Uniqueness of 2-D compressible vortex sheets

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Abstract

We consider compressible vortex sheets for the isentropic Euler equations of gas dynamics in two space dimensions. Under a supersonic condition that precludes violent instabilities, in previous papers [3, 4] we have studied the linearized stability and proved the local existence of piecewise smooth solutions to the nonlinear problem. This is a free boundary nonlinear hyperbolic problem with two main difficulties: the free boundary is characteristic, and the so-called Lopatinskii condition holds only in a weak sense, which yields losses of derivatives. In the present paper we prove that sufficiently smooth solutions are unique.

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1 Introduction and main results

We consider Euler equations of isentropic gas dynamics in the whole plane $\mathbb{R}^2$. Denoting by $\mathbf{u}$ the velocity of the fluid and ρ the density, the equations read:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho\, \mathbf{u}) = 0\,, \\ \partial_t (\rho\, \mathbf{u}) + \nabla \cdot (\rho\, \mathbf{u} \otimes \mathbf{u}) + \nabla\, p = 0\,, \end{cases} \tag{1}$$

where $p = p(\rho)$ is the pressure law. In all this paper, p is assumed to be a strictly increasing function of ρ, defined on $]0, +\infty[$. We also assume that p is a $\mathcal{C}^\infty$ function of ρ. The speed of sound $c(\rho)$ in the fluid is then defined by the relation:

$$c(\rho) := \sqrt{p'(\rho)}\,.$$

In this paper, we are interested in solutions to (1) that are smooth on either side of a curve $\Gamma(t) := \{y_2 = \varphi(t, y_1)\,, t \in [0, T_0]\,, y_1 \in \mathbb{R}\}$, and such that, at each time $t \in [0, T_0]$, the tangential velocity (with respect to $\Gamma(t)$) is the only quantity that experiments a jump across this curve. The density, and the normal velocity should be continuous across $\Gamma(t)$. For such solutions, the jump conditions across $\Gamma(t)$ read (see [3, 4]):

$$\begin{aligned} \partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu\,, \\ \rho^+ = \rho^-\,, \end{aligned} \tag{2}$$

where $\pm$ denote the states on the two sides of the curve and $\nu := (-\partial_{y_1}\varphi, 1)$ is a (space) normal vector to $\Gamma(t)$. As detailed in [3], for the isentropic Euler equations (1), these solutions are exactly the contact discontinuities in the sense of Lax [6]. Due to the jump of tangential velocity, the vorticity is concentrated along $\Gamma(t)$. For this reason tangential discontinuities are also called vortex sheets.

The interface $\Gamma(t)$, or equivalently the function φ, is part of the unknowns of the problem. We thus deal with a free boundary problem.

Problem (1), (2) has simple solutions like the stationary rectilinear vortex sheets:

$$(\rho, \mathbf{u}) = \begin{cases} \overline{U}^+ = (\overline{\rho}, \overline{v}, 0)\,, & \text{if } y_2 > 0, \\ \overline{U}^- = (\overline{\rho}, -\overline{v}, 0)\,, & \text{if } y_2 < 0, \end{cases} \tag{3}$$

where $\overline{\rho}, \overline{v} \in \mathbb{R}$, $\overline{\rho} > 0$. Up to Galilean transformations, every rectilinear vortex sheet has this form and, without loss of generality, one may also assume $\overline{v} > 0$.

The uniform Kreiss-Lopatinski stability condition is never satisfied by contact discontinuities in two or three space dimensions, see e.g. [7, 5] or [8, page 222]. In three space dimensions, every contact discontinuity is even violently unstable (this violent instability is the analogue of the Kelvin-Helmholtz instability for incompressible fluids), while in two space dimensions a large jump of the tangential velocity makes the contact discontinuity weakly stable. A precise study of this *weak stability* has been performed in [3], where it is shown that for such *weakly stable* vortex sheets, the linearized equations satisfy an a priori L^2 energy estimate with a loss of one derivative. In [4] the existence of contact discontinuity solutions to the Euler equations is proved by extending the linear well-posedness result to Sobolev spaces and then by introducing a suitable Nash-Moser iteration scheme.

In the present paper we prove the uniqueness of smooth enough vortex sheets.

Given two solutions $V_1 = (\rho_1, \mathbf{u}_1), \varphi_1$, and $V_2 = (\rho_2, \mathbf{u}_2), \varphi_2$, let us denote the two sides determined by each unknown front φ_i by $\Omega_i^\pm = \{(t, y) \in [0, T_0] \times \mathbb{R}^2 : \pm(y_2 - \varphi_i(t, y_1)) > 0\}$. Each solution has states $V_i^\pm$ on $\Omega_i^\pm$, $i = 1, 2$.

The main result of the paper is the following Theorem:

Theorem 1. *Let $T_0 > 0$. Assume that the stationary solution defined by* (3) *satisfies the "supersonic"condition:*

$$\overline{v} > \sqrt{2}\, c(\overline{\rho})\,. \tag{4}$$

Let us consider two solutions V_1, φ_1 and V_2, φ_2 of (1), (2) *that have the form $V_i^\pm = \overline{U}^\pm + \dot{V}_i^\pm$ with $\dot{V}_i^\pm \in H^7(\Omega_i^\pm)$, and $\varphi_i \in H^8(]0, T_0[\times\mathbb{R})$. There exists a constant $K > 0$ such that, if:*

$$\|\dot{V}_2^\pm\|_{W^{2,\infty}(\Omega_2^\pm)} + \|\varphi\|_{W^{3,\infty}(]0,T_0[\times\mathbb{R})} \le K\,, \tag{5}$$

and if $V_1 = V_2$ and $\varphi_1 = \varphi_2$ at the initial time $t = 0$, then $V_1 = V_2, \varphi_1 = \varphi_2$ for all $t \in [0, T_0]$.

