The Stability of Compressible Vortex Sheets in Two Space Dimensions

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ABSTRACT. We study the linear stability of compressible vortex sheets in two space dimensions. Under a supersonic condition that precludes violent instabilities, we prove an energy estimate for the linearized boundary value problem. Since the problem is characteristic, the estimate we prove exhibits a loss of control on the trace of the solution. Furthermore, the failure of the uniform Kreiss-Lopatinskii condition yields a loss of derivatives in the energy estimate.

1. INTRODUCTION

A velocity discontinuity in an inviscid flow is called a *vortex sheet*. In three-space dimensions, a vortex sheet has vorticity concentrated along a surface in the space. In two-space dimensions, the vorticity is concentrated along a curve in the plane. The present paper deals with compressible vortex sheets, i.e., vortex sheets in a compressible flow.

If the solution is piecewise constant on either side of the interface of discontinuity, one has planar vortex sheets in the three dimensional case and rectilinear vortex sheets in the two dimensional case, respectively. The linear stability of planar and rectilinear compressible vortex sheets has been analyzed a long time ago, see [12, 27]. In three space dimensions, planar vortex sheets are known to be violently unstable (see e.g. [30]). In the two dimensional case, subsonic vortex sheets are also violently unstable, while supersonic vortex sheets are neutrally linearly stable, see e.g. [27, 30]. This result formally agrees with the theory of incompressible vortex sheets. In fact, in the incompressible limit, the speed of sound tends to infinity, with the result that two-dimensional vortex sheets are always unstable. This kind of instability is usually referred to as the Kelvin-Helmholtz instability. For the incompressible theory of two-dimensional vortex sheets, we refer the reader to the books [7, 22]. Moreover, we refer to [14] for the study of the instability of vortex sheets when heat conduction is taken into account.

However, the normal modes analysis performed to derive the linear stability of supersonic vortex sheets is by far not sufficient to guarantee the existence of nonconstant vortex sheets (that is, contact discontinuities) solutions to the compressible isentropic Euler equations. In this paper, we first show that supersonic constant vortex sheets are linearly stable, in the sense that the linearized system (around these particular piecewise constant solutions) obeys an energy estimate. Then we consider the linearized equations around a perturbation of a constant vortex sheet, and we show that these linearized equations obey the same energy estimate. This is a first crucial step towards proving the existence of nonconstant compressible vortex sheets.

Several points need to be highlighted. First of all, the existence of compressible vortex sheets is a free boundary nonlinear hyperbolic problem. Moreover, the free boundary is characteristic with respect to both left and right states since we deal with contact discontinuities. This is one of the reasons why one can not apply Majda's analysis on shock waves (see [20, 21]), that are noncharacteristic interfaces. In some previous works devoted to weakly stable shock waves, see [10, 11], the first author has considered noncharacteristic hyperbolic Initial Boundary Value Problems that did not meet the uniform Kreiss-Lopatinskii condition. In the case of vortex sheets, the analysis is closely related, with the additional difficulty that the boundary is characteristic (the present analysis thus relies more on the work of Majda and Osher [23] rather than on the work of Kreiss [6, 17]). The connection with [10, 11] is that in both cases, the analogue of the Kreiss-Lopatinskii condition is fulfilled but not in a uniform way. Furthermore, in the case of vortex sheets as in the case of shock waves, the linearized Rankine-Hugoniot conditions form an elliptic system for the unknown front. This property is a key point in our work since it allows to *eliminate* the unknown front and to consider a standard Boundary Value Problem with a symbolic boundary condition (this ellipticity property is also crucial in Majda's analysis on shock waves [20, 21]).

Regarding the energy estimates for the linearized problems, the failure of the uniform Kreiss-Lopatinskii condition yields a loss of derivatives with respect to the source terms. Furthermore, because the boundary is characteristic, we expect to lose some control on the trace of the solution at the boundary. As a matter of fact, we shall see that the only loss of control is on the tangential velocity (which corresponds to the "characteristic part" of the solution). The good point is that the ellipticity of the boundary conditions for the unknown front enables us to gain one derivative for it, as in Majda's work on shock waves [20]. Going slightly more into the details, we shall prove that the only frequencies for which we lose some control on the solution correspond to bicharacteristic curves. Those curves originate from those points at the boundary of the space domain where the so-called Lopatinskii determinant vanishes. In the interior of the space domain, these singularities propagate along two bicharacteristics associated with the *incoming* modes.

Let us now describe the content of the paper. In Section 2, we present the nonlinear equations describing the evolution of compressible vortex sheets and introduce some notations. Then, in Section 3, we shall consider the linearized equations around a constant (stationary) vortex sheet. The main result for the constant coefficient linearized problem is given in Theorems 3.1 and 3.2. After several reductions, we shall detail in Section 4 the normal modes analysis of the linearized problem and construct a *degenerate* Kreiss' symmetrizers in order to derive our energy estimate. In Section 5, we first present the variable coefficients linearized problem and introduce Alinhac's *good unknown*. Then we paralinearize the equations, in order to use symbolic calculus and derive the energy estimate. A precise estimate of the paralinearization errors is given. Eventually, we show how to control the different pieces of the solution, depending on their microlocalization. The main result for the variable coefficients linearized problem is given in Theorem 5.1. In Section 6, we make some remarks about possible future achievements. Appendix A is devoted to the proof of several technical lemmas and Appendix B gathers the main results on paradifferential calculus that are used throughout Section 5.

2. The Nonlinear Equations

We consider Euler equations of isentropic gas dynamics in the whole plane $\mathbb{R}^2$. Denoting by $\mathbf{u}$ the velocity of the fluid and ρ the density, the equations read:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \end{cases} \tag{2.1}$$

where $p = p(\rho)$ is the pressure law. In all this paper, p is assumed to be a strictly increasing function of ρ, defined on $]0, +\infty[$. We also assume that p is a C^∞ function of ρ. The speed of sound $c(\rho)$ in the fluid is then defined by the relation

$$c(\rho) := \sqrt{p'(\rho)}.$$

Let $(\rho, \mathbf{u})(t, x_1, x_2)$ be a smooth function on either side of a smooth hypersurface $\Gamma := \{x_2 = \varphi(t, x_1)\}$. Then $(\rho, \mathbf{u})$ is a (weak) solution of (2.1) if and only if $(\rho, \mathbf{u})$ is a classical solution of (2.1) on both sides of Γ and the Rankine-Hugoniot conditions hold at each point of Γ:

$$\partial_t \varphi [\rho] - [\rho \mathbf{u} \cdot \nu] = 0, \tag{2.2a}$$

$$\partial_t \varphi [\rho \mathbf{u}] - [(\rho \mathbf{u} \cdot \nu) \mathbf{u}] - [p] \nu = 0, \tag{2.2b}$$

where $\nu := (-\partial_{x_1} \varphi, 1)$ is a (space) normal vector to Γ. As usual, $[q] = q^+ - q^-$ denotes the jump of a quantity q across the interface Γ (see [29]).

Following Lax [18], we shall say that $(\rho, \mathbf{u})$ is a contact discontinuity if the Rankine-Hugoniot conditions (2.2) are satisfied in the following way:

$$\partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu, \quad p^+ = p^-.$$

Because p is monotone, the previous equalities read

$$\partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu, \quad \rho^+ = \rho^- . \tag{2.3}$$

Since the density and the normal velocity are continuous across the interface Γ, the only jump experimented by the solution is on the tangential velocity. (Here, normal and tangential mean normal and tangential with respect to Γ). For this reason, a contact discontinuity is a vortex sheet and we shall make no distinction in the terminology we use.

Note that the first two equalities above are nothing but eikonal equations: if $x_2 = \varphi(t, x_1)$ is the equation of the interface Γ, then φ satisfies

$$\partial_t \varphi + \lambda_2(\rho^+, \mathbf{u}^+, \partial_{x_1}\varphi) = 0 \quad \text{and} \quad \partial_t \varphi + \lambda_2(\rho^-, \mathbf{u}^-, \partial_{x_1}\varphi) = 0,$$

on $\{x_2 = 0\}$, where

$$\lambda_2(\rho, \mathbf{u}, \xi) := \mathbf{u} \cdot \begin{pmatrix} \xi \\ -1 \end{pmatrix}, \quad \xi \in \mathbb{R},$$

is the second characteristic field of the system (2.1). It is linearly degenerate since the corresponding eigenvector, in the quasilinear form of (2.1), is given by

$$r_2(\rho, \mathbf{u}, \xi) := \begin{pmatrix} 0 \\ 1 \\ \xi \end{pmatrix}.$$

Recall that the space dimension equals 2.

The interface Γ, or equivalently the function φ, is part of the unknowns of the problem. We thus deal with a free boundary problem. As it is common in this kind of situation, we first straighten the unknown front in order to work in a fixed domain. More precisely, the unknowns $(\rho, \mathbf{u})$, that are smooth on either side of $\{x_2 = \varphi(t, x_1)\}$, are replaced by the functions

$$(\rho_\sharp^+, \mathbf{u}_\sharp^+)(t, x_1, x_2) := (\rho, \mathbf{u})(t, x_1, \Phi(t, x_1, x_2)),$$
$$(\rho_\sharp^-, \mathbf{u}_\sharp^-)(t, x_1, x_2) := (\rho, \mathbf{u})(t, x_1, \Phi(t, x_1, -x_2)),$$

where Φ is a smooth function satisfying

$$\partial_{x_2}\Phi(t, x_1, x_2) \geq \kappa > 0, \quad \Phi(t, x_1, 0) = \varphi(t, x_1).$$

With these requirements for Φ, all functions $\rho_\sharp^\pm, \mathbf{u}_\sharp^\pm$ are smooth on the fixed domain $\{x_2 > 0\}$. For convenience, we drop the $\sharp$ index and only keep the $+$ and $-$ exponents. We also define the functions

$$\Phi^\pm(t, x_1, x_2) := \Phi(t, x_1, \pm x_2),$$

which are both smooth on the half-space $\{x_2 > 0\}$.

Let us denote by v and u the two components of the velocity, that is, $\mathbf{u} = (v, u)$. Then the existence of compressible vortex sheets amounts to proving the existence of smooth solutions to the following system:

$$\partial_t \rho^+ + v^+ \partial_{x_1} \rho^+ + (u^+ - \partial_t \Phi^+ - v^+ \partial_{x_1} \Phi^+) \frac{\partial_{x_2} \rho^+}{\partial_{x_2} \Phi^+} + \rho^+ \partial_{x_1} v^+ + \rho^+ \frac{\partial_{x_2} u^+}{\partial_{x_2} \Phi^+} - \rho^+ \frac{\partial_{x_1} \Phi^+}{\partial_{x_2} \Phi^+} \partial_{x_2} v^+ = 0, \tag{2.4a}$$

$$\partial_t v^+ + v^+ \partial_{x_1} v^+ + (u^+ - \partial_t \Phi^+ - v^+ \partial_{x_1} \Phi^+) \frac{\partial_{x_2} v^+}{\partial_{x_2} \Phi^+} + \frac{p'(\rho^+)}{\rho^+} \partial_{x_1} \rho^+ - \frac{p'(\rho^+)}{\rho^+} \frac{\partial_{x_1} \Phi^+}{\partial_{x_2} \Phi^+} \partial_{x_2} \rho^+ = 0, \tag{2.4b}$$

$$\partial_t u^+ + v^+ \partial_{x_1} u^+ + (u^+ - \partial_t \Phi^+ - v^+ \partial_{x_1} \Phi^+) \frac{\partial_{x_2} u^+}{\partial_{x_2} \Phi^+} + \frac{p'(\rho^+)}{\rho^+} \frac{\partial_{x_2} \rho^+}{\partial_{x_2} \Phi^+} = 0, \tag{2.4c}$$

$$\partial_t \rho^- + v^- \partial_{x_1} \rho^- + (u^- - \partial_t \Phi^- - v^- \partial_{x_1} \Phi^-) \frac{\partial_{x_2} \rho^-}{\partial_{x_2} \Phi^-} + \rho^- \partial_{x_1} v^- + \rho^- \frac{\partial_{x_2} u^-}{\partial_{x_2} \Phi^-} - \rho^- \frac{\partial_{x_1} \Phi^-}{\partial_{x_2} \Phi^-} \partial_{x_2} v^- = 0, \tag{2.4d}$$

$$\partial_t v^- + v^- \partial_{x_1} v^- + (u^- - \partial_t \Phi^- - v^- \partial_{x_1} \Phi^-) \frac{\partial_{x_2} v^-}{\partial_{x_2} \Phi^-} + \frac{p'(\rho^-)}{\rho^-} \partial_{x_1} \rho^- - \frac{p'(\rho^-)}{\rho^-} \frac{\partial_{x_1} \Phi^-}{\partial_{x_2} \Phi^-} \partial_{x_2} \rho^- = 0, \tag{2.4e}$$

$$\partial_t u^- + v^- \partial_{x_1} u^- + (u^- - \partial_t \Phi^- - v^- \partial_{x_1} \Phi^-) \frac{\partial_{x_2} u^-}{\partial_{x_2} \Phi^-} + \frac{p'(\rho^-)}{\rho^-} \frac{\partial_{x_2} \rho^-}{\partial_{x_2} \Phi^-} = 0, \tag{2.4f}$$

in the fixed domain $\{x_2 > 0\}$, together with the boundary conditions

$$\begin{aligned}
&\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi,\\
&\partial_t \varphi = -v^+_{|x_2=0}\, \partial_{x_1}\varphi + u^+_{|x_2=0} = -v^-_{|x_2=0}\, \partial_{x_1}\varphi + u^-_{|x_2=0},\\
&\rho^+_{|x_2=0} = \rho^-_{|x_2=0}.
\end{aligned}$$

For convenience, we rewrite the boundary conditions in the following way:

$$\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi, \tag{2.5a}$$

$$(v^+ - v^-)_{|x_2=0}\, \partial_{x_1}\varphi - (u^+ - u^-)_{|x_2=0} = 0, \tag{2.5b}$$

$$\partial_t \varphi + v^+_{|x_2=0}\, \partial_{x_1}\varphi - u^+_{|x_2=0} = 0, \tag{2.5c}$$

$$(\rho^+ - \rho^-)_{|x_2=0} = 0. \tag{2.5d}$$

The functions Φ^+ and Φ^- should also satisfy

$$\partial_{x_2}\Phi^+(t,x_1,x_2) \geq \kappa, \quad \partial_{x_2}\Phi^-(t,x_1,x_2) \leq -\kappa, \tag{2.6}$$

for a suitable constant $\kappa > 0$.

In [20, 21], Majda makes the particular choice

$$\Phi^\pm(t,x_1,x_2) := \pm x_2 + \varphi(t,x_1).$$

This choice is appropriate in the study of shock waves because these are noncharacteristic discontinuities. In the study of contact discontinuities, it seems rather natural to choose the change of variables $\Phi^\pm$ such that the eikonal equations

$$\begin{aligned}
\partial_t \Phi^+ + \lambda_2(\rho^+, \mathbf{u}^+, \partial_{x_1}\Phi^+) = \partial_t \Phi^+ + v^+\, \partial_{x_1}\Phi^+ - u^+ = 0,\\
\partial_t \Phi^- + \lambda_2(\rho^-, \mathbf{u}^-, \partial_{x_1}\Phi^-) = \partial_t \Phi^- + v^-\, \partial_{x_1}\Phi^- - u^- = 0,
\end{aligned}$$

are satisfied in the whole closed half-space $\{x_2 \geq 0\}$. This choice, that is inspired from [13], has several advantages. First, it simplifies much the expression of the nonlinear equations (2.4). But it also implies that the so-called *boundary matrix* has constant rank in the whole space domain $\{x_2 \geq 0\}$, and not only on the boundary $\{x_2 = 0\}$. This will enable us to develop a Kreiss' symmetrizers technique, in the spirit of [23]. We shall go back to this feature later on.

The problem is thus the construction of (local in time) smooth solutions to (2.4)–(2.5)–(2.6), once initial data have been prescribed. Of course, such initial data will have to fulfill a certain number of compatibility conditions. The first step in proving such an existence result is the study of the linearized problem around a particular constant solution, and this is our first main result, see Theorems 3.1 and 3.2. The second step is the study of the linearized problem around a (variable coefficients) perturbation of the constant solution. The extension to the variable coefficients linearized problem is addressed in the second part of the paper. Our second main result states that the constant coefficients energy estimate still holds when one considers a variable coefficients linearized problem, see Theorem 5.1.

To avoid overloading the paper, we introduce some compact notations for the nonlinear equations (2.4). For all $U := (\rho, v, u)^T$, we define

$$A_1(U) := \begin{pmatrix} v & \rho & 0 \\ p'(\rho)/\rho & v & 0 \\ 0 & 0 & v \end{pmatrix}, \quad A_2(U) := \begin{pmatrix} u & 0 & \rho \\ 0 & u & 0 \\ p'(\rho)/\rho & 0 & u \end{pmatrix}.$$

Then the nonlinear equations (2.4) read

$$\begin{aligned} \partial_t U^+ &+ A_1(U^+)\partial_{x_1} U^+ \\ &+ \frac{1}{\partial_{x_2}\Phi^+}\left(A_2(U^+) - \partial_t\Phi^+ - \partial_{x_1}\Phi^+ A_1(U^+)\right)\partial_{x_2} U^+ = 0, \end{aligned} \tag{2.7a}$$

$$\begin{aligned} \partial_t U^- &+ A_1(U^-)\partial_{x_1} U^- \\ &+ \frac{1}{\partial_{x_2}\Phi^-}\left(A_2(U^-) - \partial_t\Phi^- - \partial_{x_1}\Phi^- A_1(U^-)\right)\partial_{x_2} U^- = 0. \end{aligned} \tag{2.7b}$$

With an obvious definition for the differential operator L, the system (2.7) also reads

$$L(U^+, \nabla\Phi^+)U^+ = 0, \quad L(U^-, \nabla\Phi^-)U^- = 0. \tag{2.8}$$

When no confusion is possible, we also write this system under the form

$$L(U, \nabla\Phi)U = 0,$$

where U stands for the vector (U^+, U^-) and Φ for (Φ^+, Φ^-). One should always remember that the interior equations (2.7) are entirely decoupled. The coupling between the right and left states is made by the boundary conditions (2.5).

There exist many simple solutions of (2.8)–(2.5)–(2.6), that correspond, in the original variables, to rectilinear vortex sheets:

$$(\rho, \mathbf{u}) = \begin{cases} (\rho, \mathbf{u}_r) & \text{if } x_2 > \sigma t + n x_1, \\ (\rho, \mathbf{u}_l) & \text{if } x_2 < \sigma t + n x_1. \end{cases}$$

Here above, $\mathbf{u}_r$, $\mathbf{u}_l$ are fixed vectors in $\mathbb{R}^2$, and $\rho > 0$, σ and n are fixed real numbers. These quantities are linked by the Rankine-Hugoniot conditions:

$$\sigma = -v_r n + u_r = -v_l n + u_l.$$

Changing observer if necessary, we may assume without loss of generality

$$\sigma = n = u_r = u_l = 0 \quad \text{and} \quad v_r + v_l = 0 \quad (v_r \neq 0).$$

In the new variables, this corresponds to the following regular solution of (2.8)–(2.5)–(2.6):

$$U_r \equiv \begin{pmatrix} \rho \\ v_r \\ 0 \end{pmatrix}, \quad U_l \equiv \begin{pmatrix} \rho \\ v_l \\ 0 \end{pmatrix}, \quad \Phi_{r,l}(t, x_1, x_2) \equiv \pm x_2, \tag{2.9}$$

with the relation $v_r + v_l = 0$. We only consider the case $v_r \neq 0$, and without loss of generality, we assume $v_r > 0$. In the next section, we study the linearized equations around the particular solution defined by (2.9). Under a certain "supersonic" assumption, we shall show that the linearized equations satisfy an a priori energy estimate.

3. THE CONSTANT COEFFICIENTS LINEARIZED SYSTEM

3.1. The linearized equations. Let us denote by $\dot\rho_\pm, \dot{\mathbf{u}}_\pm, \Psi_\pm$ some small perturbations of the exact solution given by (2.9). Up to second order, the perturbations $\dot U_\pm = (\dot\rho_\pm, \dot v_\pm, \dot u_\pm)^T$ satisfy

$$\partial_t \dot U_+ + A_1(U_r)\,\partial_{x_1} \dot U_+ + A_2(U_r)\,\partial_{x_2} \dot U_+ = 0, \tag{3.1a}$$
$$\partial_t \dot U_- + A_1(U_l)\,\partial_{x_1} \dot U_- - A_2(U_l)\,\partial_{x_2} \dot U_- = 0, \tag{3.1b}$$

in the domain $\{x_2 > 0\}$, together with the linearized Rankine-Hugoniot relations

$$\Psi_+ = \Psi_- = \psi, \tag{3.2a}$$
$$(v_r - v_l)\,\partial_{x_1}\psi - (\dot u_+ - \dot u_-) = 0, \tag{3.2b}$$
$$\partial_t \psi + v_r\,\partial_{x_1}\psi - \dot u_+ = 0, \tag{3.2c}$$
$$\dot\rho_+ - \dot\rho_- = 0, \tag{3.2d}$$

on the boundary $\{x_2 = 0\}$. In short, equations (3.1)–(3.2) read

$$\begin{cases} L'\dot U = 0, & \text{if } x_2 > 0, \\ B(\dot U, \psi) = 0, & \text{if } x_2 = 0, \end{cases} \tag{3.3}$$

with $\dot{U} := (\dot{U}_+, \dot{U}_-)$, and obvious definitions for the operators L' and B:

$$L'\dot{U} := \partial_t \begin{pmatrix} \dot{U}_+ \\ \dot{U}_- \end{pmatrix} + \begin{pmatrix} A_1(U_r) & 0 \\ 0 & A_1(U_l) \end{pmatrix} \partial_{x_1} \begin{pmatrix} \dot{U}_+ \\ \dot{U}_- \end{pmatrix} + \begin{pmatrix} A_2(U_r) & 0 \\ 0 & -A_2(U_l) \end{pmatrix} \partial_{x_2} \begin{pmatrix} \dot{U}_+ \\ \dot{U}_- \end{pmatrix},$$

$$B(\dot{U}, \psi) := \begin{pmatrix} (v_r - v_l)\,\partial_{x_1}\psi - (\dot{u}_+ - \dot{u}_-) \\ \partial_t\psi + v_r\,\partial_{x_1}\psi - \dot{u}_+ \\ \dot{\rho}_+ - \dot{\rho}_- \end{pmatrix}.$$

It is important to note that the interior equations do not involve the perturbation ψ, so L' is an operator that only acts on $\dot{U}$. This property also holds when one studies the linearized shock wave equations around a planar shock, see [20, 25].

Proving an energy estimate for the linearized equations amounts to working with source terms, both in the interior domain and on the boundary. From now on, we thus consider the linear equations

$$\begin{cases} L'\dot{U} = f, & \text{if } x_2 > 0, \\ B(\dot{U}, \psi) = g, & \text{if } x_2 = 0, \end{cases} \tag{3.4}$$

and try to estimate $\dot{U}$ and ψ in terms of f and g (in appropriate functional spaces). In order to simplify the subsequent calculations, we introduce some new unknown functions, and define the following quantities:

$$\begin{aligned} &W_1 := \dot{v}_+, \quad W_2 := \tfrac{1}{2}\left(-\dot{\rho}_+/\rho + \dot{u}_+/c\right), \quad W_3 := \tfrac{1}{2}\left(\dot{\rho}_+/\rho + \dot{u}_+/c\right), \\ &W_4 := \dot{v}_-, \quad W_5 := \tfrac{1}{2}\left(-\dot{\rho}_-/\rho + \dot{u}_-/c\right), \quad W_6 := \tfrac{1}{2}\left(\dot{\rho}_-/\rho + \dot{u}_-/c\right). \end{aligned} \tag{3.5}$$

We also define the following vectors:

$$\begin{aligned} W &:= (W_1, W_2, W_3, W_4, W_5, W_6)^T, \\ W^{\mathrm{c}} &:= (W_1, W_4)^T, \\ W^{\mathrm{nc}} &:= (W_2, W_3, W_5, W_6)^T. \end{aligned}$$

The notations W^{c} and W^{nc} are introduced in order to separate the "characteristic part" of the vector W and the "noncharacteristic part" of W. We shall go back to this decomposition later on. It is obvious that estimating W is equivalent to estimating $\dot{U}$.

