *Geometric side.* Here we quickly explain [25, Proposition 3.4]. The heat kernel of M is written as

$$K_t^p[M](x,y) = \sum_{\gamma\in\Gamma} k_t^p(x, \gamma y);$$

the sum over $\gamma \in \Gamma$ can be separated into three summands: the one corresponding to $\gamma = 1$, the sum over elements in Γ with trace in $]2, +\infty[$ (usually called loxodromic elements, so we'll denote the set of them by Γ_{lox}) and the sum over nontrivial unipotents elements. The two first sums yield integrable functions on Γ: the integral of $\operatorname{tr} k_t^p(x,x)$ over M equals $\operatorname{Tr} e^{-t\Delta^p[\mathbb{H}^3]} \cdot \operatorname{vol} M$ (where x_0 is any point in $\mathbb{H}^3$). We have an inequality similar to (4.7) for the sum over loxodromics:

$$\sum_{\gamma\in\Gamma_{\text{lox}}} \operatorname{tr} k_t^p(x, \gamma x) \leq Cet^{-\frac{3}{2}} \ell(x)^{-3} e^{-\frac{\ell_x(M)^2}{10t}}$$

where C depends on Γ and is uniform for t in a compact set, and $\ell(x)$ is the smallest length of an essential closed curve going through x (in particular it is bounded below for $x \in M$). It follows that this summand is bounded on M and thus integrable; we will denote its integral by $G_t^p(x)$:

$$G_t^p(x) = \int_M \sum_{\gamma\in\Gamma_{\text{lox}}} \operatorname{tr} k_t^p(x, \gamma x) dx. \tag{4.9}$$

The sum over unipotent terms is not integrable over M, because the displacement of a unipotent element goes to 0 as one gets closer to its fixed point; however it is not hard to quantify the divergence: there is a smooth function h_t^p on $[0, +\infty)$ such that $\operatorname{tr} k_t^p(x, nx) = h_t^p(d(x, nx))$ for any $x \in \mathbb{H}^3$ and any unipotent $n \in \mathrm{SL}_2(\mathbb{C})$; moreover $d(x, nx)$ is given by $\ell(|n|)$ where $|n|$ is the Euclidean norm of n in an horosphere passing through x. Now if $\{\Lambda\}$ is the conjugacy class of a unipotent subgroup of Γ then we have when $\min_j Y_j \to +\infty$ the following asymptotic expansion

$$\int_{M^Y} \sum_{\gamma\in\{\Lambda\},\gamma\neq 1} \operatorname{tr} k_t^p(x, \gamma x) = \sum_{j=1}^h \log(Y_j) \int_0^{+\infty} r \log(r) h_t^p(\ell(r)) + \sum_{j=1}^h \kappa_j \operatorname{vol} \Lambda_j \int_0^{+\infty} r h_t^p(\ell(r)) dr + o(1)$$

where κ_j is a constant which depends only on the geometry of T_j. More precisely, we have

$$\kappa_j \operatorname{vol} \Lambda_j = \kappa'_j + \log \alpha_1(T_j)$$

where κ'_j depends only on the conformal structure of T_j, thus only on M and not on the choice of M^1, and $\alpha_1(T_j)$ is the (Euclidean) systole of T_j (and thus does depend on M^1). The geometric expression for the "trace" of the heat kernel is finally given by

$$\mathrm{Tr}_R K_t^p[M] = \mathrm{Tr}\, e^{-t\Delta^p[\mathbb{H}^3]} \cdot \mathrm{vol}\, M + \int_M G_t^p(x)dx + \sum_{j=1}^h \kappa_j \,\mathrm{vol}\, \Lambda_j \int_0^{+\infty} rh_t^p(\ell(r))dr \tag{4.10}$$

(the index R in Tr_R stands for "regularized" since this is not a bona fide trace).

4.2.3. *Spectral side.* The spectral decomposition for the space $L^2\Omega^p(M;E)$ can be written in its roughest form as

$$L^2\Omega^p(M;E) = L^2_{\mathrm{disc}}\Omega^p(M;E) \oplus L^2_{\mathrm{cont}}\Omega^p(M;E)$$

where $L^2_{\mathrm{disc}}\Omega^p(M;E)$ is the closure of the space generated by eigenforms of $\Delta^p[M]$ and $L^2_{\mathrm{cont}}\Omega^p(M;E)$ its orthogonal complement. If $\lambda_1 \leq \ldots \leq \lambda_k \leq \ldots$ are the positive eigenvalues of $\Delta^p[M]$ in $L^2_{\mathrm{disc}}\Omega^p(M;E)$ it is known that the sum $\sum_{k\geq 1} e^{-t\lambda_k}$ is convergent for $t > 0$. On the other hand there is an exact description of the continuous part in terms of the so-called Eisenstein series, and it yields an expansion