Let us observe that our method to prove uniqueness also applies to weakly stable noncharacteristic waves such as shock waves or phase transitions (whose existence follows from the analysis in [4]). We shall focus here on contact discontinuities since the major technical problems arise in this situation. As usual, the analysis of noncharacteristic discontinuities presents some simplifications so we shall not deal with this case.

In order to solve problem (1), (2), it is convenient to straighten the unknown front in order to work in a fixed domain, see [3, 4]. To do so, the unknowns $(\rho, \mathbf{u})$, that are smooth functions on either side of $\{y_2 = \varphi(t, y_1)\}$, are replaced by the functions:

$$(\rho_\sharp^\pm, \mathbf{u}_\sharp^\pm)(t, x_1, x_2) := (\rho, \mathbf{u})(t, x_1, \Phi(t, x_1, \pm x_2))\,, \tag{6}$$

where Φ is any continuous function, smooth away from $\{x_2 = 0\}$, and satisfying the following conditions:

$$\partial_{x_2}\Phi(t, x_1, x_2) \geq \kappa > 0 \,, \quad \Phi(t, x_1, 0) = \varphi(t, x_1) \,.$$

With these requirements for Φ, all functions $\rho_\sharp^\pm, \mathbf{u}_\sharp^\pm$ are smooth on the fixed domain $\{x_2 > 0\}$. We also define the functions:

$$\Phi^\pm(t, x_1, x_2) := \Phi(t, x_1, \pm x_2) \,,$$

that are both smooth on the half-space $\{x_2 > 0\}$.

Let us denote by $v_\sharp^\pm$ and $u_\sharp^\pm$ the two components of the velocity field, that is, $\mathbf{u}_\sharp^\pm = (v_\sharp^\pm, u_\sharp^\pm)$. Let us also set $U^\pm = (\rho_\sharp^\pm, v_\sharp^\pm, u_\sharp^\pm)$. After the change of variables (6) the nonlinear equations (1) read:

$$\begin{aligned}
&\partial_t U^+ + A_1(U^+)\,\partial_{x_1} U^+ + \frac{1}{\partial_{x_2}\Phi^+}\left(A_2(U^+) - \partial_t\Phi^+ - \partial_{x_1}\Phi^+\,A_1(U^+)\right)\partial_{x_2} U^+ = 0 \,, \\
&\partial_t U^- + A_1(U^-)\,\partial_{x_1} U^- + \frac{1}{\partial_{x_2}\Phi^-}\left(A_2(U^-) - \partial_t\Phi^- - \partial_{x_1}\Phi^-\,A_1(U^-)\right)\partial_{x_2} U^- = 0 \,,
\end{aligned} \tag{7}$$

in the domain $Q_{T_0} := \{t \in [0, T_0] \,, x_1 \in \mathbb{R} \,, x_2 > 0\}$, where we have set:

$$A_1(U) := \begin{pmatrix} v_\sharp & \rho_\sharp & 0 \\ \dfrac{p'(\rho_\sharp)}{\rho_\sharp} & v_\sharp & 0 \\ 0 & 0 & v_\sharp \end{pmatrix} \,, \quad A_2(U) := \begin{pmatrix} u_\sharp & 0 & \rho_\sharp \\ 0 & u_\sharp & 0 \\ \dfrac{p'(\rho_\sharp)}{\rho_\sharp} & 0 & u_\sharp \end{pmatrix} \,.$$

The boundary conditions (2) now read:

$$\begin{aligned}
\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} &= \varphi \,, \\
\partial_t\varphi = -v^+_{\sharp|x_2=0}\,\partial_{x_1}\varphi + u^+_{\sharp|x_2=0} &= -v^-_{\sharp|x_2=0}\,\partial_{x_1}\varphi + u^-_{\sharp|x_2=0} \,, \\
\rho^+_{\sharp|x_2=0} &= \rho^-_{\sharp|x_2=0} \,,
\end{aligned} \tag{8}$$

at $\Sigma_{T_0} := \{t \in [0, T_0] \,, x_1 \in \mathbb{R}\}$. With an obvious definition for the differential operator L, system (7) also reads:

$$L(U^+, \Phi^+)U^+ = 0 \,, \quad L(U^-, \Phi^-)U^- = 0 \,. \tag{9}$$

When no confusion is possible, we also write this system under the form $L(U, \Phi)U = 0$, where U stands for the vector (U^+, U^-) and Φ for (Φ^+, Φ^-). In the same way, the boundary conditions (8) can be rewritten in the compact form:

$$\begin{aligned}
&\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi \,, \\
&\mathbb{B}(U^+_{|x_2=0}, U^-_{|x_2=0}, \varphi) = 0 \,.
\end{aligned} \tag{10}$$

In [4] we have shown how to construct a solution U, Φ to (9), (10), where Φ has the form $\Phi^\pm = \pm x_2 + \dot{\Phi}^\pm$ and satisfies:

$$\partial_{x_2}\Phi^+(t, x_1, x_2) \geq \kappa \,, \quad \partial_{x_2}\Phi^-(t, x_1, x_2) \leq -\kappa \,, \tag{11}$$

for a suitable constant $\kappa > 0$, and the eikonal equations:

$$\partial_t\Phi^+ + v_\sharp^+\,\partial_{x_1}\Phi^+ - u_\sharp^+ = \partial_t\Phi^- + v_\sharp^-\,\partial_{x_1}\Phi^- - u_\sharp^- = 0 \,, \tag{12}$$

in the whole domain Q_{T_0}.