Let us define the following 6×6 symmetric matrices:

$$\mathcal{A}_0 := \begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & 2c^2 & 0 & & \mathbf{O} & \\ 0 & 0 & 2c^2 & & & \\ & & & 1 & 0 & 0 \\ & \mathbf{O} & & 0 & 2c^2 & 0 \\ & & & 0 & 0 & 2c^2 \end{pmatrix}, \tag{3.6}$$

$$\mathcal{A}_1 := \begin{pmatrix} v_r & -c^2 & c^2 & & & \\ -c^2 & 2c^2 v_r & 0 & & \mathbf{O} & \\ c^2 & 0 & 2c^2 v_r & & & \\ & & & v_l & -c^2 & c^2 \\ & \mathbf{O} & & -c^2 & 2c^2 v_l & 0 \\ & & & c^2 & 0 & 2c^2 v_l \end{pmatrix}, \tag{3.7}$$

$$\mathcal{A}_2 := \begin{pmatrix} 0 & 0 & 0 & & & \\ 0 & -2c^3 & 0 & & \mathbf{O} & \\ 0 & 0 & 2c^3 & & & \\ & & & 0 & 0 & 0 \\ & \mathbf{O} & & 0 & 2c^3 & 0 \\ & & & 0 & 0 & -2c^3 \end{pmatrix}, \tag{3.8}$$

where $\mathbf{O}$ stands for the 3×3 null matrix, as well as the following

$$\underline{b} := \begin{pmatrix} 0 & v_r - v_l \\ 1 & v_r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2v_r \\ 1 & v_r \\ 0 & 0 \end{pmatrix}, \quad \underline{M} := \begin{pmatrix} -c & -c & c & c \\ -c & -c & 0 & 0 \\ -1 & 1 & 1 & -1 \end{pmatrix}. \tag{3.9}$$

Then, using (3.5)–(3.6)–(3.9), the linear equations (3.4) equivalently read

$$\begin{cases} \mathcal{L}W = f, & \text{if } x_2 > 0, \\ \mathcal{B}(W^{\text{nc}}, \psi) = g, & \text{if } x_2 = 0, \end{cases} \tag{3.10}$$

with new f and g, and where we have let

$$\mathcal{L}W := \mathcal{A}_0\,\partial_t W + \mathcal{A}_1\,\partial_{x_1} W + \mathcal{A}_2\,\partial_{x_2} W,$$

$$\mathcal{B}(W^{\text{nc}}, \psi) := \underline{M} W^{\text{nc}}{}_{|x_2=0} + \underline{b} \begin{pmatrix} \partial_t \psi \\ \partial_{x_1} \psi \end{pmatrix}.$$

Note that the kernel of $\mathcal{A}_2$ is exactly the set of those W such that $W^{\text{nc}} = 0$ (and W^{c} is arbitrary). The boundary $\{x_2 = 0\}$ is thus characteristic with multiplicity 2. As already noted in earlier works, see e.g. [19, 23], we expect to lose control of the trace of W^{c}, that is, on the trace of the tangential velocities $(\dot{v}_+, \dot{v}_-)$. At the opposite, we expect to control the trace of W^{nc} on $\{x_2 = 0\}$, that is, we expect to control the trace of $(\dot{\rho}_+, \dot{\rho}_-, \dot{u}_+, \dot{u}_-)$.

3.2. The main result for the constant coefficients case. Before stating our energy estimate for the system (3.10), we need to introduce some Sobolev weighted norms. First define the half-space

$$\Omega := \{(t, x_1, x_2) \in \mathbb{R}^3 \text{ s.t. } x_2 > 0\} = \mathbb{R}^2 \times \mathbb{R}^+.$$

The boundary $\partial\Omega$ is identified to $\mathbb{R}^2$.

For all real number s and all $\gamma \geq 1$, define the space

$$H^s_\gamma(\mathbb{R}^2) := \{u \in \mathcal{D}'(\mathbb{R}^2) \text{ s.t. } \exp(-\gamma t)u \in H^s(\mathbb{R}^2)\}.$$

It is equipped with the norm

$$\|u\|_{H^s_\gamma(\mathbb{R}^2)} := \|\exp(-\gamma t)u\|_{H^s(\mathbb{R}^2)}.$$

Letting $\tilde{u} := \exp(-\gamma t)u$, one has

$$\|u\|_{H^s_\gamma(\mathbb{R}^2)} \simeq \|\tilde{u}\|_{s,\gamma}, \quad \text{where } ||v||^2_{s,\gamma} := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} (\gamma^2 + |\xi|^2)^s |\hat{v}(\xi)|^2 \, d\xi,$$

where $\hat{v}$ is the Fourier transform of any function v defined on $\mathbb{R}^2$.

For all integers k, one can define the space $H^k_\gamma(\Omega)$ in an entirely similar way.

The space $L^2(\mathbb{R}^+; H^s_\gamma(\mathbb{R}^2))$ is equipped with the norm

$$|||\, v \,|||^2_{L^2(H^s_\gamma)} := \int_0^{+\infty} ||v(\cdot, x_2)||^2_{H^s_\gamma(\mathbb{R}^2)} \, dx_2.$$

In the sequel, the variable in $\mathbb{R}^2$ is (t, x_1), while x_2 is the variable in $\mathbb{R}^+$.

Our first main result is stated as follows.

Theorem 3.1. *Assume that the particular solution defined by* (2.9) *satisfies*

$$v_r - v_l > 2\sqrt{2}\, c. \tag{3.11}$$

Then there exists a positive constant C such that for all $\gamma \geq 1$ and for all $(W, \psi) \in H^2_\gamma(\Omega) \times H^2_\gamma(\mathbb{R}^2)$, the following estimate holds:

$$\begin{aligned} \gamma\, |||\, W \,|||^2_{L^2_\gamma(\Omega)} &+ ||W^{\mathrm{nc}}{}_{|x_2=0}||^2_{L^2_\gamma(\mathbb{R}^2)} + ||\psi||^2_{H^1_\gamma(\mathbb{R}^2)} \\ &\leq C\left(\frac{1}{\gamma^3}\, |||\, \mathcal{L}W \,|||^2_{L^2(H^1_\gamma)} + \frac{1}{\gamma^2}||\mathcal{B}(W^{\mathrm{nc}}, \psi)||^2_{H^1_\gamma(\mathbb{R}^2)}\right). \end{aligned} \tag{3.12}$$

Introducing $\widetilde{W} := \exp(-\gamma t)W$ and $\tilde{\psi} := \exp(-\gamma t)\psi$, we easily find that (3.10) is equivalent to

$$
\begin{cases}
\mathcal{L}^\gamma \widetilde{W} := \gamma \mathcal{A}_0 \widetilde{W} + \mathcal{L}\widetilde{W} & \\
\qquad = \exp(-\gamma t) f, & \text{if } x_2 > 0, \\
\mathcal{B}^\gamma(\widetilde{W}^{\rm nc}, \tilde{\psi}) := \underline{M} \widetilde{W}^{\rm nc}_{|x_2=0} + \underline{b} \begin{pmatrix} \gamma \tilde{\psi} + \partial_t \tilde{\psi} \\ \partial_{x_1} \tilde{\psi} \end{pmatrix} & \\
\qquad = \exp(-\gamma t) g, & \text{if } x_2 = 0.
\end{cases}
\tag{3.13}
$$

Consequently, Theorem 3.1 admits the following equivalent formulation.

Theorem 3.2. *Assume that* (3.11) *holds. Then there exists a positive constant C such that for all $\gamma \geq 1$ and for all $(\widetilde{W}, \tilde{\psi}) \in H^2(\Omega) \times H^2(\mathbb{R}^2)$, the following estimate holds:*

$$
\gamma ||| \widetilde{W} |||_0^2 + || \widetilde{W}^{\rm nc}_{|x_2=0} ||_0^2 + || \tilde{\psi} ||_{1,\gamma}^2 \leq C \left(\frac{1}{\gamma^3} ||| \mathcal{L}^\gamma \widetilde{W} |||_{1,\gamma}^2 + \frac{1}{\gamma^2} || \mathcal{B}^\gamma(\widetilde{W}^{\rm nc}, \tilde{\psi}) ||_{1,\gamma}^2 \right). \tag{3.14}
$$

In (3.14), we have used the following notations for any $v \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^2))$:

$$
||| v |||_{s,\gamma}^2 := \int_0^{+\infty} || v(\cdot, x_2) ||_{s,\gamma}^2 \, dx_2 .
$$

For instance, $||| \cdot |||_{0,\gamma}$ is the usual norm on $L^2(\Omega)$ and does not involve γ, so we shall denote it by $||| \cdot |||_0$. The norm $||| \cdot |||_{1,\gamma}$ is the weighted norm on $L^2(\mathbb{R}^+, H^1(\mathbb{R}^2))$.

4. PROOF OF THEOREM 3.2

4.1. Some preliminary reductions. In this paragraph, we show that it is sufficient to prove Theorem 3.2 in the particular case $\mathcal{L}^\gamma \widetilde{W} \equiv 0$. This first reduction simplifies many subsequent calculations. The argument we use was introduced in [23, page 636].

In order to simplify notations, we drop the tildas. Let $W \in H^2(\Omega)$ and $\psi \in H^2(\mathbb{R}^2)$. Then we define:

$$
f := \mathcal{L}^\gamma W \in H^1(\Omega), \quad g := \mathcal{B}^\gamma(W^{\rm nc}, \psi) \in H^1(\mathbb{R}^2).
$$

Consider the following auxiliary problem:

$$
\begin{cases}
\mathcal{L}^\gamma W_1 = f, & \text{if } x_2 > 0, \\
M^{\rm aux} W^{\rm nc}_{1\,|x_2=0} = 0, & \text{if } x_2 = 0,
\end{cases}
\tag{4.1}
$$

where M^{aux} is defined by

$$M^{\mathrm{aux}} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In the auxiliary problem (4.1), the boundary conditions are maximally dissipative (see the formula (3.6) defining the matrix $\mathcal{A}_2$). Since $f \in L^2(\mathbb{R}^+; H^1(\mathbb{R}^2))$, it follows from [19] that there exists a function $W_1 \in L^2(\mathbb{R}^+; H^1(\mathbb{R}^2))$, such that the trace of W_1^{nc} on $\{x_2 = 0\}$ belongs to $H^1(\mathbb{R}^2)$, and that is a solution to (4.1). In particular, the function W_1 satisfies the following estimates:

$$\gamma \, ||| W_1 |||_0^2 \le \frac{C}{\gamma} ||| f |||_0^2, \tag{4.2a}$$

$$\| W_1^{\mathrm{nc}}{}_{|x_2=0} \|_{1,\gamma}^2 \le \frac{C}{\gamma} ||| f |||_{1,\gamma}^2 = \frac{C}{\gamma} \int_0^{+\infty} \| f(\cdot, x_2) \|_{1,\gamma}^2 \, dx_2. \tag{4.2b}$$

Let us define $W_2 := W - W_1$. It satisfies

$$\begin{cases} \mathcal{L}^\gamma W_2 = 0, & \text{if } x_2 > 0, \\ \mathcal{B}^\gamma (W_2^{\mathrm{nc}}, \psi) = g - \underline{M} W_1^{\mathrm{nc}}{}_{|x_2=0}, & \text{if } x_2 = 0. \end{cases}$$

Consequently, if we manage to prove that Theorem 3.2 holds true as long as the interior source term is zero, we shall obtain

$$\begin{aligned} \gamma \, ||| W_2 |||_0^2 + \| W_2^{\mathrm{nc}}{}_{|x_2=0} \|_0^2 + \| \psi \|_{1,\gamma}^2 &\le \frac{C}{\gamma^2} \| g - \underline{M} W_1^{\mathrm{nc}}{}_{|x_2=0} \|_{1,\gamma}^2 \\ &\le \frac{C}{\gamma^2} \left(\| g \|_{1,\gamma}^2 + \| W_1^{\mathrm{nc}}{}_{|x_2=0} \|_{1,\gamma}^2 \right). \end{aligned}$$

Then using (4.2) to estimate the H^1 norm of the trace of W_1^{nc} as well as the L^2 norm of W_1, we shall derive our main energy estimate (3.14). Without loss of generality, we thus assume from now on that W and ψ satisfy

$$\gamma \mathcal{A}_0 W + \mathcal{A}_0 \, \partial_t W + \mathcal{A}_1 \, \partial_{x_1} W + \mathcal{A}_2 \, \partial_{x_2} W = 0 \tag{4.3}$$

in the interior domain Ω, as well as the following boundary conditions

$$\underline{M} W^{\mathrm{nc}}{}_{|x_2=0} + \underline{b} \begin{pmatrix} \gamma \psi + \partial_t \psi \\ \partial_{x_1} \psi \end{pmatrix} = g, \quad x_2 = 0. \tag{4.4}$$

With slight abuse of notations, we still denote the source term in the boundary conditions by g, instead of $g - \underline{M} W_1^{\mathrm{nc}}$. This is of pure convenience.

Recall that all matrices $\mathcal{A}_j$ are symmetric, and that $\mathcal{A}_0$ is positive definite, see (3.6). Taking the scalar product of (4.3) with W and integrating over Ω yields the following inequality:

$$\gamma\, |||W|||_0^2 \le C \| W^{\mathrm{nc}}|_{x_2=0} \|_0^2 .$$

Consequently, it is sufficient to derive an estimate of the form

$$\| W^{\mathrm{nc}}|_{x_2=0} \|_0^2 + \|\psi\|_{1,\gamma}^2 \le \frac{C}{\gamma^2} \|g\|_{1,\gamma}^2 \tag{4.5}$$

in order to obtain (3.14).

We shall derive (4.5) by means of a Kreiss' symmetrizer, whose construction is detailed in the next paragraphs. Once performed a Fourier transform in (t, x_1), the first step consists in "eliminating" the front ψ in the boundary conditions (4.4). We emphasize that this operation is possible, even though the vortex sheet is a characteristic boundary. Then we shall detail the normal modes analysis and construct a symbolic symmetrizer.

4.2. Eliminating the front. As mentionned above, we focus on (4.3)–(4.4), and perform a Fourier transform in (t, x_1). The dual variables are denoted by (δ, η). We also define $\tau := \gamma + i\delta$. This is the Laplace dual variable (indeed, the previous manipulations amount to performing a Laplace transform with respect to t). We obtain the following system of differential equations:

$$(\tau \mathcal{A}_0 + i\eta \mathcal{A}_1)\widehat{W} + \mathcal{A}_2 \frac{d\widehat{W}}{dx_2} = 0, \quad x_2 > 0, \tag{4.6a}$$

$$b(\tau, \eta)\hat{\psi} + \underline{M}\,\widehat{W^{\mathrm{nc}}}(0) = \hat{g}, \tag{4.6b}$$

where $b(\tau, \eta)$ is simply defined by

$$b(\tau, \eta) := \underline{b} \begin{pmatrix} \tau \\ i\eta \end{pmatrix} = \begin{pmatrix} 2iv_r\eta \\ \tau + iv_r\eta \\ 0 \end{pmatrix}. \tag{4.7}$$

Recall that $\underline{b}$ and $\underline{M}$ are defined by (3.9). Observe that $b(\tau, \eta)$ is homogeneous (of degree 1) with respect to (τ, η). In order to take such homogeneity properties into account, we define the hemisphere

$$\Sigma := \{(\tau, \eta) \in \mathbb{C} \times \mathbb{R} \text{ s.t. } |\tau|^2 + v_r^2\eta^2 = 1 \text{ and } \Re\tau \ge 0\}.$$

Recall that τ is the Laplace dual variable of the time variable t, while η is the Fourier dual variable of x_1, so τ/η is a velocity. Our definition of Σ takes this property into account.

We denote by Ξ the set

$$\Xi := \{(\gamma,\delta,\eta) \in [0,+\infty[\,\times\mathbb{R}^2 \text{ s.t. } (\gamma,\delta,\eta)\neq(0,0,0)\} = \,]0,+\infty[\,\cdot\Sigma.$$

It is the set of "frequencies" we shall consider in the sequel. We always identify $(\gamma,\delta)\in\mathbb{R}^2$ and $\tau=\gamma+i\delta\in\mathbb{C}$.

One crucial remark is that the symbol $b(\tau,\eta)$ is elliptic, that is, it does not vanish on the closed hemisphere Σ. More precisely, we have the following lemma.

Lemma 4.1. *There exists a C^∞ mapping Q defined on Ξ such that Q has values in $GL_3(\mathbb{C})$, is homogeneous of degree* 0, *and satisfies*

$$\forall\,(\tau,\eta)\in\Xi,\quad Q(\tau,\eta)b(\tau,\eta)=\begin{pmatrix}0\\0\\\vartheta(\tau,\eta)\end{pmatrix},$$

where ϑ is C^∞, homogeneous of degree 1, *and has the additional property:*

$$\min_{(\tau,\eta)\in\Sigma}|\vartheta(\tau,\eta)|>0.$$

Proof. We shall define the mapping Q on Σ and then extend Q by homogeneity. If we define

$$\forall\,(\tau,\eta)\in\Sigma,\quad Q(\tau,\eta):=\begin{pmatrix}0&0&1\\\tau+iv_r\eta&-2iv_r\eta&0\\-2iv_r\eta&\bar\tau-iv_r\eta&0\end{pmatrix},$$

then we check that Q has all the required properties. The corresponding ϑ is given by

$$\forall\,(\tau,\eta)\in\Sigma,\quad \vartheta(\tau,\eta):=|\tau+iv_r\eta|^2+4v_r^2\eta^2.$$

Note that the last row of $Q(\tau,\eta)$ is nothing but $b(\tau,\eta)^*$, when $(\tau,\eta)\in\Sigma$. $\square$

Let us multiply the boundary conditions in (4.6) by the matrix $Q(\tau,\eta)$. We obtain:

$$\begin{pmatrix}0\\0\\\vartheta(\tau,\eta)\end{pmatrix}\hat\psi+\begin{pmatrix}\beta(\tau,\eta)\\\ell(\tau,\eta)\end{pmatrix}\widehat{W^{\mathrm{nc}}}(0)=Q(\tau,\eta)\hat g, \tag{4.8}$$

where β has two rows while ℓ has one row and

$$\begin{pmatrix}\beta(\tau,\eta)\\\ell(\tau,\eta)\end{pmatrix}:=Q(\tau,\eta)\underline{M}.$$

The exact expression of ℓ is useless, but we shall use the expression of β:

(4.9)
$$\beta(\tau,\eta) = \begin{pmatrix} -1 & 1 & 1 & -1 \\ -c(\tau+iv_l\eta) & -c(\tau+iv_l\eta) & c(\tau+iv_r\eta) & c(\tau+iv_r\eta) \end{pmatrix}, \quad \forall\,(\tau,\eta)\in\Sigma.$$

Both β and ℓ are homogeneous of degree 0 and C^∞ on Ξ.

The last equation in (4.8) is

$$\forall\,(\tau,\eta)\in\Sigma,\quad \vartheta(\tau,\eta)\hat{\psi} + \ell(\tau,\eta)\widehat{W^{\mathrm{nc}}}(0) = b(\tau,\eta)^*\hat{g},$$

since b^* is the last row of Q. Using the ellipticity property of ϑ (see Lemma 4.1) together with a uniform bound for ℓ and b^* on Σ, we obtain

$$\forall\,(\tau,\eta)\in\Xi,\quad (|\tau|^2+v_r^2\eta^2)|\hat{\psi}|^2 \le C(|\widehat{W^{\mathrm{nc}}}(0)|^2+|\hat{g}|^2).$$

Let us now integrate this last inequality with respect to $(\delta,\eta)\in\mathbb{R}^2$. (Recall that δ is the imaginary part of τ). Using Plancherel's Theorem, we obtain

$$\begin{aligned} ||\psi||^2_{1,\gamma} &\le C(||W^{\mathrm{nc}}{}_{|x_2=0}||_0^2 + ||g||_0^2) \\ &\le C\left(||W^{\mathrm{nc}}{}_{|x_2=0}||_0^2 + \frac{1}{\gamma^2}||g||^2_{1,\gamma}\right). \end{aligned} \tag{4.10}$$

In order to derive (4.5), it is therefore sufficient to derive an estimate of the trace of W^{nc}. Consequently, we focus on the reduced problem

$$\begin{aligned} (\tau\mathcal{A}_0 + i\eta\mathcal{A}_1)\widehat{W} + \mathcal{A}_2\frac{d\widehat{W}}{dx_2} &= 0, \quad x_2>0, \\ \beta(\tau,\eta)\widehat{W^{\mathrm{nc}}}(0) &= \hat{h}, \end{aligned}$$

and try to derive an estimate for $\widehat{W^{\mathrm{nc}}}(0)$. One has to remember that the source term $\hat{h}\in\mathbb{C}^2$ is easily estimated by $\hat{g}$, see (4.8).

In the next paragraph, we recall that under the assumption made in Theorems 3.1 and 3.2, the above boundary problem satisfies the Kreiss-Lopatinskii condition but violates the uniform Kreiss-Lopatinskii condition.

4.3. The normal modes analysis. Writing $W=(W_1,W_2,W_3,W_4,W_5,W_6)^T$, the two first equations in (4.6) are:

$$\begin{aligned} (\tau+iv_r\eta)\widehat{W_1} - ic^2\eta\widehat{W_2} + ic^2\eta\widehat{W_3} &= 0, \\ (\tau+iv_l\eta)\widehat{W_4} - ic^2\eta\widehat{W_5} + ic^2\eta\widehat{W_6} &= 0. \end{aligned}$$

They do not involve derivation with respect to the normal variable x_2. For $\Re\tau > 0$, we obtain an expression for $\widehat{W_1}$ and $\widehat{W_4}$ that we plug in the last four equations. This operation yields a system of ordinary differential equations of the following form:

$$\begin{cases} \dfrac{d\widehat{W^{\mathrm{nc}}}}{dx_2} = \mathcal{A}(\tau,\eta)\widehat{W^{\mathrm{nc}}}, & \text{if } x_2 > 0, \\ \beta(\tau,\eta)\widehat{W^{\mathrm{nc}}}(0) = \hat{h}, & \text{if } x_2 = 0. \end{cases} \tag{4.11}$$

The matrix $\mathcal{A}(\tau,\eta)$ in (4.11) is given by

$$\mathcal{A}(\tau,\eta) := \begin{pmatrix} \mu_r & -m_r & 0 & 0 \\ m_r & -\mu_r & 0 & 0 \\ 0 & 0 & -\mu_l & m_l \\ 0 & 0 & -m_l & \mu_l \end{pmatrix}, \tag{4.12}$$

$$\mu_{r,l} := \frac{(1/c)(\tau + iv_{r,l}\eta)^2 + (c/2)\eta^2}{\tau + iv_{r,l}\eta},$$

$$m_{r,l} := \frac{(c/2)\eta^2}{\tau + iv_{r,l}\eta}.$$

A well-known result, that is due to Hersh [15] in the noncharacteristic case (see [23] for the extension to the characteristic case), asserts that the matrix $\mathcal{A}(\tau,\eta)$ has no purely imaginary eigenvalue as long as $\Re\tau > 0$. As a consequence, the stable subspace of $\mathcal{A}(\tau,\eta)$ has constant dimension when $\Re\tau > 0$. This dimension equals the number of characteristics going out of the discontinuity. In our case, those theoretical results can be checked directly by computing the eigenvalues and the stable subspace of $\mathcal{A}(\tau,\eta)$. The following lemma gives an expression of the stable subspace.

Lemma 4.2. *Let $\tau \in \mathbb{C}$ and $\eta \in \mathbb{R}$, with $\Re\tau > 0$ and $(\tau,\eta) \in \Sigma$. The eigenvalues of $\mathcal{A}(\tau,\eta)$ are the roots ω of the dispersion relations*

$$\omega^2 = \mu_r^2 - m_r^2 = \frac{1}{c^2}(\tau + iv_r\eta)^2 + \eta^2, \tag{4.13a}$$

$$\omega^2 = \mu_l^2 - m_l^2 = \frac{1}{c^2}(\tau + iv_l\eta)^2 + \eta^2. \tag{4.13b}$$

In particular, (4.13a) (resp. (4.13b)) admits a unique root ω_r (resp. ω_l) of negative real part. The other root of (4.13a) (resp. (4.13b)) is $-\omega_r$ (resp. $-\omega_l$), and has positive real part. The stable subspace $\mathcal{E}^-(\tau,\eta)$ of $\mathcal{A}(\tau,\eta)$ has dimension 2, and is *spanned by the two following vectors:*

$$E_r(\tau,\eta) := \left(\frac{c}{2}\eta^2, \frac{1}{c}(\tau+iv_r\eta)^2+\frac{c}{2}\eta^2-(\tau+iv_r\eta)\omega_r, 0, 0\right)^T, \tag{4.14a}$$

$$E_l(\tau,\eta) := \left(0, 0, \frac{1}{c}(\tau+iv_l\eta)^2+\frac{c}{2}\eta^2-(\tau+iv_l\eta)\omega_l, \frac{c}{2}\eta^2\right)^T. \tag{4.14b}$$

Both ω_r and ω_l admit a continuous extension to any point (τ,η) such that $\Re\tau = 0$ and $(\tau,\eta) \in \Sigma$. This allows to extend both vectors E_r and E_l in (4.14) to the whole hemisphere Σ. Those two vectors are linearly independent for any value of $(\tau,\eta) \in \Sigma$.

The symbol $\mathcal{A}(\tau,\eta)$ is diagonalizable as long as both ω_r and ω_l do not vanish, that is, when $\tau \neq (\pm v_r \pm c)i\eta$. Away from such points, $\mathcal{A}$ admits a C^∞ basis of eigenvectors.

