Workshop on Fukaya categories

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Orientation and Floer homology

Part I. Motivation

Kontsevich's Homological Mirror Symmetry Conjecture:

(M,ω) symplectic Calabi-Yau

$(\check{M},\check{J})$ mirror complex Calabi-Yau

Then as triangulated categories

$$\boxed{D^b(\mathrm{Fuk}(M,\omega)) \simeq D^b(\mathrm{coh}(\check{M},\check{J}))}$$

(derived Fukaya category) (derived cat. of coherent sheaves)

Recollections: Let: L_0, L_1 compact Lagrangian in a symplectic manifold (M,ω) such that L_0 and L_1 intersects transversely

Novikov ring with base field k:

$$\Lambda_k := \left\{ \sum_{i=0}^{+\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in k,\ \lambda_i \in \mathbb{R},\ \lim_{i \to +\infty} \lambda_i = +\infty \right\}$$

Floer complex as a Λ_k-vector space:

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda_k \cdot p$$

To define the differential ∂, we equip M with an ω-compatible almost complex structure J (this is a contractible choice)

$$\boxed{\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u] \in \pi_2(M, L_0 \cap L_1) \\ \mathrm{ind}([u]) = 1}} \# \mathcal{M}(p,q,[u],J) \cdot T^{\omega([u])}\, q}$$

where • $\mathcal{M}(p,q,[u],J)$ is the moduli space of pseudo-holomorphic strips of finite energy and fixed topological type $[u]$ modulo reparametrization $s \mapsto s-a$.

• $\mathrm{ind}\,[u]$ is the Maslov index

- $\omega([u])$ is the energy/symplectic area
- $\# \mathcal{M}$ = signed sum of points when $\mathcal{M}$ is "oriented"...

What does that mean?

Part II. General definitions

<u>Def</u>: V vector space of $\dim_{\mathbb{R}} V = n$
an <u>orientation</u> of V is an equivalence class of non-zero elements of the line $\Lambda^n V$
where $x \sim y$ if $\exists \lambda > 0, x = \lambda y$

(n^{th} alternative power)

More generally for fiber bundle ...

A manifold X is <u>orientable</u> if the tangent bundle TX is orientable.

<u>Obstruction</u>: the first Stiefel-Whitney class is zero.

Def: The spin group Spin(n) (for $n \geq 1$) is the double cover of $SO(n)$:

(which is non-trivial if $n \geq 2$)

$$0 \to \mathbb{Z}/2 \longrightarrow \mathrm{Spin}(n) \longrightarrow SO(n) \longrightarrow 1$$

Rk: There is a Whitehead tower for $O(n)$:

$$\cdots \to \mathrm{Fivebrane}(n) \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \xrightarrow{\xi} SO(n) \to O(n)$$

Def: An orientable real bundle E/X admits a spin structure if there is a principal spin(n)-bundle with 2-fold covering map: $P_{\mathrm{Spin}(n)}(E) \xrightarrow{\eta} P_{SO(n)}(E)$ such that the diagram commutes:

$$\begin{tikzcd}
P_{\mathrm{Spin}(n)}(E) \times \mathrm{Spin}(n) \arrow[r] \arrow[d, "\eta \times \xi"] & P_{\mathrm{Spin}(n)}(E) \arrow[d] \arrow[dr] & \\
P_{SO(n)}(E) \times SO(n) \arrow[r] & P_{SO(n)}(E) \arrow[r] & X
\end{tikzcd}$$

Equivalently, the second Stiefel-Whitney class $w_2(E)$ is zero.

Rk: A spin structure on a Kähler manifold X is:

- the choice of a square root $\sqrt{\Omega^{n,0}}$ of the canonical line bundle $\Omega^{n,0}$ ($n = \dim_{\mathbb{C}} X$).
- (equivalently) a trivialization of the first Chern class $c_1(TX)$ of the tangent bundle.