The eikonal equations (12), that are clearly imposed on the boundary $\{x_2 = 0\}$ by (8) ensure that the matrices $A_2(U^\pm) - \partial_t \Phi^\pm - \partial_{x_1}\Phi^\pm A_1(U^\pm)$ have a constant rank in the whole domain $\{x_2 \geq 0\}$, and not only on the boundary. This constant rank property is crucial in [3, 4] to perform a Kreiss symmetrizers construction and to derive a priori estimates.

Theorem 1 is a consequence of the following auxiliary result:

Theorem 2. *Let $T_0 > 0$. Assume that the stationary solution defined by* (3) *satisfies* (4). *Let us consider two solutions of* (9), (10), (11), (12), *U_1, Φ_1, φ_1, and U_2, Φ_2, φ_2 of the form $U_i^\pm = \overline{U}^\pm + \dot{U}_i^\pm$ with $\dot{U}_i^\pm \in H^5(Q_{T_0})$, $\Phi_i = \pm x_2 + \dot{\Phi}_i^\pm$ with $\dot{\Phi}_i^\pm \in H^5(Q_{T_0})$, and $\varphi_i \in H^4(\Sigma_{T_0})$. There exists a constant $K > 0$ such that, if:*

$$\|\dot{U}_2^\pm\|_{W^{2,\infty}(Q_{T_0})} + \|\dot{\Phi}_2^\pm\|_{W^{3,\infty}(Q_{T_0})} \leq K\,,$$

and if $U_1 = U_2$, $\Phi_1 = \Phi_2$, and $\varphi_1 = \varphi_2$ at the initial time $t = 0$, then $U_1 = U_2, \Phi_1 = \Phi_2, \varphi_1 = \varphi_2$ for all $t \in [0, T_0]$.

Theorem 2 is a uniqueness result in the straightened variables (t, x_1, x_2), for a particular choice of the *lifting function* Φ (this lifting function is supposed to satisfy the eikonal equations (12)). Let us note that the physical problem is in the original variables (t, y_1, y_2) so the "physically relevant" result is Theorem 1. However energy estimates and therefore uniqueness of solutions are more easily derived in a fixed domain so we shall use Theorem 2 as an intermediate step in the proof of Theorem 1.

The rest of this paper is devoted to the proof of Theorem 2 then to the proof of Theorem 1. The proof of Theorem 2 follows the arguments of [2] with several modifications that arise from our functional framework (that is not the same as in [2]). However, the proof of Theorem 1 differs from the corresponding analysis in [2] because our choice of lifting function is not as explicit as in [2]. We shall therefore need to develop new arguments that do not appear in [2] to handle the construction of an appropriate lifting function. This is due to the characteristic nondissipative nature of the boundary conditions in our problem.

2 Proof of Theorem 2

For convenience, in this section we drop the $\sharp$ index and only keep the $+$ and $-$ exponents when a confusion is possible. Given two solutions U_1, Φ_1, φ_1 and U_2, Φ_2, φ_2 with the regularity stated in Theorem 2, we denote:

$$U := U_1 - U_2\,, \qquad \Phi := \Phi_1 - \Phi_2\,, \qquad \varphi := \varphi_1 - \varphi_2\,.$$

For the sake of shortness, we also introduce the notation:

$$\tilde{A}(U, \Phi) := \frac{1}{\partial_{x_2}\Phi} \left(A_2(U) - \partial_t \Phi - \partial_{x_1}\Phi\, A_1(U) \right)\,.$$

The calculation of the difference $L(U_1, \Phi_1)U_1 - L(U_2, \Phi_2)U_2$ gives (see e.g. [2]):

$$\begin{aligned} & L(U_2, \Phi_2)U + [\mathrm{d}A_1(U_2)U]\, \partial_{x_1} U_2 + [\mathrm{d}\tilde{A}(U_2, \Phi_2)(U, \Phi)]\, \partial_{x_2} U_2 \\ & +[A_1(U_1) - A_1(U_2)]\, \partial_{x_1} U + [A_1(U_1) - A_1(U_2) - \mathrm{d}A_1(U_2)U]\, \partial_{x_1} U_2 \quad (13) \\ & +[\tilde{A}(U_1, \Phi_1) - \tilde{A}(U_2, \Phi_2)]\, \partial_{x_2} U + [\tilde{A}(U_1, \Phi_1) - \tilde{A}(U_2, \Phi_2) - \mathrm{d}\tilde{A}(U_2, \Phi_2)(U, \Phi)]\, \partial_{x_2} U_2 = 0\,. \end{aligned}$$

Such an equation may be simplified by introducing the "good unknown" as in [1]:

$$\dot{U} := U - \frac{\partial_{x_2} U_2}{\partial_{x_2} \Phi_2}\, \Phi\,. \tag{14}$$

The substitution of (14) in (13) yields (see again [1]) the system of equations[1]:

$$L(U_2, \Phi_2)\dot{U} + C(U_2, \Phi_2)\dot{U} = f\,, \tag{15}$$

where we have set:

$$C(U_2, \Phi_2)\dot{U} := [\mathrm{d}A_1(U_2)\dot{U}]\, \partial_{x_1} U_2 + \frac{1}{\partial_{x_2}\Phi_2}\, [\mathrm{d}A_2(U_2)\dot{U} - \partial_{x_1}\Phi_2\, \mathrm{d}A_1(U_2)\dot{U}]\, \partial_{x_2} U_2\,,$$

and where the right-hand side in (15) is the quadratic term $f = -\sum_{i=1}^4 f_i$ with:

$$f_1 := [\mathrm{d}A_1(U_2)U]\, \partial_{x_1} U\,, \tag{16a}$$

$$f_2 := [A_1(U_1) - A_1(U_2) - \mathrm{d}A_1(U_2)U]\, \partial_{x_1} U_1\,, \tag{16b}$$

$$f_3 := [\mathrm{d}\tilde{A}(U_2, \Phi_2)(U, \Phi)]\, \partial_{x_2} U\,, \tag{16c}$$

$$f_4 := [\tilde{A}(U_1, \Phi_1) - \tilde{A}(U_2, \Phi_2) - \mathrm{d}\tilde{A}(U_2, \Phi_2)(U, \Phi)]\, \partial_{x_2} U_1\,. \tag{16d}$$

