On maximal inequalities for stable stochastic integrals

Aldéric Joulin*
Laboratoire de Mathématiques et Applications
Université de La Rochelle
Avenue Michel Crépeau
17042 La Rochelle Cedex 1
France

May 9, 2006

Abstract

Sharp maximal inequalities in large and small range are derived for stable stochastic integrals. In order to control the tail of a stable process, we introduce a truncation level in the support of its Lévy measure: we show that the contribution of the compound Poisson stochastic integral is negligible as the truncation level is large, so that the study is reduced to establish maximal inequalities for the martingale part with a suitable choice of truncation level. The main problem addressed in this paper is to give upper bounds which remain bounded as the parameter of stability of the underlying stable process goes to 2. Applications to estimates of first passage times of symmetric stable processes above positive continuous curves complete this work.

Key words: Stable processes, stable stochastic integrals, maximal inequalities, first passage times.
Mathematics Subject Classification. 60E15, 60G52, 60H05.

1 Introduction

Given a filtered probability space $\mathbf{\Omega} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, consider on $\mathbf{\Omega}$ a càdlàg real stable process $Z = (Z_t)_{t\geq 0}$ of index $\alpha \in (0,2)$ without Gaussian component

*ajoulin@univ-lr.fr

and let $H = (H_t)_{t\geq 0}$ be a sufficiently integrable predictable càdlàg process. The purpose of this paper is to give maximal inequalities for stable stochastic integrals $H \cdot Z = (\int_0^t H_s dZ_s)_{t\geq 0}$. We show that their decay in the bilateral case is

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\left|\int_0^s H_\tau dZ_\tau\right| \geq x\right) \leq \frac{K}{\alpha x^\alpha}\,\|H\|^\alpha_{L^{\alpha+p}(\mathbf{\Omega}\times[0,t])}, \quad x \geq x_\alpha, \quad p > 2-\alpha, \tag{1.1}$$

whereas in the unilateral case, if Z is symmetric and $\alpha \in (1,2)$, it is

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\int_0^s H_\tau dZ_\tau \geq x\right) \leq L_\alpha \, \exp\left(-M_\alpha\left(\frac{x}{\|H\|_{L^\infty(\mathbf{\Omega},L^\alpha([0,t]))}}\right)^{\alpha/(\alpha-1)}\right), \quad x \leq \widetilde{x}_\alpha. \tag{1.2}$$

Here $L_\alpha, M_\alpha, x_\alpha$ and $\widetilde{x}_\alpha$ stand for positive numbers depending explicitly on α, whereas K is a positive constant independent of α.

It is known since the early 80's that stable stochastic integrals inherit regularly varying tails from the underlying stable process. For example, in order to prove the central limit theorem for stable stochastic integrals in the Skorohod space, Giné and Marcus established in [12] the maximal inequality

$$\sup_{x>0} x^\alpha\,\mathbb{P}\left(\sup_{0\leq t\leq 1}\left|\int_0^t H_s dZ_s\right| \geq x\right) \leq \frac{D}{\alpha(2-\alpha)^2}\,\|H\|^\alpha_{L^\alpha(\mathbf{\Omega}\times[0,t])}, \tag{1.3}$$

where D is a universal constant independent of α. However, as α tends to 2, the upper bound in their maximal inequality (1.3) goes to infinity. On the other hand, the extremal behavior of stochastic integrals driven by multivariate Lévy processes with regularly varying tails have been studied recently in [14] by Hult and Lindskog, and by Applebaum, see [3]. In particular, if Z is symmetric and H is square-integrable and satisfies further the uniform integrability condition $\mathbb{E}\left[\sup_{t\in[0,1]}|H_t|^{\alpha+p}\right] < +\infty$ for some $p > 0$, then Example 3.2 in [14] yields the extremal behavior

$$\lim_{x\to+\infty} x^\alpha\,\mathbb{P}\left(\sup_{0\leq s\leq t}\left|\int_0^s H_\tau dZ_\tau\right| \geq x\right) = C_\alpha\,\|H\|^\alpha_{L^\alpha(\mathbf{\Omega}\times[0,t])}, \quad t \in [0,1], \tag{1.4}$$

where C_α depends on α and remains bounded as $\alpha \in (0,2]$. Therefore, as α gets close to 2, the maximal inequality (1.3) of Giné and Marcus does not recover the non-explosive asymptotic estimate (1.4).

Our approach to establish maximal inequalities for stable stochastic integrals is based on stochastic calculus for jump processes and allows us to avoid the limiting explosion of the upper bound described above. Following Pruitt in [19] for Lévy processes and more recently Houdré and Marchal in [13] in the specific case of stable random vectors, the method relies on the use of the Lévy-Itô decomposition of Z with a truncation level R in the support of its Lévy measure, in order to control the jump size of the martingale part: Z is split into the sum of a square-integrable martingale with infinitely many jumps bounded by R on each compact time interval, and a compound Poisson process which represents the large jumps of Z, plus a drift part. Constructing then the stable stochastic integral $H \cdot Z$ with respect to the above semimartingale decomposition, we show that the contribution of the compound Poisson stochastic integral in both bilateral and unilateral cases is negligible as the truncation level is large, reducing the study to the proof of maximal inequalities for the martingale part of $H \cdot Z$. Using stochastic calculus for Poisson random measures, sharp estimates follow by choosing suitably the truncation level R.

Let us describe the content of the paper. In Section 2, some notation and basic properties of stable processes are introduced. Then we apply a truncation method somewhat similar to that of Pruitt to derive maximal inequalities for stable stochastic integrals, and compare them with the corresponding results of Giné and Marcus, and Hult and Lindskog, see [12] and [14]. In particular, Proposition 2.4 slightly improves the estimate in [12, Theorem 3.5] when the index of stability α of the underlying stable process lies in $(1, 2)$ and under some integrability conditions. The main contribution of this paper is contained in Section 3, Theorem 3.2, where large range inequalities are given in the bilateral case (1.1), freeing us from the explosion of the upper bound as α goes to 2. Section 4 is devoted to small range tail estimates in the unilateral case (1.2). As a result, we recover the classical maximal Gaussian inequality via Theorem 4.2 and a limiting procedure in the Skorohod space. Finally, we apply in Section 5 the results of Section 2 and 3 to estimate first passage times of a symmetric stable process above several positive continuous curves. The method relies on an extension to the stable case of the results of [1, 18] established for Brownian motions.

2 Notation and preliminaries

Let $\mathbf{\Omega} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and let Z be a càdlàg real stable process on $\mathbf{\Omega}$ of index $\alpha \in (0,2)$ without Gaussian component. For the sake of briefness, by a *stable process* we will implicitly mean an $(\mathscr{F}_t)_{t\geq 0}$*-adapted real càdlàg stable process* in the remainder of this paper. Recall that its characteristic function is defined by

$$\varphi_{Z_t}(u) = \exp t \left(iub + \int_{-\infty}^{+\infty} \left(e^{iuy} - 1 - iuy\, 1_{\{|y|\leq 1\}}\right) \nu(dy) \right), \tag{2.1}$$

where ν stands for the stable Lévy measure on $\mathbb{R}$:

$$\nu(dy) = \left(c_-\, 1_{\{y<0\}} + c_+\, 1_{\{y>0\}}\right) \frac{dy}{|y|^{\alpha+1}}, \quad c_-, c_+ \geq 0, \quad c_- + c_+ > 0. \tag{2.2}$$

As a Lévy process, Z is a semimartingale whose Lévy-Itô decomposition is given by

$$Z_t = bt + \int_0^t \int_{|y|\leq 1} y\,(\mu - \sigma)(dy, ds) + \int_0^t \int_{|y|>1} y\,\mu(dy, ds), \quad t \geq 0, \tag{2.3}$$

where μ is a Poisson random measure on $\mathbb{R} \times [0, +\infty)$ with intensity $\sigma(dy, dt) = \nu(dy) \otimes dt$ and b is the drift. In particular, if $\alpha < 1$, then Z is a finite variation process whereas when $\alpha \geq 1$, we have a.s.

$$\sum_{s\leq t} |\Delta Z_s| = +\infty, \quad t > 0,$$

where ΔZ_s denotes the jump size of Z at time $s > 0$.

Z is said to be strictly stable if we have the self-similarity property

$$(Z_{kt})_{t\geq 0} \overset{(d)}{=} (k^{\frac{1}{\alpha}} Z_t)_{t\geq 0},$$

where $k > 0$ and the equality $\overset{(d)}{=}$ is in the sense of finite dimensional distributions. If moreover $c := c_+ = c_-$, then Z is symmetric and its characteristic function (2.1) is computed to be

$$\varphi_{Z_t}(u) = e^{-t\rho_\alpha |u|^\alpha}, \tag{2.4}$$

where

$$\rho_\alpha := \frac{\sqrt{\pi}\Gamma((2-\alpha)/2)}{\alpha 2^\alpha \Gamma((1+\alpha)/2)} 2c.$$

2.1 The truncation method

In order to control the jump size of the martingale part of the stable stochastic integral, let us introduce the truncation method of the stable Lévy measure (2.2). For some truncation level $R > 1$, let $Z^{(R+)}$ and $Z^{(R-)}$ be the independent Lévy processes defined by

$$Z_t^{(R-)} := \int_0^t \int_{|y|\leq R} y\,(\mu - \sigma)(dy, ds), \quad Z_t^{(R+)} := \int_0^t \int_{|y|>R} y\,\mu(dy, ds), \quad t \geq 0.$$