The proof follows from straightforward computations, and we shall omit it.

We point out that the stable subspace $\mathcal{E}^-(\tau,\eta)$ is defined for all $(\tau,\eta) \in \Sigma$, while the matrix $\mathcal{A}(\tau,\eta)$ has some poles on the boundary of Σ, see (4.12). The poles are exactly those points $(\tau,\eta) \in \Sigma$ verifying $\tau = -iv_{r,l}\eta = \mp iv_r\eta$ (recall that we have $v_r = -v_l \neq 0$).

Following Majda and Osher [23], we define the Lopatinskii determinant associated with the boundary conditions β in the following way:

$$\Delta(\tau,\eta) := \det\left[\beta(\tau,\eta)\begin{pmatrix}E_r(\tau,\eta) & E_l(\tau,\eta)\end{pmatrix}\right], \tag{4.15}$$

with β defined by (4.9) and (E_r, E_l) defined by (4.14). We emphasize that the Lopatinskii determinant Δ is defined on the whole hemisphere Σ and is continuous with respect to (τ,η). The first step in proving an energy estimate for (4.11) consists in determining whether Δ vanishes on Σ. The answer is given in the following result.

Proposition 4.3. *Assume that (3.11) holds. Then there exists a positive number V_1 such that for any $(\tau,\eta) \in \Sigma$, one has $\Delta(\tau,\eta) = 0$ if and only if*

$$\tau = 0 \quad \text{or} \quad \tau = \pm iV_1\eta.$$

Each of these roots is simple. For instance, there exists a neighborhood $\mathcal{V}$ of $(0, 1/v_r)$ in Σ and a C^∞ function h defined on $\mathcal{V}$ such that

$$\forall\,(\tau,\eta) \in \mathcal{V}, \quad \Delta(\tau,\eta) = \tau h(\tau,\eta) \quad \text{and} \quad h(0, 1/v_r) \neq 0.$$

A similar result holds near $(0, -1/v_r)$ or near the points $(\pm iV_1\eta, \eta) \in \Sigma$.

The number V_1 is given by $V_1^2 := c^2 + v_r^2 - c\sqrt{c^2+4v_r^2}$. In particular, one has: $0 < V_1 < v_r - c$. When $\tau = 0$ or $\tau = \pm iV_1\eta$, both eigenmodes ω_r and ω_l are purely imaginary.

We postpone the proof of Proposition 4.3 to Appendix A. We simply note here that the three *critical* speeds $-V_1$, 0, V_1 are exactly the speeds of the kink modes exhibited in the work by Artola and Majda [2]. As a matter of fact, Artola and Majda used a "geometric optics approach," while we have followed here a "normal modes analysis approach." However, the calculations are similar in both cases (in [2], the number η equals 1).

4.4. Constructing a symmetrizer: the interior points. We now turn to the construction of our degenerate Kreiss' symmetrizer. The construction is microlocal and is achieved near any point $(\tau_0, \eta_0) \in \Sigma$. The analysis is rather long since one has to distinguish between five different cases. In the end, we shall consider a partition of unity to patch things together and derive our energy estimate.

In all the rest of the article, the letter κ denotes a generic positive constant (typically, though not necessarily, a rather small one).

We first consider the case $(\tau_0, \eta_0) \in \Sigma$ with $\Re \tau_0 > 0$. Then the matrix $\mathcal{A}(\tau, \eta)$ is diagonalizable for all (τ, η) close to (τ_0, η_0). A smooth (that is, C^∞) basis of eigenvectors is given by the following family:

$$E_r(\tau, \eta), \quad E_l(\tau, \eta),$$
$$\left(\frac{c}{2}\eta^2, \frac{1}{c}(\tau + iv_r\eta)^2 + \frac{c}{2}\eta^2 + (\tau + iv_r\eta)\omega_r, 0, 0\right)^T,$$
$$\left(0, 0, \frac{1}{c}(\tau + iv_l\eta)^2 + \frac{c}{2}\eta^2 + (\tau + iv_l\eta)\omega_l, \frac{c}{2}\eta^2\right)^T.$$

Both vectors E_r and E_l are defined by (4.14). Therefore, there exists a C^∞ mapping $T(\tau, \eta)$, defined on a neighborhood $\mathcal{V}$ of (τ_0, η_0) in Σ, with values in $GL_4(\mathbb{C})$, and such that

$$\forall\, (\tau, \eta) \in \mathcal{V}, \quad T(\tau, \eta)\mathcal{A}(\tau, \eta)T(\tau, \eta)^{-1} = \begin{pmatrix} \omega_r & 0 & 0 & 0 \\ 0 & \omega_l & 0 & 0 \\ 0 & 0 & -\omega_r & 0 \\ 0 & 0 & 0 & -\omega_l \end{pmatrix}.$$

The first two columns of the matrix $T(\tau, \eta)^{-1}$ are the vectors E_r and E_l. In the neighborhood $\mathcal{V}$, we define the symmetrizer r in the usual way:

$$\forall\, (\tau, \eta) \in \mathcal{V}, \quad r(\tau, \eta) := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix},$$

where $K \geq 1$ is a positive real number, to be fixed large enough. In what follows, we use the standard notation

$$\Re M := \frac{M + M^*}{2}$$

for all square matrix M with complex entries (M^* is the classical adjoint matrix).

The matrix r defined just above is hermitian, and we have

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad \Re\big(r(\tau,\eta)T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1}\big)\geq\kappa I\geq\kappa\gamma I, \tag{4.16}$$

for some positive constant κ. This is because ω_r and ω_l have negative real part when $(\tau,\eta)\in\mathcal{V}$, and $\gamma\leq 1$ when $(\tau,\eta)\in\Sigma$.

We now simply need to fix $K\geq 1$ in order to recover an estimate for the trace of W^{nc}. We show that for K sufficiently large, the following inequality holds:

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad r(\tau,\eta)+C(\tilde{\beta}(\tau,\eta))^*\tilde{\beta}(\tau,\eta)\geq I, \tag{4.17}$$

where C is a positive constant and $\tilde{\beta}(\tau,\eta):=\beta(\tau,\eta)T(\tau,\eta)^{-1}$.

Let $Z=(Z^-,Z^+)^T\in\mathbb{C}^4$, with $Z^-,Z^+\in\mathbb{C}^2$. We write

$$\tilde{\beta}(\tau,\eta)Z=\tilde{\beta}(\tau,\eta)\begin{pmatrix}Z^-\\0\end{pmatrix}+\tilde{\beta}(\tau,\eta)\begin{pmatrix}0\\Z^+\end{pmatrix},$$

and recall that the first two columns of $T(\tau,\eta)^{-1}$ are E_r and E_l. Because the Lopatinskii determinant does not vanish at (τ_0,η_0), see (4.15) and Proposition 4.3, we obtain an estimate of the form

$$|Z^-|^2\leq C_0(|Z^+|^2+|\tilde{\beta}(\tau,\eta)Z|^2),$$

for a suitable constant C_0 that is independent of $(\tau,\eta)\in\mathcal{V}$. With C_0 satisfying this inequality, the definition of r yields

$$\begin{aligned}\langle r(\tau,\eta)Z,Z\rangle_{\mathbb{C}^4}+2C_0|\tilde{\beta}(\tau,\eta)Z|^2&=-|Z^-|^2+K|Z^+|^2+2C_0|\tilde{\beta}(\tau,\eta)Z|^2\\&\geq|Z^-|^2+(K-2C_0)|Z^+|^2.\end{aligned}$$

This gives (4.17) for K large enough (e.g., $K=2C_0+1$).

4.5. Constructing a symmetrizer: the boundary points (case 1). We now turn to the construction of the symmetrizer near those points $(\tau,\eta)\in\Sigma$ such that $\Re\tau=0$. We first prove a general result on the behavior of the eigenmodes $\omega_{r,l}$ in the neighborhood of such points.

Lemma 4.4. *Let $(\tau_0,\eta_0)\in\Sigma$ so that $\Re\tau_0=0$ and $\tau_0\neq(-v_r\pm c)i\eta_0$. In particular, ω_r depends analytically on (τ,η) near (τ_0,η_0). Then the two following cases may occur:*

(1) *The eigenmode ω_r has negative real part at (τ_0,η_0), and, in a suitable neighborhood $\mathcal{V}$ of (τ_0,η_0), one has*

$$\Re\omega_r\leq-\kappa<0.$$

(2) *The eigenmode* ω_r *is purely imaginary at* (τ_0, η_0). *In this case, the derivative* $\partial_\gamma \omega_r$ *calculated at* (τ_0, η_0) *is a nonzero real number. In a suitable neighborhood* $\mathcal{V}$ *of* (τ_0, η_0) *in* Σ, *we have*

$$\Re \omega_r \leq -\kappa \gamma.$$

A completely similar result holds for ω_l *near all points* $(\tau_0, \eta_0) \in \Sigma$ *satisfying* $\Re \tau_0 = 0$ *and* $\tau_0 \neq (-v_l \pm c) i\eta_0$.

Proof. Because $\tau_0 \neq (-v_r \pm c) i\eta_0$, the eigenmode ω_r is not zero at (τ_0, η_0) and it depends analytically on (τ, η) by the implicit functions theorem.

The first case in Lemma 4.4 simply follows from the continuity of ω_r with respect to (τ, η). In the second case, we use (4.13a) to derive

$$\omega_r \frac{\partial \omega_r}{\partial \gamma} = \frac{1}{c^2} (\tau + i v_r \eta).$$

When $\tau = \tau_0$ and $\eta = \eta_0$, one has $\omega_r \in i\mathbb{R} \setminus \{0\}$ and $(\tau_0 + i v_r \eta_0) \in i\mathbb{R}$. This proves that the derivative $\partial_\gamma \omega_r$ is real. We now remark that $\tau_0 \neq -i v_r \eta_0$ for, in such a case, ω_r has negative real part. Consequently, the derivative $\partial_\gamma \omega_r$ is not zero. The estimate on $\Re \omega_r$ is obtained by performing a Taylor expansion of ω_r at (τ_0, η_0). $\square$

According to Proposition 4.3, there are exactly four types of points on the boundary of Σ:

(1) Those points (τ_0, η_0) where $\mathcal{A}(\tau_0, \eta_0)$ is diagonalizable and the Lopatinskii condition is satisfied at (τ_0, η_0).
(2) Those points (τ_0, η_0) where $\mathcal{A}(\tau_0, \eta_0)$ is diagonalizable and the Lopatinskii condition breaks down at (τ_0, η_0).
(3) Those points (τ_0, η_0) where $\mathcal{A}(\tau_0, \eta_0)$ is not diagonalizable, that is, $\tau_0 = (\pm v_r \pm c) i\eta_0$. In this case, Proposition 4.3 asserts that the Lopatinskii condition is satisfied at (τ_0, η_0).
(4) Those points (τ_0, η_0) that are the poles of $\mathcal{A}$, that is, $\tau_0 = \pm i v_r \eta_0$. At those points, the Lopatinskii condition is satisfied.

As a matter of fact, an immediate consequence of Proposition 4.3 is that whenever the Lopatinskii condition fails at (τ_0, η_0), then (τ_0, η_0) is not a pole and the symbol $\mathcal{A}(\tau, \eta)$ is (smoothly) diagonalizable in a neighborhood of (τ_0, η_0). The three first categories of boundary points can thus be treated as in [6, 10, 17, 28], provided the technical assumptions of [10] near instability points hold. We are going to show that such technical assumptions hold true. The last category of boundary points (the poles of the symbol $\mathcal{A}$) requires special attention.

We now deal with the construction of our symmetrizer in case 1: $(\tau_0, \eta_0) \in \Sigma$ is such that $\mathcal{A}(\tau_0, \eta_0)$ is diagonalizable and the Lopatinskii condition is satisfied at (τ_0, η_0), that is, $\Delta(\tau_0, \eta_0) \neq 0$.

Because $\mathcal{A}(\tau_0,\eta_0)$ is diagonalizable, we have $\tau_0 \neq (\pm v_r \pm c)i\eta_0$. Hence there exists a neighborhood $\mathcal{V}$ of (τ_0,η_0) in Σ and a smooth basis of eigenvectors of $\mathcal{A}$ defined on $\mathcal{V}$. The smooth basis is the same as in the case of interior points (see the preceding paragraph). We thus have

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1}=\begin{pmatrix}\omega_r & 0 & 0 & 0\\ 0 & \omega_l & 0 & 0\\ 0 & 0 & -\omega_r & 0\\ 0 & 0 & 0 & -\omega_l\end{pmatrix},$$

where, once again, the two first columns of $T(\tau,\eta)^{-1}$ are exactly E_r and E_l. We choose r as in the case of interior points:

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad r(\tau,\eta):=\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & K & 0\\ 0 & 0 & 0 & K\end{pmatrix},$$

with $K\geq 1$ to be fixed large enough. Clearly, r is a hermitian matrix. Using Lemma 4.4, we can already conclude that

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad \Re\big(r(\tau,\eta)T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1}\big)\geq \kappa\gamma I. \tag{4.18}$$

Because the Lopatinskii condition is satisfied at (τ_0,η_0), it is possible to choose K large enough so that the following estimate holds:

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad r(\tau,\eta)+C(\tilde\beta(\tau,\eta))^*\tilde\beta(\tau,\eta)\geq I. \tag{4.19}$$

In (4.19), we have let, as usual, $\tilde\beta(\tau,\eta):=\beta(\tau,\eta)T(\tau,\eta)^{-1}$. The analysis is the same as what we have done for interior points. The estimates (4.18)–(4.19) are similar (but not exactly identical) to (4.16)–(4.17).

4.6. Constructing a symmetrizer: the boundary points (case 2). In this paragraph, we consider a point $(\tau_0,\eta_0)\in\Sigma$ such that the Lopatinskii determinant Δ vanishes at (τ_0,η_0). From Proposition 4.3, we know that the symbol $\mathcal{A}$ is smoothly diagonalizable on a neighborhood $\mathcal{V}$ of (τ_0,η_0) in Σ. In other words, there exists a smooth mapping $T(\tau,\eta)$ with values in $GL_4(\mathbb{C})$, and such that

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1}=\begin{pmatrix}\omega_r & 0 & 0 & 0\\ 0 & \omega_l & 0 & 0\\ 0 & 0 & -\omega_r & 0\\ 0 & 0 & 0 & -\omega_l\end{pmatrix}.$$

The first two columns of $T(\tau,\eta)^{-1}$ are E_r and E_l. In this case, we define our symmetrizer r in the following (degenerate) way:

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad r(\tau,\eta):=\begin{pmatrix}-\gamma^2 & 0 & 0 & 0\\ 0 & -\gamma^2 & 0 & 0\\ 0 & 0 & K & 0\\ 0 & 0 & 0 & K\end{pmatrix},$$

with $K\geq 1$ to be fixed large enough. The matrix $r(\tau,\eta)$ is hermitian and we have

$$\Re(r(\tau,\eta)T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1})\geq\kappa\gamma\begin{pmatrix}\gamma^2 & 0 & 0 & 0\\ 0 & \gamma^2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\quad \forall\,(\tau,\eta)\in\mathcal{V}. \tag{4.20}$$

It only remains to fix K appropriately in order to recover an estimate on the boundary. The choice of K relies on the following lemma.

Lemma 4.5. *Let $(\tau_0,\eta_0)\in\Sigma$ be a point where the Lopatinskii determinant vanishes. Then there exists a neighborhood $\mathcal{V}$ of (τ_0,η_0) in Σ and a constant $\kappa_0>0$ such that the following estimate holds for all $(\tau,\eta)\in\mathcal{V}$:*

$$\forall\,Z^-\in\mathbb{C}^2,\quad \left|\beta(\tau,\eta)\begin{pmatrix}E_r(\tau,\eta) & E_l(\tau,\eta)\end{pmatrix}Z^-\right|^2\geq\kappa_0\gamma^2|Z^-|^2.$$

Before proving Lemma 4.5, let us first show that it enables us to obtain an estimate between r and the boundary matrix β. More precisely, we are going to show the following estimate:

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad r(\tau,\eta)+C(\tilde{\beta}(\tau,\eta))^*\tilde{\beta}(\tau,\eta)\geq\gamma^2 I, \tag{4.21}$$

for an appropriate positive constant C. The definition of $\tilde{\beta}(\tau,\eta)$ is the same as in the preceding cases.

Let $Z=(Z^-,Z^+)^T\in\mathbb{C}^4$, where Z^- and Z^+ belong to $\mathbb{C}^2$. Once again, we write

$$\tilde{\beta}(\tau,\eta)Z=\tilde{\beta}(\tau,\eta)\begin{pmatrix}Z^-\\ 0\end{pmatrix}+\tilde{\beta}(\tau,\eta)\begin{pmatrix}0\\ Z^+\end{pmatrix},$$

and we recall that the first two columns of $T(\tau,\eta)^{-1}$ are E_r and E_l. Using Lemma 4.5, we obtain

$$\kappa_0\gamma^2|Z^-|^2\leq C_0(|Z^+|^2+|\tilde{\beta}(\tau,\eta)Z|^2).$$

We thus derive

$$\begin{aligned}\langle r(\tau,\eta)Z, Z\rangle_{\mathbb{C}^4} + \frac{2C_0}{\kappa_0}|\tilde{\beta}(\tau,\eta)Z|^2 &= -\gamma^2|Z^-|^2 + K|Z^+|^2 + \frac{2C_0}{\kappa_0}|\tilde{\beta}(\tau,\eta)Z|^2 \\ &\geq \gamma^2|Z^-|^2 + \left(K - \frac{2C_0}{\kappa_0}\right)|Z^+|^2.\end{aligned}$$

Choosing $K = 2C_0/\kappa_0 + 1$ yields

$$\langle r(\tau,\eta)Z, Z\rangle_{\mathbb{C}^4} + \frac{2C_0}{\kappa_0}|\tilde{\beta}(\tau,\eta)Z|^2 \geq \gamma^2|Z^-|^2 + |Z^+|^2 \geq \gamma^2|Z|^2.$$

Here we have much weakened our estimate for the last two components Z^+. However, an inequality like (4.21) is simpler to use since it does not distinguish between the different coordinates of the vector Z. One should remember that the real loss of control is on the modes ω_r and ω_l, and not on the modes $-\omega_r$ and $-\omega_l$.

The proof of Lemma 4.5 is postponed to Appendix A. It relies on the fact that the roots of the Lopatinskii determinant Δ are simple (see Proposition 4.3).

4.7. Constructing a symmetrizer: the boundary points (case 3). In this paragraph, we consider a point $(\tau_0,\eta_0) \in \Sigma$ such that $\tau_0 = -iv_r\eta_0 \pm ic\eta_0$. (The case $\tau_0 = -iv_l\eta_0 \pm ic\eta_0$ is entirely similar, and we shall not detail it). Because $v_l = -v_r$ and $v_r > c\sqrt{2}$, we have $\tau_0 \neq -iv_l\eta_0 \pm ic\eta_0$, and therefore, the eigenmode ω_l depends smoothly on (τ,η) in a neighborhood $\mathcal{V}$ of (τ_0,η_0). Oppositely, ω_r is only continuous with respect to (τ,η) near (τ_0,η_0), but ω_r^2 is C^∞ near (τ_0,η_0). This is because (4.13a) has a double root when $(\tau,\eta) = (\tau_0,\eta_0)$.

When (τ,η) is close to (τ_0,η_0), the following family is a C^∞ basis of $\mathbb{C}^4$:

$$(m_r, -m_r, 0, 0)^T, \quad (-i, 0, 0, 0)^T, \quad E_l(\tau,\eta),$$

$$\left(0, 0, \frac{1}{c}(\tau + iv_l\eta)^2 + \frac{c}{2}\eta^2 + (\tau + iv_l\eta)\omega_l, \frac{c}{2}\eta^2\right)^T.$$

Recall that m_r is defined by (4.12). Let $T(\tau,\eta)^{-1}$ be the (regular) matrix whose columns are those four vectors. We compute

$$T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1} = \begin{pmatrix} a_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \omega_l & 0 \\ \mathbf{0} & 0 & -\omega_l \end{pmatrix},$$

where a_r is the 2×2 matrix defined as follows:

$$a_r(\tau,\eta) := \begin{pmatrix} \dfrac{-c\omega_r^2}{\tau + iv_r\eta} & i \\ \dfrac{2im_r c\omega_r^2}{\tau + iv_r\eta} & \dfrac{c\omega_r^2}{\tau + iv_r\eta} \end{pmatrix}.$$

In particular, we have

$$a_r(\tau_0,\eta_0) = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} =: iN.$$

Moreover, we make the following observations:
– For all (τ,η) close to (τ_0,η_0) so that $\tau \in i\mathbb{R}$, a_r has purely imaginary coefficients.
– The lower left coefficient ϑ_r of a_r satisfies

$$\frac{\partial \vartheta_r}{\partial \gamma}(\tau_0,\eta_0) \in \mathbb{R} \setminus \{0\}.$$

Here we have exhibited a basis in which Ralston's result [28] applies. Readers who are familiar with the theory will have recognized the "block structure condition," that is a consequence here of the strict hyperbolicity of (2.1), see [24].

We are looking for a symmetrizer r under the form

$$r(\tau,\eta) = \begin{pmatrix} s(\tau,\eta) & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & -1 & 0 \\ \mathbf{O} & 0 & K \end{pmatrix},$$

where $K \geq 1$ is a real number, to be fixed large enough, and s is some 2×2 hermitian matrix, depending smoothly on (τ,η). More precisely, we are looking for the matrix s under the following form

$$s(\tau,\eta) = \underbrace{\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}}_{E} + \underbrace{\begin{pmatrix} f(\tau,\eta) & 0 \\ 0 & 0 \end{pmatrix}}_{F(\tau,\eta)} - i\gamma \underbrace{\begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix}}_{G},$$

where e_1, e_2 and g are real numbers, and f is a real valued C^∞ mapping that vanishes at (τ_0,η_0), see [6, 17, 28]. We choose e_1 in the following way:

$$e_1 := \left(\frac{\partial \vartheta_r}{\partial \gamma}(\tau_0,\eta_0) \right)^{-1} \in \mathbb{R} \setminus \{0\},$$

where ϑ_r is defined just above. This choice may look surprising but it will be justified later on.

Now we observe that our choice of s yields ($\gamma_0 = 0$):

$$r(\tau_0,\eta_0) = \begin{pmatrix} 0 & e_1 & 0 & 0 \\ e_1 & e_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & K \end{pmatrix}.$$

Moreover, the first and third columns of $T(\tau_0,\eta_0)^{-1}$ are nothing but $E_r(\tau_0,\eta_0)$ and $E_l(\tau_0,\eta_0)$. This is due to the equality $\mu_r = -m_r$ when $(\tau,\eta) = (\tau_0,\eta_0)$. Because Lopatinskii's condition is satified at (τ_0,η_0), we can choose e_2 and K large enough such that the following estimate holds:

$$r(\tau_0,\eta_0) + C(\tilde{\beta}(\tau_0,\eta_0))^*\tilde{\beta}(\tau_0,\eta_0) \geq I.$$

As was done before, we have let $\tilde{\beta}(\tau,\eta) := \beta(\tau,\eta)T(\tau,\eta)^{-1}$. Up to shrinking $\mathcal{V}$, we have thus derived the following estimate

$$\forall\ (\tau,\eta) \in \mathcal{V}, \quad r(\tau,\eta) + C(\tilde{\beta}(\tau,\eta))^*\tilde{\beta}(\tau,\eta) \geq \tfrac{1}{2}I, \tag{4.22}$$

for a suitable constant C.

