Notions of A_∞-categories

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Fix a field $\mathbb{k}$.

Def.: A small non-unital A_∞-cat. $\mathcal{A}$ consists of

① $\mathrm{obj}.\ \mathcal{A}$ = set

② $\forall\ X_0, X_1 \in \mathrm{obj}$, $\hom_{\mathcal{A}}(X_0, X_1)$ graded $\mathbb{k}$-space.

③ $\forall\ d \geq 1$, $\mu^d_{\mathcal{A}} : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{A}}(X_0, X_d)[2-d]$

s.t. the following A_∞-associativity equations are satisfied

$$\sum_{n,m} (-1)^{\dagger_n}\ \mu^{d-m+1}_{\mathcal{A}}\big(a_d, \ldots, \mu^m_{\mathcal{A}}(\ldots), \ldots, a_0\big) = 0 \qquad (*)$$

$\dagger_n = |a_1| + \cdots + |a_n| - n$

A Quiver E: $\left|\begin{array}{l} \mathrm{obj}\ E \\ \hom_E(X_0, X_1) \in E \end{array}\right.$

$\mathrm{Quiv}_{\mathbb{k}}$: has a monoidal struct. $\otimes$.

$E, F \in \mathrm{Quiv}_{\mathbb{k}}$

$E \otimes F$: $\left|\begin{array}{l} \mathrm{obj}.\ E\otimes F = \mathrm{obj}\ E \times \mathrm{obj}\ F \\ \hom_{E\otimes F}((X_0, Y_0), (X_1, Y_1)) = \hom_E(X_0, Y_0) \otimes_{\mathbb{k}} \hom_E(X_1, Y_1) \end{array}\right.$

$\mathrm{forget} : \mathrm{coCat}_{\mathbb{k}} \rightleftarrows \mathrm{Quiv}_{\mathbb{k}} : T^{\geq 1}$

$\left|\begin{array}{l} \mathrm{obj}.\ T^{\geq 1}(E) = \mathrm{obj}\ E \\ \hom_{T^{\geq 1}(E)}(X, Y) = \prod\limits_{X = X_0, X_1, \ldots, X_d = Y} \hom(X_{d-1}, X_d) \otimes \cdots \otimes \hom(X_0, X_1) \end{array}\right.$

the decomposition map structure

Bar construction (de $\mathcal{A}$): $T^{\geq 1}(s\mathcal{A})$ (with a differential)

Motivation: A_∞-struct. $\longleftrightarrow$ $T^{\geq 1}(s\mathcal{A}) \longrightarrow T^{\geq 1}(s\mathcal{A})$ in $\mathrm{coCat}_{\mathbb{k}}$.

Examples of A_∞-cat:

① Non-unital dg cat.

② A A_∞-alg.

Mod_A is an A_∞-cat.

③ Fukaya cat. of a symplectic manifold.

④ $\mathcal{A}$ A_∞-cat. $\rightsquigarrow$ $\mathcal{A}^{op}$ by $\begin{cases} \mathrm{obj}\,\mathcal{A}^{op} = \mathrm{obj}\,\mathcal{A} \\ \hom_{\mathcal{A}^{op}}(X_0, X_1) = \hom_{\mathcal{A}}(X_1, X_0) \\ \mu^d_{\mathcal{A}^{op}} = (-1)^{+d} \mu^d_{\mathcal{A}} \end{cases}$.

Cohomological cat. $H(\mathcal{A})$

$\left| \begin{array}{l} \mathrm{obj}\,H(\mathcal{A}) := \mathrm{obj}\,\mathcal{A} \\ \hom_{H(\mathcal{A})}(X_0, X_1) = H^*(\hom_{\mathcal{A}}(X_0, X_1), \mu^1_{\mathcal{A}}) \end{array} \right.$

is a differential by ⊛

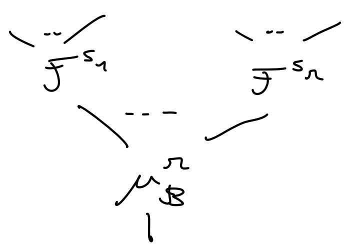
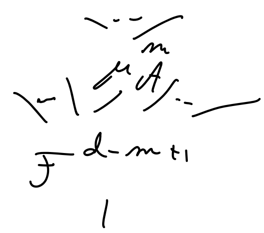
Def: • Let $\mathcal{A}, \mathcal{B}$ A_∞-cat. A (nu A_∞-)functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is

① $\mathcal{F} : \mathrm{obj}\,\mathcal{A} \to \mathrm{obj}\,\mathcal{B}$

② $(\mathcal{F}^d)_{d \geq 1}$ family of maps: $\forall X_0, X_1, \dots, X_d \in \mathrm{obj}\,\mathcal{A}$,

$\mathcal{F}^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{A}}(\mathcal{F}(X_0), \mathcal{F}(X_d))[1-d]$

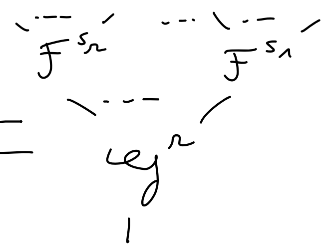
s.t.