<u>Def</u>: <u>Relative spin structure</u>

$L \subset M$ be an oriented Lagrangian submanifold

$st \in H^2(M, \mathbb{Z}/2)$ such that $st|_L = w_2(L)$

Fix a triangulation of M such that L is a subcomplex

Choose an oriented real vector bundle V on the 3-skeleton $M_{[3]}$ of M such that $w_2(V) = st$.

Then since $w_2(TL|_{L_{[2]}} \oplus V|_{L_{[2]}}) = 0$

it follows that $TL|_{L_{[2]}} \oplus V|_{L_{[2]}}$ has a spin structure.

The choice of

- an orientation of L,
- an oriented real vector bundle V on $M_{[3]}$,
- a cohomology class $st \in H^2(M, \mathbb{Z}/2)$,
- and a spin structure σ on $TL \oplus V$

is called a <u>relative spin structure</u>.

Rk:
- spin $\Longrightarrow$ relative spin
- depends on the choice of V and of a triangulation of M but: not in practice (up to conjugacy)

<u>Part III. Clean self-intersections</u>

(X,ω) $2n$-dim. symp. manifold (smooth without boundary)

L smooth n-dim. manifold

$\iota : L \hookrightarrow X$ propre Lagrangian immersion ($\iota^*\omega = 0$)

$\iota(L) \subset X$

Consider the fiber product
$$\begin{tikzcd}
L \times_\iota L \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, "pr_X"] & L \arrow[d, "\iota"] \\
L \arrow[r, "\iota"'] & X
\end{tikzcd}$$

we have $L \times_\iota L = \Delta \sqcup R$

(diagonal) (information about self-intersection)

Def: $\iota : L \looparrowright X$ intersect <u>itself cleanly</u> if

$L \times_\iota L$ is a smooth submanifold of $L \times L$

and $\forall (x,y) \in L \times_\iota L$, $d\iota_x(T_x L) \cap d\iota_y(T_y L)$

$$= d(\iota \circ pr_1)_{(x,y)}\, T_{(x,y)}(L \times_\iota L)$$

as subbundles of $(\iota \circ pr_1)^* TX$.

Ex: n even $\mathbb{RP}^n$ not orientable

$$\begin{tikzcd} S^n \arrow[r, "\text{(double cover)}"'] \arrow[d, equal] & \mathbb{RP}^n \arrow[r, hook] \arrow[d, equal] & \mathbb{CP}^n \arrow[d, equal] \\ L \arrow[r] & \iota(L) \arrow[r, hook] & X \end{tikzcd}$$

$L \cong \mathbb{R}$ (diffeomorphism)

Part IV : Moduli spaces

Fix J an almost complex structure on X

Def: $(D^2, \vec{z}, u)$ is called a "<u>marked J-holomorphic disk with boundary over the almost complex (X,J)</u>"

if

$u : D^2 \to (X, J)$ continuous $L^{1,2}$ map pseudo-holomorph in the interior

$\vec{z} = (z_0, \dots, z_m)$ points on ∂D^2 ← (boundary on D^2)

α_{jk} are induced by $z_j \to z_k$ (consistent with the induced orientation on the boundary)

Assume J compatible with ω (almost Kähler manifold (X, ω, J))

<u>Def</u>: Fix $I \subset [\![0, m]\!]$
A marked J-holomorphic disk $(D^2, \vec{z}, u)$ satisfies

"<u>the Lagrangian boundary condition of type I</u>" if

there exist continuous functions $\tilde{u}_{jk} : \alpha_{jk} \to L$

(for all arcs $\{\alpha_{jk}\}$ defined by $\vec{z}$) such that:

① $\iota \circ \tilde{u}_{jk} = u|_{\alpha_{jk}}$

②&③ $(\tilde{u}_{*j}(z_j), \tilde{u}_{j\heartsuit}(z_j)) \in \begin{cases} R & \text{if } j \in I \\ \Delta_L & \text{if } j \notin I \end{cases}$ ↵ "z_j is a jump point"