The first one has a compactly supported Lévy measure and is a square-integrable martingale with infinitely many jumps bounded by R on each compact time interval, whereas the second one is a compound Poisson process. The Lévy-Itô decomposition (2.3) rewrites as

$$Z_t = b_R t + Z_t^{(R-)} + Z_t^{(R+)}, \quad t \geq 0, \tag{2.5}$$

where $b_R := b + \int_{1<|y|\leq R} y\,\nu(dy)$ is a drift depending on R.
Given a predictable càdlàg process H, let

$$\|H\|_{(p,t)} := \|H\|_{L^p(\mathbf{\Omega}\times[0,t])} = \left(\int_0^t \mathbb{E}\left[|H_s|^p\right] ds\right)^{\frac{1}{p}}, \quad t \geq 0, \quad p > 0,$$

and define $\mathscr{P}_p$ (resp. $\mathscr{B}_p$) as the space of predictable càdlàg process H such that for all $t \geq 0$, $\|H\|_{(p,t)} < +\infty$ (resp. $\|H\|_{L^\infty(\mathbf{\Omega},L^p([0,t]))} < +\infty$). In particular, H is said integrable if $H \in \mathscr{P}_1$ and square-integrable if $H \in \mathscr{P}_2$.
Following [2, Chapter 4], we construct the stable stochastic integral of a square-integrable predictable process H as the sum of L^2-type and Lebesgue-Stieltjes stochastic integrals: letting

$$X_t^{(R-)} := \int_0^t H_s dZ_s^{(R-)}, \quad X_t^{(R+)} := \int_0^t H_s dZ_s^{(R+)}, \quad A_t^R := b_R \int_0^t H_s ds, \quad t \geq 0,$$

the first integral $X^{(R-)} = H \cdot Z^{(R-)}$ is a square-integrable martingale, whereas the integrals $X^{(R+)} = H \cdot Z^{(R+)}$ and A^R are constructed in the Lebesgue-Stieltjes sense, and we define the stable stochastic integral as

$$X_t := \int_0^t H_s dZ_s = A_t^R + X_t^{(R-)} + X_t^{(R+)}, \quad t \geq 0. \tag{2.6}$$

We denote respectively by $a \vee b$ and $a \wedge b$ the maximum and the minimum between two real numbers a and b.
We finish by making two remarks on the maximal inequalities of type (1.1) or (1.2) we will establish in the remainder of this paper:

Remark 2.1 The truncation level R is related to the deviation level x and to some L^p-norm of the process H, and is chosen each time equal to its optimal value.

Remark 2.2 Although they can be computed, the constants appearing in the upper bounds are not given explicitly in general, since their numerical value is not of crucial importance in our study.

2.2 A first maximal inequality

In order to study the rates of growth of Lévy processes, Pruitt established in [19] some maximal inequalities whose proofs are based on a truncation method for general Lévy measures, with a particular choice of truncation level.
Inspired by this work, we derive in this part a first maximal inequality for stable stochastic integrals by using the semimartingale decomposition (2.6).
Fix $t \geq 0$ and $x > \|H\|_{(2,t)}$. Using the above notation, we have by (2.6):

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq s\leq t}|X_s| \geq x\right) \\
&\quad \leq \mathbb{P}\left(\sup_{0\leq s\leq t}\left|A_s^R\right| + \sup_{0\leq s\leq t}\left|X_s^{(R-)}\right| + \sup_{0\leq s\leq t}\left|X_s^{(R+)}\right| \geq x\right) \\
&\quad \leq \mathbb{P}\left(\sup_{0\leq s\leq t}\left|A_s^R\right| \geq \frac{x}{2}\right) + \mathbb{P}\left(\sup_{0\leq s\leq t}|X_s^{(R-)}| \geq \frac{x}{2}\right) + \mathbb{P}\left(\sup_{0\leq s\leq t}|X_s^{(R+)}| > 0\right).
\end{aligned} \tag{2.7}$$

First, we investigate the absolutely continuous part A^R. By Chebychev's inequality,

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leq s\leq t}\left|A_s^R\right| \geq \frac{x}{2}\right) &\leq \mathbb{P}\left(\int_0^t |H_\tau|\, d\tau \geq \frac{x}{2|b_R|}\right) \\
&\leq \frac{4{b_R}^2}{x^2}\,\mathbb{E}\left[\left(\int_0^t |H_\tau|\, d\tau\right)^2\right].
\end{aligned}$$

Using the elementary inequality $(a+b)^2 \le 2\left(a^2+b^2\right)$, $a, b \in \mathbb{R}$, and then Cauchy-Schwarz' inequality,

$$\begin{aligned}
{b_R}^2 &= \left(b + \int_{1<|y|\le R} y\,\nu(dy)\right)^2 \\
&\le 2b^2 + 2\left(\int_{1<|y|\le R} y\,\nu(dy)\right)^2 \\
&\le 2b^2 + 2\nu\left(\{y\in\mathbb{R} : 1<|y|\le R\}\right)\int_{1<|y|\le R} y^2\,\nu(dy) \\
&\le 2b^2 + 2\nu\left(\{y\in\mathbb{R} : |y|>1\}\right)\int_{|y|\le R} y^2\,\nu(dy) \\
&= 2\left(b^2 + \frac{(c_- + c_+)^2}{\alpha(2-\alpha)}R^{2-\alpha}\right).
\end{aligned}$$

By Cauchy-Schwarz' inequality again and since $x > \|H\|_{(2,t)}$, we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le s\le t}\left|A_s^R\right| \ge \frac{x}{2}\right) &\le \frac{8t}{x^2}\left(b^2 + \frac{(c_- + c_+)^2}{\alpha(2-\alpha)}R^{2-\alpha}\right)\|H\|^2_{(2,t)} \\
&< \frac{8tb^2\|H\|^{\alpha}_{(2,t)}}{x^{\alpha}} + \frac{8t(c_- + c_+)^2R^{2-\alpha}\|H\|^2_{(2,t)}}{\alpha(2-\alpha)x^2}. \qquad (2.8)
\end{aligned}$$

Now, we show that the contribution of the compound Poisson stochastic integral $X^{(R+)}$ is negligible as the truncation level R is sufficiently large. Recall that the integral $X^{(R+)}$, and so its supremum process $(\sup_{0\le s\le t}|X_s^{(R+)}|)_{t\ge 0}$, has piecewise constant sample paths and its distribution at any time has an atom at 0. Now, denote by T_1^R the first jump time of the Poisson process $\left(\mu\left(\{y\in\mathbb{R} : |y|>R\}\times[0,t]\right)\right)_{t\ge 0}$ on the set $\{y\in\mathbb{R} : |y|>R\}$. If a.s. T_1^R occurs after time t, then the compound Poisson stochastic integral $X^{(R+)}$ (and so its supremum process) is identically 0 on the interval $[0,t]$. Thus we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le s\le t}|X_s^{(R+)}| > 0\right) &= 1 - \mathbb{P}\left(\sup_{0\le s\le t}|X_s^{(R+)}| = 0\right) \\
&\le 1 - \mathbb{P}\left(T_1^R > t\right) \\
&= 1 - \exp\left(-t\nu\left(\{y\in\mathbb{R} : |y|>R\}\right)\right) \\
&\le t\nu\left(\{y\in\mathbb{R} : |y|>R\}\right) \\
&= \frac{(c_- + c_+)t}{\alpha R^{\alpha}}, \qquad (2.9)
\end{aligned}$$

where we used in the second equality above that T_1^R is exponentially distributed with parameter $\nu(\{y \in \mathbb{R} : |y| > R\})$, see e.g. [22, Theorem 21.3].
Recall now that $X^{(R-)}$ is a square-integrable martingale involving the small jumps of Z. By Doob's inequality together with the isometry formula for Poisson stochastic integrals,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le s\le t}|X_s^{(R-)}| \geq \frac{x}{2}\right) &\leq \frac{4}{x^2}\,\mathbb{E}\left[\left|\int_0^t\int_{|y|\le R} H_\tau y\,(\mu-\sigma)(dy,d\tau)\right|^2\right] \\ &= \frac{4}{x^2}\,\mathbb{E}\left[\int_0^t\int_{|y|\le R} H_\tau^2 y^2\,\nu(dy)d\tau\right] \\ &= \frac{4}{x^2}\int_{|y|\le R} y^2\,\nu(dy)\int_0^t \mathbb{E}\left[H_\tau^2\right]d\tau,\end{aligned}$$

that is to say

$$\mathbb{P}\left(\sup_{0\le s\le t}|X_s^{(R-)}| \geq \frac{x}{2}\right) \leq \frac{4(c_- + c_+)\|H\|_{(2,t)}^2 R^{2-\alpha}}{(2-\alpha)x^2}. \tag{2.10}$$

Finally, using (2.7) and choosing the truncation level

$$R = \frac{x}{\|H\|_{(2,t)}} > 1$$

in (2.8), (2.9) and (2.10) show that there exists $K := K(b, c_-, c_+, t) > 0$, independent of α, such that

$$\mathbb{P}\left(\sup_{0\le s\le t}\left|\int_0^s H_\tau dZ_\tau\right| \geq x\right) \leq \frac{K\|H\|_{(2,t)}^\alpha}{\alpha(2-\alpha)x^\alpha}, \quad x > \|H\|_{(2,t)}. \tag{2.11}$$