$\sum_r \sum_{s_1, \dots, s_r}$  $= \sum (-1)^{+m}$ 

Then: $H(\mathcal{F}) : H(\mathcal{A}) \to H(\mathcal{B})$ functor of graded $\mathbb{k}$-cat.

$\left| \begin{array}{l} X \longmapsto \mathcal{F}(X) \\ H(\mathcal{F})(a) = [a] \end{array} \right.$

- **Composition of A_∞-functors**

$$(\mathcal{G} \circ \mathcal{F})(a_d, \dots, a_1) = \sum \sum$$  (associative)

$\hookrightarrow$ A_∞-cat. forms a category.

$$\text{Forget} : \text{coCat}_{\mathbb{k}} \rightleftarrows \text{Quiv}_{\mathbb{k}} : T^{\geq 1}(-)$$

Fact: both cat. have internal hom-obj.

$$\hom_{\text{coCat}_{\mathbb{k}}}(\mathcal{C}, T^{\geq 1}(E)) \cong T^{\geq 1}(\hom_{\text{Quiv}_{\mathbb{k}}}(\text{forget}(\mathcal{C}), E))$$

in $\text{coCat}_{\mathbb{k}}$

$$\hom_{\text{coCat}_{\mathbb{k}}}(T^{\geq 1}(s\mathcal{A}), T^{\geq 1}(s\mathcal{B})) \cong T^{\geq 1}(\hom_{\text{Quiv}_{\mathbb{k}}}(\text{forget}(T^{\geq 1}(s\mathcal{A})), s\mathcal{B}))$$

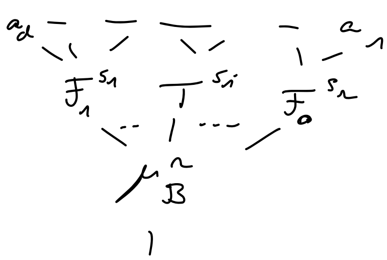
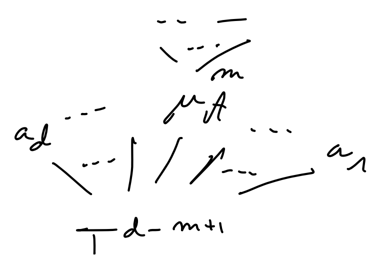
Def: Let $\mathcal{A} \underset{\mathcal{F}_1}{\overset{\mathcal{F}_0}{\rightrightarrows}} \mathcal{B}$

- **Pre-natural transformation** (T of degree g)

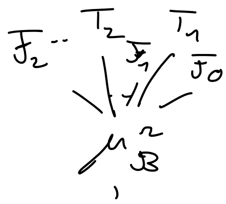
$\hom_2^g(\mathcal{F}_0, \mathcal{F}_1)$ consists of family of multilinear maps (T^d), $d \geq 0$, defined $\forall\, X_0, \dots, X_d$

$$T^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(\mathcal{F}_0 X_0, \mathcal{F}_1 X_1)[-d+g]$$

- $\mu_2^1(T)(a_d, \dots, a_1) :=$

$$\sum_r \sum_{s_1, \dots, s_r}$$  $$- \sum_{m,n} (-1)^{\maltese_n + |T| - 1}$$ 

Natural transformation T if $\mu_2^1(T) = 0$.

- $\mu_2^2(T_2, T_1)(a_d, \dots, a_1) := \sum_{i,j} \sum_{s_1, \dots, s_r}$  .

$(+\text{ similar } \mu_2^d,\ d \geq 3)$

A_∞-cat. graded $\mathbb{k}$-cat.

$$H(nu\text{-}fun(\mathcal{A}, \mathcal{B})) \longrightarrow Nu\text{-}fun(H(\mathcal{A}), H(\mathcal{B})).$$

Question: Is it fully faithful?

Length filtration: $F^r(hom_2^\bullet(\mathcal{F}_0, \mathcal{F}_1)) \subset hom_2^\bullet(\mathcal{F}_0, \mathcal{F}_1)$

$= \{(T^\bullet) \text{ s.t. } T^d = 0 \text{ for } d < r\}$.

Associated spectral sequence

$$E_1^{r,s}(X,Y) = \prod_{X = X_0, X_1, \dots, X_r = Y} Hom\left(hom_{\mathcal{A}}(X_{r-1}, X_r) \otimes \cdots \otimes hom_{\mathcal{A}}(X_0, X_1),\ hom_{\mathcal{B}}(\mathcal{F}_0 X_0, \mathcal{F}_1 X_r)\right)$$

A_∞-mod.