The set $\{\tilde{u}_{*\heartsuit}\}$ is called <u>a lift</u> of u and induces

$$\begin{tikzcd} \partial D^2 \arrow[r, "\tilde{u}"] \arrow[d, dash] & L \times_\iota L \arrow[d, "pr_X"] \\ (D^2, \partial D^2) \arrow[r, "u"] & (X, \iota(L)) \end{tikzcd}$$

Any biholomorphic map $\varphi : D^2 \to D^2$ acts by:

$$\varphi \cdot (D^2, \vec{z}, u, \tilde{u}) = (D^2, \varphi^{-1}(\vec{z}), u \circ \varphi, \tilde{u} \circ \varphi)$$

We talk about <u>automorphism</u> if φ fixes $(D^2, \vec{z}, u, \tilde{u})$

<u>Def</u>: $(D^2, \vec{z}, u, \tilde{u})$ is <u>stable</u> if

$$|\mathrm{Aut}(D^2, \vec{z}, u, \tilde{u})| < +\infty$$

<u>Def</u>: Let $\beta \in H^2(X, \iota(L), \mathbb{Z})$. The moduli space $\mathcal{M}_{m+1}(J, I, \beta)$ is the set of isomorphism classes

$$\left\{ (D^2, \vec{z}, u, \tilde{u}) \mid u_*[D^2] = \beta, \text{ stable} \right\} /\sim$$

where $(D^2, \vec{z}, u, \tilde{u}) \sim (D^2, \vec{z}\,', u', \tilde{u}')$ if $\exists \varphi : D^2 \to D^2$ bihol. such that $\varphi \cdot (D^2, \vec{z}, u, \tilde{u}) = (D^2, \vec{z}\,', u', \tilde{u}')$

Denote its class by $[D^2, \vec{z}, u, \tilde{u}]$.

$$ev_j : \mathcal{M}_{m+1}(J, I, \beta) \longrightarrow \iota(L)$$
$$[D^2, \vec{z}, u, \tilde{u}] \longmapsto u(z_j).$$

$$ev_j : \mathcal{M}_{m+1}(J, I, \beta) \longrightarrow L \times_2 L$$
$$[D^2, \vec{z}, u, \tilde{u}] \longmapsto \tilde{u}(z_j) = (\tilde{u}_{*j}(z_j), \tilde{u}_{j*}(z_j))$$

Gromov's compactification :

$$\overline{\mathcal{M}}_{m+1}(J, I, \beta) := \left\{ (\overline{\Sigma}, \vec{z}, u, \tilde{u}) \;\middle|\; \begin{array}{l} u_*[D^2] = \beta \\ (\overline{\Sigma}, \vec{z}) \text{ genus } 0 \\ \text{prestable curve} \\ (\overline{\Sigma}, \vec{z}, u, \tilde{u}) \text{ stable} \end{array} \right\} \Big/ \sim$$

Part V Linearization of Cauchy-Riemann operators

(X, ω, J) as before, $p > 2$

$I \subset [\![0, m]\!]$, $m+1$ = number of marked pts.

$$\boxed{\bar{\partial}_J : \widetilde{W}_\delta^{1,p}(X, \iota(L), I) \times ((\partial D^2)^{m+1} \setminus \Delta) \longrightarrow \mathcal{E}}$$

where $\cdot\ \Delta \subset (\partial D^2)^{m+1}$ in which 2 marked pts coincide

- $\mathcal{E} \to \widetilde{W}^{1,p}_\delta(X, \iota(L), I)$

is the Banach space bundle whose fiber at $\underline{v} = (\Delta, v, \tilde{v}) \in \widetilde{W}^{1,p}_\delta(X, \iota(L), I)$ is the Banach space

$$\mathcal{E}_{\underline{v}} := L^p_\delta(\Delta, \Omega^{0,1}(T^*\Delta) \otimes v^*(TX))$$