Let us comment the estimate (2.11).
If Z is symmetric with Lévy measure $\nu(dy) = c|y|^{-\alpha-1}dy$, $c > 0$, and H satisfies further the uniform integrability condition $\mathbb{E}\left[\sup_{0\le t\le 1}|H_t|^{\alpha+p}\right] < +\infty$, $p > 0$, then Example 3.2 in [14] entails the asymptotic estimate

$$\lim_{x\to+\infty} x^\alpha\,\mathbb{P}\left(\sup_{0\le s\le t}\left|\int_0^s H_\tau dZ_\tau\right| \geq x\right) = \frac{K_{\alpha,c}}{\alpha}\int_0^t |H_\tau|^\alpha\,d\tau, \quad t \in [0,1], \tag{2.12}$$

where

$$K_{\alpha,c} := \begin{cases} \frac{2c\sqrt{\pi}(1-\alpha)\Gamma((2-\alpha)/2)}{2^{\alpha+1}\Gamma(2-\alpha)\Gamma((1+\alpha)/2)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1, \\ c & \text{if } \alpha = 1, \end{cases}$$

which remains bounded as $\alpha \in [0, 2]$. It shows that (2.11) is sharp for $\alpha \in (0, 2)$ and also as α converges to 0, but goes to infinity as α tends to 2, in contrast to (2.12). On the other hand, assuming $H \in \mathscr{P}_\alpha$, then a combination of Theorem 3.5 and Example 3.7 in [12] implies the maximal inequality

$$\sup_{x>0} x^\alpha \, \mathbb{P}\left(\sup_{0\le t\le 1} \left| \int_0^t H_s dZ_s \right| \ge x \right) \le \frac{D}{\alpha(2-\alpha)^2} \int_0^1 \mathbb{E}\left[|H_t|^\alpha \right] dt, \tag{2.13}$$

where D is a universal constant independent of α. Therefore, the speed of explosion in (2.11) is better than in (2.13) since it is linear in α and not quadratic, but it is worse in terms of L^p-norm of H, since the L^2-norm is involved instead of the optimal L^α-norm. Before avoiding in Section 3 the explosion of its upper bound as α gets close to 2, let us now improve (2.11) in terms of L^p-norm of H.

2.3 A maximal inequality in optimal L^α-norm

First, we quote [4, Proposition 2.1], up to a minor modification related to the integrability property of H:

Lemma 2.3 *Consider a stable stochastic integral $X := H \cdot Z$, where Z is a symmetric stable process of index $\alpha \in (1, 2)$ with generator $\mathscr{L}$, and H is square-integrable. Let f be a $C^2(\mathbb{R})$-function with bounded first and second derivatives. Then the process M^f given by*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t |H_s|^\alpha \mathscr{L} f(X_{s-}) ds, \quad t \ge 0,$$

is a martingale.

Now, we improve the upper bound in (2.11) in terms of L^p-norm of H. Actually, the estimate in Proposition 2.4 below recovers via a different proof the inequality (2.13) of Giné and Marcus, and slightly improves it as α tends to 2, since the speed of the explosion of the upper bound is not quadratic but only linear in α:

Proposition 2.4 *Let Z be a symmetric stable process of index $\alpha \in (1, 2)$ and Lévy measure $\nu(dz) = c|z|^{-\alpha-1}dz$, $c > 0$, and let H be square-integrable. Then there exists $K_{\alpha,c} > 0$, finite as α tends to 2, such that*

$$\sup_{x>0} x^\alpha \, \mathbb{P}\left(\sup_{0\le t\le 1} \left| \int_0^t H_s dZ_s \right| \ge x \right) \le \frac{K_{\alpha,c}}{2-\alpha} \int_0^1 \mathbb{E}\left[|H_t|^\alpha \right] dt.$$

Proof. The present proof is an adaptation to the case of stable stochastic integrals of that of Bass in [5, Proposition 3.1]. Denote by $\mathscr{L}$ the infinitesimal generator of Z. Let f be a non-negative $C^2(\mathbb{R})$-function such that $f(0) = 0$, $f(y) = 1$ if $|y| \geq 1$ and whose first and second derivatives are bounded above in absolute value respectively by $c_1 > 0$ and $c_2 > 0$. Let $x > 0$, $f_x(y) := f(y/x)$ and let

$$\tau_x := \inf\{t \geq 0 : |X_t| \geq x\}$$

be the first exit time of the stable stochastic integral $X = H \cdot Z$ of the centered ball of radius x. If the process exits the ball before time 1, then $f_x(X_{1\wedge\tau_x}) = 1$ and by Lemma 2.3 and a conditioning argument,

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leq t\leq 1} |X_t| \geq x\right) &= \mathbb{P}(\tau_x \leq 1) \\
&\leq \mathbb{E}\left[f_x(X_{1\wedge\tau_x})\right] \\
&= \mathbb{E}\left[\int_0^{1\wedge\tau_x} |H_t|^\alpha \mathscr{L} f_x(X_{t-})dt\right] \\
&\leq \int_0^1 \mathbb{E}\left[|H_t|^\alpha \left|\mathscr{L} f_x(X_{t-})\right|\right] dt.
\end{aligned}$$

Therefore,

$$\mathbb{P}\left(\sup_{0\leq t\leq 1} |X_t| \geq x\right) \leq \|\mathscr{L} f_x\|_{L^\infty(\mathbb{R})} \int_0^1 \mathbb{E}\left[|H_t|^\alpha\right] dt. \tag{2.14}$$

By the symmetry of ν,

$$\begin{aligned}
\mathscr{L} f_x(y) &= \int_{\mathbb{R}} (f_x(y+z) - f_x(y) - z f_x'(y))\,\nu(dz) \\
&\leq \int_{|z|\leq R} \frac{c_2 z^2}{2x^2}\nu(dz) + \int_{|z|>R} \frac{2c_1|z|}{x}\nu(dz) \\
&= \frac{c_2 c R^{2-\alpha}}{(2-\alpha)x^2} + \frac{4c_1 c R^{1-\alpha}}{(\alpha-1)x}.
\end{aligned}$$

If we choose the truncation level $R = x$, then denoting $K_{\alpha,c} := c_2 c + 4c_1 c(2-\alpha)/(\alpha-1)$, the calculus above implies the bound

$$\|\mathscr{L} f_x\|_{L^\infty(\mathbb{R})} \leq \frac{K_{\alpha,c}}{(2-\alpha)x^\alpha}.$$

Finally, plugging this into (2.14), the proof is complete. ∎

3 Large range estimates for α close to 2

The purpose of the present part is to control the upper bound in (2.11), freeing us from its explosion as α tends to 2. The price to pay is to require stronger integrability conditions on the process H and to reduce the range interval of the deviation level x. First, we recall Bihari's inequality, which is a Gronwall-type inequality. See e.g. [11, Chapter 1] for a proof of such an inequality.

Lemma 3.1 *Let T be a positive time horizon and let ρ, ψ and g be positive measurable functions such that ρ is monotone-increasing, $s \mapsto \psi(s)\rho(g(s))$ is integrable on $[0,T]$ and*

$$g(s) \leq K_T + \int_0^s \psi(\tau)\, \rho(g(\tau))\, d\tau, \quad s \in [0,T], \tag{3.1}$$

where $K_T \geq 0$. Then the Bihari inequality

$$g(T) \leq \phi^{-1}\left(\phi(K_T) + \int_0^T \psi(s) ds\right)$$

holds, where $\phi(x) := \int_0^x \frac{dy}{\rho(y)}$.

We can now state the main result of this paper:

Theorem 3.2 *Let Z be a stable process of index $\alpha \in (1,2)$ and Lévy measure ν given by (2.2). Let $p > 2-\alpha$, $\epsilon > 0$ and let $H \in \mathscr{P}_{\alpha+p}$. Then for all $t \geq 0$, there exists $K := K(b, c_-, c_+, t, p, \epsilon) > 0$, independent of α, such that for all*

$$x^\alpha \quad > \quad \|H\|^\alpha_{(\alpha+p,t)} \max\left\{1, \left(\frac{(2p)^{\frac{2}{\alpha+p}}}{\epsilon(2-\alpha)(\alpha+p)^{\frac{2}{\alpha+p}}}\right)^{\frac{\alpha+p}{\alpha+p-2}} \left(2^{\frac{\alpha+p-4}{2}} \vee 1\right)(c_- + c_+)t\right\},$$

we have the maximal inequality

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\left|\int_0^s H_\tau dZ_\tau\right| \geq x\right) \leq \frac{K\|H\|^\alpha_{(\alpha+p,t)}}{x^\alpha}. \tag{3.2}$$

Proof. We proceed as in the proof of inequality (2.11) and investigate first the absolutely continuous part A^R analogously to (2.8). Fix $t \geq 0$ and $x > \|H\|_{(\alpha+p,t)}$. By the elementary inequality $(a+b)^q \leq 2^{q-1}(|a|^q + |b|^q)$, $a, b \in \mathbb{R}$, $q \geq 1$, applied with $q = \alpha + p$, together with Hölder's inequality, we get