Now, we show how to choose the real valued function f and the real number g. We first write

$$a_r(\tau,\eta) = a_r(\tau_0,\eta_0) + (a_r(i\delta,\eta) - a_r(\tau_0,\eta_0)) + (a_r(\tau,\eta) - a_r(i\delta,\eta)),$$

and then use Taylor's formula. We obtain

$$a_r(\tau,\eta) = iN + (a_r(i\delta,\eta) - a_r(\tau_0,\eta_0)) + \gamma\frac{\partial a_r}{\partial\gamma}(i\delta,\eta) + \gamma^2 M(\tau,\eta),$$

for a suitable continuous function M. Because a_r has purely imaginary coefficients when τ is purely imaginary, we have

$$a_r(i\delta,\eta) - a_r(\tau_0,\eta_0) = \begin{pmatrix} ib_1(i\delta,\eta) & 0 \\ ib_2(i\delta,\eta) & ib_3(i\delta,\eta) \end{pmatrix} =: iB_r(i\delta,\eta),$$

for some C^∞, real valued mappings b_1, b_2, b_3. Those three mappings obviously vanish at (τ_0,η_0). We choose

$$f(\tau,\eta) := e_1(b_1(i\delta,\eta) - b_3(i\delta,\eta)) + e_2 b_2(i\delta,\eta),$$

so that f is a C^∞ real valued mapping that vanishes at (τ_0,η_0). Moreover, this choice of f implies that the matrix

$$(E + F(\tau,\eta))(N + B_r(i\delta,\eta))$$

is real and symmetric for all (τ,η). Consequently, we obtain

$$\Re(s(\tau,\eta)a_r(\tau,\eta)) = \gamma\Re\left(GN + E\frac{\partial a_r}{\partial\gamma}(i\delta,\eta)\right) + \gamma L(\tau,\eta),$$

where L is continuous (it is even C^∞) and $L(\tau_0,\eta_0)=0$. Our choice for e_1 yields

$$\Re\left(GN+E\frac{\partial a_r}{\partial\gamma}(\tau_0,\eta_0)\right)=\begin{pmatrix}0&0\\0&g\end{pmatrix}+\begin{pmatrix}1&*\\ *&*\end{pmatrix},$$

where the $*$ are coefficients that only depend on e_1, e_2 (that have already been fixed) and (τ_0,η_0). Choosing g large enough, and shrinking $\mathcal{V}$ if necessary, we end up with

$$\Re(s(\tau,\eta)a_r(\tau,\eta))\geq\tfrac{1}{4}\gamma I,$$

and we thus obtain

$$\forall\,(\tau,\eta)\in\mathcal{V},\quad \Re(r(\tau,\eta)T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1})\geq\kappa\gamma I. \tag{4.23}$$

4.8. Constructing a symmetrizer: the boundary points (case 4). We now consider the last case, which is $(\tau_0,\eta_0)\in\Sigma$ with $\tau_0=-iv_r\eta_0$. (We shall not detail the case $\tau_0=-iv_l\eta_0$ that is entirely similar). The symbol $\mathcal{A}$ is not defined at (τ_0,η_0), while the stable subspace $\mathcal{E}^-$ of $\mathcal{A}$ admits a continuous extension at this point. The family (E_r,E_l) is a C^∞ basis of $\mathcal{E}^-$ near (τ_0,η_0), see Lemma 4.2, and Lopatinskii's condition is satisfied near (τ_0,η_0), see Proposition 4.3.

The eigenmode ω_r has negative real part when (τ,η) is close to (τ_0,η_0), see (4.13a). Oppositely, the eigenmode ω_l is purely imaginary when (τ,η) is close to (τ_0,η_0) and $\Re\tau=0$, see (4.13b). Furthermore, both ω_r and ω_l depend analytically on (τ,η) in a neighborhood $\mathcal{V}$ of (τ_0,η_0).

The matrix $\mathcal{A}$ is not smoothly diagonalizable on $\mathcal{V}$. This is because the eigenvector associated with the eigenvalue $-\omega_r$ tends to be parallel to the eigenvector associated with the eigenvalue ω_r. Consequently, Majda and Osher's construction of a symmetrizer in this case involves a singularity in the symmetrizer, see [23]. We prefer to avoid this singularity and construct a smooth (that is, C^∞) symmetrizer in the whole neighborhood $\mathcal{V}$. This is possible if we go back to the original system:

$$(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\widehat{W}+\mathcal{A}_2\frac{d\widehat{W}}{dx_2}=0,\quad x_2>0, \tag{4.24a}$$

$$\beta(\tau,\eta)\widehat{W^{\mathrm{nc}}}(0)=\hat{h}, \tag{4.24b}$$

The following analysis is inspired from [4]. For (τ,η) in a neighborhood $\mathcal{V}$ of (τ_0,η_0), the following matrix is regular (that is, invertible):

(4.25) $\mathcal{T}(\tau,\eta) :=$

$$\begin{pmatrix} 1 & -ic\eta(\tau+iv_r\eta-c\omega_r) & 0 & & & \\ 0 & \dfrac{c}{2}\eta^2 & 0 & & \mathbf{O} & \\ 0 & (\tau+iv_r\eta)(\mu_r-\omega_r) & 1 & & & \\ & & & 1 & ic\eta(\tau+iv_l\eta-c\omega_l) & ic\eta(\tau+iv_l\eta+c\omega_l) \\ & \mathbf{O} & & 0 & (\tau+iv_l\eta)(\mu_l-\omega_l) & (\tau+iv_l\eta)(\mu_l+\omega_l) \\ & & & 0 & \dfrac{c}{2}\eta^2 & \dfrac{c}{2}\eta^2 \end{pmatrix}.$$

To avoid overloading the equations, we shall simply denote by $\mathbf{O}$ the 3×3 zero matrix. The determinant of $\mathcal{T}(\tau,\eta)$ is given by

$$\det \mathcal{T}(\tau,\eta) = \frac{c^2}{2}\eta^4(\tau+iv_l\eta)\omega_l,$$

and it is easy to check that this quantity does not vanish near (τ_0,η_0). The matrix $\mathcal{T}$ depends smoothly on (τ,η) in the whole neighborhood $\mathcal{V}$. It has no pole.

Let us define

$$\xi_r := (\tau+iv_r\eta)(\mu_r-\omega_r) = \frac{1}{c}(\tau+iv_r\eta)^2 + \frac{c}{2}\eta^2 - (\tau+iv_r\eta)\omega_r, \tag{4.26a}$$

$$\xi_l^{\pm} := (\tau+iv_l\eta)(\mu_l\pm\omega_l) = \frac{1}{c}(\tau+iv_l\eta)^2 + \frac{c}{2}\eta^2 \pm (\tau+iv_l\eta)\omega_l. \tag{4.26b}$$

The main idea now is that (4.24a) has a simple expression if we decompose the vector $\widehat{W}$ in the basis defined by $\mathcal{T}$. In other words, the matrices $(\tau\mathcal{A}_0 + i\eta\mathcal{A}_1)\mathcal{T}(\tau,\eta)$ and $\mathcal{A}_2\mathcal{T}(\tau,\eta)$ have a rather similar structure. Performing some manipulations on the rows of these two matrices, we shall transform (4.24a) into an "almost diagonal" system of differential equations.

After some simplifications, we obtain the following expressions:

$$(\tau\mathcal{A}_0 + i\eta\mathcal{A}_1)\mathcal{T}$$
$$= \begin{pmatrix} \tau+iv_r\eta & 0 & ic^2\eta & & & \\ -ic^2\eta & c^4\eta^2\omega_r & 0 & & \mathbf{O} & \\ ic^2\eta & -2c^3\omega_r\xi_r & 2c^2(\tau+iv_r\eta) & & & \\ 3pt] & & & \tau+iv_l\eta & 0 & 0 \\ 3pt] & \mathbf{O} & & -ic^2\eta & -2c^3\omega_l\xi_l^- & 2c^3\omega_l\xi_l^+ \\ & & & ic^2\eta & c^4\eta^2\omega_l & -c^4\eta^2\omega_l \end{pmatrix},$$

$$\mathcal{A}_2\mathcal{T} = \begin{pmatrix} 0 & 0 & 0 & & & \\ 0 & -c^4\eta^2 & 0 & & \mathbf{O} & \\ 0 & 2c^3\xi_r & 2c^3 & & & \\ & & & 0 & 0 & 0 \\ & \mathbf{O} & & 0 & 2c^3\xi_l^- & 2c^3\xi_l^+ \\ & & & 0 & -c^4\eta^2 & -c^4\eta^2 \end{pmatrix},$$

where ξ_r and $\xi_l^\pm$ are defined by (4.26a)–(4.26b). These expressions may look somehow complicated, but it is not so hard to simplify them multiplying on the left by a suitable symmetrizer. Indeed, for all (τ,η) in a neighborhood $\mathcal{V}$ of (τ_0,η_0), let us define the following matrix:

(4.27) $R_1(\tau,\eta) :=$

$$\begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & \mathbf{O} & \\ 2ic^4\eta(\tau+iv_r\eta-c\omega_r) & 2c^3\xi_r & c^4\eta^2 & & & \\ & & & 1 & 0 & 0 \\ & \mathbf{O} & & -2ic^4\eta(\tau+iv_l\eta+c\omega_l) & c^4\eta^2 & 2c^3\xi_l^+ \\ & & & -2ic^4\eta(\tau+iv_l\eta-c\omega_l) & c^4\eta^2 & 2c^3\xi_l^- \end{pmatrix}.$$

First we obtain

$$R_1\mathcal{A}_2\mathcal{T} = \operatorname{diag}(0,-c^4\eta^2,2c^7\eta^2,0,-4c^7\eta^2(\tau+iv_l\eta)\omega_l,4c^7\eta^2(\tau+iv_l\eta)\omega_l),$$

where $\operatorname{diag}(a_1,\dots,a_p)$ stands for the diagonal matrix whose diagonal elements are $a_1,\dots,a_p$. We also obtain the following expression:

(4.28) $R_1(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\mathcal{T} =$

$$\begin{pmatrix} \tau+iv_r\eta & 0 & ic^2\eta & & & \\ -ic^2\eta & c^4\eta^2\omega_r & 0 & & \mathbf{O} & \\ 0 & 0 & 2c^7\eta^2\omega_r & & & \\ & & & \tau+iv_l\eta & 0 & 0 \\ & \mathbf{O} & & 0 & 4c^7\eta^2\omega_l^2(\tau+iv_l\eta) & 0 \\ & & & 0 & 0 & 4c^7\eta^2\omega_l^2(\tau+iv_l\eta) \end{pmatrix}.$$

Recall that ω_l, η and $(\tau+iv_l\eta)$ do not vanish in the neighborhood $\mathcal{V}$ of (τ_0,η_0), up to shrinking $\mathcal{V}$. The following matrix is therefore a C^∞ mapping of (τ,η) on $\mathcal{V}$:

$$(4.29)\quad R_2(\tau,\eta) := \operatorname{diag}\left(1,1,\frac{1}{2c^7\eta^2},1,\frac{1}{4c^7\eta^2\omega_l(\tau+iv_l\eta)},\frac{1}{4c^7\eta^2\omega_l(\tau+iv_l\eta)}\right).$$

We define

$$S(\tau,\eta) := R_2(\tau,\eta)R_1(\tau,\eta),$$

where R_1 and R_2 are defined by (4.27)–(4.29). It is now easy to derive the following equalities for all (τ,η) in $\mathcal{V}$:

$$(4.30a)\qquad S(\tau,\eta)\mathcal{A}_2\mathcal{T}(\tau,\eta) = \operatorname{diag}(0,-c^4\eta^2,1,0,-1,1),$$

$$(4.30b)\qquad S(\tau,\eta)(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\mathcal{T}(\tau,\eta) = \begin{pmatrix} \tau+iv_r\eta & 0 & ic^2\eta & & & \\ -ic^2\eta & c^4\eta^2\omega_r & 0 & & \mathbf{O} & \\ 0 & 0 & \omega_r & & & \\ & & & \tau+iv_l\eta & 0 & 0 \\ & \mathbf{O} & & 0 & \omega_l & 0 \\ & & & 0 & 0 & \omega_l \end{pmatrix}.$$

In particular, it is important to observe that both matrices S and $\mathcal{T}$ are C^∞ on the whole neighborhood $\mathcal{V}$. Up to shrinking $\mathcal{V}$, we may assume that ω_r has negative real part for all $(\tau,\eta)\in\mathcal{V}$. This is possible because $\omega_r = -|\eta|$ at (τ_0,η_0).

Though a little complicated, the preceding calculations are based on the simple idea that the differential equations (4.24a) have an easy expression if we replace the standard coordinates by the coordinates on the stable subspace. The difficulty comes from the fact that we deal with the differential system satisfied by $\widehat{W}$ and not with the system satisfied by $\widehat{W^{\mathrm{nc}}}$. Because the boundary is characteristic, (4.24a) is not an *ordinary* differential equation, and we thus need to diagonalize simultaneously $(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)$ and $\mathcal{A}_2$. Even though the matrix $S(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\mathcal{T}$ is not diagonal, we shall see that the reduced expressions (4.30a)–(4.30b) are sufficient to derive energy estimates in such a neighborhood $\mathcal{V}$ of the pole (τ_0,η_0).

4.9. Derivation of the energy estimate. We now turn to the derivation of the estimate (4.5). Recall that we are considering a function $W\in H^1(\Omega)$ such that

$$(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\widehat{W}+\mathcal{A}_2\frac{d\widehat{W}}{dx_2}=0,\quad x_2>0,$$
$$\beta(\tau,\eta)\widehat{W^{\mathrm{nc}}}(0)=\hat{h},$$

where $\hat{h}$ is obtained from the source term $\hat{g}$ in (4.6) after eliminating the unknown front:

$$\forall\,(\tau,\eta)\in\Sigma,\quad \hat{h} = \begin{pmatrix} 0 & 0 & 1 \\ \tau+iv_r\eta & -2iv_r\eta & 0\end{pmatrix}\hat{g}.$$

Thanks to (4.10), it is sufficient to get an estimate of the trace of W^{nc} on $\{x_2 = 0\}$ in order to derive (4.5).

The previous analysis shows that for all $(\tau_0, \eta_0) \in \Sigma$, there exists a neighborhood $\mathcal{V}$ of (τ_0, η_0) in Σ and mappings defined on this neighborhood that satisfy certain properties. Because Σ is a C^∞ compact manifold, there exists a finite covering $(\mathcal{V}_1, \dots, \mathcal{V}_I)$ of Σ by such neighborhoods, and a smooth partition of unity $(\chi_1, \dots, \chi_I)$ associated with this covering. Namely, the χ_i's are nonnegative, C^∞, and satisfy

$$\operatorname{Supp} \chi_i \subset \mathcal{V}_i \quad \text{and} \quad \sum_{i=1}^{I} \chi_i^2 \equiv 1.$$

There are three different cases.

In the first case, $\mathcal{V}_i$ is a neighborhood of an interior point or a neighborhood of a boundary point corresponding to cases 1 and 3 above (boundary points that are not poles and for which the Lopatinskii condition is satisfied). On such a neighborhood, there exist two C^∞ mappings r_i and T_i such that

– r_i is hermitian,
– T_i has values in $GL_4(\mathbb{C})$,
– the following estimates hold for all $(\tau, \eta) \in \mathcal{V}_i$:

$$\Re\big(r_i(\tau,\eta)T_i(\tau,\eta)\mathcal{A}(\tau,\eta)T_i(\tau,\eta)^{-1}\big) \ge \kappa_i \gamma I, \tag{4.31a}$$

$$r_i(\tau,\eta) + C_i\big(\beta(\tau,\eta)T_i(\tau,\eta)^{-1}\big)^* \beta(\tau,\eta)T_i(\tau,\eta)^{-1} \ge I. \tag{4.31b}$$

The inequalities (4.31) are direct consequences of (4.16)–(4.17)–(4.18)–(4.19)–(4.22)–(4.23).

We define

$$U_i(\tau,\eta,x_2) := \chi_i(\tau,\eta)T_i(\tau,\eta)\widehat{W^{\mathrm{nc}}}(\delta,\eta,x_2).$$

Both mappings r_i and T_i are not defined on Σ but only on the neighborhood $\mathcal{V}_i$. However, only the values of these mappings on the support of χ_i will be involved in the subsequent calculations, so we choose for convenience to extend these mappings to the whole hemishpere Σ. Then we extend χ_i, r_i, T_i to the whole set of frequencies Ξ as homogeneous mappings of degree 0 with respect to (τ, η).

Because $\mathcal{V}_i$ does not contain any pole of the symbol $\mathcal{A}$, which is defined by (4.12), one easily shows that U_i satisfies

$$\frac{dU_i}{dx_2} = T_i(\tau,\eta)\mathcal{A}(\tau,\eta)T_i(\tau,\eta)^{-1}U_i, \quad x_2 > 0,$$
$$\beta(\tau,\eta)T_i(\tau,\eta)^{-1}U_i(0) = \chi_i \hat{h}.$$

We take the scalar product of this later ordinary differential equation with $r_i U_i$ and integrate with respect to x_2 on $\mathbb{R}^+$. Then we take the real part of the obtained equality and use (4.31). These operations yield the classical Kreiss' estimate:

$$\kappa_i \gamma \int_0^{+\infty} |U_i(\tau,\eta,x_2)|^2\, dx_2 + \frac{1}{2}|U_i(\tau,\eta,0)|^2 \leq C_i \chi_i(\tau,\eta)^2 |\hat{h}(\tau,\eta)|^2.$$

Now we use the definition of U_i and a uniform bound for $\|T_i(\tau,\eta)^{-1}\|$ on the support of χ_i, to derive

$$\begin{aligned}\gamma\chi_i(\tau,\eta)^2 \int_0^{+\infty} |\widehat{W^{\mathrm{nc}}}(\delta,\eta,x_2)|^2\, dx_2 + \chi_i(\tau,\eta)^2 |\widehat{W^{\mathrm{nc}}}(\delta,\eta,0)|^2 \\ \leq C_i \chi_i(\tau,\eta)^2 |\hat{h}|^2.\end{aligned} \tag{4.32}$$

In the second case, $\mathcal{V}_i$ is a neighborhood of a zero of the Lopatinskii determinant. On such a neighborhood, there exist two C^∞ mappings r_i and T_i such that

- r_i is hermitian,
- T_i has values in $GL_4(\mathbb{C})$,
- the following estimates hold for all $(\tau,\eta) \in \mathcal{V}_i$:

$$\Re(r_i(\tau,\eta) T_i(\tau,\eta) \mathcal{A}(\tau,\eta) T_i(\tau,\eta)^{-1}) \geq \kappa_i \gamma^3 I, \tag{4.33a}$$

$$r_i(\tau,\eta) + C_i (\beta(\tau,\eta) T_i(\tau,\eta)^{-1})^* \beta(\tau,\eta) T_i(\tau,\eta)^{-1} \geq \gamma^2 I. \tag{4.33b}$$

These inequalities correspond to (4.20)–(4.21).

As was done before, we first extend r_i and T_i as C^∞ mappings on the whole hemisphere Σ. Then we extend T_i and χ_i as homogeneous mappings of degree 0 with respect to (τ,η), and we extend r_i as a homogeneous mapping of degree 2 with respect to (τ,η). Thus (4.33) reads

$$\Re(r_i(\tau,\eta) T_i(\tau,\eta) \mathcal{A}(\tau,\eta) T_i(\tau,\eta)^{-1}) \geq \kappa_i \gamma^3 I, \tag{4.34a}$$

$$\begin{aligned} r_i(\tau,\eta) + C_i(|\tau|^2 + v_r^2\eta^2)(\beta(\tau,\eta) T_i(\tau,\eta)^{-1})^* \beta(\tau,\eta) T_i(\tau,\eta)^{-1} \\ \geq \gamma^2 I,\end{aligned} \tag{4.34b}$$

for all $(\tau,\eta) \in \mathbb{R}^+ \cdot \mathcal{V}_i$. Once again, we define

$$U_i(\tau,\eta,x_2) := \chi_i(\tau,\eta) T_i(\tau,\eta) \widehat{W^{\mathrm{nc}}}(\delta,\eta,x_2).$$

Because $\mathcal{V}_i$ does not contain any pole of $\mathcal{A}$, we still have

$$\frac{dU_i}{dx_2} = T_i(\tau,\eta) \mathcal{A}(\tau,\eta) T_i(\tau,\eta)^{-1} U_i, \quad x_2 > 0,$$
$$\beta(\tau,\eta) T_i(\tau,\eta)^{-1} U_i(0) = \chi_i \hat{h}.$$

We perform the same calculations as above (in the first case), but now we use (4.34) instead of (4.31). We obtain

$$\gamma \chi_i(\tau,\eta)^2 \int_0^{+\infty} |\widehat{W^{\mathrm{nc}}}(\delta,\eta,x_2)|^2\, dx_2 + \chi_i(\tau,\eta)^2 |\widehat{W^{\mathrm{nc}}}(\delta,\eta,0)|^2 \leq \frac{C_i}{\gamma^2} \chi_i(\tau,\eta)^2 |\hat{h}|^2 (|\tau|^2 + v_r^2\eta^2). \tag{4.35}$$

In the third and last case, $\mathcal{V}_i$ is a neighborhood of a pole of the symbol $\mathcal{A}$. For instance, $\mathcal{V}_i$ is a neighborhood of a point $(-iv_r\eta_0, \eta_0) \in \Sigma$. In this case, we have shown that there exists C^∞ mappings $\mathcal{T}_i$ and S_i defined on $\mathcal{V}_i$ such that

- $\mathcal{T}_i$ has values in $GL_6(\mathbb{C})$,
- both relations (4.30a)–(4.30b) hold on $\mathcal{V}_i$.

Recall that ω_r has negative real part on $\mathcal{V}_i$.

Here, we extend χ_i, $\mathcal{T}_i$ and S_i as homogeneous mappings of degree 0 with respect to (τ,η). Moreover, we shall make as if $\mathcal{T}_i$ and S_i were defined on the whole hemisphere Σ (this is of pure convenience since only the values on the support of χ_i are involved in what follows). We define

$$U_i(\tau,\eta,x_2) := \chi_i(\tau,\eta)\mathcal{T}_i(\tau,\eta)^{-1}\widehat{W}(\delta,\eta,x_2) \in \mathbb{C}^6.$$

The components of the vector U_i are denoted as follows

$$U_i = (U_{i,1}, U_{i,2}, \dots, U_{i,6})^T,$$

and we also define

$$U_i^{\mathrm{nc}} := (U_{i,2}, U_{i,3}, U_{i,5}, U_{i,6})^T \in \mathbb{C}^4.$$

Using the definition (4.25) of $\mathcal{T}_i(\tau,\eta)$, it is clear that the vector U_i^{nc} is given by a relation of the form

$$U_i^{\mathrm{nc}} = T_i(\tau,\eta)^{-1}\widehat{W^{\mathrm{nc}}}, \quad \text{with } T_i(\tau,\eta) \in GL_4(\mathbb{C}).$$

We also note that the first and third column vectors of the matrix $T_i(\tau,\eta)$ are exactly the vectors $E_r(\tau,\eta)$ and $E_l(\tau,\eta)$. With these notations and definitions, we get

$$\begin{aligned}
&(\tau\mathcal{A}_0 + i\eta\mathcal{A}_1)\mathcal{T}_i(\tau,\eta)U_i + \mathcal{A}_2\mathcal{T}_i(\tau,\eta)\frac{dU_i}{dx_2} = 0, \quad x_2 > 0,\\
&\beta(\tau,\eta)T_i(\tau,\eta)U_i^{\mathrm{nc}}(0) = \chi_i(\tau,\eta)\hat{h}.
\end{aligned}$$

Multiplying the equation in $\{x_2 > 0\}$ by $S_i(\tau, \eta)$ and using (4.30a)–(4.30b), we obtain the following system:

$$(\tau + iv_r\eta)U_{i,1} + ic^2\eta U_{i,3} = 0, \tag{4.36a}$$

$$-ic^2\eta U_{i,1} + \frac{c^4\eta^2}{|\tau|^2 + v_r^2\eta^2}\omega_r U_{i,2} - \frac{c^4\eta^2}{|\tau|^2 + v_r^2\eta^2}\frac{dU_{i,2}}{dx_2} = 0, \tag{4.36b}$$

$$\omega_r U_{i,3} + \frac{dU_{i,3}}{dx_2} = 0, \tag{4.36c}$$

$$(\tau + iv_l\eta)U_{i,4} = 0, \tag{4.36d}$$

$$\omega_l U_{i,5} - \frac{dU_{i,5}}{dx_2} = 0, \tag{4.36e}$$

$$\omega_l U_{i,6} + \frac{dU_{i,6}}{dx_2} = 0. \tag{4.36f}$$

Recall that when (τ, η) belongs to the conical set $\mathbb{R}^+ \cdot \mathcal{V}_i$, one has

$$\Re\omega_r \le -\kappa(|\tau|^2 + v_r^2\eta^2)^{1/2} \quad \text{and} \quad \Re\omega_l \le -\kappa\gamma,$$

for a suitable constant $\kappa > 0$. Because $U_{i,3}(x_2)$ and $U_{i,6}(x_2)$ belong to $L^2(\mathbb{R}^+)$, for $\gamma > 0$, (4.36c) and (4.36f) imply $U_{i,3} \equiv 0$ *and* $U_{i,6} \equiv 0$. Using (4.36a) and (4.36d), we also obtain $U_{i,1} \equiv 0$ *and* $U_{i,4} \equiv 0$. Eventually, (4.36b) and (4.36e) reduce to

$$\omega_r U_{i,2} - \frac{dU_{i,2}}{dx_2} = 0 \quad \text{and} \quad \omega_l U_{i,5} - \frac{dU_{i,5}}{dx_2} = 0.$$

Because the first and third columns of $T_i(\tau, \eta)$ are nothing but E_r and E_l defined in Lemma 4.2, the boundary conditions for $U_{i,2}$ and $U_{i,5}$ read:

$$\beta(\tau, \eta)\begin{pmatrix} E_r(\tau, \eta) & E_l(\tau, \eta)\end{pmatrix}\begin{pmatrix} U_{i,2}(0) \\ U_{i,5}(0)\end{pmatrix} = \chi_i(\tau, \eta)\hat{h}. \tag{4.37}$$