Cauchy - Riemann operator:

$$\boxed{\bar{\partial}_J(v) := \frac{1}{2}(dv + J \circ dv \circ j)}$$

Then $\mathcal{M}_{m+1}(J, I, \beta) = \ker(\bar{\partial}_J) = \bar{\partial}_J(0)$

$$\subset \widetilde{W}^{1,p}_\delta(X, \iota(L), I)$$

The linearization of $\bar{\partial}_J$ at a pseudo-holomorphic map $v \in \underline{v} = (\Delta, v, \tilde{v})$ gives a Fredholm operator:

$$\boxed{(**)\ D_\delta := D_{\underline{v}} \bar{\partial}_{\delta, J} : W^{1,p}_\delta(\Delta, v^*TX, \tilde{v}^*TL) \to L^p_\delta(\Delta, \Omega^{0,1}(T^*\Delta) \otimes v^*TX)}$$

(Interlude)

Def: A Fredholm operator is a bounded linear operator $T: X \to Y$ between Banach spaces with finite dimensional $\ker T$ and $\operatorname{coker} T$

The index of T is

$$\boxed{\operatorname{ind} T := \dim \ker T - \dim \operatorname{coker} T}$$

We say that (the index of) T is <u>oriented</u> if the determinant line $\det T = \det(\ker T) \otimes \det(\operatorname{coker} T)^*$ is oriented.

$\mathcal{E} = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \cup \{ z \in \mathbb{C} \mid Re(z) \geq 0, |Im(z)| \leq 1 \}$

$\lambda_k : \partial\mathcal{E} \to \mathcal{L}ag(T_{p_k} X)$ a path between the two Lagrangian subspaces $\Lambda_k^{\pm} := d\iota(T_{p_k^{\pm}} L_k^{\pm})$

(+ assumptions...)

$$\boxed{(\heartsuit\heartsuit)\quad \overline{\partial}_{\delta,\lambda_k} := \partial_\tau + J_{p_k}\partial_t : W^{1,p}_\delta(\mathcal{E}, T_{p_k}X, \lambda_k) \to L^p_\delta(\mathcal{E}, \Omega^{0,1}(T^*\mathcal{E}) \otimes T_{p_k}X)}$$

is a Fredholm operator which consists of the highest order terms of the linearized Cauchy-Riemann operator $D_{v_k}\overline{\partial}_J$.

We can glue $(**)$ and $(\heartsuit\heartsuit)$!
Their indices add up to the index of the glued-up operator ([Lockhart – McOwen, '85])

Part VI. Orientability of the moduli space

Assume the Lagrangian L relatively spin and R orientable

Fix $\vec{p} = (x,y) \in R$, $p := \iota(\vec{p})$

Let
$$\Omega_{\vec{p}} := \Omega\left(\mathcal{L}ag^{ori}(T_pX),\ \mathcal{L}_x, \mathcal{L}_y\right)$$
be the space of paths of oriented Lagrangian subspaces of the symplectic vector space T_pX originating at $\mathcal{L}_x := d\iota_x(T_xL)$ and terminating at $\mathcal{L}_y := d\iota_y(T_yL)$.

The local orientations of $\mathcal{L}_x$ and $\mathcal{L}_y$ as Lagrangian subspaces of T_pX are taken to be the one induced by the respective oriented tangent spaces of L at x and y.

$$\mathcal{L}ag^{ori}(T_pX) \simeq U(n)/SO(n)$$

The relative Maslov index of the oriented Lagrangian paths gives an isomorphism:

$$\boxed{\mu_{\lambda_0,\vec{p}} : \pi_0(\Omega_{\vec{p}}) \xrightarrow{\sim} \mathbb{Z}}$$

where $\lambda_{0,\vec{p}}$ is a chosen path in $\Omega_{\vec{p}}$

Put $$\boxed{\Omega := \bigcup_{\vec{p} \in \mathcal{R}} \Omega_{\vec{p}}}$$

Two paths $\lambda_{\vec{p}}$ and $\lambda'_{\vec{p}}$ in Ω are equivalent if the Radov index of the loop $\lambda_{\vec{p}} \bar{\lambda}'_{\vec{p}}$ is even.