Next we consider the boundary conditions. Let us introduce the matrices:

$$b(t, x_1) := \begin{pmatrix} 0 & (v_2^+ - v_2^-)_{|x_2=0} \\ 1 & v_2^+{}_{|x_2=0} \\ 0 & 0 \end{pmatrix}\,,$$

$$M(t, x_1) := \begin{pmatrix} 0 & \partial_{x_1}\varphi_2 & -1 & 0 & -\partial_{x_1}\varphi_2 & 1 \\ 0 & \partial_{x_1}\varphi_2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}\,.$$

We take the difference of the boundary conditions (8) for the two solutions and we obtain:

$$\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi\,,$$
$$b\, \nabla\varphi + M\, U_{|x_2=0} = g\,,$$

where we have set $\nabla\varphi = (\partial_t\varphi, \partial_{x_1}\varphi)$ and:

$$g := -\begin{pmatrix} (v^+ - v^-)_{|x_2=0}\, \partial_{x_1}\varphi \\ v^+{}_{|x_2=0}\, \partial_{x_1}\varphi \\ 0 \end{pmatrix}\,. \tag{17}$$

In terms of the good unknown defined by (14), the boundary conditions read:

$$\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi\,,$$

$$\underbrace{b\, \nabla\varphi + M \begin{pmatrix} \partial_{x_2} U_2^+ / \partial_{x_2}\Phi_2^+ \\ \partial_{x_2} U_2^- / \partial_{x_2}\Phi_2^- \end{pmatrix}_{|x_2=0}}_{b_\sharp} \varphi + M\, \dot{U}_{|x_2=0} = g\,. \tag{18}$$

We wish to apply the L^2 a priori estimate for (15), (18) that we have derived in [3]. Given any $0 < T \leq T_0$, we assume that the solution (U_2, Φ_2) satisfies:

$$\|\dot{U}_2^\pm\|_{W^{2,\infty}(Q_T)} + \|\dot{\Phi}_2^\pm\|_{W^{3,\infty}(Q_T)} \leq K\,, \tag{19}$$

where $Q_T := \{t \in [0, T]\,, x_1 \in \mathbb{R}\,, x_2 > 0\}$ and K is a positive constant. Then the following result holds:

[1]The computations with the "good unknown" in [1] require $\mathcal{C}^2$ smoothness of the solutions, which is obtained here from the Sobolev imbedding Theorem.

Lemma 1. *Assume that the particular solution* (3) *satisfies the "supersonic"condition* (4), *and that the solution* $U_2^{\pm}, \Phi_2^{\pm}$ *satisfies* (19). *There exist positive constants* $K_0 > 0$, $C_0 > 0$ *independent of* T*, such that if* $K \leq K_0$*, then the following estimate holds:*

$$\|\dot{U}\|_{L^2(Q_T)} + \|\varphi\|_{H^1(\Sigma_T)} \leq C_0 \left(\|f\|_{H^1(Q_T)} + \|g\|_{H^1(\Sigma_T)}\right). \tag{20}$$

Proof. From $U_1 = U_2$, $\Phi_1 = \Phi_2$, and $\varphi_1 = \varphi_2$ at the initial time $t = 0$, we infer $f = g = 0$ at $t = 0$. We then extend both f and g by 0 for $t < 0$ and we obtain $f \in H^1(]-\infty, T] \times \mathbb{R}^2_+)$, $g \in H^1(]-\infty, T] \times \mathbb{R})$ from the classical Theorems on products of Sobolev functions. We can also extend U, Φ, φ, and therefore $\dot{U}$ by zero for $t < 0$. The initial conditions for $U_{1,2}, \Phi_{1,2}$ show that $U, \Phi, \dot{U}$ belong to $H^1(]-\infty, T] \times \mathbb{R}^2_+)$ and φ belongs to $H^1(]-\infty, T] \times \mathbb{R})$. Moreover, extending the coefficients (U_2, Φ_2) for negative times in an arbitrary way, the equations (15), (18) are satisfied for all $t \in]-\infty, T_0]$, and not only for $t \in [0, T_0]$.

If K is sufficiently small, then (U_2, Φ_2) satisfy all the assumptions of Theorem 3 in [4]. We apply Theorem 3 in [4] and infer that the unique solution $(\dot{U}, \varphi)$ of (15), (18) satisfies (20). □

We notice that from $U, \Phi \in H^5(Q_{T_0})$ and equation (7) we may deduce $\partial_t^j U = \partial_t^j \Phi = 0$ at time $t = 0$, for $j = 0, \ldots, 4$. Therefore we may extend U, Φ by 0 for $t < 0$ and we obtain $U, \Phi, \dot{U} \in H^5(]-\infty, T] \times \mathbb{R}^2_+)$. This property ensures that all constants in the Sobolev imbeddings and interpolation inequalities used in the following calculations will not depend on T.