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\left|A^R_s\right| \geq \frac{x}{2}\right)$$

$$
\begin{aligned}
&\le \frac{2^{2\alpha+2p-1}t^{\alpha+p-1}\|H\|^{\alpha+p}_{(\alpha+p,t)}}{x^{\alpha+p}}\left(|b|^{\alpha+p}+\nu\left(\{y\in\mathbb{R}:|y|>1\}\right)^{\alpha+p-1}\int_{|y|\le R} y^{\alpha+p}\nu(dy)\right)\\
&= \frac{2^{2\alpha+2p-1}t^{\alpha+p-1}\|H\|^{\alpha+p}_{(\alpha+p,t)}}{x^{\alpha+p}}\left(|b|^{\alpha+p}+\frac{(c_-+c_+)^{\alpha+p}R^p}{p\,\alpha^{\alpha+p-1}}\right)\\
&\le \frac{2^{2\alpha+2p-1}t^{\alpha+p-1}\|H\|^{\alpha}_{(\alpha+p,t)}}{x^{\alpha}}\left(|b|^{\alpha+p}+\frac{(c_-+c_+)^{\alpha+p}R^p\|H\|^{p}_{(\alpha+p,t)}}{p\,\alpha^{\alpha+p-1}x^p}\right), \qquad (3.3)
\end{aligned}
$$

where we used in the last inequality $x>\|H\|_{(\alpha+p,t)}$.
Now, let us control the martingale part $X^{(R-)}=H\cdot Z^{(R-)}$. By Doob's and Burkholder's inequalities for martingales with jumps, see e.g. pp. 303-4 in [10], we have

$$
\begin{aligned}
\mathbb{P}\left(\sup_{0\le s\le t}|X^{(R-)}_s|\ge\frac{x}{2}\right) &\le \frac{2^{\alpha+p}}{x^{\alpha+p}}\,\mathbb{E}\left[\left|\int_0^t H_s dZ^{(R-)}_s\right|^{\alpha+p}\right]\\
&\le \frac{2^{\alpha+p}C_{\alpha+p}}{x^{\alpha+p}}\,\mathbb{E}\left[\left[\int_0^\cdot H_s dZ^{(R-)}_s,\int_0^\cdot H_s dZ^{(R-)}_s\right]_t^{\frac{\alpha+p}{2}}\right]\\
&= \frac{2^{\alpha+p}C_{\alpha+p}}{x^{\alpha+p}}\,\mathbb{E}\left[\left(\int_0^t\int_{|y|\le R} H_s^2\,y^2\,\mu(dy,ds)\right)^{\frac{\alpha+p}{2}}\right]. \qquad (3.4)
\end{aligned}
$$

Let $(Y_s)_{s\in[0,t]}$ be the finite variation process defined by

$$
Y_s := \int_0^s\int_{|y|\le R} H_\tau^2\,y^2\,\mu(dy,d\tau),\quad 0\le s\le t.
$$

By Itô's formula for jump processes and the inequality $(a+b)^q-a^q\le qb(a+b)^{q-1}$, $0\le a\le b$, $q\ge 1$, applied with $q=(\alpha+p)/2$, we have

$$
\begin{aligned}
Y_s^{\frac{\alpha+p}{2}} &= \int_0^s\int_{|y|\le R}\left(\left(Y_{\tau-}+H_\tau^2\,y^2\right)^{\frac{\alpha+p}{2}}-Y_{\tau-}^{\frac{\alpha+p}{2}}\right)\mu(dy,d\tau)\\
&\le \frac{\alpha+p}{2}\int_0^s\int_{|y|\le R} H_\tau^2\,y^2\left(Y_{\tau-}+H_\tau^2\,y^2\right)^{\frac{\alpha+p-2}{2}}\mu(dy,d\tau)\\
&\le \frac{\alpha+p}{2}\,(2^{\frac{\alpha+p-4}{2}}\vee 1)\int_0^s\int_{|y|\le R} H_\tau^2\,y^2\left(Y_{\tau-}^{\frac{\alpha+p-2}{2}}+|H_\tau|^{\alpha+p-2}\,|y|^{\alpha+p-2}\right)\mu(dy,d\tau).
\end{aligned}
$$

where we used in the last inequality the elementary bound $(a+b)^q\le(2^{q-1}\vee 1)\,(a^q+b^q)$, $a,b\ge 0$, $q\ge 0$, applied with $q=(\alpha+p-2)/2$. Denote $D_{\alpha,p}=2^{\frac{\alpha+p-4}{2}}\vee 1$. Taking expectations and using Hölder's inequality, we get

$$
\mathbb{E}\left[Y_s^{\frac{\alpha+p}{2}}\right]
$$

$$\le D_{\alpha,p}\frac{(\alpha+p)(c_-+c_+)}{2}\left(\frac{R^{2-\alpha}}{2-\alpha}\int_0^s \mathbb{E}\left[H_\tau^2 Y_\tau^{\frac{\alpha+p-2}{2}}\right]d\tau+\frac{R^p}{p}\|H\|_{(\alpha+p,s)}^{\alpha+p}\right)$$
$$\le D_{\alpha,p}\frac{(\alpha+p)(c_-+c_+)}{2}\left(\frac{R^{2-\alpha}}{2-\alpha}\int_0^s \mathbb{E}\left[|H_\tau|^{\alpha+p}\right]^{\frac{2}{\alpha+p}}\mathbb{E}\left[Y_\tau^{\frac{\alpha+p}{2}}\right]^{\frac{\alpha+p-2}{\alpha+p}}d\tau+\frac{R^p}{p}\|H\|_{(\alpha+p,t)}^{\alpha+p}\right).$$

Applying Lemma 3.1 with $T=t$,

$$g(s):=\mathbb{E}\left[Y_s^{\frac{\alpha+p}{2}}\right],\quad \psi(\tau):=\mathbb{E}\left[|H_\tau|^{\alpha+p}\right]^{\frac{2}{\alpha+p}},\quad K_t:=D_{\alpha,p}\frac{(\alpha+p)(c_-+c_+)R^p}{2p}\|H\|_{(\alpha+p,t)}^{\alpha+p}$$

and

$$\rho(x):=D_{\alpha,p}\frac{(\alpha+p)(c_-+c_+)R^{2-\alpha}}{2(2-\alpha)}x^{\frac{\alpha+p-2}{\alpha+p}},$$

and by using Hölder's inequality to estimate $\int_0^t\psi(\tau)d\tau$, we obtain

$$g(t)\le\Phi^{-1}\left(\Phi\left(D_{\alpha,p}\frac{(\alpha+p)(c_-+c_+)R^p}{2p}\|H\|_{(\alpha+p,t)}^{\alpha+p}\right)+t^{\frac{\alpha+p-2}{\alpha+p}}\|H\|_{(\alpha+p,t)}^2\right),$$

where

$$\begin{aligned}\Phi(x)&:=\int_0^x\frac{dy}{\rho(y)}\\&=\frac{2-\alpha}{D_{\alpha,p}(c_-+c_+)R^{2-\alpha}}x^{\frac{2}{\alpha+p}}.\end{aligned}$$

Hence we have

$$\mathbb{E}\left[Y_t^{\frac{\alpha+p}{2}}\right]\le$$
$$\frac{D_{\alpha,p}^{\frac{\alpha+p}{2}}(c_-+c_+)^{\frac{\alpha+p}{2}}R^{\frac{(2-\alpha)(\alpha+p)}{2}}}{(2-\alpha)^{\frac{\alpha+p}{2}}}\left(\frac{(2-\alpha)(\alpha+p)^{\frac{2}{\alpha+p}}R^{\frac{\alpha(\alpha+p-2)}{\alpha+p}}}{D_{\alpha,p}^{\frac{\alpha+p-2}{\alpha+p}}(c_-+c_+)^{\frac{\alpha+p-2}{\alpha+p}}(2p)^{\frac{2}{\alpha+p}}}+t^{\frac{\alpha+p-2}{\alpha+p}}\right)^{\frac{\alpha+p}{2}}\|H\|_{(\alpha+p,t)}^{\alpha+p}.$$

Now, choose the truncation level

$$R=\frac{x}{\|H\|_{(\alpha+p,t)}}>1.$$

Since the assumption on x claims that

$$x^{\frac{\alpha(\alpha+p-2)}{\alpha+p}}>\frac{\|H\|_{(\alpha+p,t)}^{\frac{\alpha(\alpha+p-2)}{\alpha+p}}t^{\frac{\alpha+p-2}{\alpha+p}}D_{\alpha,p}^{\frac{\alpha+p-2}{\alpha+p}}(c_-+c_+)^{\frac{\alpha+p-2}{\alpha+p}}(2p)^{\frac{2}{\alpha+p}}}{\epsilon(2-\alpha)(\alpha+p)^{\frac{2}{\alpha+p}}},$$

we establish the following bound on moments

$$\mathbb{E}\left[Y_t^{\frac{\alpha+p}{2}}\right] \leq \frac{D_{\alpha,p}(c_-+c_+)(\alpha+p)(1+\epsilon)^{\frac{\alpha+p}{2}}\|H\|^\alpha_{(\alpha+p,t)}}{2p}\, x^p.$$

Finally, plugging the latter inequality into (3.4) yields

$$\mathbb{P}\left(\sup_{0\leq s\leq t}|X_s^{(R-)}| \geq \frac{x}{2}\right) \leq \frac{2^{\alpha+p}C_{\alpha+p}D_{\alpha,p}(c_-+c_+)(\alpha+p)(1+\epsilon)^{\frac{\alpha+p}{2}}\|H\|^\alpha_{(\alpha+p,t)}}{2px^\alpha},$$

and together with (2.7) and the choice of truncation level $R = x/\|H\|_{(\alpha+p,t)}$ in (2.9) and (3.3), Theorem 3.2 is proved. ∎

Under further assumptions on Z and H, the process $H \cdot Z$ is a time-changed stable process and we get the following maximal inequality, which is asymptotically optimal in terms of L^α-norm when $\|H\|_{L^\alpha([0,t])}$ is bounded on $\mathbf{\Omega}$ for all $t \geq 0$:

Corollary 3.3 *Let Z be a symmetric stable process of index $\alpha \in (1,2)$ and Lévy measure $\nu(dy) = c|y|^{-\alpha-1}dy$, $c > 0$. Let $H \in \mathscr{B}_\alpha$ with a.s. $\lim_{t\to+\infty}\int_0^t |H_s|^\alpha ds = +\infty$. Let $p > 2-\alpha$ and $\epsilon > 0$. Then there exists $K := K(c,p,\epsilon) > 0$, independent of α, such that for all $t \geq 0$ and for all*

$$x^\alpha > \left\|\int_0^t |H_s|^\alpha\, ds\right\|_{L^\infty(\Omega)} \left(\frac{2p^{\frac{2}{\alpha+p}}}{\epsilon(2-\alpha)(\alpha+p)^{\frac{2}{\alpha+p}}}\right)^{\frac{\alpha+p}{\alpha+p-2}} \left(2^{\frac{\alpha+p-4}{2}} \vee 1\right) c,$$

we have the estimate

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\left|\int_0^s H_\tau dZ_\tau\right| \geq x\right) \leq \frac{K}{x^\alpha}\left\|\int_0^t |H_s|^\alpha\, ds\right\|_{L^\infty(\Omega)}. \tag{3.5}$$