(reverse path)

Define $\mathcal{P}_{\vec{p}} := \Omega_{\vec{p}}/\sim$

and the double cover

$$\mathcal{P} := \bigcup_{\vec{p} \in \mathcal{R}} \mathcal{P}_{\vec{p}} \longrightarrow \mathcal{R}$$

<u>Lem</u>:

$\mathcal{P} \to \mathcal{R}$ is trivial on each connected components.

For a path $\lambda_{\vec{p}} \in \Omega_{\vec{p}}$, consider the trivial real vector bundle

$$F_{\lambda_{\vec{p}}} := \bigcup_{t \in [0,1]} \{t\} \times \lambda_{\vec{p}}(t) \longrightarrow [0,1]$$

Let $\sigma_{\lambda_{\vec{p}}} : [0,1] \times \mathbb{R}^n \xrightarrow{\sim} F_{\lambda_{\vec{p}}}$ be a trivialization

The map $(d\iota)^{-1} \circ \sigma_{\lambda_{\vec{p}}|\{0,1\}}$ induces a framing on $\mathcal{L}_x$ and $\mathcal{L}_y$, giving an embedding

$$(\sigma_{\lambda_{\vec{p}}})_* : SO(n) \times SO(\iota^* V_{\vec{p}}) \hookrightarrow P_{SO(n)}(TL \oplus \iota^* V)_{|\vec{p}}$$

so that:

$$\begin{tikzcd}
Spin(n) \times Spin(\iota^* V_{\vec{p}}) / \{\pm 1\} \arrow[r, "\varphi_{\vec{p}}"] \arrow[d] & P_{Spin}(TL \oplus \iota^* V)_{|\vec{p}} \arrow[d] \\
SO(n) \times SO(\iota^* V_{\vec{p}}) \arrow[r, "(\sigma_{\lambda_{\vec{p}}})_*"] & P_{SO}(TL \oplus \iota^* V)_{|\vec{p}}
\end{tikzcd}$$

There are two such $\varphi_{\vec{p}}$ lifting $\sigma_{\lambda_{\vec{p}}}$.

Denote by $\mathcal{T}_{\lambda_{\vec{p}}}$ the space of equivalence classes $[\sigma_{\lambda_{\vec{p}}}]$ of trivializations of $F_{\lambda_{\vec{p}}}$.

Set
$$\mathcal{T} := \bigcup_{\lambda_{\vec{p}}} \mathcal{T}_{\lambda_{\vec{p}}} \longrightarrow \mathcal{P}$$

$\widetilde{\mathcal{T}}_{\lambda_{\vec{p}}}$ space of equivalence classes $[\varphi_{\lambda_{\vec{p}}}]$

$$\widetilde{\mathcal{T}}_{\vec{p}} := \bigcup_{\lambda_{\vec{p}}} \widetilde{\mathcal{T}}_{\lambda_{\vec{p}}} \longrightarrow \mathcal{P}$$

This is the "space of spin stuctures on the Lagrangian path space Ω"

We have:

$$\widetilde{\mathcal{T}} \xrightarrow{\pi_3} \mathcal{T} \xrightarrow{\pi_2} \mathcal{P} \xrightarrow{\pi_1} \mathcal{R} \xrightarrow{pr_X} pr_X(\mathcal{R}) \subset X$$

(double cover)

("self-intersection of the immersed Lagrangian)

Family of operators (PP) can be pullbacked:

$$\begin{tikzcd}
\mathcal{D} := \overline{\partial}_{\delta,\lambda} \arrow[r] \arrow[d] & \widetilde{\mathcal{T}} \arrow[d] \\
\overline{\partial}_{\delta,\lambda} \arrow[r] & \mathcal{P}
\end{tikzcd}$$

Prop: The determinant line $\det \mathcal{D}$ descends to a real line bundle

$$\Theta \longrightarrow pr_X \mathcal{R}$$

which we pullback:

$$\ell_{\mathcal{R}} := pr_X^* \Theta \longrightarrow \mathcal{R}.$$

Set

$$\mathcal{L} := \det(D_\delta) \otimes \bigotimes_{j \in I} evl_j^*(\ell_{\mathcal{R}})$$

$$\big\downarrow \leftarrow \text{line bundle}$$

$$\widetilde{W}_\delta^{1,p}(X, \iota(L), I) \times \left((\partial D^2)^{m+1} \setminus \Delta \right)$$

where $D_\delta := \{ D_{\delta, p} \}_{p \in \mathcal{R}^{|I|}}$ family of $(**)$.