Using the properties of ω_r and ω_l on the conical set $\mathbb{R}^+ \cdot \mathcal{V}_i$, we derive

$$(|\tau|^2 + v_r^2\eta^2)^{1/2}\int_0^{+\infty} |U_{i,2}(\tau, \eta, x_2)|^2\, dx_2 \le C|U_{i,2}(\tau, \eta, 0)|^2,$$
$$\gamma\int_0^{+\infty} |U_{i,5}(\tau, \eta, x_2)|^2\, dx_2 \le C|U_{i,5}(\tau, \eta, 0)|^2.$$

Because the uniform Lopatinskii condition is satisfied on $\mathcal{V}_i$, (4.37) yields the following estimate:

$$\left|\begin{pmatrix} U_{i,2}(\tau, \eta, 0) \\ U_{i,5}(\tau, \eta, 0)\end{pmatrix}\right|^2 \le C\chi_i(\tau, \eta)^2|\hat{h}|^2.$$

Eventually, we obtain

$$\gamma \int_0^{+\infty} |U_i^{\mathrm{nc}}(\tau,\eta,x_2)|^2\, dx_2 + |U_i^{\mathrm{nc}}(\tau,\eta,0)|^2 \leq C\chi_i(\tau,\eta)^2|\hat{h}|^2.$$

We now use the definition of the vector U_i^{nc} to derive

$$\gamma\chi_i^2 \int_0^{+\infty} |\widehat{W^{\mathrm{nc}}}(\delta,\eta,x_2)|^2\, dx_2 + \chi_i^2|\widehat{W^{\mathrm{nc}}}(\delta,\eta,0)|^2 \leq C\chi_i^2|\hat{h}|^2. \tag{4.38}$$

We now add up (4.32)–(4.35)–(4.38), and use that the χ_i's form a partition of unity. We obtain

$$\gamma \int_0^{+\infty} |\widehat{W^{\mathrm{nc}}}(\delta,\eta,x_2)|^2\, dx_2 + |\widehat{W^{\mathrm{nc}}}(\delta,\eta,0)|^2 \leq \frac{C}{\gamma^2}|\hat{h}|^2(|\tau|^2 + v_r^2\eta^2).$$

We have already recalled that $\hat{h}$ is simply obtained from the source term $\hat{g}$ in (4.6) by multiplying by a uniformly bounded matrix. Thus integrating the previous inequality with respect to $(\delta,\eta) \in \mathbb{R}^2$ and using Plancherel's theorem yields the desired estimate:

$$\gamma |||W^{\mathrm{nc}}|||_0^2 + ||W^{\mathrm{nc}}{}_{|x_2=0}||_0^2 \leq \frac{C}{\gamma^2}||g||_{1,\gamma}^2.$$

Combining with (4.10), we have finished to prove (4.5). ◻

5. The Variable Coefficients Linearized Problem

5.1. The linearized equations and the main result. We introduce the linearized equations around a state given by a perturbation of the constant solution in (2.9). More precisely, let us consider the functions

$$U_r = \begin{pmatrix} \rho \\ \bar{v}_r \\ 0 \end{pmatrix} + \dot{U}_r, \quad U_l = \begin{pmatrix} \rho \\ -\bar{v}_r \\ 0 \end{pmatrix} + \dot{U}_l, \quad \Phi_r, \quad \Phi_l, \tag{5.1}$$

where ρ, $\bar{v}_r$ are fixed positive constants (in this section we introduce the small change of notation $v_r \to \bar{v}_r$ for the piecewise constant solution) and where

$$U_r(t,x_1,x_2) = \begin{pmatrix} \rho_r(t,x_1,x_2) \\ v_r(t,x_1,x_2) \\ u_r(t,x_1,x_2) \end{pmatrix}, \quad \dot{U}_r(t,x_1,x_2) = \begin{pmatrix} \dot{\rho}_r(t,x_1,x_2) \\ \dot{v}_r(t,x_1,x_2) \\ \dot{u}_r(t,x_1,x_2) \end{pmatrix},$$

$$U_l(t,x_1,x_2) = \begin{pmatrix} \rho_l(t,x_1,x_2) \\ v_l(t,x_1,x_2) \\ u_l(t,x_1,x_2) \end{pmatrix}, \quad \dot{U}_l(t,x_1,x_2) = \begin{pmatrix} \dot{\rho}_l(t,x_1,x_2) \\ \dot{v}_l(t,x_1,x_2) \\ \dot{u}_l(t,x_1,x_2) \end{pmatrix}.$$

The functions Φ_r, Φ_l are perturbations of the change of variables. The index r (resp. l) denotes the state on the right (resp. on the left) of the interface (after the change of variables). We assume that

$$U_r,\ U_l,\ \nabla\Phi_r,\ \nabla\Phi_l \in W^{2,\infty}(\Omega), \tag{5.2a}$$

$$\|(U_r, U_l)\|_{W^{2,\infty}(\Omega)} + \|(\nabla\Phi_r, \nabla\Phi_l)\|_{W^{2,\infty}(\Omega)} \leq K_0, \tag{5.2b}$$

where K_0 is a suitable positive constant, and that the perturbations $\dot{U}_r$, $\dot{U}_l$ have compact support. These quantities are linked by the Rankine-Hugoniot conditions and the continuity condition for the functions $\Phi_{r,l}$ that, written in the form of (2.5), become

$$\Phi_r(t, x_1, 0) = \Phi_l(t, x_1, 0) = \varphi(t, x_1), \tag{5.3a}$$

$$(v_r - v_l)_{|x_2=0}\, \partial_{x_1}\varphi - (u_r - u_l)_{|x_2=0} = 0, \tag{5.3b}$$

$$\partial_t\varphi + v_{r|x_2=0}\, \partial_{x_1}\varphi - u_{r|x_2=0} = 0, \tag{5.3c}$$

$$(\rho_r - \rho_l)_{|x_2=0} = 0. \tag{5.3d}$$

The functions Φ_r and Φ_l should also satisfy

$$\partial_t\Phi_r + v_r\, \partial_{x_1}\Phi_r - u_r = 0, \tag{5.4a}$$

$$\partial_t\Phi_l + v_l\, \partial_{x_1}\Phi_l - u_l = 0, \tag{5.4b}$$

together with

$$\partial_{x_2}\Phi_r \geq \kappa_0, \qquad \partial_{x_2}\Phi_l \leq -\kappa_0, \tag{5.5}$$

for a suitable constant $\kappa_0 > 0$, in the whole closed half-space $\{x_2 \geq 0\}$.

Let us consider the families $U_s^\pm = U_{r,l} + sU_\pm$, $\Phi_s^\pm = \Phi_{r,l} + s\Psi_\pm$, where s is a small parameter. We compute the linearized equations

$$L'(U_{r,l}, \nabla\Phi_{r,l})(U_\pm, \Psi_\pm) := \frac{d}{ds} L(U_s^\pm, \nabla\Phi_s^\pm) U^\pm_{s|s=0} = f_\pm.$$

We obtain

$$\begin{aligned}
&\partial_t U_+ + A_1(U_r)\, \partial_{x_1} U_+ + \frac{1}{\partial_{x_2}\Phi_r}(A_2(U_r) - \partial_t\Phi_r - \partial_{x_1}\Phi_r A_1(U_r))\, \partial_{x_2} U_+ \\
&\quad + [dA_1(U_r)U_+]\partial_{x_1} U_r - \frac{\partial_{x_2}\Psi_+}{(\partial_{x_2}\Phi_r)^2}(A_2(U_r) - \partial_t\Phi_r - \partial_{x_1}\Phi_r A_1(U_r))\, \partial_{x_2} U_r \\
&\quad + \frac{1}{\partial_{x_2}\Phi_r}[dA_2(U_r)U_+ - \partial_t\Psi_+ - \partial_{x_1}\Psi_+ A_1(U_r) - \partial_{x_1}\Phi_r\, dA_1(U_r)U_+]\partial_{x_2} U_r = f_+,
\end{aligned} \tag{5.6}$$

in the domain $\{x_2 > 0\}$, and a similar equation with U_-, Ψ_-, U_l, Φ_l, f_- instead of U_+, Ψ_+, U_r, Φ_r, f_+. Recall that, according to the definition in (2.7), (2.8), the first row in (5.6) may be simply denoted by $L(U_r, \nabla\Phi_r)U_+$, namely we set:

$$L(U_r, \nabla\Phi_r)U_+ := \partial_t U_+ + A_1(U_r)\,\partial_{x_1}U_+ \\ + \frac{1}{\partial_{x_2}\Phi_r}(A_2(U_r) - \partial_t\Phi_r - \partial_{x_1}\Phi_r A_1(U_r))\,\partial_{x_2}U_+.$$

The equation (5.6) and the corresponding one for (U_-, Ψ_-) may be simplified by the introduction of "la bonne inconnue" (the good unknown) as in [1]:

$$\dot{U}_+ := U_+ - \frac{\Psi_+}{\partial_{x_2}\Phi_r}\partial_{x_2}U_r, \quad \dot{U}_- := U_- - \frac{\Psi_-}{\partial_{x_2}\Phi_l}\partial_{x_2}U_l. \tag{5.7}$$

A direct calculation shows that $\dot{U}_+$, $\dot{U}_-$ satisfy

$$L(U_r, \nabla\Phi_r)\dot{U}_+ + C(U_r, \nabla U_r, \nabla\Phi_r)\dot{U}_+ + \frac{\Psi_+}{\partial_{x_2}\Phi_r}\partial_{x_2}[L(U_r, \nabla\Phi_r)U_r] = f_+,$$
$$L(U_l, \nabla\Phi_l)\dot{U}_- + C(U_l, \nabla U_l, \nabla\Phi_l)\dot{U}_- + \frac{\Psi_-}{\partial_{x_2}\Phi_l}\partial_{x_2}[L(U_l, \nabla\Phi_l)U_l] = f_-,$$

where

$$C(U_r, \nabla U_r, \nabla\Phi_r)\dot{U}_+ := [dA_1(U_r)\dot{U}_+]\partial_{x_1}U_r \\ + \frac{1}{\partial_{x_2}\Phi_r}[dA_2(U_r)\dot{U}_+ - \partial_{x_1}\Phi_r dA_1(U_r)\dot{U}_+]\,\partial_{x_2}U_r,$$

and with a similar expression for $C(U_l, \nabla U_l, \nabla\Phi_l)\dot{U}_-$. In view of the results in [1, 13], we neglect the zeroth order term in Ψ_+, Ψ_- in the linearized equations and thus consider the linear equations

$$L'_r\dot{U}_+ := L(U_r, \nabla\Phi_r)\dot{U}_+ + C(U_r, \nabla U_r, \nabla\Phi_r)\dot{U}_+ = f_+, \tag{5.8a}$$
$$L'_l\dot{U}_- := L(U_l, \nabla\Phi_l)\dot{U}_- + C(U_l, \nabla U_l, \nabla\Phi_l)\dot{U}_- = f_-. \tag{5.8b}$$

We easily verify, using (5.2), that the coefficients of the operators $L(U_r, \nabla\Phi_r)$ and $L(U_l, \nabla\Phi_l)$ are in $W^{2,\infty}(\Omega)$, that is

$$A_1(U_r) \in W^{2,\infty}(\Omega), \quad \frac{1}{\partial_{x_2}\Phi_r}(A_2(U_r) - \partial_t\Phi_r - \partial_{x_1}\Phi_r A_1(U_r)) \in W^{2,\infty}(\Omega),$$
$$A_1(U_l) \in W^{2,\infty}(\Omega), \quad \frac{1}{\partial_{x_2}\Phi_l}(A_2(U_l) - \partial_t\Phi_l - \partial_{x_1}\Phi_l A_1(U_l)) \in W^{2,\infty}(\Omega).$$

Moreover, we have $C(U_{r,l}, \nabla U_{r,l}, \nabla\Phi_{r,l}) \in W^{1,\infty}(\Omega)$. It is clear that the linearized equations (5.8) form a symmetrizable hyperbolic system. For instance, a Friedrichs symmetrizer for the operator L'_r is given by

$$S_r := \begin{pmatrix} (p'(\rho_r)/\rho_r) & 0 & 0 \\ 0 & \rho_r & 0 \\ 0 & 0 & \rho_r \end{pmatrix}.$$

Using (5.4), we compute

$$\begin{aligned} &\frac{S_r}{\partial_{x_2}\Phi_r}(A_2(U_r) - \partial_t\Phi_r - \partial_{x_1}\Phi_r A_1(U_r)) \\ &\qquad = \frac{1}{\partial_{x_2}\Phi_r} \begin{pmatrix} 0 & -p'(\rho_r)\,\partial_{x_1}\Phi_r & p'(\rho_r) \\ -p'(\rho_r)\,\partial_{x_1}\Phi_r & 0 & 0 \\ p'(\rho_r) & 0 & 0 \end{pmatrix}, \end{aligned}$$

and we thus expect to control the traces of $\dot{U}_{+,1}$ and $(\dot{U}_{+,3} - \partial_{x_1}\Phi_r\dot{U}_{+,2})$ on the boundary $\{x_2 = 0\}$. These considerations motivate the introduction of the following operator:

$$\mathbb{P}\dot{U}_{\pm|x_2=0} := \begin{pmatrix} \dot{U}_{\pm,1} \\ \dot{U}_{\pm,3} - \partial_{x_1}\varphi\dot{U}_{\pm,2} \end{pmatrix}_{|x_2=0}. \tag{5.9}$$

We now turn to the linearized boundary conditions. The linearization of (2.5) gives

$$\begin{aligned} &\Psi_{+|x_2=0} = \Psi_{-|x_2=0} = \psi, \\ &\qquad (v_r - v_l)\,\partial_{x_1}\psi + (v_+ - v_-)\,\partial_{x_1}\varphi - (u_+ - u_-) = g_1, \\ &\qquad\qquad \partial_t\psi + v_r\,\partial_{x_1}\psi + v_+\partial_{x_1}\varphi - u_+ = g_2, \\ &\qquad\qquad\qquad \rho_+ - \rho_- = g_3, \end{aligned}$$

on the boundary $\{x_2 = 0\}$. Let us introduce the vector $\mathbf{b}_0 = (0,1,0)^T$ and the matrices

$$\underline{b}(t,x_1) := \begin{pmatrix} 0 & (v_r - v_l)_{|x_2=0} \\ 1 & v_{r\,|x_2=0} \\ 0 & 0 \end{pmatrix},$$

$$\underline{M}(t,x_1) := \begin{pmatrix} 0 & \partial_{x_1}\varphi & -1 & 0 & -\partial_{x_1}\varphi & 1 \\ 0 & \partial_{x_1}\varphi & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let us also denote $U = (U_+, U_-)^T$, $\nabla\psi = (\partial_t\psi, \partial_{x_1}\psi)^T$ and $g = (g_1, g_2, g_3)^T$. Then the boundary conditions equivalently read

$$\Psi_{+|x_2=0} = \Psi_{-|x_2=0} = \psi, \quad \underline{b}\nabla\psi + \underline{M}U_{|x_2=0} = g.$$

In terms of the *good unknown* $\dot{U}$ defined by (5.7), the linearized boundary conditions read

$$\Psi_{+|x_2=0} = \Psi_{-|x_2=0} = \psi, \tag{5.10a}$$

$$B'(\dot{U},\psi) := \underline{b}\nabla\psi + \underbrace{\underline{M}\begin{pmatrix}\partial_{x_2}U_r/\partial_{x_2}\Phi_r \\ \partial_{x_2}U_l/\partial_{x_2}\Phi_l\end{pmatrix}}_{\check{\mathbf{b}}}\psi + \underline{M}\dot{U}_{|x_2=0} = g. \tag{5.10b}$$

We observe that the linearized boundary conditions only involve the traces of $\mathbb{P}\dot{U}_+$ and $\mathbb{P}\dot{U}_-$, with $\mathbb{P}$ defined by (5.9). With this notation, we can state our main result (the norms are the weighted norms defined in Section 3).

Theorem 5.1. *Assume that the particular solution defined by* (5.1) *satisfies*

$$\bar{v}_r > \sqrt{2}c(\rho), \tag{5.11}$$

and that the perturbations $\dot{U}_{r,l}, \nabla\Phi_{r,l}$ have compact support and are small enough in $W^{2,\infty}(\Omega)$. Then there exist some constants C_1 and $\gamma_1 \geq 1$, that only depend on K_0 and κ_0 (defined in (5.2), (5.5)*), such that for all $\gamma \geq \gamma_1$ and for all $(\dot{U},\psi) \in H^2_\gamma(\Omega) \times H^2_\gamma(\mathbb{R}^2)$ the following estimate holds:*

$$\begin{aligned}\gamma |||\dot{U}|||^2_{L^2_\gamma(\Omega)} + ||\mathbb{P}\dot{U}_{|x_2=0}||^2_{L^2_\gamma(\mathbb{R}^2)} + ||\psi||^2_{H^1_\gamma(\mathbb{R}^2)} \\ \leq C_1\left(\frac{1}{\gamma^3}|||L'\dot{U}|||^2_{L^2(H^1_\gamma)} + \frac{1}{\gamma^2}||B'(\dot{U},\psi)||^2_{H^1_\gamma(\mathbb{R}^2)}\right).\end{aligned} \tag{5.12}$$

The linearized operators L' and B' are defined in (5.8) *and* (5.10).

The remaining part of this section is devoted to the proof of Theorem 5.1.

5.2. Some preliminary transformation. Let us consider again the linearized equations (5.8). After multiplication by the Friedrichs' symmetrizer defined above and a straightforward integration by parts, we easily prove the following lemma.

Lemma 5.2. *There exist two constants $C > 0$ and $\gamma_0 \geq 1$ such that for all $\gamma \geq \gamma_0$, the following estimate holds:*

$$\gamma |||\dot{U}_+|||^2_{L^2_\gamma(\Omega)} \leq \frac{C}{\gamma}|||L'_r\dot{U}_+|||^2_{L^2_\gamma(\Omega)} + C||\mathbb{P}\dot{U}_{+|x_2=0}||^2_{L^2_\gamma(\mathbb{R}^2)},$$

where the operator $\mathbb{P}$ is defined in (5.9). *The estimate for $\dot{U}_-$ is the same, namely:*

$$\gamma |||\dot{U}_-|||^2_{L^2_\gamma(\Omega)} \leq \frac{C}{\gamma}|||L'_l\dot{U}_-|||^2_{L^2_\gamma(\Omega)} + C||\mathbb{P}\dot{U}_{-|x_2=0}||^2_{L^2_\gamma(\mathbb{R}^2)}.$$

As was done in the constant coefficients case, it remains to show an estimate of the traces $\mathbb{P}\dot{U}_{\pm|_{x_2=0}}$ and the front function ψ in terms of the source terms in the interior domain and on the boundary. In order to prove such an estimate, it is convenient to transform further the interior equations (5.8) in order to deal with a problem with a constant and diagonal boundary matrix (i.e., the matrix coefficient of ∂_{x_2} in the differential operators $L'_{r,l}$). This is possible because the boundary matrix has constant rank in the whole closed half-space. Namely, let us consider the coefficients of $\partial_{x_2}\dot{U}_\pm$ in (5.8). The coefficients are equal to

$$\frac{1}{\partial_{x_2}\Phi}(A_2(U) - \partial_t\Phi - \partial_{x_1}\Phi A_1(U)),$$

where we forget for the moment the indices r, l. Under the assumption (5.4), this coefficient reduces to the matrix

$$A'_2(U, \nabla\Phi) = \frac{1}{\partial_{x_2}\Phi}\begin{pmatrix} 0 & -\rho\partial_{x_1}\Phi & \rho \\ -(p'(\rho)/\rho)\partial_{x_1}\Phi & 0 & 0 \\ (p'(\rho)/\rho) & 0 & 0 \end{pmatrix}$$

which has eigenvalues

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \pm\frac{c(\rho)\langle\partial_{x_1}\Phi\rangle}{\partial_{x_2}\Phi}.$$

Here we have introduced the notation $\langle\partial_{x_1}\Phi\rangle = \sqrt{1 + (\partial_{x_1}\Phi)^2}$. In order to diagonalize the above matrix we compute the eigenvectors associated to the above eigenvalues. We obtain respectively the vectors

$$\begin{pmatrix} 0 & 1 & \partial_{x_1}\Phi \end{pmatrix}^T, \quad \begin{pmatrix} \langle\partial_{x_1}\Phi\rangle & -\frac{c(\rho)}{\rho}\partial_{x_1}\Phi & \frac{c(\rho)}{\rho} \end{pmatrix}^T,$$
$$\begin{pmatrix} \langle\partial_{x_1}\Phi\rangle & \frac{c(\rho)}{\rho}\partial_{x_1}\Phi & -\frac{c(\rho)}{\rho} \end{pmatrix}^T.$$

Observe that these eigenvectors are not orthonormal (because A'_2 is not symmetric). Thus, we may define the following (non orthogonal) matrix

$$T(U, \nabla\Phi) := \begin{pmatrix} 0 & \langle\partial_{x_1}\Phi\rangle & \langle\partial_{x_1}\Phi\rangle \\ 1 & -\frac{c(\rho)}{\rho}\partial_{x_1}\Phi & \frac{c(\rho)}{\rho}\partial_{x_1}\Phi \\ \partial_{x_1}\Phi & \frac{c(\rho)}{\rho} & -\frac{c(\rho)}{\rho} \end{pmatrix},$$

which permits to diagonalize the above matrix $A_2'(U,\nabla\Phi)$:

$$T^{-1}(U,\nabla\Phi)A_2'(U,\nabla\Phi)T(U,\nabla\Phi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

In order to obtain a constant boundary matrix in the differential operators, we also introduce the matrix

$$A_0(U,\nabla\Phi) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\partial_{x_2}\Phi}{c(\rho)\langle\partial_{x_1}\Phi\rangle} & 0 \\ 0 & 0 & -\dfrac{\partial_{x_2}\Phi}{c(\rho)\langle\partial_{x_1}\Phi\rangle} \end{pmatrix}.$$

It follows that $A_0 T^{-1} A_2' T = \mathbf{I}_2 := \operatorname{diag}(0,1,1)$. Let us define the new unknown functions $W^+ := T^{-1}(U_r,\nabla\Phi_r)\dot U_+$, $W^- := T^{-1}(U_l,\nabla\Phi_l)\dot U_-$, and set

$$T_{r,l} := T(U_{r,l},\nabla\Phi_{r,l}), \quad \mathbf{A}_0^{r,l} := A_0(U_{r,l},\nabla\Phi_{r,l}).$$

After multiplication on the left side of the equations in (5.8) by $\mathbf{A}_0^{r,l}T_{r,l}^{-1}$, we see that $W^\pm$ solve the equations

$$\mathbf{A}_0^r\,\partial_t W^+ + \mathbf{A}_1^r\,\partial_{x_1} W^+ + \mathbf{I}_2\,\partial_{x_2} W^+ + \mathbf{A}_0^r\mathbf{C}^r W^+ = F^+, \tag{5.13a}$$

$$\mathbf{A}_0^l\,\partial_t W^- + \mathbf{A}_1^l\,\partial_{x_1} W^- + \mathbf{I}_2\,\partial_{x_2} W^- + \mathbf{A}_0^l\mathbf{C}^l W^- = F^-, \tag{5.13b}$$

where we have set (with slight abuse of notation)

$$\begin{aligned}
\mathbf{A}_1^{r,l} &:= \mathbf{A}_0^{r,l} T^{-1} A_1 T(U_{r,l},\nabla\Phi_{r,l}),\\
\mathbf{C}^{r,l} &:= [T^{-1}\partial_t T + T^{-1}A_1\partial_{x_1}T + T^{-1}A_2'\partial_{x_2}T + T^{-1}CT](U_{r,l},\nabla U_{r,l},\nabla\Phi_{r,l}),\\
F^\pm &= \mathbf{A}_0^{r,l} T_{r,l}^{-1} f^\pm.
\end{aligned}$$

The above equations (5.13) are equivalent to the linearized equations (5.8). Introducing $\widetilde W^\pm := e^{-\gamma t}W^\pm$, the equations (5.13) become equivalent to

$$\begin{aligned}
\mathcal{L}_r^\gamma \widetilde W^+ := \gamma \mathbf{A}_0^r \widetilde W^+ + \mathbf{A}_0^r\,\partial_t \widetilde W^+ \\
+ \mathbf{A}_1^r\,\partial_{x_1}\widetilde W^+ + \mathbf{I}_2\,\partial_{x_2}\widetilde W^+ + \mathbf{A}_0^r\mathbf{C}^r\widetilde W^+ = e^{-\gamma t}F^+,
\end{aligned} \tag{5.14a}$$

$$\begin{aligned}
\mathcal{L}_l^\gamma \widetilde W^- := \gamma \mathbf{A}_0^l \widetilde W^- + \mathbf{A}_0^l\,\partial_t \widetilde W^- \\
+ \mathbf{A}_1^l\,\partial_{x_1}\widetilde W^- + \mathbf{I}_2\,\partial_{x_2}\widetilde W^- + \mathbf{A}_0^l\mathbf{C}^l\widetilde W^- = e^{-\gamma t}F^-.
\end{aligned} \tag{5.14b}$$