Now, we estimate the right-hand side f of (15):

Lemma 2. *Let* f *be defined by* (16). *Under the assumptions of Theorem 2 there exists a positive constant* C *independent of* T *such that:*

$$\|f\|_{H^1(Q_T)} \leq C \left(\|U\|_{H^{9/4}(Q_T)} + \|\Phi\|_{H^{9/4}(Q_T)}\right) 2\,. \tag{21}$$

Proof. We estimate the four terms given in (16). The most critical cases concern f_3 and f_4 and we show them in detail. The estimates of f_1 and f_2 are proved in a similar way. Denoting a generic space or time derivative by ∂, from (16c) one has:

$$\begin{aligned}\|f_3\|_{H^1(Q_T)} \leq C\,(&\|U\,\partial U\|_{L^2(Q_T)} + \|\partial U\,\partial U\|_{L^2(Q_T)} + \|U\,\partial^2 U\|_{L^2(Q_T)} \\ &+ \|\partial\Phi\,\partial U\|_{L^2(Q_T)} + \|\partial\Phi\,\partial^2 U\|_{L^2(Q_T)} + \|\partial^2\Phi\,\partial U\|_{L^2(Q_T)})\,.\end{aligned}$$

By the Hölder inequality:

$$\|w_1\, w_2\|_{L^2(Q_T)} \leq \|w_1\|_{L^{12/5}(Q_T)}\, \|w_2\|_{L^{12}(Q_T)}\,,$$

and the Sobolev imbeddings $H^{1/4}(Q_T) \hookrightarrow L^{12/5}(Q_T)$, $H^{5/4}(Q_T) \hookrightarrow L^{12}(Q_T)$, we infer:

$$\|f_3\|_{H^1(Q_T)} \leq C\, \|U\|_{H^{9/4}(Q_T)} \left(\|U\|_{H^{5/4}(Q_T)} + \|\Phi\|_{H^{9/4}(Q_T)}\right). \tag{22}$$

Using a similar Hölder inequality and Sobolev imbeddings, from (16d) one has:

$$\begin{aligned}\|f_4\|_{H^1(Q_T)} &\leq C\,(\|U\,U\|_{L^2(Q_T)} + \|U\,\partial U\|_{L^2(Q_T)} + \|\partial\Phi\,\partial U\|_{L^2(Q_T)} \\ &\qquad + \|U\,\partial^2\Phi\|_{L^2(Q_T)} + \|\partial\Phi\,\partial^2\Phi\|_{L^2(Q_T)}) \\ &\leq C\,(\|U\|_{H^{5/4}(Q_T)} + \|\Phi\|_{H^{9/4}(Q_T)})2\,.\end{aligned} \tag{23}$$

From (16a), (16b) we also have (the details are omitted):

$$\|f_1\|_{H^1(Q_T)} + \|f_2\|_{H^1(Q_T)} \leq C \left(\|U\|_{H^{5/4}(Q_T)} + \|U\|_{H^{9/4}(Q_T)}\right) \|U\|_{H^{5/4}(Q_T)}\,. \tag{24}$$

From (22), (23), (24), and the obvious estimate $\|U\|_{H^{5/4}(Q_T)} \leq \|U\|_{H^{9/4}(Q_T)}$, we get (21). □

Now we estimate the right-hand side g of (18):

Lemma 3. *Let g be defined by* (17)*. Under the assumptions of Theorem 2 there exists a positive constant C independent of T such that:*

$$\|g\|_{H^1(\Sigma_T)} \leq C \, \|U\|_{H^2(Q_T)} \, \|\varphi\|_{H^2(\Sigma_T)} \, . \tag{25}$$

Proof. From the definition of g given in (17) we have:

$$\|g\|_{H^1(\Sigma_T)} \leq C \, (\|U_{|x_2=0} \, \partial\varphi\|_{L^2(\Sigma_T)} + \|U_{|x_2=0} \, \partial^2\varphi\|_{L^2(\Sigma_T)} + \|\partial U_{|x_2=0} \, \partial\varphi\|_{L^2(\Sigma_T)}) \, .$$

The thesis easily follows from the imbeddings $H^2(Q_T) \hookrightarrow L^\infty(Q_T)$ and $H^{1/2}(\Sigma_T) \hookrightarrow L^4(\Sigma_T)$. ☐

The next step is to estimate Φ in terms of U or $\dot{U}$. We first compute the equation satisfied by Φ. The difference of the two eikonal equations (12) for Φ_1 and Φ_2 gives (for each state $+$ and $-$):

$$\partial_t \Phi + v_2 \, \partial_{x_1} \Phi = u - v \, \partial_{x_1} \Phi_1 \, , \tag{26}$$

where $u = u_1 - u_2$ and $v = v_1 - v_2$. By the introduction of the "the good unknown" (14), we also have:

$$\partial_t \Phi + v_2 \, \partial_{x_1} \Phi + \frac{1}{\partial_{x_2} \Phi_2} \, (\partial_{x_2} v_2 \, \partial_{x_1} \Phi_1 - \partial_{x_2} u_2) \; \Phi = \dot{u} - \dot{v} \, \partial_{x_1} \Phi_1 \, . \tag{27}$$

The result is the following:

Lemma 4. *Under the assumptions of Theorem 2 there exists a positive constant C independent of T such that:*

$$\|\Phi\|_{H^{9/4}(Q_T)} \leq C \, \|U\|_{H^{9/4}(Q_T)} \, , \tag{28a}$$

$$\|\Phi\|_{L^2(Q_T)} \leq C \, \|\dot{U}\|_{L^2(Q_T)} \, . \tag{28b}$$

Proof. By taking derivatives of order two and three of (26) and integrating by parts over Q_T we easily show:

$$\|\Phi\|_{H^2(Q_T)} \leq C \, \|U\|_{H^2(Q_T)} \, , \quad \text{and} \quad \|\Phi\|_{H^3(Q_T)} \leq C \, \|U\|_{H^3(Q_T)} \, .$$

Therefore the linear map $U \mapsto \Phi$ is bounded in $H^2(Q_T)$ and $H^3(Q_T)$. By interpolation we infer (28a). Next we consider (27). Multiplying by Φ and integrating by parts over Q_T immediately yields (28b). Recall that Φ vanishes for $t < 0$. ☐