Proof. By [21, Theorem 3.1], the process $H \cdot Z$ is a time-changed process of Z, i.e. we have the identity a.s.

$$\int_0^t H_s dZ_s = \hat{Z}_{\tau_t}, \quad t \geq 0,$$

where $\tau = (\tau_t)_{t\geq 0}$ given by $\tau_t := \int_0^t |H_s|^\alpha ds$ is a time change process, and $\hat{Z}$ is a symmetric stable process defined on $\mathbf{\Omega}$ and having the same distribution as Z. Since the symmetry of $\hat{Z}$ implies it is self-similar of index α, then so is the supremum process:

$$\left(\sup_{0\leq s\leq kt}\hat{Z}_s\right)_{t\geq 0} \stackrel{(d)}{=} \left(k^{\frac{1}{\alpha}}\sup_{0\leq s\leq t}\hat{Z}_s\right)_{t\geq 0}, \quad k > 0.$$

Thus, denoting $\beta(t) := \|\tau_t\|_{L^\infty(\Omega)}^{1/\alpha}$, we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le s\le t}\left|\int_0^s H_\tau dZ_\tau\right| \ge x\right) &= \mathbb{P}\left(\sup_{0\le s\le t}|\hat{Z}_{\tau_s}| \ge x\right) \\
&\le \mathbb{P}\left(\sup_{0\le s\le \tau_t}|\hat{Z}_s| \ge x\right) \\
&\le \mathbb{P}\left(\sup_{0\le s\le \beta(t)^\alpha}|\hat{Z}_s| \ge x\right) \\
&= \mathbb{P}\left(\sup_{0\le s\le 1}|\hat{Z}_s| \ge \frac{x}{\beta(t)}\right).
\end{aligned}$$

Finally, applying Theorem 3.2 with $H_s = 1$ for all $0 \le s \le t = 1$, the proof is complete. ∎

Remark 3.4 If Z is a non-symmetric strictly stable process and H is positive and satisfies further the hypothesis of Corollary 3.3 (resp. that of Theorem 4.2 below), then the stable stochastic integral $H \cdot Z$ is still a time-changed process of Z. Thus, applying in the proof above (resp. in the proof of Theorem 4.2) Theorem 3 in [17] instead of Theorem 3.1 in [21], an estimate somewhat similar to that of Corollary 3.3 (resp. Theorem 4.2) can be established.

4 Small range maximal inequalities

In this part, we derive small range estimates in the unilateral case (1.2). Recently, Breton and Houdré investigated in [8] small and intermediate range concentration for stable random vectors. In particular, the small range behavior is covered by their Theorem 1, whose small deviation rate is of order $\exp\left(-c_\alpha x^{\alpha/(\alpha-1)}\right)$ for some positive c_α depending on α. Before proving a similar rate for suprema of stable stochastic integrals, let us establish first the result for symmetric stable processes via Proposition 4.1 below. We point out that using the scaling property, it is sufficient to get the result on the time interval $[0, 1]$.

Proposition 4.1 *Let Z be a symmetric stable process of index $\alpha \in (1, 2)$ and Lévy measure $\nu(dy) = c|y|^{-\alpha-1}dy$, $c > 0$. Then for all $\lambda > \lambda_0(\alpha)$, where $\lambda_0(\alpha)$ is the unique* *solution of the equation*

$$\lambda \log\left(1+\frac{(2-\alpha)\lambda}{2c}\right)=\frac{4c}{\alpha},$$

there exists $x_0(\alpha,\lambda)>0$ *such that for all* $0\le x\le x_0(\alpha,\lambda)$,

$$\mathbb{P}\left(\sup_{0\le t\le 1} Z_s \ge x\right) \le \frac{2c}{\alpha}\left(\frac{x}{\lambda}\right)^{\frac{\alpha}{\alpha-1}} + \exp\left(-\frac{\lambda\log\left(1+\frac{(2-\alpha)\lambda}{2c}\right)}{2}\left(\frac{x}{\lambda}\right)^{\frac{\alpha}{\alpha-1}}\right). \qquad (4.1)$$

Proof. As in the proof of inequality (2.9), we have

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le 1} Z_s \ge x\right) &\le \mathbb{P}\left(\sup_{0\le t\le 1} Z_s^{(R+)} > 0\right) + \mathbb{P}\left(\sup_{0\le t\le 1} Z_s^{(R-)} \ge x\right)\\ &\le \frac{2c}{\alpha R^\alpha} + \mathbb{P}\left(\sup_{0\le t\le 1} Z_s^{(R-)} \ge x\right). \qquad (4.2)\end{aligned}$$

The Lévy process $Z^{(R-)}$ is a martingale with jumps bounded by R, hence has exponential moments, see e.g. [9, Proposition 3.14]. Moreover, the angle bracket process $< Z^{(R-)}, Z^{(R-)} >$ is computed to be

$$\begin{aligned}< Z^{(R-)}, Z^{(R-)} >_t &= \int_0^t \int_{|y|\le R} y^2\,\nu(dy)ds\\ &= \frac{2ct}{2-\alpha} R^{2-\alpha}\\ &= v_t(R)^2.\end{aligned}$$

Let $\phi(z) := z^{-2}\left(e^z - z - 1\right)$, $z>0$, and define for all $\beta>0$ the process $S^{(\beta,R)}$ by

$$S_t^{(\beta,R)} = \exp\left(\beta Z_t^{(R-)} - \beta^2\phi(\beta R) < Z^{(R-)}, Z^{(R-)} >_t\right), \quad t\ge 0.$$

By [16, Lemma 23.19], $S^{(\beta,R)}$ is a supermartingale for all $\beta>0$. Thus, the exponential Markov's inequality yields

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le 1} Z_s^{(R-)} \ge x\right) &\le \inf_{\beta>0} \mathbb{P}\left(\sup_{0\le t\le 1} S_t^{(\beta,R)} \ge \exp\left(\beta x - \beta^2 v_1(R)^2\phi(\beta R)\right)\right)\\ &\le \inf_{\beta>0} \exp\left(-\beta x + \beta^2 v_1(R)^2 \phi(\beta R)\right)\\ &= \exp\left(\frac{x}{R} - \left(\frac{x}{R} + \frac{v_1(R)^2}{R^2}\right)\log\left(1+\frac{Rx}{v_1(R)^2}\right)\right)\\ &\le \exp\left(-\frac{x}{2R}\log\left(1+\frac{Rx}{v_1(R)^2}\right)\right)\end{aligned}$$

$$= \exp\left(-\frac{x}{2R}\log\left(1+\frac{(2-\alpha)R^{\alpha-1}x}{2c}\right)\right),$$

where in the latter inequality we used $(1+u)\log(1+u) - u \geq \frac{u}{2}\log(1+u)$, $u \geq 0$, which is equivalent to $(1+u/2)\log(1+u) \geq u$, $u \geq 0$, established by a standard convexity argument. Now, let the truncation level R be such that $x = \lambda R^{1-\alpha}$ for some $\lambda > 0$. Plugging the last inequality into (4.2), we get

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq t\leq 1} Z_s \geq x\right) &\leq \frac{2c}{\alpha}\left(\frac{x}{\lambda}\right)^{\frac{\alpha}{\alpha-1}} + \exp\left(-\frac{\lambda\log\left(1+\frac{(2-\alpha)\lambda}{2c}\right)}{2}\left(\frac{x}{\lambda}\right)^{\frac{\alpha}{\alpha-1}}\right)\\ &=: F\left(\left(\frac{x}{\lambda}\right)^{\frac{\alpha}{\alpha-1}}\right). \end{aligned} \tag{4.3}$$

A necessary condition for the upper bound in (4.3) to make sense is that the real number $F\left((x/\lambda)^{\alpha/(\alpha-1)}\right)$ has to be smaller than 1, which is the case in a neighborhood of 0_+ if $\lambda > \lambda_0(\alpha)$. Finally, choose $x_0(\alpha,\lambda) > 0$ such that $F\left((x_0(\alpha,\lambda)/\lambda)^{\alpha/(\alpha-1)}\right) = 1$ to obtain the maximum range of validity for the result. ■

Now, we can establish a small range maximal inequality for stable stochastic integrals:

Theorem 4.2 *Let Z be a symmetric stable process of index $\alpha \in (1,2)$ and Lévy measure $\nu(dy) = c|y|^{-\alpha-1}dy$, $c > 0$, and let $H \in \mathscr{B}_\alpha$ with a.s. $\lim_{t\to+\infty}\int_0^t |H_s|^\alpha\, ds = +\infty$. Then for all $\lambda > \lambda_0(\alpha)$, where $\lambda_0(\alpha)$ is the unique solution of the equation*

$$\lambda\log\left(1+\frac{(2-\alpha)\lambda}{2c}\right) = \frac{4c}{\alpha},$$

there exists $x_1(\alpha,\lambda) > 0$ such that for all $0 \leq x \leq x_1(\alpha,\lambda)$ and all $t \geq 0$,