Recall that we have $\mathbf{A}_j^{r,l} \in W^{2,\infty}(\Omega)$, and $\mathbf{C}^{r,l} \in W^{1,\infty}(\Omega)$. Using the vector $W = (W^+, W^-)^T$ as defined above, the boundary conditions (5.10) become equivalent to

$$\Psi_{+|x_2=0} = \Psi_{-|x_2=0} = \psi, \tag{5.15a}$$

$$\underline{b}\nabla\psi + \check{\mathbf{b}}\psi + \underline{M}\begin{pmatrix} T_r & 0 \\ 0 & T_l \end{pmatrix} W_{|x_2=0} = g. \tag{5.15b}$$

Introducing $\widetilde{W}^{\pm}$ and $\widetilde{\Psi}_{\pm} := e^{-\gamma t}\Psi_{\pm}$, $\widetilde{\psi} := e^{-\gamma t}\psi$, the equations (5.15) are also equivalent to

$$\widetilde{\Psi}_{+} = \widetilde{\Psi}_{-} = \widetilde{\psi}, \tag{5.16a}$$

$$\mathcal{B}^{\gamma}(\widetilde{W}, \widetilde{\psi}) := \gamma\mathbf{b}_0\widetilde{\psi} + \underline{b}\nabla\widetilde{\psi} + \check{\mathbf{b}}\widetilde{\psi} + \underline{M}\begin{pmatrix} T_r & 0 \\ 0 & T_l \end{pmatrix} \widetilde{W}_{|x_2=0} = e^{-\gamma t} g. \tag{5.16b}$$

From (5.2) we have

$$\underline{b} \in W^{2,\infty}(\mathbb{R}^2), \qquad \check{\mathbf{b}} \in W^{1,\infty}(\mathbb{R}^2),$$
$$\underline{M} \in W^{2,\infty}(\mathbb{R}^2), \quad T_{r,l|x_2=0} \in W^{2,\infty}(\mathbb{R}^2).$$

The next step is to look for an a priori estimate of the solution to the (weighted) linearized problem (5.14), (5.16). In view of Lemma 5.2, we are looking for an estimate of $\mathbb{P}\dot{U}_+$ and $\mathbb{P}\dot{U}_-$. Of course, we shall derive this estimate using the new function W. The reader should keep in mind the relations

$$\mathbb{P}\dot{U}_{+|x_2=0} = \begin{pmatrix} \langle\partial_{x_1}\varphi\rangle(W_2^+ + W_3^+)_{|x_2=0} \\ \dfrac{c_r}{\rho_r}\langle\partial_{x_1}\varphi\rangle^2(W_2^+ - W_3^+)_{|x_2=0} \end{pmatrix},$$

$$\mathbb{P}\dot{U}_{-|x_2=0} = \begin{pmatrix} \langle\partial_{x_1}\varphi\rangle(W_2^- + W_3^-)_{|x_2=0} \\ \dfrac{c_l}{\rho_l}\langle\partial_{x_1}\varphi\rangle^2(W_2^- - W_3^-)_{|x_2=0} \end{pmatrix},$$

from which we easily deduce the estimate

$$\begin{aligned} &\|\mathbb{P}\dot{U}_{+|x_2=0}\|_{L^2_\gamma(\mathbb{R}^2)} + \|\mathbb{P}\dot{U}_{-|x_2=0}\|_{L^2_\gamma(\mathbb{R}^2)} \\ &\qquad \le C(\|(W_2^+, W_3^+)_{|x_2=0}\|_{L^2_\gamma(\mathbb{R}^2)} + \|(W_2^-, W_3^-)_{|x_2=0}\|_{L^2_\gamma(\mathbb{R}^2)}). \end{aligned} \tag{5.17}$$

We are thus led to estimating the trace of the vector $(\widetilde{W}_2^+, \widetilde{W}_3^+, \widetilde{W}_2^-, \widetilde{W}_3^-)$, when $\widetilde{W}$ is a solution to the (weighted) linearized equations (5.14), (5.16). From now on, for the sake of simplicity, we drop the tildas and write $W^{\pm}$, $\Psi_{\pm}$, ψ instead of $\widetilde{W}^{\pm}$, $\widetilde{\Psi}_{\pm}$, $\widetilde{\psi}$. Observe that $\Psi_{\pm}$, ψ are coupled to $W^{\pm}$ only through the boundary conditions.

5.3. Paralinearization. We refer to Appendix B for the definition of paradifferential symbols and operators, where the reader will also find the main results on paralinearization and symbolic calculus. We recall that the Fourier dual variables of (t, x_1) are (δ, η), and that we always denote $\tau = \gamma + i\delta$ the Laplace dual variable of t. Recall we have introduced the positive constants K_0, κ_0 in (5.2), (5.5). We now turn to the paralinearization of the linearized equations.

(1) *The boundary conditions.* Define the following symbols:

$$\mathbf{b}_0 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{b}_1(t, x_1) := \begin{pmatrix} v_r - v_l \\ v_r \\ 0 \end{pmatrix}(t, x_1, 0),$$

$$\mathbf{b}(t, x_1, \delta, \eta, \gamma) := \tau \mathbf{b}_0 + i\eta \mathbf{b}_1(t, x_1).$$

Because $\mathbf{b}_0$ is constant, we have

$$\gamma \mathbf{b}_0 \psi + \mathbf{b}_0 \, \partial_t \psi = T^{\gamma}_{\tau \mathbf{b}_0} \psi.$$

The main paralinearization estimate (Theorem B.9) yields

$$\|\mathbf{b}_1 \, \partial_{x_1} \psi - T^{\gamma}_{i\eta \mathbf{b}_1} \psi\|_{1,\gamma} \leq C \|\mathbf{b}_1\|_{W^{2,\infty}(\mathbb{R}^2)} \, \|\psi\|_0 \leq \frac{C(K_0)}{\gamma} \|\psi\|_{1,\gamma}.$$

We now easily obtain

$$\|\gamma \mathbf{b}_0 \psi + \mathbf{b}_0 \, \partial_t \psi + \mathbf{b}_1 \, \partial_{x_1} \psi - T^{\gamma}_{\mathbf{b}} \psi\|_{1,\gamma} \leq \frac{C(K_0)}{\gamma} \|\psi\|_{1,\gamma}. \tag{5.18}$$

We also have the following inequalities:

$$\|\check{\mathbf{b}} \psi - T^{\gamma}_{\check{\mathbf{b}}} \psi\|_{1,\gamma} \leq C \|\check{\mathbf{b}}\|_{W^{1,\infty}(\mathbb{R}^2)} \, \|\psi\|_0 \leq \frac{C(K_0, \kappa_0)}{\gamma} \|\psi\|_{1,\gamma}, \tag{5.19a}$$

$$\|T^{\gamma}_{\check{\mathbf{b}}} \psi\|_{1,\gamma} \leq C \|\check{\mathbf{b}}\|_{L^\infty(\mathbb{R}^2)} \, \|\psi\|_{1,\gamma} \leq C(K_0, \kappa_0) \|\psi\|_{1,\gamma}, \tag{5.19b}$$

where $\check{\mathbf{b}}$ is defined by (5.10). Eventually, we define the symbol

$$\mathbf{M}(t, x_1) := \underline{M}(t, x_1, 0) \begin{pmatrix} T_r & 0 \\ 0 & T_l \end{pmatrix}(t, x_1, 0),$$

with the matrices $\underline{M}$, T_r, T_l defined above. Recall that the state around which the equations are linearized satisfies

$$\Phi_r(t,x_1,0) = \Phi_l(t,x_1,0) = \varphi(t,x_1), \quad \rho_r(t,x_1,0) = \rho_l(t,x_1,0).$$

A direct calculation yields

$$\mathbf{M} = \begin{pmatrix} 0 & -\dfrac{c_r}{\rho_r}\langle\partial_{x_1}\varphi\rangle^2 & \dfrac{c_r}{\rho_r}\langle\partial_{x_1}\varphi\rangle^2 & 0 & \dfrac{c_l}{\rho_l}\langle\partial_{x_1}\varphi\rangle^2 & -\dfrac{c_l}{\rho_l}\langle\partial_{x_1}\varphi\rangle^2 \\ 0 & -\dfrac{c_r}{\rho_r}\langle\partial_{x_1}\varphi\rangle^2 & \dfrac{c_r}{\rho_r}\langle\partial_{x_1}\varphi\rangle^2 & 0 & 0 & 0 \\ 0 & \langle\partial_{x_1}\varphi\rangle & \langle\partial_{x_1}\varphi\rangle & 0 & -\langle\partial_{x_1}\varphi\rangle & -\langle\partial_{x_1}\varphi\rangle \end{pmatrix}.$$

Thus the matrix $\mathbf{M}$ only acts on the noncharacteristic part of the vector $W = (W^+, W^-)$, that is, $W^{\mathrm{nc}} := (W_2^+, W_3^+, W_2^-, W_3^-)$. Since $\mathbf{M} \in W^{2,\infty}(\mathbb{R}^2)$, we have

$$\begin{aligned} (5.20) \qquad \|\mathbf{M}\, W_{|x_2=0} - T^\gamma_{\mathbf{M}} W_{|x_2=0}\|_{1,\gamma} &\le \frac{C}{\gamma}\, \|\mathbf{M}\|_{W^{2,\infty}(\mathbb{R}^2)}\, \|W^{\mathrm{nc}}_{|x_2=0}\|_0 \\ &\le \frac{C(K_0)}{\gamma}\, \|W^{\mathrm{nc}}_{|x_2=0}\|_0. \end{aligned}$$

Adding (5.18)–(5.19)–(5.20), we obtain the paralinearization estimate for the boundary operator:

$$(5.21) \quad \|\mathcal{B}^\gamma(W,\psi) - T^\gamma_{\mathbf{b}}\psi - T^\gamma_{\mathbf{M}} W_{|x_2=0}\|_{1,\gamma} \le C(K_0,\kappa_0)\left(\|\psi\|_{1,\gamma} + \frac{1}{\gamma}\|W^{\mathrm{nc}}_{|x_2=0}\|_0\right).$$

We recall that the boundary operator $\mathcal{B}^\gamma$ is defined by (5.16). Observe that in the paralinearized version of $\mathcal{B}^\gamma$, there is no more zeroth order term in ψ.

(2) *The interior equations.* We first estimate the paralinearization error for fixed x_2, and then integrate with respect to x_2. For instance, we have

$$\begin{aligned} &||| \gamma \mathbf{A}_0^r W^+ - T^\gamma_{\gamma\mathbf{A}_0^r} W^+ |||^2_{1,\gamma} \\ &\quad = \int_0^{+\infty} \gamma^2 \|\mathbf{A}_0^r W^+(\cdot,x_2) - T^\gamma_{\mathbf{A}_0^r} W^+(\cdot,x_2)\|^2_{1,\gamma}\, dx_2 \\ &\quad \le C \int_0^{+\infty} \|\mathbf{A}_0^r(\cdot,x_2)\|^2_{W^{2,\infty}(\mathbb{R}^2)} \|W^+(\cdot,x_2)\|_0^2\, dx_2 \\ &\quad \le C\|\mathbf{A}_0^r\|^2_{W^{2,\infty}(\Omega)}\, |||W^+|||_0^2 \le C(K_0)\, |||W^+|||_0^2. \end{aligned}$$

In a completely similar way, we obtain the following estimates

$$\begin{aligned} |||\mathbf{A}_0^r\, \partial_t W^+ - T^\gamma_{i\delta\mathbf{A}_0^r} W^+|||_{1,\gamma} &\le C(K_0)|||W^+|||_0, \\ |||\mathbf{A}_1^r\, \partial_{x_1} W^+ - T^\gamma_{i\eta\mathbf{A}_1^r} W^+|||_{1,\gamma} &\le C(K_0)|||W^+|||_0, \\ |||\mathbf{A}_0^r \mathbf{C}^r W^+ - T^\gamma_{\mathbf{A}_0^r\mathbf{C}^r} W^+|||_{1,\gamma} &\le C(K_0,\kappa_0)|||W^+|||_0. \end{aligned}$$

Adding these inequalities, we end up with the paralinearization estimate for the interior equations:

$$
\begin{aligned}
(5.22)\qquad & |||\mathcal{L}_r^\gamma W^+ - T^\gamma_{\tau \mathbf{A}_0^r + i\eta \mathbf{A}_1^r + \mathbf{A}_0^r \mathbf{C}^r} W^+ - \mathbf{I}_2\, \partial_{x_2} W^+|||_{1,\gamma} \\
& \quad \le C(K_0, \kappa_0)\, |||W^+|||_0,
\end{aligned}
$$

where the linearized operator $\mathcal{L}_r^\gamma$ is defined by (5.14). The estimate for the equation on W^- is identical:

$$
\begin{aligned}
(5.23)\qquad & |||\mathcal{L}_l^\gamma W^- - T^\gamma_{\tau \mathbf{A}_0^l + i\eta \mathbf{A}_1^l + \mathbf{A}_0^l \mathbf{C}^l} W^- - \mathbf{I}_2\, \partial_{x_2} W^-|||_{1,\gamma} \\
& \quad \le C(K_0, \kappa_0)\, |||W^-|||_0.
\end{aligned}
$$

(3) *Eliminating the front.* We proceed as in the constant coefficients case, and show how to eliminate the front ψ in the (paralinearized) boundary conditions. If the perturbation is small enough (in the L^∞ norm), there exists a constant $c > 0$ (depending only on K_0) such that

$$
|\mathbf{b}(t, x_1, \delta, \eta, \gamma)|^2 \ge c(\gamma^2 + \delta^2 + \eta^2).
$$

Applying Gårding's inequality (Theorem B.7), we obtain

$$
\Re \langle T^\gamma_{\mathbf{b}^* \mathbf{b}} \psi, \psi \rangle_{L^2(\mathbb{R}^2)} \ge \frac{c}{2} \|\psi\|^2_{1,\gamma},
$$

for all $\gamma \ge \gamma_0$ (where γ_0 only depends on K_0). Using the rules of symbolic calculus (Theorem B.6), we have $T^\gamma_{\mathbf{b}^*\mathbf{b}} = (T^\gamma_{\mathbf{b}})^* T^\gamma_{\mathbf{b}} + R^\gamma$, where R^γ is of order ≤ 1. Consequently, we have an estimate of the form

$$
\|\psi\|_{1,\gamma} \le C(K_0) \|T^\gamma_{\mathbf{b}} \psi\|_0.
$$

For all $\gamma \ge \gamma_0$, we thus obtain

$$
\begin{aligned}
(5.24)\qquad \|\psi\|_{1,\gamma} & \le C(K_0)\left(\|T^\gamma_{\mathbf{b}} \psi + T^\gamma_{\mathbf{M}} W_{|x_2=0}\|_0 + \|W^{\mathrm{nc}}{}_{|x_2=0}\|_0\right) \\
& \le C(K_0)\left(\frac{1}{\gamma}\|T^\gamma_{\mathbf{b}} \psi + T^\gamma_{\mathbf{M}} W_{|x_2=0}\|_{1,\gamma} + \|W^{\mathrm{nc}}{}_{|x_2=0}\|_0\right).
\end{aligned}
$$

From (5.21) and (5.24) we deduce for $\gamma \ge \gamma_0$ large enough (depending on K_0) the estimate

$$
\|\psi\|_{1,\gamma} \le C(K_0)\left(\frac{1}{\gamma}\|\mathcal{B}^\gamma(W, \psi)\|_{1,\gamma} + \|W^{\mathrm{nc}}{}_{|x_2=0}\|_0\right),
$$

which shows that it only remains to prove an estimate of $W^{\mathrm{nc}}{}_{|x_2=0}$ in terms of the source terms.

For all (τ, η) in the hemisphere Σ, we define the matrix

$$\Pi(t, x_1, \delta, \eta, \gamma) := \begin{pmatrix} 0 & 0 & 1 \\ \tau + i\eta v_r(t, x_1, 0) & -i\eta(v_r - v_l)(t, x_1, 0) & 0 \end{pmatrix},$$

and we extend Π as a homogeneous mapping of degree 0 with respect to (τ, η). We have $\Pi \mathbf{b} \equiv 0$, and $\Pi \in \Gamma_2^0$. Applying Theorem B.6, we thus obtain:

$$\begin{aligned} \|T_\Pi^\gamma T_{\mathbf{b}}^\gamma \psi\|_{1,\gamma} &= \|T_\Pi^\gamma T_{\mathbf{b}}^\gamma \psi - T_{\Pi\mathbf{b}}^\gamma \psi\|_{1,\gamma} \leq C(K_0)\|\psi\|_{1,\gamma}, \\ \|T_{\Pi\mathbf{M}}^\gamma W_{|x_2=0} - T_\Pi^\gamma T_{\mathbf{M}}^\gamma W_{|x_2=0}\|_{1,\gamma} &\leq C(K_0)\|W^{\mathrm{nc}}{}_{|x_2=0}\|_0. \end{aligned}$$

Using the decomposition

$$T_{\Pi\mathbf{M}}^\gamma W_{|x_2=0} = (T_{\Pi\mathbf{M}}^\gamma - T_\Pi^\gamma T_{\mathbf{M}}^\gamma) W_{|x_2=0} + T_\Pi^\gamma (T_{\mathbf{M}}^\gamma W_{|x_2=0} + T_{\mathbf{b}}^\gamma \psi) - T_\Pi^\gamma T_{\mathbf{b}}^\gamma \psi,$$

we get the following estimate

$$\begin{aligned} &\|T_{\Pi\mathbf{M}}^\gamma W_{|x_2=0}\|_{1,\gamma} \\ &\qquad \leq C(K_0)(\|W^{\mathrm{nc}}{}_{|x_2=0}\|_0 + \|T_{\mathbf{b}}^\gamma \psi + T_{\mathbf{M}}^\gamma W_{|x_2=0}\|_{1,\gamma} + \|\psi\|_{1,\gamma}). \end{aligned} \tag{5.25}$$

As was done in the constant coefficients case, we define the symbol $\boldsymbol{\beta}$ of the reduced boundary conditions:

$$\forall\ (t, x_1, \delta, \eta, \gamma) \in \mathbb{R}^4 \times \mathbb{R}^+, \quad \boldsymbol{\beta}(t, x_1, \delta, \eta, \gamma) := \Pi(t, x_1, \delta, \eta, \gamma)\mathbf{M}(t, x_1).$$

We now focus on the paralinearized system with reduced boundary conditions:

$$\begin{cases} T^\gamma_{\tau\mathbf{A}_0^r + i\eta\mathbf{A}_1^r + \mathbf{A}_0^r\mathbf{C}^r} W^+ + \mathbf{I}_2\,\partial_{x_2} W^+ = \widetilde{F_+}, & x_2 > 0, \\ T^\gamma_{\tau\mathbf{A}_0^l + i\eta\mathbf{A}_1^l + \mathbf{A}_0^l\mathbf{C}^l} W^- + \mathbf{I}_2\,\partial_{x_2} W^- = \widetilde{F_-}, & x_2 > 0, \\ T^\gamma_{\boldsymbol{\beta}} W_{|x_2=0} = \tilde{G}, & x_2 = 0. \end{cases} \tag{5.26}$$

Our aim is to prove an energy estimate for the paralinearized equations (5.26). Once this is done, we shall obtain an energy estimate for the linearized equations. More precisely, we have the following proposition.

Proposition 5.3. *Assume that there exists a constant C_0, depending only on K_0 and κ_0, such that the solution W to (5.26) satisfies*

$$\|W^{\mathrm{nc}}{}_{|x_2=0}\|_0^2 \leq C_0 \left(\frac{1}{\gamma^3} |||\tilde{F}|||_{1,\gamma}^2 + \frac{1}{\gamma^2} \|\tilde{G}\|_{1,\gamma}^2 \right), \tag{5.27}$$

for all $\gamma \geq \gamma_0$ (where γ_0 only depends on K_0 and κ_0). Then the thesis of Theorem 5.1 holds.

Proof. The proof is straightforward. We first write

$$T^{\gamma}_{\tau \mathbf{A}_0^r + i\eta \mathbf{A}_1^r + \mathbf{A}_0^r \mathbf{C}^r} W^+ + \mathbf{I}_2\, \partial_{x_2} W^+ = \mathcal{L}^{\gamma}_r W^+ + \text{error},$$
$$T^{\gamma}_{\tau \mathbf{A}_0^l + i\eta \mathbf{A}_1^l + \mathbf{A}_0^l \mathbf{C}^l} W^- + \mathbf{I}_2\, \partial_{x_2} W^- = \mathcal{L}^{\gamma}_l W^- + \text{error},$$

and estimate the error terms with the help of (5.22)–(5.23). We use (5.27) to derive

$$\|W^{\mathrm{nc}}{}_{|x_2=0}\|_0^2 \leq C_0' \left(\frac{1}{\gamma^3} |||\, \mathcal{L}^{\gamma} W \,|||_{1,\gamma}^2 + \frac{1}{\gamma^3} |||\, W \,|||_0^2 + \frac{1}{\gamma^2} \|T^{\gamma}_{\boldsymbol{\beta}} W_{|x_2=0}\|_{1,\gamma}^2 \right),$$

where, as usual, $\mathcal{L}^{\gamma} W = (\mathcal{L}^{\gamma}_r W^+, \mathcal{L}^{\gamma}_l W^-)$. Using (5.24) and (5.25), and choosing γ large enough, we obtain the following inequality:

$$\begin{aligned}
&\|W^{\mathrm{nc}}{}_{|x_2=0}\|_0^2 + \|\psi\|_{1,\gamma}^2 \\
&\quad \leq C_0'' \left(\frac{1}{\gamma^3} |||\, \mathcal{L}^{\gamma} W \,|||_{1,\gamma}^2 + \frac{1}{\gamma^3} |||\, W \,|||_0^2 + \frac{1}{\gamma^2} \|T^{\gamma}_{\mathbf{b}} \psi + T^{\gamma}_{\mathbf{M}} W_{|x_2=0}\|_{1,\gamma}^2 \right).
\end{aligned}$$

Eventually, we use (5.21) to derive (up to choosing γ large enough):

$$\begin{aligned}
&\|W^{\mathrm{nc}}{}_{|x_2=0}\|_0^2 + \|\psi\|_{1,\gamma}^2 \\
&\quad \leq C_0''' \left(\frac{1}{\gamma^3} |||\, \mathcal{L}^{\gamma} W \,|||_{1,\gamma}^2 + \frac{1}{\gamma^3} |||\, W \,|||_0^2 + \frac{1}{\gamma^2} \|\mathcal{B}^{\gamma}(W, \psi)\|_{1,\gamma}^2 \right).
\end{aligned}$$

Then one uses the definitions

$$e^{-\gamma t} \dot{U}_+ = T_r W^+, \quad e^{-\gamma t} \dot{U}_- = T_l W^-,$$
$$e^{-\gamma t} \mathbf{A}_0^r T_r^{-1} L_r' \dot{U}_+ = \mathcal{L}^{\gamma}_r W^+, \quad e^{-\gamma t} \mathbf{A}_0^l T_l^{-1} L_l' \dot{U}_- = \mathcal{L}^{\gamma}_l W^-,$$

as well as (5.17) and Lemma 5.2 to derive (5.12). The reader will easily check that the constants C_0', C_0'' etc. involved in the energy estimates only depend on K_0 and κ_0. □

Thanks to Proposition 5.3, we only need to prove the estimate (5.27) for the paralinearized system (5.26). This will be done in the next paragraphs.

Recall that the boundary matrix $\boldsymbol{\beta}$ in (5.26) only acts on $W^{\mathrm{nc}} = (W_2^+, W_3^+, W_2^-, W_3^-)$ and not on the full vector W. Namely, the first and fourth columns of $\boldsymbol{\beta}$ vanish. Consequently, we feel free to write the boundary conditions under the form $T^{\gamma}_{\boldsymbol{\beta}} W^{\mathrm{nc}}_{|x_2=0} = \tilde{G}$, that is, we consider $\boldsymbol{\beta}$ as a matrix with only four columns and two rows.