From the definition of "the good unknown" given in (14) and from (28b) we get:

$$\|U\|_{L^2(Q_T)} \leq C \, \|\dot{U}\|_{L^2(Q_T)} \, .$$

Then, from (20), (21), (25) and (28) we may deduce:

$$\|U\|_{L^2(Q_T)} + \|\varphi\|_{H^1(\Sigma_T)} \leq C \, (\|U\|^2_{H^{9/4}(Q_T)} + \|\varphi\|^2_{H^2(\Sigma_T)}) \, .$$

Using the interpolation inequalities:

$$\|U\|_{H^{9/4}(Q_T)} \leq C \, \|U\|^{1/2}_{H^{9/2}(Q_T)} \, \|U\|^{1/2}_{L^2(Q_T)} \, ,$$

$$\|\varphi\|_{H^2(\Sigma_T)} \leq C \, \|\varphi\|^{1/2}_{H^3(\Sigma_T)} \, \|\varphi\|^{1/2}_{H^1(\Sigma_T)} \, ,$$

we eventually get the estimate:

$$\|U\|_{L^2(Q_T)} + \|\varphi\|_{H^1(\Sigma_T)} \leq C \, (\|U\|_{H^{9/2}(Q_T)} + \|\varphi\|_{H^3(\Sigma_T)}) \, (\|U\|_{L^2(Q_T)} + \|\varphi\|_{H^1(\Sigma_T)}) \, . \tag{29}$$

We remark that the constant C in (29) does not depend on T.

By the elementary inequality $\|w\|_{L^2(0,T)} \le T\, \|\partial_t w\|_{L^2(0,T)}$ if $w_{t=0} = 0$, we deduce:

$$\|U\|_{H^4(Q_T)} \le C\,T\, \|U\|_{H^5(Q_T)} \, .$$

Another interpolation argument gives $\|U\|_{H^{9/2}(Q_T)} \le C\,T^{1/2}\, \|U\|_{H^5(Q_T)}$, where C is still independent of T. Similarly we have $\|\varphi\|_{H^3(\Sigma_T)} \le C\,T\, \|\varphi\|_{H^4(\Sigma_T)}$, with C independent of T. Therefore from (29) we infer:

$$\|U\|_{L^2(Q_T)} + \|\varphi\|_{H^1(\Sigma_T)} \le C\,T^{1/2} \left(\|U\|_{H^5(Q_T)} + T_0^{1/2}\, \|\varphi\|_{H^4(\Sigma_T)} \right) \left(\|U\|_{L^2(Q_T)} + \|\varphi\|_{H^1(\Sigma_T)} \right) , \quad (30)$$

with C independent of T. Since U belongs to $H^5(Q_{T_0})$ and φ belongs to $H^4(\Sigma_{T_0})$, (30) yields $U = 0, \varphi = 0$ for $t \in [0,T]$, provided that T is taken sufficiently small. From (28a), we also have $\Phi = 0$ for $t \in [0,T]$.

By repeating the same argument on the intervals $[T, 2T], [2T, 3T], \dots,$ we prove that $U = 0, \Phi = 0, \varphi = 0$ on the whole time interval $[0, T_0]$. This completes the proof of Theorem 2.

Let us note that the regularity required for the uniqueness Theorem 2 is less than the minimal regularity of solutions constructed in [4].

3 Proof of Theorem 1

We consider a solution $V = (\rho, v, u), \varphi$ of the equations (1), (2) in the original variables (t, y_1, y_2). Our goal is to construct a lifting function Φ and to apply Theorem 2 in the straightened variables.

Let us first develop some formal arguments to understand how Φ should be constructed. The function Φ should satisfy (12), that is:

$$\partial_t \Phi(t, x_1, x_2) + v(t, x_1, \Phi(t, x_1, x_2))\, \partial_{x_1} \Phi(t, x_1, x_2) - u(t, x_1, \Phi(t, x_1, x_2)) = 0 \, ,$$

where the coefficients v, u are known. This is a quasilinear equation with coefficients that depend on (t, x_1). The construction and the regularity analysis of solutions for such equations are not very practical. Instead of constructing Φ, we are going to show that it is by far easier to construct its "inverse". More precisely, let us assume that Φ is known and satisfies (11). Let $\Psi(t, y_1, y_2)$ be defined by the relations:

$$\Phi(t, y_1, \Psi(t, y_1, y_2)) = y_2 \, , \quad \Psi(t, x_1, \Phi(t, x_1, x_2)) = x_2 \, .$$

We compute the equation satisfied by Ψ starting from (12). We use the relations:

$$\begin{aligned}
&\partial_t \Psi(t, x_1, \Phi(t, x_1, x_2)) + \partial_{y_2} \Psi(t, x_1, \Phi(t, x_1, x_2))\, \partial_t \Phi(t, x_1, x_2) = 0 \, , \\
&\partial_{y_1} \Psi(t, x_1, \Phi(t, x_1, x_2)) + \partial_{y_2} \Psi(t, x_1, \Phi(t, x_1, x_2))\, \partial_{x_1} \Phi(t, x_1, x_2) = 0 \, , \\
&\partial_{y_2} \Psi(t, x_1, \Phi(t, x_1, x_2))\, \partial_{x_2} \Phi(t, x_1, x_2) = 1 \, ,
\end{aligned}$$

and obtain:

$$\partial_t \Psi(t, y_1, y_2) + v(t, y_1, y_2)\, \partial_{y_1} \Psi(t, y_1, y_2) + u(t, y_1, y_2)\, \partial_{y_2} \Psi(t, y_1, y_2) = 0 \, . \quad (31)$$

The main point is that (31) is a linear transport equation for Ψ that should hold on the domain $\Omega^- \cup \Omega^+$, together with the continuity condition:

$$\Psi(t, y_1, \varphi(t, y_1)) = 0 \, . \quad (32)$$

We shall see that the equations (31), (32) can be solved by applying the method of characteristics. Once Ψ is constructed, we can obtain Φ by inverting Ψ provided that $\partial_{y_2}\Psi$ is uniformly positive. Below we are therefore going to show how to construct Ψ with a sufficiently large Sobolev index (in order to apply Theorem 2).