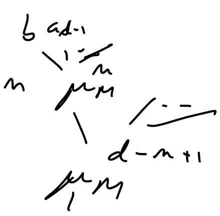
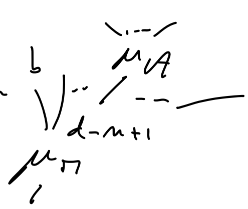
Def.: $\mathcal{A}$ A_∞-cat.

- **right-$\mathcal{A}$-mod** are $Mod_{\mathcal{A}}^{nu} := nu\text{-}fun(\mathcal{A}^{op}, Ch_{\mathbb{k}})$

Concretely: $M \in Mod_{\mathcal{A}}^{nu}$: $M(X)$ a chain complex $\forall X \in \mathcal{A}$

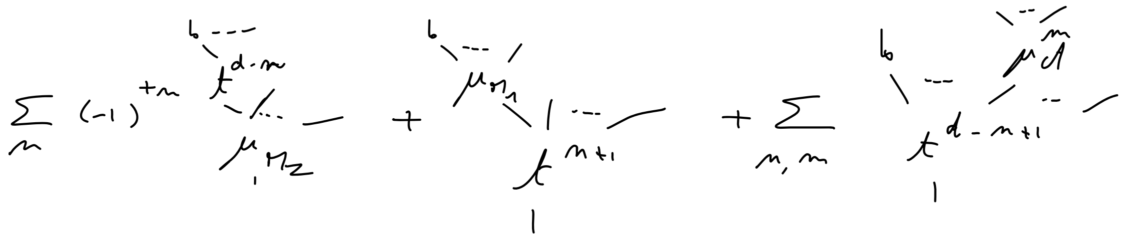
together w/ $\mu_M^d(b, a_{d-1}, \dots, a_1) := M^{d-1}(a_1, \dots, a_{d-1})(b)$.

s.t.

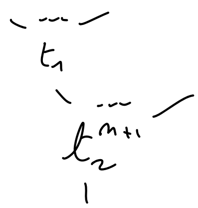
$\sum_n (-1)^{+n}$  $+ (-1)^{+n}$  $= 0$

• $M_1 \xrightarrow{t} M_2$ in $\mathrm{Mod}^{nu}_{\mathcal{A}}$ are called **pre-module homomorphisms** $(\leftrightarrow T)$

$\times \; (\mu^1_{\mathcal{Q}}(t))^d(b, a_{d-1}, \dots, a_1) :=$



Module homomorphisms: t s.t. $\mu^1_{\mathcal{Q}}(t) = 0$.

$\times \; (\mu^2_{\mathcal{Q}}(t_1, t_2))^d := \sum (-1)^+$  , $\mu^d_{\mathcal{Q}} = 0 \;\; \forall d \geq 3$.

$\hookrightarrow \mathrm{Mod}^{nu}_{\mathcal{A}}$ dg cat.

Yoneda functor

$h_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathrm{Mod}^{nu}_{\mathcal{A}}$ nu A_∞-functor

- $h_{\mathcal{A}}(Y)(X) := \hom_{\mathcal{A}}(X, Y)$
- $\forall c \in \hom_{\mathcal{A}}(Y_0, Y_1)$, $h^1(c) \in \hom_{\mathcal{A}}(h_{\mathcal{A}}(Y_0), h_{\mathcal{A}}(Y_1))$, that is

$$h^1(c): h_{\mathcal{A}}(Y_0)(X_{d-1}) \otimes \hom(X_{d-2}, X_{d-1}) \otimes \dots \otimes \hom(X_0, X_1) \xrightarrow{\mu^{r+1}_{\mathcal{A}}} h_{\mathcal{A}}(Y_1)(X_0)$$

$\underline{r \geq 1}$

$$h^r(c_r, \dots, c_1)(b, a_{d-1}, \dots, a_1) := \mu^{r+d}_{\mathcal{A}}(c_r, \dots, c_1, b, a_{d-1}, \dots, a_1)$$

Pullback:

$\mathcal{G} : \mathcal{A} \to \mathcal{B}$ nu A_∞-functor

$M \in \mathrm{Mod}^{nu}_{\mathcal{B}}$,

$\mathcal{G}_*(M)(X) := M(\mathcal{G}(X))$

$\mu^d_{\mathcal{G}^*(M)}(b, a_{d-1}, \dots, a_1) := \sum\sum \mu^r_M(b, \mathcal{G}^{s_r}(-), \dots, \mathcal{G}^{s_1}(-))$

$\mathcal{G}^*(t)^d(b, a_{d-1}, \dots, a_1) := \sum\sum t(b, \mathcal{G}^{s_r}(-), \dots, \mathcal{G}^{s_1}(-))$

$\hookrightarrow \mathcal{G}^* : \mathrm{Mod}_{\mathcal{B}}^{nu} \longrightarrow \mathrm{Mod}^{nu}_{\mathcal{A}}$ nu dg functor between dg categories.

The Yoneda embedding is compatible w/ pullback in the sense that

$T : h_B \longrightarrow \mathcal{G}^* h_A \mathcal{G}$ natural transformation.