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leq s\leq t}\int_0^s H_\tau dZ_\tau \geq x\right)\\ &\leq \frac{2c}{\alpha}\left(\frac{x}{\lambda\|H\|_{L^\infty(\Omega,L^\alpha([0,t]))}}\right)^{\frac{\alpha}{\alpha-1}} + \exp\left(-\frac{\lambda\log\left(1+\frac{(2-\alpha)\lambda}{2c}\right)}{2}\left(\frac{x}{\lambda\|H\|_{L^\infty(\Omega,L^\alpha([0,t]))}}\right)^{\frac{\alpha}{\alpha-1}}\right).\end{aligned} \tag{4.4}$$

Proof. Following the proof of Corollary 3.3, we have by time change and scaling

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\int_0^s H_\tau dZ_\tau \geq x\right) \leq \mathbb{P}\left(\sup_{0\leq s\leq 1}\hat{Z}_s \geq \frac{x}{\|H\|_{L^\infty(\Omega,L^\alpha([0,t]))}}\right),$$

where $\hat{Z}$ is a symmetric stable process defined on $\mathbf{\Omega}$ and having the same law as Z. Finally, Proposition 4.1 applied to $\hat{Z}$ achieves the proof. ■

Remark 4.3 For all $\epsilon > 0$, let x_ϵ be the unique solution of the equation

$$\frac{2c}{\alpha}\left(\frac{x}{\lambda\|H\|_{L^\infty(\Omega,L^\alpha([0,t]))}}\right)^{\frac{\alpha}{\alpha-1}} = \epsilon \exp\left(-\frac{\lambda\log\left(1+\frac{(2-\alpha)\lambda}{2c}\right)}{2}\left(\frac{x}{\lambda\|H\|_{L^\infty(\Omega,L^\alpha([0,t]))}}\right)^{\frac{\alpha}{\alpha-1}}\right).$$

Then for all $0 \le x \le x_\epsilon$, the inequality (4.4) implies

$$\mathbb{P}\left(\sup_{0\le s\le t}\int_0^s H_\tau dZ_\tau \ge x\right) \le (1+\epsilon)\exp\left(-\frac{\lambda\log\left(1+\frac{(2-\alpha)\lambda}{2c}\right)}{2}\left(\frac{x}{\lambda\|H\|_{L^\infty(\Omega,L^\alpha([0,t]))}}\right)^{\frac{\alpha}{\alpha-1}}\right).$$

Thus, the order of the upper bound in (4.4) is $\exp\left(-c_\alpha\left(x/\|H\|_{L^\infty(\Omega,L^\alpha([0,t]))}\right)^{\alpha/(\alpha-1)}\right)$, and is comparable to that in [8, Theorem 1] for Lipschitz functions of stable random vectors.

Remark 4.4 The quantity $x_1(\alpha,\lambda)$ in Theorem 4.2 can be given explicitly. Indeed, let $x_0^*(\alpha,\lambda) > 0$ be the real number where the function F in (4.3) reaches its unique minimum, i.e.

$$x_0^*(\alpha,\lambda)^{\frac{\alpha}{\alpha-1}} = \frac{2\lambda^{\frac{1}{\alpha-1}}}{\log\left(1+\frac{(2-\alpha)\lambda}{2c}\right)}\log\left(\frac{\alpha\lambda\log\left(1+\frac{(2-\alpha)\lambda}{2c}\right)}{4c}\right) < x_0(\alpha,\lambda)^{\frac{\alpha}{\alpha-1}},$$

then choose $x_1(\alpha,\lambda) = \|H\|_{L^\infty(\Omega,L^\alpha([0,t]))}\, x_0^*(\alpha,\lambda)$.

Remark 4.5 There is no optimal choice for the parameter λ in Theorem 4.2: on the one hand, $\lambda = \lambda_0(\alpha)$ achieves the best maximal inequality (4.4) but in this case the domain for the deviation level x is empty; on the other hand, as λ increases, the domain expands but in this case the maximal inequality (4.4) is the worst.

As an application of Theorem 4.2, let us recover the classical maximal inequality in the Gaussian case, cf. Proposition 1.8 p.55 in [20].

Corollary 4.6 *Let $(B_t)_{t\ge 0}$ be a standard Brownian motion. Then the following maximal inequality holds*

$$\mathbb{P}\left(\sup_{0\le s\le t} B_s \ge x\right) \le \exp\left(-\frac{x^2}{2t}\right), \quad x > 0, \quad t \ge 0.$$

Proof. Let $(X^n)_{n\geq 2}$ be a sequence of symmetric stable processes of index $\alpha_n = 2-1/n$ and Lévy measure $\nu_n(dy) = (2n)^{-1}dy/|y|^{\alpha_n+1}$. Applying Theorem 4.2 to X^n, $n\geq 2$, the inequality (4.4) becomes for all $0\leq x\leq x_1(\alpha_n,\lambda)$, all $\lambda > \lambda_0(\alpha_n)$ and all $t\geq 0$

$$\mathbb{P}\left(\sup_{0\leq s\leq t} X_s^n \geq x\right) \leq \frac{1}{2n-1}\left(\frac{x}{\lambda t^{\frac{n}{2n-1}}}\right)^{\frac{2n-1}{n-1}} + \exp\left(-\frac{\lambda\log(1+\lambda)}{2}\left(\frac{x}{\lambda t^{\frac{n}{2n-1}}}\right)^{\frac{2n-1}{n-1}}\right), \tag{4.5}$$

where

$$x_1(\alpha,\lambda)^{\frac{2n-1}{n-1}} = \frac{2(t\lambda)^{\frac{n}{n-1}}}{\log(1+\lambda)}\log\left((n-\frac{1}{2})\lambda\log(1+\lambda)\right),$$

and $\lambda_0(\alpha_n)$ is the unique solution of the equation

$$\lambda\log(1+\lambda) = \frac{2}{2n-1}.$$

Note that $\lambda_0(\alpha_n)$ converges to 0 and $x_1(\alpha_n,\lambda)$ to infinity as n goes to infinity. Denoting $D[0,+\infty)$ the Skorohod space of real-valued càdlàg functions on $[0,+\infty)$ equipped with the Skorohod topology, the sequence of processes $(X^n)_{n\geq 2}$ converges weakly in $D[0,+\infty)$ as $n\to+\infty$ to a standard Brownian motion $(B_t)_{t\geq 0}$ (say), see e.g. Section 3 of Chapter VII in [15]. Since the supremum functional is continuous on $D[0,+\infty)$, then the Continuous Mapping Theorem p.20 in [7] implies

$$\lim_{n\to+\infty}\mathbb{P}\left(\sup_{0\leq s\leq t} X_s^n\geq x\right) = \mathbb{P}\left(\sup_{0\leq s\leq t} B_s\geq x\right), \quad x>0, \quad t\geq 0.$$

Finally, letting n going to infinity and then λ to 0 in the right-hand-side of (4.5) yield the result. ∎

5 Estimates of first passage times of symmetric stable processes above positive continuous curves

In [1, 18], the authors investigate functional transformations related to first crossing problems for self-similar diffusions. More precisely, they show via a time change transformation how the distribution of the first passage time of a Gauss-Markov process of Ornstein-Uhlenbeck type can be deduced from the law of the first crossing time of a

continuous curve by a Brownian motion. In this part, we adapt this method in order to estimate the first passage time of a symmetric stable process above several positive continuous curves, by using the maximal inequalities of Section 2 and 3.
To do so, let X^ϕ be a stable-Markov process of Ornstein-Uhlenbeck type of index $\alpha \in (0,2)$ and parameter ϕ, i.e. X^ϕ has the integral representation

$$X_t^\phi := \phi(t) \int_0^t \frac{dZ_s}{\phi(s)}, \quad t \in [0,T), \quad T \in (0,+\infty],$$

where Z is a symmetric stable process of index α and Lévy measure $\nu(dy) = c|y|^{-\alpha-1}dy$, $c > 0$, and ϕ is a positive $C^\infty([0,T))$-function. Let also

$$T_x^\phi := \inf\{t \in [0,T) : |X_t^\phi| \geq x\}$$

be its first exit time of the centered ball of radius x. Given a positive continuous function f such that $f(0) \neq 0$, define

$$T^{(f)} := \inf\{t \geq 0 : |Z_t| \geq f(t)\}$$

as the first passage time of $|Z|$ above f. Let us give a first lemma which states an identity in law between first passage times:

Lemma 5.1 *Let X^ϕ be a stable-Markov process of Ornstein-Uhlenbeck type of index $\alpha \in (0,2)$ and parameter ϕ. Assume that $\tau_t := \int_0^t \frac{ds}{\phi(s)^\alpha} < +\infty$ for all $t \in [0,T)$ and that $\lim_{t\to T} \tau_t = +\infty$. Denote by τ^{-1} the inverse of τ and let $h_{\phi,\tau}$ be the function defined on $(0,+\infty)$ by $h_{\phi,\tau}(t) = 1/(\phi \circ \tau^{-1}(t))$. Then for all $x > 0$, we have the identity in distribution*

$$\mathbb{P}\left(T_x^\phi \in dr\right) = \mathbb{P}\left(\tau^{-1}\left(T^{(xh_{\phi,\tau})}\right) \in dr\right), \quad r \in [0,T).$$

Proof. By [21, Theorem 3.1], the process X^ϕ rewrites as a time-changed symmetric stable process, i.e. we have a.s.

$$X_t^\phi = \phi(t)\hat{Z}_{\tau_t}, \quad t \in [0,T),$$

where $\hat{Z}$ is a symmetric stable process defined on the same probability space as Z and having the same distribution. Thus, we have for all $r \in [0,T)$

$$\mathbb{P}\left(T_x^\phi \leq r\right) = \mathbb{P}\Big(\inf\{t \in [0,T) : |X_t^\phi| \geq x\} \leq r\Big)$$

$$= \mathbb{P}\left(\inf\left\{t \in [0,T) : |\hat{Z}_{\tau_t}| \geq \frac{x}{\phi(t)}\right\} \leq r\right)$$
$$= \mathbb{P}\left(\tau^{-1}\left(T^{(xh_{\phi,\tau})}\right) \leq r\right).$$