We thus consider a solution V, φ of the equations (1), (2) in the original variables, and assume that this solution has the form $V^\pm = \overline{U}^\pm + \dot{V}^\pm$, with $\dot{V}^\pm \in H^7(\Omega^\pm)$, and $\varphi \in H^8(]0,T_0[\times\mathbb{R})$. Moreover, as in (5) $\dot{V}^\pm$ may be sufficiently small in $W^{2,\infty}(\Omega^\pm)$ and φ may be sufficiently small in $W^{3,\infty}(]0,T_0[\times\mathbb{R})$. Recall that $\Omega^\pm$ denote the sets $\{(t,y) \in [0,T_0]\times\mathbb{R}^2 : \pm(y_2 - \varphi(t,y_1)) > 0\}$.

We let $\varphi_0 \in H^{15/2}(\mathbb{R})$ denote the initial condition for φ. Then we consider a function $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that χ equals 1 on the interval $[-\|\varphi_0\|_{L^\infty}, \|\varphi_0\|_{L^\infty}]$. The initial condition for Ψ is defined in the following way:

$$\Psi_0(y_1,y_2) = y_2 - \varphi_0(y_1)\,\chi(y_2)\,. \tag{33}$$

It is clear that $\Psi_0(y_1,\varphi_0(y_1)) = 0$ for all y_1. The Sobolev imbedding Theorem shows that Ψ_0 belongs to $\mathcal{C}^5(\mathbb{R}^2)$ and the derivatives of Ψ_0 up to order 5 are bounded. We can also choose χ such that the following inequalities hold:

$$\partial_{y_2}\Psi_0(y_1,y_2) \geq \frac{1}{2}\,, \quad |\partial_{y_1}\Psi_0(y_1,y_2)| \leq \|\varphi_0'\|_{L^\infty}\,.$$

Moreover, if φ_0 is sufficiently small in $W^{3,\infty}(\mathbb{R})$, then $\Psi_0 - y_2$ is small in $W^{3,\infty}(\mathbb{R}^2)$. The choice of χ, and therefore the construction of Ψ_0, only depends on φ_0.

The proof of Theorem 1 relies on two lemmas that we give now:

Lemma 5. *Let p,k denote some positive integers. Let $\mathcal{O}_1$, $\mathcal{O}_2$ denote two open sets of $\mathbb{R}^p$ and let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a $\mathcal{C}^k$-diffeomorphism such that the jacobian J_f of f satisfies:*

$$\inf_{x\in\mathcal{O}_1} |J_f(x)| > 0\,,$$

and such that all derivatives $\mathrm{d}^1 f, \dots, \mathrm{d}^k f$ belong to $L^\infty(\mathcal{O}_1)$. Then the inverse f^{-1} of f satisfies:

$$\inf_{x\in\mathcal{O}_2} |J_{f^{-1}}(x)| > 0\,,$$

and the derivatives $\mathrm{d}^1 f^{-1}, \dots, \mathrm{d}^k f^{-1}$ belong to $L^\infty(\mathcal{O}_2)$.

Lemma 6. *Under the same assumptions and notations as in Lemma 5, one has $w \in H^k(\mathcal{O}_2)$ if and only if $w \circ f \in H^k(\mathcal{O}_1)$.*

The proofs of Lemmata 5 and 6 follow from classical calculus arguments and are therefore omitted.

Our first goal is to construct a solution Ψ of (31), (32) with the initial condition Ψ_0. Let us introduce the integral curves of the velocity field $\mathbf{u}$. From the Sobolev imbedding Theorem, we know that $\mathbf{u}$ belongs to $\mathcal{C}^5(\Omega^+)$ and has bounded derivatives up to order 5. Let $y_2 > \varphi_0(y_1)$ and let $X(t,y_1,y_2)$ denote the solution to the ordinary differential equation:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(t,y_1,y_2) = \mathbf{u}(t, X(t,y_1,y_2))\,, \quad t \in [0,T_0]\,,\\ X(0,y_1,y_2) = (y_1,y_2)\,. \end{cases}$$

The vector field $\mathbf{u}$ is tangent to the boundary of Ω^+, so the integral curve $X(t,y_1,y_2)$ is well-defined for all $t \in [0,T_0]$ and stays in Ω^+. Moreover, standard arguments of ordinary differential equations theory show that the flow:

$$\begin{aligned} \mathcal{O}^+ := \{(t,y_1,y_2) \in [0,T_0]\times\mathbb{R}^2 : y_2 > \varphi_0(y_1)\} &\longrightarrow \Omega^+ \\ (t,y_1,y_2) &\longmapsto (t, X(t,y_1,y_2))\,, \end{aligned} \tag{34}$$

is a $\mathcal{C}^5$-diffeomorphism. Differentiating the differential equation with respect to the initial conditions, we obtain uniform bounds of the form:

$$|\mathrm{d}_y X(t,y_1,y_2) - I| \le C\,t\,, \quad |\mathrm{d}_y^2 X(t,y_1,y_2)| + \cdots + |\mathrm{d}_y^5 X(t,y_1,y_2)| \le C\,.$$

Such bounds imply that up to restricting the final time T_0, the $\mathcal{C}^5$-diffeomorphism (34) satisfies the assumptions of Lemma 5. Its inverse has the form:

$$\begin{aligned} \Omega^+ &\longrightarrow \mathcal{O}^+ \\ (t,y_1,y_2) &\longmapsto (t,\mathbb{Y}(t,y_1,y_2))\,, \end{aligned} \tag{35}$$

and satisfies the conclusions of Lemma 5. The function Ψ is now defined on Ω^+ as follows:

$$\forall\,(t,y_1,y_2) \in \Omega^+\,, \quad \Psi(t,y_1,y_2) = \Psi_0(\mathbb{Y}(t,y_1,y_2))\,. \tag{36}$$

It is clear that Ψ belongs to $\mathcal{C}^5$ and satisfies the transport equation (31) together with the condition (32) on the boundary of Ω^+ thanks to the properties of Ψ_0. The derivatives of Ψ up to order 5 are bounded and, up to restricting T_0 once again, we have the bound $\partial_{y_2}\Psi \ge 1/4$.