5.4. Microlocalization. To derive the desired energy estimate for (5.26), we follow the general strategy of the constant coefficients case. Namely, we first consider the two equations that do not involve any x_2 derivative:

$$\begin{aligned}
&T^{\gamma}_{\tau+iv_r\eta}W_1^+ + T^{\gamma}_{i\eta c_r^2/\rho_r\langle\partial_{x_1}\Phi_r\rangle}W_2^+ \\
&\qquad\qquad + T^{\gamma}_{i\eta c_r^2/\rho_r\langle\partial_{x_1}\Phi_r\rangle}W_3^+ + \text{order } 0 \text{ terms} = F_1^+, \\
&T^{\gamma}_{\tau+iv_l\eta}W_1^- + T^{\gamma}_{i\eta c_l^2/\rho_l\langle\partial_{x_1}\Phi_l\rangle}W_2^- \\
&\qquad\qquad + T^{\gamma}_{i\eta c_l^2/\rho_l\langle\partial_{x_1}\Phi_l\rangle}W_3^- + \text{ order } 0 \text{ terms} = F_1^-.
\end{aligned}$$

Formally, the idea is to invert the operators $T^{\gamma}_{\tau+iv_{r,l}\eta}$ and to substitute the corresponding value of $W_1^{\pm}$ into the four remaining equations. We shall thus get a system of the form

$$\begin{cases}\partial_{x_2}W^{\mathrm{nc}} = T^{\gamma}_{\mathbb{A}}W^{\mathrm{nc}} + T^{\gamma}_{\mathbb{E}}W^{\mathrm{nc}} + \text{source term}, & x_2 > 0, \\ T^{\gamma}_{\boldsymbol{\beta}}W^{\mathrm{nc}}_{|x_2=0} = \text{source term}, & x_2 = 0,\end{cases}$$

where $\mathbb{A}$ is of degree 1 and $\mathbb{E}$ is of degree 0. (Both matrices $\mathbb{A}$ and $\mathbb{E}$ are block diagonal since the equations for W^+ and W^- are decoupled). An important issue is to show that this operation can be achieved. Namely, the zeroth order terms in the two scalar equations above involve W_1^+ and W_1^-. When inverting the operators $T^{\gamma}_{\tau+iv_{r,l}\eta}$, one needs to take the zeroth order terms into account, in order to avoid introducing $W_1^{\pm}$ in the final equation for W^{nc}. We shall show that such an inversion is possible. But in this paragraph, we focus on the first order term and explicit the symbol $\mathbb{A}$. Consider the following 2×2 matrix:

$$\mathbb{A}^r := \begin{pmatrix}\mathbb{A}_1^r & -\mathbb{A}_3^r \\ \mathbb{A}_3^r & \mathbb{A}_2^r\end{pmatrix}, \tag{5.28}$$

with

$$\mathbb{A}_1^r := -\frac{c_r\eta^2\,\partial_{x_2}\Phi_r}{2(\tau+iv_r\eta)\langle\partial_{x_1}\Phi_r\rangle^3} - \frac{(\tau+iv_r\eta)\,\partial_{x_2}\Phi_r}{c_r\langle\partial_{x_1}\Phi_r\rangle} + \frac{\partial_{x_2}\Phi_r\,\partial_{x_1}\Phi_r\,i\eta}{\langle\partial_{x_1}\Phi_r\rangle^2}, \tag{5.29a}$$

$$\mathbb{A}_2^r := \frac{c_r\eta^2\,\partial_{x_2}\Phi_r}{2(\tau+iv_r\eta)\langle\partial_{x_1}\Phi_r\rangle^3} + \frac{(\tau+iv_r\eta)\,\partial_{x_2}\Phi_r}{c_r\langle\partial_{x_1}\Phi_r\rangle} + \frac{\partial_{x_2}\Phi_r\,\partial_{x_1}\Phi_r\,i\eta}{\langle\partial_{x_1}\Phi_r\rangle^2}, \tag{5.29b}$$

$$\mathbb{A}_3^r := \frac{c_r\eta^2\,\partial_{x_2}\Phi_r}{2(\tau+iv_r\eta)\langle\partial_{x_1}\Phi_r\rangle^3}. \tag{5.29c}$$

The definition of $\mathbb{A}^l$ is completely similar, changing the r index by l. The symbol $\mathbb{A}$ mentionned above is nothing but the block diagonal matrix

$$\mathbb{A} := \begin{pmatrix} \mathbb{A}^r & 0 \\ 0 & \mathbb{A}^l \end{pmatrix}. \tag{5.30}$$

The set of poles of $\mathbb{A}$ is denoted by $\Upsilon_{\mathbf{p}}$, that is,

$$\Upsilon_{\mathbf{p}} := \{(t, x_1, x_2, \tau, \eta) \in \bar{\Omega} \times \Xi \text{ such that } \tau = -i\eta\, v_{r,l}(t, x_1, x_2)\}.$$

As was done in the constant coefficients case, we denote by $\mathcal{E}^-(t, x_1, x_2, \tau, \eta)$ the stable subspace of $\mathbb{A}(t, x_1, x_2, \tau, \eta)$. This stable subspace is well defined when $\Re\tau > 0$, and admits a continuous extension up to any (τ, η) such that $\tau \in i\mathbb{R}$ and $(\tau, \eta) \neq (0, 0)$.

At each point $(t, x_1, 0)$ of the boundary $\partial\Omega$, the subspace

$$\{Z \in \mathcal{E}^-(t, x_1, 0, \tau, \eta) \text{ s.t. } \boldsymbol{\beta}(t, x_1, \tau, \eta) Z = 0\}$$

is nontrivial (that is, not reduced to $\{0\}$) if and only if

$$\tau = -i\eta \frac{v_r(t, x_1, 0) + v_l(t, x_1, 0)}{2} \quad \text{or} \quad \tau = i\eta V_1(t, x_1) \quad \text{or} \quad \tau = i\eta V_2(t, x_1),$$

for suitable functions $V_{1,2} \in W^{2,\infty}(\mathbb{R}^2)$. This is just because when one freezes the coefficients on the boundary and computes the associated Lopatinskii determinant, the calculations yield the same result as in Section 4. Recall that the state $(U_{r,l}, \nabla\Phi_{r,l})$ around which the equations are linearized satisfy the Rankine-Hugoniot conditions, see (5.3). As in [11], we define the *critical set* of space-frequency variables in the following way:

$$\Upsilon_{\mathbf{c}}^0 := \Big\{(t, x_1, \tau, \eta) \in \partial\Omega \times \Xi \\ \text{s.t. } \tau \in \Big\{ -i\eta \frac{(v_r + v_l)(t, x_1, 0)}{2}, i\eta V_1(t, x_1), i\eta V_2(t, x_1) \Big\}\Big\}.$$

This is exactly the set of space variables on the boundary $\partial\Omega$ and frequencies so that the Lopatinskii determinant vanishes. Moreover, if the perturbation $(\dot{U}_{r,l}, \nabla\Phi_{r,l})$ is sufficiently small (in the L^∞ norm), we have

$$-v_l(t, x_1, 0) > V_1(t, x_1) > -\frac{(v_r + v_l)(t, x_1, 0)}{2} > V_2(t, x_1) > -v_r(t, x_1, 0), \\ \forall\ (t, x_1) \in \partial\Omega.$$

Therefore, the critical set $\Upsilon_{\mathbf{c}}^0$ does not intersect the set of poles on the boundary of the space domain:

$$\Upsilon_{\mathbf{c}}^0 \cap (\Upsilon_{\mathbf{p}} \cap \{x_2 = 0\}) = \emptyset.$$

Another important feature of the critical set $\Upsilon_{\mathbf{c}}^0$ is that it admits a neighborhood in which the symbol $\mathbb{A}$ is diagonalizable. To be more precise, there exists a neighborhood $\mathcal{V}_{\mathbf{c}}^0$ of $\Upsilon_{\mathbf{c}}^0$ in $\mathbb{R}^2 \times \Xi$ and a mapping Q_0 on $\mathcal{V}_{\mathbf{c}}^0$ (with values in the set of 4×4 invertible matrices and homogeneous of degree 0 with respect to (τ, η)) such that

$$(5.31) \quad Q_0(z)\mathbb{A}(z)Q_0(z)^{-1} = \mathrm{diag}(\omega_r^-(z), \omega_r^+(z), \omega_l^-(z), \omega_l^+(z)), \qquad \forall\, z = (t, x_1, \tau, \eta) \in \mathcal{V}_{\mathbf{c}}^0,$$

where ω_r^- (resp. ω_r^+) is the eigenvalue with negative (resp. positive) real part of $\mathbb{A}^r$ when $\gamma > 0$. (The definition of $\omega_l^\mp$ is similar). Note that the matrix Q_0 has the same block diagonal structure as $\mathbb{A}$, see (5.30). The symbol Q_0 belongs to the class Γ_2^0 (see Appendix B for the precise definition).

Since $\Upsilon_{\mathbf{c}}^0$ does not intersect $\Upsilon_{\mathbf{p}} \cap \{x_2 = 0\}$, we may assume that the neighborhood $\mathcal{V}_{\mathbf{c}}^0$ does not intersect $\Upsilon_{\mathbf{p}} \cap \{x_2 = 0\}$ either.

The key point in the derivation of an energy estimate is to understand how the singularities at the boundary (that is, the set $\Upsilon_{\mathbf{c}}^0$) propagate in the interior domain. Following [11], we shall show that the singularities propagate along the two bicharacteristic curves associated with the (real) symbols $\Im\omega_{r,l}^-$, provided that these curves do not reach the poles of $\mathbb{A}$ or the points where $\mathbb{A}$ stops being diagonalizable. These ideas motivate the following result.

Proposition 5.4. *Assume that the perturbation $(\dot U_{r,l}, \nabla\Phi_{r,l})$ is small in $W^{2,\infty}(\Omega)$ and has compact support. Then one can choose the neighborhood $\mathcal{V}_{\mathbf{c}}^0$ such that there exists an open set $\mathcal{V}_{\mathbf{c}} \subset \bar\Omega \times \Xi$ satisfying the following properties:*

- *$\mathcal{V}_{\mathbf{c}} \cap \{x_2 = 0\} = \mathcal{V}_{\mathbf{c}}^0$ and $\mathcal{V}_{\mathbf{c}} \cap \Upsilon_{\mathbf{p}} = \emptyset$.*
- *The symbol $\mathbb{A}$ defined by (5.28)–(5.29)–(5.30) is diagonalizable on the set $\mathcal{V}_{\mathbf{c}}$. In other words, (5.31) holds on all $\mathcal{V}_{\mathbf{c}}$, and not only on the trace $\mathcal{V}_{\mathbf{c}}^0$.*
- *For all $z = (t, x_1, x_2, \tau, \eta) \in \mathcal{V}_{\mathbf{c}}$, one has*

$$\omega_r^-(z) \neq \omega_r^+(z) \quad \text{and} \quad \omega_l^-(z) \neq \omega_l^+(z).$$

- *The solutions of the hamiltonian system of ODEs*

$$(5.32a) \qquad \frac{dt}{dx_2} = \frac{\partial h}{\partial \delta}(t, x_1, x_2, \tau, \eta),$$

$$(5.32b) \qquad \frac{dx_1}{dx_2} = \frac{\partial h}{\partial \eta}(t, x_1, x_2, \tau, \eta),$$

$$(5.32c) \qquad \frac{d\delta}{dx_2} = -\frac{\partial h}{\partial t}(t, x_1, x_2, \tau, \eta),$$

$$(5.32d) \qquad \frac{d\eta}{dx_2} = -\frac{\partial h}{\partial x_1}(t, x_1, x_2, \tau, \eta), \qquad (t, x_1, \delta, \eta, \gamma)_{|x_2=0} \in \mathcal{V}_{\mathbf{c}}^0,$$

  are defined for all $x_2 \geq 0$ and remain in $\mathcal{V}_{\mathbf{c}}$, both for $h = \Im\omega_r^-$ and $h = \Im\omega_l^-$. These solutions are referred to as bicharacteristic curves.

Following [11], we now construct a (real) weight that vanishes on the bicharacteristic curves, and that satisfies a linear transport equation. For all $z = (t, x_1, \tau, \eta) \in \mathbb{R}^2 \times \Sigma$, define

$$\sigma(z) := \left(\delta + \eta \frac{(v_r + v_l)(t, x_1, 0)}{2}\right)(\delta - \eta V_1(t, x_1))(\delta - \eta V_2(t, x_1)), \tag{5.33}$$

and extend σ to the whole set $\mathbb{R}^2 \times \Xi$ as a homogeneous mapping of degree 1 with respect to (τ, η). The velocities $V_{1,2}$ are those defined above, and correspond to the critical speeds for which the Lopatinskii determinant vanishes. The symbol σ thus belongs to the class Γ_2^1. It is straightforward to check that

$$\Upsilon_{\mathbf{c}}^0 = \{z = (t, x_1, \tau, \eta) \in \mathbb{R}^2 \times \Xi \text{ s.t. } \gamma + i\sigma(z) = 0\}.$$

Using Proposition 5.4, it is possible to construct solutions $\sigma_{r,l}$ of the linear transport equations

$$\partial_{x_2} \sigma_r + \{\sigma_r, \Im \omega_r^-\} = 0, \tag{5.34a}$$

$$\partial_{x_2} \sigma_l + \{\sigma_l, \Im \omega_l^-\} = 0, \tag{5.34b}$$

$$\sigma_r{}_{|x_2=0} = \sigma_l{}_{|x_2=0} = \sigma, \tag{5.34c}$$

where $\{a, b\}$ stands for the Poisson bracket:

$$\{a, b\} := \frac{\partial a}{\partial \delta} \frac{\partial b}{\partial t} + \frac{\partial a}{\partial \eta} \frac{\partial b}{\partial x_1} - \frac{\partial a}{\partial t} \frac{\partial b}{\partial \delta} - \frac{\partial a}{\partial x_1} \frac{\partial b}{\partial \eta}.$$

As a matter of fact, both σ_r and σ_l are well-defined in the neighborhood of the bicharacteristic curves starting from the critical set $\Upsilon_{\mathbf{c}}^0$. This is because $\sigma_{r,l}$ are constant along the bicharacteristic curves defined by (5.32), provided these curves are globally defined! Shrinking $\mathcal{V}_{\mathbf{c}}^0$ and $\mathcal{V}_{\mathbf{c}}$, if necessary, we may assume that σ_r and σ_l are defined in the whole open set $\mathcal{V}_{\mathbf{c}}$. The key point is that σ_r vanishes on the bicharacteristic curve originating from $\Upsilon_{\mathbf{c}}^0$ and associated with the symbol $\Im \omega_r^-$. (A similar result holds for σ_l). Far from these bicharacteristic curves, both $|\sigma_r|$ and $|\sigma_l|$ are bounded from below.

Up to now, the symbols Q_0, σ_r, σ_l are only defined microlocally, that is, locally in the frequency space. To circumvent this difficulty, we now introduce cut-off functions. We fix, once and for all, two nonnegative cut-off functions (with values in $[0, 1]$) $\chi_{\mathbf{c}}$ and $\chi_{\mathbf{p}}$ such that

- $\chi_{\mathbf{c}}$ and $\chi_{\mathbf{p}}$ are smooth, that is, C^∞ and homogeneous of degree 0 with respect to (τ, η). They thus belong to the class Γ_k^0 for any integer k.
- The support of $\chi_{\mathbf{c}}$ is contained in the open set $\mathcal{V}_{\mathbf{c}}$, and $\chi_{\mathbf{c}} \equiv 1$ in a neighborhood of the bicharacteristic curves originating from $\Upsilon_{\mathbf{c}}^0$.
- The support of $\chi_{\mathbf{p}}$ does not intersect the support of $\chi_{\mathbf{c}}$, that is, $\chi_{\mathbf{c}} \chi_{\mathbf{p}} \equiv 0$. Moreover, $\chi_{\mathbf{p}} \equiv 1$ in a neighborhood of the poles $\Upsilon_{\mathbf{p}}$.

Eventually, we define $\chi_{\mathbf{u}} := 1 - \chi_{\mathbf{c}} - \chi_{\mathbf{p}}$ and observe that $\chi_{\mathbf{u}}$ is supported far from the bicharacteristic curves and far from the poles. As a consequence, the support of $\chi_{\mathbf{u}}$ on the boundary $\partial\Omega$ only consists of points for which the uniform Lopatinskii condition holds. We shall therefore be able to use standard Kreiss' symmetrizers (as constructed in Section 4) to derive an energy estimate for $T^{\gamma}_{\chi_{\mathbf{u}}} W$.

The end of this section is devoted to the proof of the a priori energy estimate (5.27). We thus fix $W = (W^+, W^-) \in H^2(\Omega)$ and define the source terms

$$F^+ := T^{\gamma}_{\tau \mathbf{A}_0^r + i\eta \mathbf{A}_1^r + \mathbf{A}_0^r \mathbf{C}^r} W^+ + \mathbf{I}_2\, \partial_{x_2} W^+ \in H^1(\Omega), \tag{5.35a}$$

$$F^- := T^{\gamma}_{\tau \mathbf{A}_0^l + i\eta \mathbf{A}_1^l + \mathbf{A}_0^l \mathbf{C}^l} W^- + \mathbf{I}_2\, \partial_{x_2} W^- \in H^1(\Omega), \tag{5.35b}$$

$$G := T^{\gamma}_{\boldsymbol{\beta}} W^{\mathrm{nc}}|_{x_2=0} \in H^{3/2}(\mathbb{R}^2). \tag{5.35c}$$

We first show how to estimate the trace of $T^{\gamma}_{\chi_{\mathbf{c}}} W^{\mathrm{nc}}$; then we show how to estimate the trace of $T^{\gamma}_{\chi_{\mathbf{u}}} W^{\mathrm{nc}}$, using Kreiss' symmetrizers. Eventually, we show how to estimate the trace of $T^{\gamma}_{\chi_{\mathbf{p}}} W^{\mathrm{nc}}$. In the first two cases, the first step in the analysis consists in deriving an equation that only involves W^{nc}, that is, in eliminating $W_1^{\pm}$ in the paradifferential equations (5.26). Once we have derived this *noncharacteristic* equation, we apply the strategy of [11]. At the very end of the proof, we show how to absorb the microlocalization errors.

5.5. Derivation of energy estimates: The bad frequencies. We define

$$W^+_{\mathbf{c}} := T^{\gamma}_{\chi_{\mathbf{c}}} W^+,$$

and compute the equation satisfied by $W^+_{\mathbf{c}}$. Starting from (5.35a), we obtain

$$\mathbf{I}_2\, \partial_{x_2} W^+_{\mathbf{c}} = \mathbf{I}_2 T^{\gamma}_{\partial_{x_2}\chi_{\mathbf{c}}} W^+ + T^{\gamma}_{\chi_{\mathbf{c}}} F^+ - T^{\gamma}_{\chi_{\mathbf{c}}} (T^{\gamma}_{\tau \mathbf{A}_0^r + i\eta \mathbf{A}_1^r + \mathbf{A}_0^r \mathbf{C}^r} W^+).$$

Then we apply the rules of symbolic calculus (Theorem B.6) to get

$$T^{\gamma}_{\tau \mathbf{A}_0^r + i\eta \mathbf{A}_1^r} W^+_{\mathbf{c}} + T^{\gamma}_{\mathbf{A}_0^r \mathbf{C}^r} W^+_{\mathbf{c}} + T^{\gamma}_{r} W^+ + \mathbf{I}_2\, \partial_{x_2} W^+_{\mathbf{c}} = T^{\gamma}_{\chi_{\mathbf{c}}} F^+ + R_{-1} W^+, \tag{5.36}$$

where R_{-1} is an operator of order ≤ -1, and r is a symbol in the class Γ_1^0 that vanishes in a neighborhood of the bicharacteristic curves. Namely, r is defined by the following formula:

$$r := \frac{1}{i} \{\chi_{\mathbf{c}}, \tau \mathbf{A}_0^r + i\eta \mathbf{A}_1^r\} - \partial_{x_2}\chi_{\mathbf{c}}\, \mathbf{I}_2,$$

and is thus a linear combination of derivatives of $\chi_{\mathbf{c}}$. Therefore, r is supported far from the bicharacterstic curves originating from the critical set.

To avoid overloading the paper with unuseful notations, we shall denote by α^m a generic symbol in the class Γ_1^m, that may vary from line to line, or within the same line, and whose exact expression is not useful. Moreover, we denote by r any symbol in Γ_1^0 that vanishes in a neighborhood of the bicharacteristic curves. The notation R_m is also used to denote a generic operator of order $\le m$. At last, we denote the components of the vectors $W_{\mathbf{c}}^+$, W^+ in the following way:

$$W_{\mathbf{c}}^+ := (w_1^+, w_2^+, w_3^+)^T, \quad W^+ := (W_1^+, W_2^+, W_3^+)^T.$$

Some tedious computations lead to

$$\mathbf{A}_1^r = \begin{pmatrix} v_r & \dfrac{c_r^2}{\rho_r \langle \partial_{x_1}\Phi_r\rangle} & \dfrac{c_r^2}{\rho_r \langle \partial_{x_1}\Phi_r\rangle} \\ \dfrac{\rho_r\,\partial_{x_2}\Phi_r}{2c_r\langle \partial_{x_1}\Phi_r\rangle^2} & \dfrac{\partial_{x_2}\Phi_r}{c_r\langle \partial_{x_1}\Phi_r\rangle}\left(v_r - \dfrac{c_r\,\partial_{x_1}\Phi_r}{\langle \partial_{x_1}\Phi_r\rangle}\right) & 0 \\ \dfrac{-\rho_r\,\partial_{x_2}\Phi_r}{2c_r\langle \partial_{x_1}\Phi_r\rangle^2} & 0 & \dfrac{-\partial_{x_2}\Phi_r}{c_r\langle \partial_{x_1}\Phi_r\rangle}\left(v_r + \dfrac{c_r\,\partial_{x_1}\Phi_r}{\langle \partial_{x_1}\Phi_r\rangle}\right) \end{pmatrix}$$

and we also have

$$\mathbf{A}_0^r = \operatorname{diag}\left(1, \frac{\partial_{x_2}\Phi_r}{c_r\langle \partial_{x_1}\Phi_r\rangle}, \frac{-\partial_{x_2}\Phi_r}{c_r\langle \partial_{x_1}\Phi_r\rangle}\right).$$

The first scalar equation in (5.36) thus reads

$$T^\gamma_{\tau+i\eta v_r} w_1^+ + T^\gamma_{i\eta c_r^2/\rho_r\langle \partial_{x_1}\Phi_r\rangle}(w_2^+ + w_3^+) + \sum_{i=1}^3 T^\gamma_{\alpha^0} w_i^+ + T^\gamma_r W^+ = T^\gamma_{\chi_{\mathbf{c}}} F_1^+ + R_{-1}W^+.$$

Since the support of $\chi_{\mathbf{c}}$ is included in the open set $\mathcal{V}_{\mathbf{c}}$ (and does not intersect the poles $\Upsilon_{\mathbf{p}}$), we can choose two smooth cut-off functions χ_1 and χ_2 such that
- $\chi_1 \equiv 1$ on the support of $\chi_{\mathbf{c}}$, and $\chi_2 \equiv 1$ on the support of χ_1.
- χ_2 (and therefore χ_1) is supported in $\mathcal{V}_{\mathbf{c}}$.
- χ_1 and χ_2 are C^∞ and homogeneous of degree 0 with respect to (τ, η).

It is clear that the properties of the cut-off function χ_2 imply $\chi_2/(\tau+i\eta v_r) \in \Gamma_2^{-1}$. We apply the operator $T^\gamma_{\chi_2/(\tau+i\eta v_r)}$ to the previous equality and, after repeated applications of Theorem B.6, we obtain:

$$T^\gamma_{\chi_2} w_1^+ + T^\gamma_{\chi_2 i\eta c_r^2/(\tau+i\eta v_r)\rho_r\langle \partial_{x_1}\Phi_r\rangle}(w_2^+ + w_3^+) + \sum_{i=1}^3 T^\gamma_{\alpha^{-1}} w_i^+ + T^\gamma_{r\chi_2/(\tau+i\eta v_r)} W^+ = T^\gamma_{\chi_2/(\tau+i\eta v_r)} T^\gamma_{\chi_{\mathbf{c}}} F_1^+ + R_{-2}W^+.$$

Now, we note that

$$T^\gamma_{\chi_2} w_1^+ = T^\gamma_{\chi_2} T^\gamma_{\chi_{\mathbf{c}}} W_1^+ = T^\gamma_{\chi_{\mathbf{c}}} W_1^+ + R_{-2} W_1^+ = w_1^+ + R_{-2} W_1^+ ,$$

and we are led to the following relation:

$$(5.37) \quad w_1^+ = -T^\gamma_{\chi_2 i\eta c_r^2/(\tau+i\eta v_r)\rho_r\langle\partial_{x_1}\Phi_r\rangle}(w_2^+ + w_3^+) + \sum_{i=1}^3 T^\gamma_{\alpha^{-1}} w_i^+ \\ + T^\gamma_{\chi_2 r/(\tau+i\eta v_r)} W^+ + T^\gamma_{\chi_2/(\tau+i\eta v_r)} T^\gamma_{\chi_{\mathbf{c}}} F_1^+ + R_{-2} W^+ .$$

It is important to note that there is a term in w_1^+ (of degree -1) in the right-hand side of (5.37).