■

Now, we establish via an integration by parts formula several maximal inequalities for stable-Markov processes of Ornstein-Uhlenbeck type:

Lemma 5.2 *Let X^ϕ be a stable-Markov process of Ornstein-Uhlenbeck type of index $\alpha \in (0,2)$ and parameter ϕ. Let $t \in [0,T)$. Then we have the support estimate*

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^\phi| < y\right) \leq \exp\left(-\frac{ct}{\alpha 2^{\alpha-1} y^\alpha}\right), \quad y > 0. \tag{5.1}$$

If $\alpha \in (0,1]$, then we have the maximal inequality

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^\phi| \geq x\right) \leq \frac{4ct}{\alpha x^\alpha}\left(1 + \left\|\phi(\cdot)\int_0^\cdot \frac{\phi'(\tau)}{\phi(\tau)^2}\, d\tau\right\|_{L^\infty([0,t])}\right)^\alpha, \quad x > 0, \tag{5.2}$$

whereas if $\alpha \in (1,2)$, then for all

$$x^\alpha > \frac{tc}{(2-\alpha)^{\frac{\alpha+1}{\alpha-1}}}\left(1 + \left\|\phi(\cdot)\int_0^\cdot \frac{\phi'(\tau)}{\phi(\tau)^2}\, d\tau\right\|_{L^\infty([0,t])}\right),$$

we have

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^\phi| \geq x\right) \leq \frac{K_c t}{x^\alpha}\left(1 + \left\|\phi(\cdot)\int_0^\cdot \frac{\phi'(\tau)}{\phi(\tau)^2}\, d\tau\right\|_{L^\infty([0,t])}\right)^\alpha, \tag{5.3}$$

where $K_c > 0$ only depends on c.

Proof. Fix $t \in [0,T)$ and $y > 0$. If a.s. the path of the process X^ϕ lies in the interval $(-y,y)$ up to time t, then there are no jumps of magnitude larger than $2y$ before time t, so that we have the set inclusion $\left\{\sup_{0\leq s\leq t} |X_s^\phi| < y\right\} \subset \left\{\sup_{0\leq s\leq t} |\Delta X_s^\phi| < 2y\right\}$. Moreover, the process X^ϕ has the same jumps as the process Z by definition. Thus, if T_1^{2y} denotes the first jump time on the set $\{z \in \mathbb{R} : |z| > 2y\}$ of the Poisson process $(\mu(\{z \in \mathbb{R} : |z| > 2y\} \times [0,t]))_{t\in[0,T)}$, then we have

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^\phi| < y\right) \leq \mathbb{P}\left(\sup_{0\leq s\leq t} |\Delta X_s^\phi| < 2y\right)$$

$$
\begin{aligned}
&= \mathbb{P}\left(\sup_{0\le s\le t}|\Delta Z_s| < 2y\right)\\
&\le \mathbb{P}\left(T_1^{2y} > t\right)\\
&= \exp\left(-t\nu\left(\{z\in\mathbb{R} : |z|\ge 2y\}\right)\right)\\
&= \exp\left(-\frac{2ct}{\alpha(2y)^\alpha}\right),
\end{aligned}
$$

where in the second equality we used that T_1^{2y} is exponentially distributed with parameter $\nu\left(\{z\in\mathbb{R} : |z| > 2y\}\right)$. The support estimate (5.1) is proved.

Now, we establish (5.2) and (5.3). By the classical integration by parts formula for semimartingales, cf. [9, Proposition 8.11], we have

$$
\begin{aligned}
\int_0^t \frac{dZ_s}{\phi(s)} &= \frac{Z_t}{\phi(t)} - \int_0^t Z_{s-}d\left(\frac{1}{\phi}\right)(s)\\
&= \frac{Z_t}{\phi(t)} + \int_0^t \frac{\phi'(s)Z_s}{\phi(s)^2}ds.
\end{aligned}
$$

Hence, the process X^ϕ rewrites as

$$
X_t^\phi = Z_t + \phi(t)\int_0^t \frac{\phi'(s)}{\phi(s)^2}Z_s\,ds, \quad t\in[0,T). \tag{5.4}
$$

Denote $A_t := \left\|\phi(\cdot)\int_0^\cdot \frac{\phi'(\tau)}{\phi(\tau)^2}\,d\tau\right\|_{L^\infty([0,t])}$ and let us distinguish two cases:

- if $\alpha\in(0,1]$, then following the proof of inequality (2.11) but restricted to the symmetric stable process Z yields the inequality

$$
\mathbb{P}\left(\sup_{0\le s\le t}|Z_s|\ge x\right) \le \frac{4ct}{\alpha x^\alpha}.
$$

Thus, together with (5.4), we have

$$
\begin{aligned}
\mathbb{P}\left(\sup_{0\le s\le t}|X_s^\phi|\ge x\right) &\le \mathbb{P}\left(\sup_{0\le s\le t}|Z_s|\ge \frac{x}{1+A_t}\right)\\
&\le \frac{4ct(1+A_t)^\alpha}{\alpha x^\alpha};
\end{aligned}
$$

- if $\alpha\in(1,2)$, then Corollary 3.3 applied with e.g. $p=1$ and $\epsilon = 2^{(\alpha-1)/(\alpha+1)}$, together with (5.4) show that there exists $K_c>0$, which only depends of c, such that

$$
\begin{aligned}
\mathbb{P}\left(\sup_{0\le s\le t}|X_s^\phi|\ge x\right) &\le \mathbb{P}\left(\sup_{0\le s\le t}|Z_s|\ge \frac{x}{1+A_t}\right)\\
&\le \frac{K_c t(1+A_t)^\alpha}{x^\alpha}
\end{aligned}
$$

for all $x^\alpha > (tc(1+A_t)^\alpha)/((2-\alpha)^{(\alpha+1)/(\alpha-1)})$. ■

Remark 5.3 The support estimate (5.1) is independent of ϕ and thus is similar to that of a symmetric stable process.

Remark 5.4 No time change techniques are required in the proof of Lemma 5.2 but just the integration by parts formula which entails (5.4). However, if we assume $\tau_t := \int_0^t \frac{ds}{\phi(s)^\alpha} < +\infty$, $t \in [0,T)$, with $\tau_t \to +\infty$ as $t \to T$ and that ϕ is non-decreasing on $[0,T)$, then time change, scaling and Corollary 3.3 entail for sufficiently large x

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le s\le t} |X_s^\phi| \ge x\right) &\le \mathbb{P}\left(\sup_{0\le s\le t} |\hat{Z}_{\tau_s}| \ge \frac{x}{\phi(t)}\right) \\
&\le \mathbb{P}\left(\sup_{0\le s\le 1} |\hat{Z}_s| \ge \frac{x}{\phi(t)\tau_t^{\frac{1}{\alpha}}}\right) \\
&\le \frac{K_c}{x^\alpha}\,\phi(t)^\alpha \int_0^t \frac{ds}{\phi(s)^\alpha}.
\end{aligned}$$

Now, we are able to state the main result of this part:

Theorem 5.5 *Let Z be a symmetric stable process of index $\alpha \in (0,2)$ and Lévy measure $\nu(dy) = c|y|^{-\alpha-1}dy$, $c > 0$. Let ϕ be a positive $C^\infty([0,T))$-function such that $\tau_t := \int_0^t \frac{ds}{\phi(s)^\alpha} < +\infty$ for all $t \in [0,T)$ and that $\lim_{t\to T}\tau_t = +\infty$. Denote by τ^{-1} the inverse of τ and by $h_{\phi,\tau}$ the function defined on $(0,+\infty)$ by $h_{\phi,\tau}(t) := 1/(\phi\circ\tau^{-1}(t))$. Then for all $x > 0$,*

$$\mathbb{P}\left(T^{(xh_{\phi,\tau})} > r\right) \le \exp\left(-\frac{2c\tau_r^{-1}}{\alpha(2x)^\alpha}\right), \quad r > 0. \tag{5.5}$$

If $\alpha \in (0,1]$, then for all $x > 0$, we have

$$\mathbb{P}\left(T^{(xh_{\phi,\tau})} \le r\right) \le \frac{4c\tau_r^{-1}}{\alpha x^\alpha}\left(1 + \left\|\phi(\cdot)\int_0^\cdot \frac{\phi'(t)}{\phi(t)^2}\,dt\right\|_{L^\infty([0,\tau_r^{-1}])}\right)^\alpha, \quad r > 0, \tag{5.6}$$

whereas if $\alpha \in (1,2)$, then there exists $K_c > 0$, which only depends of c, such that for all $x > 0$ and for all $0 \le r < r_0(\alpha,x)$, we have

$$\mathbb{P}\left(T^{(xh_{\phi,\tau})} \le r\right) \le \frac{K_c\tau_r^{-1}}{x^\alpha}\left(1 + \left\|\phi(\cdot)\int_0^\cdot \frac{\phi'(t)}{\phi(t)^2}\,dt\right\|_{L^\infty([0,\tau_r^{-1}])}\right)^\alpha, \tag{5.7}$$

where $r_0(\alpha,x)$ is the unique solution of the equation

$$(2-\alpha)^{\frac{\alpha+1}{\alpha-1}}x^\alpha = c\tau_r^{-1}\left(1 + \left\|\phi(\cdot)\int_0^\cdot \frac{\phi'(t)}{\phi(t)^2}\,dt\right\|_{L^\infty([0,\tau_r^{-1}])}\right)^\alpha.$$