The end of the proof of Theorem 1 relies on the following result that establishes integrability properties for Ψ:

Lemma 7. *Let Ψ be defined by* (36)*. Then the mapping $\big((t,y_1,y_2) \mapsto \Psi(t,y_1,y_2) - y_2\big)$ belongs to $H^5(\Omega^+)$.*

Proof. We first note that on Ω^+, the velocity $\mathbf{u}$ reads $\mathbf{u} = (\overline{v},0)+\dot{\mathbf{u}}$ with $\dot{\mathbf{u}} \in H^7(\Omega^+) \subset H^5(\Omega^+)$. Applying lemma 6, we have:

$$\big((t,y_1,y_2) \mapsto \dot{\mathbf{u}}(t,X(t,y_1,y_2))\big) \in H^5(\mathcal{O}^+)\,.$$

Recall that the set $\mathcal{O}^+$ is defined in (34). Then we use the ordinary differential equation satisfied by X and deduce that:

$$\big((t,y_1,y_2) \mapsto X(t,y_1,y_2) - (y_1 + \overline{v}\,t, y_2)\big) \in H^5(\mathcal{O}^+)\,.$$

Using the notation $X = (X_1,X_2)$ and $\mathbb{Y} = (\mathbb{Y}_1,\mathbb{Y}_2)$ for the coordinates of X and $\mathbb{Y}$, we can write $X_2(t,y_1,y_2) = y_2 + \dot{X}_2(t,y_1,y_2)$ with $\dot{X}_2 \in H^5(\mathcal{O}^+)$. Then we use the relation:

$$\forall\,(t,y_1,y_2) \in \Omega^+\,, \quad y_2 = X_2(t,\mathbb{Y}(t,y_1,y_2)) = \mathbb{Y}_2(t,y_1,y_2) + \dot{X}_2(t,\mathbb{Y}(t,y_1,y_2))\,,$$

and apply Lemma 6 again to obtain:

$$\big((t,y_1,y_2) \mapsto \mathbb{Y}_2(t,y_1,y_2) - y_2\big) \in H^5(\Omega^+)\,.$$

Eventually, we have:

$$\Psi(t,y_1,y_2) = \Psi_0(\mathbb{Y}(t,y_1,y_2)) = \mathbb{Y}_2(t,y_1,y_2) - \varphi_0(\mathbb{Y}_1(t,y_1,y_2))\,\chi(\mathbb{Y}_2(t,y_1,y_2))\,,$$

and the claim of Lemma 7 follows. $\square$

Using all the properties shown above, we claim that the mapping:

$$\begin{aligned} \Omega^+ &\longrightarrow Q_{T_0} = [0,T_0] \times \mathbb{R}^2_+ \\ (t,y_1,y_2) &\longmapsto (t,y_1,\Psi(t,y_1,y_2))\,, \end{aligned}$$

is a $\mathcal{C}^5$-diffeomorphism that satisfies the assumptions of Lemma 5. Its inverse has the form:

$$\begin{aligned} Q_{T_0} &\longrightarrow \Omega^+ \\ (t,x_1,x_2) &\longmapsto (t,x_1,\Phi(t,x_1,x_2))\,, \end{aligned}$$

and satisfies the conclusions of Lemma 5. We define:

$$U^+ : (t,x_1,x_2) \longmapsto V^+(t,x_1,\Phi(t,x_1,x_2))\,,$$

so applying Lemma 6 we get $U - \overline{U}^+ \in H^5(Q_{T_0})$. Applying the same kind of argument as in the proof of Lemma 7, we can also prove that $\Phi - x_2 \in H^5(Q_{T_0})$. The initial condition for Φ only depends on the initial condition for φ. The calculations developed at the beginning of this section prove that Φ satisfies the eikonal equation (12).

Up to restricting T_0 again if necessary, the diffeomorphism (34), and consequently also its inverse (35), may be taken close enough to the identity map. If $\Psi_0 - y_2$ is sufficiently small in $W^{3,\infty}(\mathbb{R}^2)$, then $\Psi - y_2$ is small in $W^{3,\infty}(\Omega^+)$ and $\Phi - x_2$ is small in $W^{3,\infty}(Q_{T_0})$. Moreover, if $\dot{V}^\pm = V^+ - \overline{U}^+$ is taken sufficiently small in $W^{2,\infty}(\Omega^\pm)$, then $U - \overline{U}^+$ is small in $W^{2,\infty}(Q_{T_0})$.

The construction of Ψ on Ω^-, and consequently the construction of U^-, Φ^- proceed in a similar way. We have thus constructed a solution of the system (7), (8), (11), (12) with the regularity required in Theorem 2 (on a possibly shorter time interval). Theorem 1 follows by applying the construction above to two solutions V_1, φ_1 and V_2, φ_2 with the same initial conditions. The corresponding solutions in the straightened variables will share the same initial conditions and will therefore coincide on a small time interval $[0,T]$. This shows that the maximal time up to which the solutions coincide is necessarily equal to T_0. The proof of Theorem 1 is now complete.

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