$$\int_{M^Y} \mathrm{tr}\, K_t^p[M](x,x)dx = T \cdot \sum_{j=1}^h \log Y_j + \dim\ker \Delta^p[M] + \sum_{k\geq 1} e^{-t\lambda_k} + S_t^p + o(1) \tag{4.11}$$

where S_t^p, T are computed in terms of the heat kernel and certain "intertwing" operators on V (see [25, (3.15)] for $p = 0$). Thus the spectral expression for the regularized trace is given by

$$\mathrm{Tr}_R\, e^{-t\Delta^p[M]} - \dim\ker \Delta^p[M] = \sum_{k\geq 1} e^{-t\lambda_k} + S_t^p. \tag{4.12}$$

4.2.4. *Regularized analytic torsion.* Using the geometric side of the trace formula one can study the asymptotics of $\mathrm{Tr}_R\, e^{-t\Delta^p[M]}$ as $t \to 0$, and this yields an expension similar to Minakshisundaram-Pleijel with additional terms in $t^{\frac{k}{2}} \log t$ for $k \geq -1$ [26, Proposition 5.4]; it allows to deduce that the zeta function defined as in (4.4) with traces repaced by regularized traces is a meromorphic function which is regular at 0, and one then defines regularized analytic torsion as in (4.1); see [26, 5.3.1] for details. The following result then replaces Theorem 4.1 in the non-compact case.

Theorem 4.6. *Let ρ be a strongly acyclic representation of $\mathrm{SL}_2(\mathbb{C})$, Γ a Bianchi group and Γ_n a sequence of pairwise distinct torsion-free congruence subgroups of Γ. Suppose in addition that $M_n = \Gamma_n\backslash\mathbb{H}^3$ is cusp-uniform. Then we have*

$$\lim_{n\to+\infty} \frac{\log T_R(M_n; E_\rho)}{\mathrm{vol}\, M_n} = t^{(2)}(V).$$

The proof goes as in the compact case, with some additional difficulties: for the small–time part of the proof one has to deal with the terms coming from unipotent elements on the geometric side; this part of the proof applies to any sequence of hyperbolic manifolds which is BS convergent to $\mathbb{H}^3$ and cusp-uniform (the cusp-uniformity allows to control the κ_j', see the proof of Theorem 4.5 in [25]). For the large-time part one has to control the term S_t^p: this is more technical and this is where we use the fact that the manifolds M_n are congruence (which allows to compute more or less explicitely the "intertwining operators" from which this term comes). The complete proof is given in *loc. cit*, Section 5.4.

4.3. **Asymptotic Cheeger–Müller and homology growth.** This is actually the hardest part in the proof of Theorem 3.7: there is currently no extension of the Cheeger–Müller theorem to the non-compact case. The result which is proven in these two papers is that one can define a "Reidemeister" torsion $\tau(M; L)$ such that asymptotic equality with analytic torsion holds: the statement of Theorem 5.1 in [26] is that we have

$$\lim_{n \to +\infty} \frac{\log T(M_n; E) - \log \tau(M_n; L)}{\operatorname{vol} M_n} = 0 \tag{4.13}$$

for a sequence of (manifold) congruence covers of a Bianchi orbifold. We will not detail the proof here: the main ingredients are an asymptotic equality of regularized torsion with an analytic torsion defined for the compact truncated manifolds $M_n^{Y^n}$ for an explicit sequence Y^n (Theorem 6.1 in [25]), a Cheeger–Müller equality for the latter proven in the generality we need by J. Brüning and X. Ma ([6], see [25, (6.3)]), and ideas originating in [7] (Proposition 5.4 in [26]) to conclude.

The torsion τ appearing in (4.13) is given by

$$\tau(M; V_{\mathbb{Z}}) = \frac{|H^1(M; L)_{\mathrm{tors}}|}{\operatorname{vol} H^1(M; L)_{\mathrm{free}}} \times \frac{\operatorname{vol} H^2(M; L)_{\mathrm{free}}}{|H^2(M; L)_{\mathrm{tors}}|} \tag{4.14}$$

where the covolume are taken for the metric on $H^*(M; E)$ coming from the embedding $i^* : H^*(M; E) \to \bigoplus_j H^*(T_j; L)$ (that this is a sensible choice is justified by (4.13)); the last part of the proof of Theorem 3.7 is to show that all terms but $|H^2(M; L)_{\mathrm{tors}}|$ are subexponential in the volume, and again we will not give any detail about the proof here, referring the reader to [26, Section 6] instead.

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