The orientation line bundle of $\overline{\mathcal{R}}_{m+1}(J, I, \beta)$ is $\mathcal{O}_{\mathcal{R}} \simeq \det(\mathcal{D}_\delta) \otimes \mathcal{O}_{\mathcal{R}}$

(orientation line bundle of $\mathcal{R}$)

Main theorem

(X, ω, J) and $\iota : L \to X$ as before ...

Assume :

1. L is relatively spin (e.g. L spin)
2. $\mathcal{R}$ is orientable
3. the bundle $\tilde{\mathcal{C}} \to \mathcal{R}$ is trivial

Then $\boxed{\mathcal{O}_{\mathcal{M}} \simeq \left(\bigotimes_{j \in I} ev_j^* \ell_{\mathcal{R}} \right) \otimes \mathcal{O}_{\mathcal{R}}}$

Moreover, a section of this bundle is determined by fixing a choice of :

1'. a relative spin structure on L

2'. an orientation on $\mathcal{R}$

3'. a section $s : \mathcal{R} \to \tilde{\mathcal{C}}$ of $\tilde{\mathcal{C}} \to \mathcal{R}$.

Corollary : $\mathcal{M}_{m+1}(J, I, \beta)$ is orientable (canonically)

Rk: This generalizes [Fukaya - Oh - Ohta - Ono] by working with $\mathcal{R}$ instead of its image in X so that $pr_X : \mathcal{R} \to pr_X(\mathcal{R}) \subset \iota(L)$ would be a trivial cover in this case.

Proof of the main theorem:

· We have $\mathcal{O}_{\sigma} \simeq \mathcal{O}_{\mathcal{R}} \otimes \det \mathcal{D}_{\delta}$

We glue the operators "$\bar{\partial}_{\delta, p_k}$ (♡♡)" and "D_δ (∗∗)" by partition of unity to get a Fredholm operator $D_{\delta,\lambda}$.

By a generalized "index sum formula":

$$\operatorname{Ind} D_{\delta,\lambda} \simeq \prod_{j \in I} \left(\operatorname{Ind} \bar{\partial}_{\delta, \lambda_{\vec{p}_j}} \oplus T_{\vec{p}_j}\mathcal{R}\right) \times_{T_{\vec{p}_j}\mathcal{R}} \operatorname{Ind} D_\delta$$

· Each $T_{\vec{p}_j}\mathcal{R}$ appears twice so their effects on orientation cancel locally at $\vec{p}_j$.

· Since $\mathcal{R}$ is orientable by ②, any choice of such an orientation gives a consistent way to orient the tangent spaces $T_{\vec{p}'} \mathcal{R}$ at $\vec{p} \in \mathcal{R}$.

- It remains to show that $\operatorname{Ind} D_{\delta,\lambda}$ has a canonical orientation:
  - Give an orientation of $D_{\delta,\lambda}|_{w}$ for each "holomorphic disk" w.
  - Show that this orientation depends only on the relative spin structure
  - If $\{w_t\}_{0 \le t \le 1}$ is a homotopy from w_0 to $w = w_1$, then the determinant line bundle of $D_{\delta,\lambda}|_{w_t}$ is trivial since $[0,1]$ is contractible
  - Thus we obtain an orientation of $\det D_{\delta,\lambda}|_{w}$ which may depend on the homotopy
  - L is relatively spin implies that the induced orientation does not depend on the choice of a homotopy. $\square$