Proof. It is sufficient to apply Lemma 5.1 and Lemma 5.2. ∎

Thus, given ϕ, the quantity in the right-hand-side of the inequalities (5.5), (5.6) and (5.7) can be computed explicitly. Let us give two applications of Theorem 5.5.
If $\phi(t) := e^{-\lambda t}$ for $\lambda > 0$ and $T = +\infty$, then X^{ϕ} is the stable Ornstein-Uhlenbeck process of index α. Therefore, a direct computation in Theorem 5.5 implies the

Corollary 5.6 *Let Z be a symmetric stable process of index $\alpha \in (0,2)$ and Lévy measure $\nu(dy) = c|y|^{-\alpha-1}dy$, $c > 0$. Letting $f_{\alpha,x,\lambda}(t) := x(1+\lambda\alpha t)^{1/\alpha}$, $t \geq 0$, $\lambda > 0$, we have for all $x > 0$*

$$\mathbb{P}\left(\inf\{t \geq 0 : |Z_t| \geq f_{\alpha,x,\lambda}(t)\} > r\right) \leq \frac{1}{(1+\lambda\alpha r)^{\frac{c}{\lambda\alpha^2 2^{\alpha-1} x^{\alpha}}}}, \quad r > 0.$$

If $\alpha \in (0,1]$, then for all $x > 0$ and all $r > 0$,

$$\begin{aligned}\mathbb{P}\left(\inf\{t \geq 0 : |Z_t| \geq f_{\alpha,x,\lambda}(t)\} \leq r\right) &\leq \frac{4c(2-(1+\lambda\alpha r)^{-\frac{1}{\alpha}})^{\alpha}\log(1+\lambda\alpha r)}{\lambda\alpha^2 x^{\alpha}} \\ &\leq \frac{16cr}{\alpha x^{\alpha}}.\end{aligned}$$

Finally, if $\alpha \in (1,2)$, then for all $x > 0$ and for all $0 \leq r < r_0(\alpha,x,\lambda)$, we have the estimate

$$\begin{aligned}\mathbb{P}\left(\inf\{t \geq 0 : |Z_t| \geq f_{\alpha,x,\lambda}(t)\} \leq r\right) &\leq \frac{(2-(1+\lambda\alpha r)^{-\frac{1}{\alpha}})^{\alpha}\log(1+\lambda\alpha r)K_c}{\lambda\alpha x^{\alpha}} \\ &\leq \frac{4rK_c}{x^{\alpha}},\end{aligned}$$

where K_c is the constant of Theorem 5.5 and $r_0(\alpha,x,\lambda)$ is the unique solution of the equation

$$\lambda\alpha x^{\alpha} = \frac{c(2-(1+\lambda\alpha r)^{-\frac{1}{\alpha}})^{\alpha}\left(\log(1+\lambda\alpha r)\right)}{(2-\alpha)^{\frac{\alpha+1}{\alpha-1}}}.$$

Now, we present the case of the stable bridge. Given a symmetric stable process $Z = (Z_t)_{t\geq 0}$ of index $\alpha \in (0,2)$, there exists a Markov process $X^{(\mathrm{br})} = (X_t^{(\mathrm{br})})_{0\leq t\leq T}$ starting from 0 and ending in 0 at a finite time horizon T, such that its distribution $\mathbb{Q}$ is given by

$$d\mathbb{Q}_{|\mathscr{F}_t} = \frac{p_{T-t}(-X_t)}{p_T(0)} d\mathbb{P}_{|\mathscr{F}_t}, \quad t \in (0,T),$$

where p_t is a version everywhere positive of the distribution of the stable random variable Z_t, see [6, Chapter VIII]. The process $X^{(\mathrm{br})}$ is called a stable bridge. By e.g.

Exercise 12.2 in [23], $X^{(\mathrm{br})}$ is the unique solution of the linear equation

$$X_t^{(\mathrm{br})} = Z_t - \int_0^t \frac{X_s^{(\mathrm{br})}}{T-s} ds, \quad t \in (0,T),$$

which rewrites by the integration by parts formula of Proposition 8.11 in [9] as

$$X_t^{(\mathrm{br})} = (T-t) \int_0^t \frac{dZ_s}{T-s} ds, \quad t \in (0,T).$$

Hence, the stable bridge $X^{(\mathrm{br})}$ is a stable-Markov process of Ornstein-Uhlenbeck type with parameter ϕ given by $\phi(t) = T - t$, $t \in [0,T]$. Thus, using Theorem 5.5, we get the

Corollary 5.7 *Let Z be a symmetric stable process of index $\alpha \in (1,2)$ and Lévy measure $\nu(dy) = c|y|^{-\alpha-1}dy$, $c > 0$. Letting $g_{\alpha,x,T}(t) := x(T^{1-\alpha} + (\alpha-1)t)^{1/(\alpha-1)}$, $t \geq 0$, we have for all $x > 0$ and all $r > 0$*

$$\begin{aligned} \mathbb{P}\left(\inf\{t \geq 0 : |Z_t| \geq g_{\alpha,x,\lambda}(t)\} > r\right) &\leq \exp\left(-\frac{c}{\alpha 2^{\alpha-1} x^\alpha}\left(T - (T^{1-\alpha} + (\alpha-1)r)^{\frac{1}{1-\alpha}}\right)\right) \\ &= \exp\left(-\frac{c(Tg_{\alpha,x,T}(r) - x)}{\alpha 2^{\alpha-1} g_{\alpha,x,T}(r) x^\alpha}\right), \end{aligned}$$

whereas for all $x > 0$ and for all $0 \leq r < r_0(\alpha, x, T)$, we have

$$\begin{aligned} \mathbb{P}\left(\inf\{t \geq 0 : |Z_t| \geq g_{\alpha,x,T}(t)\} \leq r\right) &\leq \frac{K_c(Tg_{\alpha,x,T}(r) - x)(2Tg_{\alpha,x,T}(r) - x)^\alpha}{T^\alpha g_{\alpha,x,T}(r)^{\alpha+1} x^\alpha} \\ &\leq \frac{4TK_c}{x^\alpha}, \end{aligned}$$

where K_c is the constant of Theorem 5.5 and $r_0(\alpha, x, \lambda)$ is the unique solution of the equation

$$(2-\alpha)^{\frac{\alpha+1}{\alpha-1}} T^\alpha g_{\alpha,x,T}(r)^{\alpha+1} x^\alpha = c(Tg_{\alpha,x,T}(r) - x)(2Tg_{\alpha,x,T}(r) - x)^\alpha.$$

Remark 5.8 In the latter corollary, only the case $\alpha \in (1,2)$ is considered, since the time change techniques we use in the proof of Theorem 5.5 are not satisfied when $\alpha \in (0,1)$.

Acknowledgments

The author would like to thank N. Privault and J.C. Breton for their encouragements and helpful comments during the preparation of this paper.

References

[1] L. Alili and P. Patie. On the first crossing times of a Brownian motion and a family of continuous curves. *C. R. Math. Acad. Sci. Paris*, 340(3):225–228, 2005.

[2] D. Applebaum. *Lévy processes and stochastic calculus*, volume 93 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004.

[3] D. Applebaum. Lévy-type stochastic integrals with regularly varying tails. *Stoch. Anal. Appl.*, 23(3):595–611, 2005.

[4] R.F. Bass. Stochastic differential equations driven by symmetric stable processes. In *Séminaire de Probabilités, XXXVI*, volume 1801 of *Lecture Notes in Math.*, pages 302–313. Springer, Berlin, 2003.

[5] R.F. Bass and D.A. Levin. Harnack inequalities for jump processes. *Potential Anal.*, 17(4):375–388, 2002.

[6] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[7] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999.

[8] J.C. Breton and C. Houdré. On finite range stable-type concentration. To appear in Teor. Veroyatnost. i Primenen., 2006.

[9] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[10] C. Dellacherie and P.A. Meyer. *Probabilités et potentiel. Chapitres V à VIII*, volume 1385 of *Actualités Scientifiques et Industrielles*. Hermann, Paris, revised edition, 1980.

[11] S.S. Dragomir. *Some Gronwall type inequalities and applications.* Nova Science Publishers Inc., Hauppauge, NY, 2003.

[12] E. Giné and M.B. Marcus. The central limit theorem for stochastic integrals with respect to Lévy processes. *Ann. Probab.*, 11(1):58–77, 1983.

[13] C. Houdré and P. Marchal. On the concentration of measure phenomenon for stable and related random vectors. *Ann. Probab.*, 32(2):1496–1508, 2004.

[14] H. Hult and F. Lindskog. Extremal behavior of stochastic integrals driven by regularly varying lévy processes. To appear in Ann. Probab., 2006.

[15] J. Jacod and A.N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, second edition, 2003.

[16] O. Kallenberg. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York, 1997.

[17] J. Kallsen and A.N. Shiryaev. Time change representation of stochastic integrals. *Teor. Veroyatnost. i Primenen.*, 46(3):579–585, 2001.

[18] P. Patie. On some first passage time problems, motivated by financial applications. Ph.D. Thesis, ETH Zurich, 2004.

[19] W.E. Pruitt. The growth of random walks and Lévy processes. *Ann. Probab.*, 9(6):948–956, 1981.

[20] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, third edition, 1999.

[21] J. Rosiński and W.A. Woyczyński. On Itô stochastic integration with respect to p-stable motion: inner clock, integrability of sample paths, double and multiple integrals. *Ann. Probab.*, 14(1):271–286, 1986.

[22] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[23] M. Yor. *Some aspects of Brownian motion. Part II.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1997.