In the second equation of (5.36), w_1^+ appears both in a term of order 1, say $T^\gamma_{\theta^1} w_1^+$ with $\theta^1 \in \Gamma^1_2$, and in a term of order 0, say $T^\gamma_{\theta^0} w_1^+$ with $\theta^0 \in \Gamma^0_1$. We first use the expression (5.37) of w_1^+ in the term $T^\gamma_{\theta^1} w_1^+$. The second equation of (5.36) thus reads

$$T^\gamma_{\theta^1_2} w_2^+ + T^\gamma_{\theta^1_3} w_3^+ + \sum_{i=1}^3 T^\gamma_{\alpha^0} w_i^+ + T^\gamma_r W^+ + \partial_{x_2} w_2^+ = R_0 T^\gamma_{\chi_{\mathbf{c}}} F_1^+ + T^\gamma_{\chi_{\mathbf{c}}} F_2^+ + R_{-1} W^+ ,$$

where $\theta^1_{2,3} \in \Gamma^1_2$. We use once more the expression (5.37) in the term $T^\gamma_{\alpha^0} w_1^+$ just above (recall that $\alpha^0 \in \Gamma^0_1$ so we can apply the rules of symbolic calculus). Collecting the different terms, we are led to an equation that can be written under the following form:

$$T^\gamma_{\theta^1_2} w_2^+ + T^\gamma_{\theta^1_3} w_3^+ + \sum_{i=2}^3 T^\gamma_{\alpha^0} w_i^+ + T^\gamma_r W^+ + \partial_{x_2} w_2^+ = R_0 T^\gamma_{\chi_{\mathbf{c}}} F_1^+ + T^\gamma_{\chi_{\mathbf{c}}} F_2^+ + R_{-1} W^+ .$$

In this equation, all the first and zeroth order terms in w_1^+ have been eliminated. Performing similar computations to eliminate w_1^+ in the last equation of (5.36), we obtain a system of two equations that reads

$$(5.38) \quad \partial_{x_2} \begin{pmatrix} w_2^+ \\ w_3^+ \end{pmatrix} = T^\gamma_{\mathbb{A}^r_{\chi_2}} \begin{pmatrix} w_2^+ \\ w_3^+ \end{pmatrix} + T^\gamma_{\mathbb{E}^r} \begin{pmatrix} w_2^+ \\ w_3^+ \end{pmatrix} + T^\gamma_r W^+ + R_0 F^+ + R_{-1} W^+ ,$$

where $\mathbb{E}^r \in \Gamma^0_1$, $\mathbb{A}^r_{\chi_2} \in \Gamma^1_2$, and $\mathbb{A}^r_{\chi_2} \equiv \mathbb{A}^r$ in the region $\{\chi_2 \equiv 1\}$. Recall that the (singular) symbol $\mathbb{A}^r$ is defined by (5.28)–(5.29). Moreover, the symbol r in (5.38) belongs to Γ^0_1 and is identically zero in the region $\{\chi_{\mathbf{c}} \equiv 1\}$, and R_0 (resp. R_{-1}) is an operator of order ≤ 0 (resp. ≤ -1).

We are now reduced to the noncharacteristic case, for which we follow the analysis of [11]. Indeed, since the symbol $\mathbb{A}^r_{\chi_2}$ equals $\mathbb{A}^r$ in the region $\{\chi_2 \equiv 1\}$, we can diagonalize $\mathbb{A}^r_{\chi_2}$ in this region, and its eigenvalues are exactly $\omega_r^{\pm}$. More precisely, in the region $\{\chi_2 \equiv 1\}$, we have the following relation:

$$Q_0^r \mathbb{A}^r_{\chi_2} = \underbrace{\begin{pmatrix} \omega_r^- & 0 \\ 0 & \omega_r^+ \end{pmatrix}}_{\mathbb{D}_1^r} Q_0^r .$$

Recall that on the set $\mathcal{V}_{\mathbf{c}}$, and therefore also in the region $\{\chi_2 \equiv 1\}$, we have $\omega_r^- \neq \omega_r^+$ thanks to Proposition 5.4. The following lemma can thus be proved as in [11] (to avoid overloaded equations, we denote $x_0 := t$, $\xi_0 := \delta$, and $\xi_1 := \eta$ the variables used in the tangential symbolic calculus).

Lemma 5.5. *There exist a symbol $Q_{-1}^r \in \Gamma_1^{-1}$ and a diagonal symbol $\mathbb{D}_0^r \in \Gamma_1^0$, that are defined in the region $\{\chi_2 \equiv 1\}$, such that*

$$\begin{aligned}(Q_0^r + Q_{-1}^r)&(\mathbb{A}^r_{\chi_2} + \mathbb{E}^r) + \partial_{x_2} Q_0^r \\ &+ \frac{1}{i} \sum_{j=0}^{1} (\partial_{\xi_j} Q_0^r \, \partial_{x_j} \mathbb{A}^r_{\chi_2} - \partial_{\xi_j} \mathbb{D}_1^r \, \partial_{x_j} Q_0^r) - (\mathbb{D}_1^r + \mathbb{D}_0^r)(Q_0^r + Q_{-1}^r)\end{aligned}$$

is a symbol of degree -1 and regularity 1 (at least in the region $\{\chi_2 \equiv 1\}$).

In terms of symbolic calculus, Lemma 5.5 means nothing but

$$(Q_0^r + Q_{-1}^r) \sharp (\partial_{x_2} - \mathbb{A}^r_{\chi_2} - \mathbb{E}^r) = (\partial_{x_2} - \mathbb{D}_1^r - \mathbb{D}_0^r) \sharp (Q_0^r + Q_{-1}^r).$$

In other words, the change of basis $(Q_0^r + Q_{-1}^r)$ diagonalizes both the first order term $\mathbb{A}^r_{\chi_2}$ and the zeroth order term $\mathbb{E}^r$. We thus wish to prove an estimate for

$$Z^+ := T^\gamma_{\chi_1 (Q_0^r + Q_{-1}^r)} \begin{pmatrix} w_2^+ \\ w_3^+ \end{pmatrix},$$

since this new vector will satisfy a paradifferential equation with diagonal symbols. Observe the role of the cut-off function χ_1, whose support is contained in the region $\{\chi_2 \equiv 1\}$, and that also satisfies $\chi_1 \equiv 1$ on the support of $\chi_{\mathbf{c}}$. (We recall that the vector (w_2^+, w_3^+) is microlocalized on the support of $\chi_{\mathbf{c}}$).

Starting from (5.38) and using Lemma 5.5, as well as Theorem B.6, we compute the equation satisfied by the vector Z^+:

$$\partial_{x_2} Z^+ = T^\gamma_{\widetilde{\mathbb{D}_1^r}} Z^+ + T^\gamma_{\widetilde{\mathbb{D}_0^r}} Z^+ + T^\gamma_r W^+ + R_0 F^+ + R_{-1} W^+, \tag{5.39}$$

where $\widetilde{\mathbb{D}_1^r}$ (resp. $\widetilde{\mathbb{D}_0^r}$) is an extension to the whole set $\Omega \times \Xi$ of $\mathbb{D}_1^r$ (resp. $\mathbb{D}_0^r$). These extensions can be chosen such that

$$\widetilde{\mathbb{D}_1^r} = \begin{pmatrix} \omega_r^- & 0 \\ 0 & \omega_r^+ \end{pmatrix} = \begin{pmatrix} \gamma e_r^- + ih_r^- & 0 \\ 0 & \gamma e_r^+ + ih_r^+ \end{pmatrix},$$

$$\widetilde{\mathbb{D}_0^r} = \begin{pmatrix} d_r^- & 0 \\ 0 & d_r^+ \end{pmatrix},$$

with $e_r^-, e_r^+ \in \Gamma_2^0$, $h_r^-, h_r^+ \in \Gamma_2^1$, and $d_r^-, d_r^+ \in \Gamma_1^0$. Moreover, the symbols e_r^-, e_r^+, h_r^-, h_r^+ are real valued and there exists a constant $c > 0$ such that

$$e_r^- \leq -c < 0, \quad e_r^+ \geq c > 0.$$

The second equation in (5.39) reads

$$\partial_{x_2} Z_2^+ = T^\gamma_{\omega_r^+} Z_2^+ + T^\gamma_{d_r^+} Z_2^+ + T^\gamma_r W^+ + R_0 F^+ + R_{-1} W^+.$$

For this scalar equation, we choose $\Lambda^{2,\gamma} := \mathrm{Op}(\gamma^2 + \xi_0^2 + \xi_1^2)$ as a symmetrizer, that is, we multiply the equation by $\Lambda^{2,\gamma} Z_2^+$ and integrate over Ω. Some elementary manipulations, whose details can be found in [11], yield the $L^2(H^1)$ estimate of Z_2^+:

$$\begin{aligned} \gamma \,|||\, Z_2^+ \,|||_{1,\gamma}^2 + \|Z_2^+(0)\|_{1,\gamma}^2 & \\ \leq \frac{C}{\gamma} (\,|||\, F^+ \,|||_{1,\gamma}^2 &+ |||\, W^+ \,|||_0^2 + |||\, T_r^\gamma W^+ \,|||_{1,\gamma}^2). \end{aligned} \tag{5.40}$$

The first equation in (5.39) reads

$$\partial_{x_2} Z_1^+ = T^\gamma_{\omega_r^-} Z_1^+ + T^\gamma_{d_r^-} Z_1^+ + T^\gamma_r W^+ + R_0 F^+ + R_{-1} W^+. \tag{5.41}$$

We first choose the identity as a symmetrizer, and derive the following L^2 estimate:

$$\begin{aligned} \gamma^3 \,|||\, Z_1^+ \,|||_0^2 \leq C\gamma^2 \|Z_1^+(0)\|_0^2 & \\ + \frac{C}{\gamma} (\,|||\, F^+ \,|||_{1,\gamma}^2 &+ |||\, W^+ \,|||_0^2 + |||\, T_r^\gamma W^+ \,|||_{1,\gamma}^2). \end{aligned} \tag{5.42}$$

Recall that in the preceding paragraph, we have constructed a symbol σ_r that satisfies the transport equation:

$$\begin{cases} \partial_{x_2} \sigma_r + \{\sigma_r, h_r^-\} = 0, & x_2 > 0, \\ \sigma_{r}{}_{|x_2=0} = \sigma, \end{cases}$$

where h_r^- is the imaginary part of the eigenvalue ω_r^-. This symbol σ_r is well-defined in the open set $\mathcal{V}_{\mathbf{c}}$, thanks to Proposition 5.4. Since we have extended the eigenvalue ω_r^- (and thus h_r^-) to the whole set $\Omega \times \Xi$, we can also extend σ_r to $\Omega \times \Xi$. Of course, we do not change the value of σ_r on $\mathcal{V}_{\mathbf{c}}$, since the solution σ_r to the transport equation is constant on the bicharacteristic curves (5.32). With slight abuse of notations, we still denote σ_r the extension of σ_r to the whole set $\Omega \times \Xi$. This extension belongs to the class Γ_2^1 and $\partial_{x_2}\sigma_r \in \Gamma_1^1$.

We now choose $S := (T^\gamma_{\sigma_r})^* T^\gamma_{\sigma_r}$ as a symmetrizer for (5.41). (We reproduce below the calculations of [11] since this is really the key point in the analysis). Standard integration by parts yields first of all:

(5.43)

$$
\begin{aligned}
-\|T^\gamma_\sigma Z_1^+(0)\|_0^2 = \Re \int_0^{+\infty} \langle (\partial_{x_2} S) Z_1^+, Z_1^+ \rangle \, dx_2 + 2\Re \int_0^{+\infty} \langle S T^\gamma_{\omega_r^-} Z_1^+, Z_1^+ \rangle \, dx_2 \\
+ 2\Re \int_0^{+\infty} \langle S T^\gamma_{d_r^-} Z_1^+, Z_1^+ \rangle \, dx_2 + 2\Re \int_0^{+\infty} \langle S T^\gamma_r W^+, Z_1^+ \rangle \, dx_2 \\
+ 2\Re \int_0^{+\infty} \langle S R_0 F^+, Z_1^+ \rangle \, dx_2 + 2\Re \int_0^{+\infty} \langle S R_{-1} W^+, Z_1^+ \rangle \, dx_2,
\end{aligned}
$$

where the notation $\langle a, b\rangle$ stands for the tangential scalar product in L^2:

$$
\langle a, b \rangle := \int_{\mathbb{R}^2} a(t, x_1) \overline{b(t, x_1)} \, dt \, dx_1.
$$

The three last integrals on the right hand-side of (5.43) are easily estimated using Cauchy-Schwarz and Young's inequalities (ε is a positive number to be fixed later on):

$$
\begin{aligned}
2\Re \int_0^{+\infty} \langle T^\gamma_{\sigma_r} T^\gamma_r W^+, T^\gamma_{\sigma_r} Z_1^+ \rangle \, dx_2 &\le \frac{C}{\varepsilon\gamma} |||\, T^\gamma_r W^+ \,|||_{1,\gamma}^2 + \varepsilon\gamma \, |||\, T^\gamma_{\sigma_r} Z_1^+ \,|||_0^2, \\
2\Re \int_0^{+\infty} \langle T^\gamma_{\sigma_r} R_0 F^+, T^\gamma_{\sigma_r} Z_1^+ \rangle \, dx_2 &\le \frac{C}{\varepsilon\gamma} |||\, F^+ \,|||_{1,\gamma}^2 + \varepsilon\gamma \, |||\, T^\gamma_{\sigma_r} Z_1^+ \,|||_0^2, \\
2\Re \int_0^{+\infty} \langle T^\gamma_{\sigma_r} R_{-1} W^+, T^\gamma_{\sigma_r} Z_1^+ \rangle \, dx_2 &\le \frac{C}{\varepsilon\gamma} |||\, W^+ \,|||_0^2 + \varepsilon\gamma \, |||\, T^\gamma_{\sigma_r} Z_1^+ \,|||_0^2.
\end{aligned}
$$

The rules of symbolic calculus give $T^\gamma_{\sigma_r} T^\gamma_{d_r^-} = T^\gamma_{d_r^-} T^\gamma_{\sigma_r} + R_0$, which yields the upper bound

$$
\begin{aligned}
2\Re \int_0^{+\infty} \langle S T^\gamma_{d_r^-} Z_1^+, Z_1^+ \rangle \, dx_2 &\le C \, |||\, T^\gamma_{\sigma_r} Z_1^+ \,|||_0^2 + C |||Z_1^+|||_0 \, |||T^\gamma_{\sigma_r} Z_1^+|||_0 \\
&\le (C + \varepsilon\gamma) \, |||\, T^\gamma_{\sigma_r} Z_1^+ \,|||_0^2 + \frac{C}{\varepsilon\gamma} |||\, Z_1^+ \,|||_0^2 \\
&\le (C + \varepsilon\gamma) \, |||\, T^\gamma_{\sigma_r} Z_1^+ \,|||_0^2 + \frac{C}{\varepsilon\gamma} |||\, W^+ \,|||_0^2.
\end{aligned}
$$

Using the definition of S, we obtain

$$\partial_{x_2} S = (T^\gamma_{\partial_{x_2}\sigma_r})^* T^\gamma_{\sigma_r} + (T^\gamma_{\sigma_r})^* T^\gamma_{\partial_{x_2}\sigma_r}.$$

From the basic estimates above, we already get the relation

$$\begin{aligned}(5.44)\qquad -2\Re \int_0^{+\infty} \langle T^\gamma_{\partial_{x_2}\sigma_r} Z_1^+ + T^\gamma_{\sigma_r} T^\gamma_{\omega_r^-} Z_1^+, T^\gamma_{\sigma_r} Z_1^+ \rangle \, dx_2 \\ \leq \| T^\gamma_\sigma Z_1^+(0) \|_0^2 + (C + 4\varepsilon\gamma) |\!|\!| T^\gamma_{\sigma_r} Z_1^+ |\!|\!|_0^2 \\ + \frac{C}{\varepsilon\gamma} (|\!|\!| F^+ |\!|\!|_{1,\gamma}^2 + |\!|\!| T^\gamma_r W^+ |\!|\!|_{1,\gamma}^2 + |\!|\!| W^+ |\!|\!|_0^2).\end{aligned}$$

Now, we decompose ω_r^- as $\omega_r^- = \gamma e_r^- + i h_r^-$, where e_r^-, h_r^- take real values, $e_r^- \in \Gamma_2^0$, and $h_r^- \in \Gamma_2^1$. Because the symbol σ_r satisfies the transport equation

$$\partial_{x_2} \sigma_r + \{\sigma_r, h_r^-\} = 0,$$

equation (5.44) yields

$$\begin{aligned}-2\gamma\Re \int_0^{+\infty} \langle T^\gamma_{e_r^-} T^\gamma_{\sigma_r} Z_1^+, T^\gamma_{\sigma_r} Z_1^+ \rangle \, dx_2 - 2\gamma\Re \int_0^{+\infty} \langle T^\gamma_{-i\{\sigma_r, e_r^-\}} Z_1^+, T^\gamma_{\sigma_r} Z_1^+ \rangle \, dx_2 \\ - 2\gamma\Re \int_0^{+\infty} \langle R_{-1} Z_1^+, T^\gamma_{\sigma_r} Z_1^+ \rangle \, dx_2 - 2\Re \int_0^{+\infty} \langle T^\gamma_{ih_r^-} T^\gamma_{\sigma_r} Z_1^+, T^\gamma_{\sigma_r} Z_1^+ \rangle \, dx_2 \\ - 2\Re \int_0^{+\infty} \langle R_0 Z_1^+, T^\gamma_{\sigma_r} Z_1^+ \rangle \, dx_2 \leq \text{right-hand side of (5.44)}.\end{aligned}$$

The first term in the left-hand side is bounded from below thanks to Gårding's inequality (Theorem B.7). All the other terms are put on the right-hand side and estimated using Cauchy-Schwarz and Young's inequalities. In the end, we choose an appropriate ε and obtain

$$\begin{aligned}\gamma |\!|\!| T^\gamma_{\sigma_r} Z_1^+ |\!|\!|_0^2 \leq C \| T^\gamma_\sigma Z_1^+(0) \|_0^2 + C\gamma |\!|\!| Z_1^+ |\!|\!|_0^2 \\ + \frac{C}{\gamma} (|\!|\!| F^+ |\!|\!|_{1,\gamma}^2 + |\!|\!| T^\gamma_r W^+ |\!|\!|_{1,\gamma}^2 + |\!|\!| W^+ |\!|\!|_0^2).\end{aligned}$$

We use (5.42) to estimate the term $\gamma |\!|\!| Z_1^+ |\!|\!|_0^2$ in the right-hand side. Eventually, we derive

$$\begin{aligned}(5.45)\qquad \gamma |\!|\!| T^\gamma_{\sigma_r} Z_1^+ |\!|\!|_0^2 \leq C (\| T^\gamma_\sigma Z_1^+(0) \|_0^2 + \| Z_1^+(0) \|_0^2) \\ + \frac{C}{\gamma} (|\!|\!| F^+ |\!|\!|_{1,\gamma}^2 + |\!|\!| T^\gamma_r W^+ |\!|\!|_{1,\gamma}^2 + |\!|\!| W^+ |\!|\!|_0^2).\end{aligned}$$

Recall that σ_r is homogeneous of degree 1 with respect to the frequencies (τ, η), so (5.45) has to be understood as a $L^2(H^1)$ estimate of Z_1^+ far from the bicharacteristic curve (which is exactly the set where σ_r vanishes).

A similar analysis enables us to derive an energy estimate for the vector

$$Z^- := T^\gamma_{\chi_1(Q_0^l + Q_{-1}^l)} T^\gamma_{\chi_\mathbf{c}} \begin{pmatrix} W_2^- \\ W_3^- \end{pmatrix},$$

where Q_0^l diagonalizes the symbol $\mathbb{A}^l_{\chi_2}$:

$$Q_0^l \mathbb{A}^l_{\chi_2} = \begin{pmatrix} \omega_l^- & 0 \\ 0 & \omega_l^+ \end{pmatrix} Q_0^l,$$

and Q_{-1}^l is defined as in Lemma 5.5, *mutatis mutandis*. The final estimates are

$$\gamma \,|||\, Z_2^- \,|||^2_{1,\gamma} + ||Z_2^-(0)||^2_{1,\gamma} \leq \frac{C}{\gamma}\left(|||\, F^- \,|||^2_{1,\gamma} + |||\, W^- \,|||^2_0 + |||\, T^\gamma_r W^- \,|||_{1,\gamma}{}^2 \right), \tag{5.46}$$

$$\begin{aligned} &\gamma^3 \,|||\, Z_1^- \,|||^2_0 + \gamma \,|||\, T^\gamma_{\sigma_l} Z_1^- \,|||^2_0 \\ &\leq C\left(\gamma^2 ||Z_1^-(0)||^2_0 + ||T^\gamma_\sigma Z_1^-(0)||^2_0\right) + \frac{C}{\gamma}\left(|||\, F^- \,|||^2_{1,\gamma} + |||\, W^- \,|||^2_0 + |||\, T^\gamma_r W^- \,|||^2_{1,\gamma}\right). \end{aligned}$$

The remaining part of the job is to estimate the traces of the incoming modes (Z_1^+ and Z_1^-) in terms of the outgoing modes (Z_2^+ and Z_2^-) and G, knowing that we have the relation

$$T^\gamma_{\boldsymbol{\beta}} \begin{pmatrix} W_2^+ & W_3^+ & W_2^- & W_3^- \end{pmatrix}^T_{|x_2=0} = G.$$

This is nothing but the definition of G, see (5.35c). Observe that the first column vector of Q_0^r and the first column vector of Q_0^l span the stable subspace $\mathcal{E}^-(t, x_1, x_2, \tau, \eta)$ (at least when the space-frequency variables belong to the set $\mathcal{V}_\mathbf{c}$). These column vectors are denoted by E_r and E_l. Following the proof of Lemma 4.5 (see Appendix A), one can show that there exist some 2×2 invertible matrices P_1 and P_2 such that

$$P_1 \boldsymbol{\beta} \begin{pmatrix} E_r & E_l \end{pmatrix} P_2 = \begin{pmatrix} \gamma + i\sigma & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, repeating the arguments of [11], one can show the following estimate for the boundary terms:

$$\begin{aligned} &\gamma^2\left(||Z_1^-(0)||^2_0 + ||Z_1^+(0)||^2_0\right) + ||T^\gamma_\sigma Z_1^-(0)||^2_0 + ||T^\gamma_\sigma Z_1^+(0)||^2_0 \\ &\qquad \leq C\left(||G||^2_{1,\gamma} + ||Z_2^-(0)||^2_{1,\gamma} + ||Z_2^+(0)||^2_{1,\gamma} + ||W^{\mathrm{nc}}{}_{|x_2=0}||^2_0\right). \end{aligned} \tag{5.47}$$

Recall that the main ingredient of the proof is the microlocalized Gårding's inequality (Theorem B.8), and the fact that σ is real valued.

Collecting (5.40), (5.42), (5.45), (5.46), and (5.47), we end up with the final estimate near the critical set:

$$
\begin{aligned}
(5.48)\quad & \gamma(|||Z_2^-|||_{1,\gamma}^2 + |||Z_2^+|||_{1,\gamma}^2 + |||T^\gamma_{\sigma_l} Z_1^-|||_0^2 + |||T^\gamma_{\sigma_r} Z_1^+|||_0^2 \\
& + \gamma^2 |||Z_1^-|||_0^2 + \gamma^2 |||Z_1^+|||_0^2) + (\|Z_2^-(0)\|_{1,\gamma}^2 + \|Z_2^+(0)\|_{1,\gamma}^2 \\
& + \|T^\gamma_\sigma Z_1^-(0)\|_0^2 + \|T^\gamma_\sigma Z_1^+(0)\|_0^2 + \gamma^2\|Z_1^-(0)\|_0^2 + \gamma^2\|Z_1^+(0)\|_0^2) \\
& \leq \frac{C}{\gamma}(|||F|||_{1,\gamma}^2 + |||W|||_0^2 + |||T^\gamma_r W|||_{1,\gamma}^2) + \|G\|_{1,\gamma}^2 + \|W^{\mathrm{nc}}{}_{|x_2=0}\|_0^2.
\end{aligned}
$$

Recall, for later use, that the vectors $Z^\pm$ are defined by the formulas

$$
Z^+ := T^\gamma_{\chi_1(Q_0^r + Q_{-1}^r)} T^\gamma_{\chi_{\mathbf{c}}} \begin{pmatrix} W_2^+ \\ W_3^+ \end{pmatrix}, \quad Z^- := T^\gamma_{\chi_1(Q_0^l + Q_{-1}^l)} T^\gamma_{\chi_{\mathbf{c}}} \begin{pmatrix} W_2^- \\ W_3^- \end{pmatrix},
$$

and the matrices $Q_0^{r,l}$ are invertible on a neighborhood of the support of χ_1. We also recall the relation $\chi_1 \chi_{\mathbf{c}} \equiv \chi_{\mathbf{c}}$.

Recall also that the components $T^\gamma_{\chi_{\mathbf{c}}} W_1^\pm$ are given in terms of $T^\gamma_{\chi_{\mathbf{c}}} W_{2,3}^\pm$ by the relation (5.37). In particular, this relation yields an L^2 estimate for $T^\gamma_{\chi_{\mathbf{c}}} W_1^\pm$, and an $L^2(H^1)$ estimate far from the bicharacteristic curves. Namely, we can add the norms

$$
\gamma^3 |||T^\gamma_{\chi_{\mathbf{c}}} W_1^+|||_0^2 + \gamma^3 |||T^\gamma_{\chi_{\mathbf{c}}} W_1^-|||_0^2 + \gamma |||T^\gamma_{\sigma_r} T^\gamma_{\chi_{\mathbf{c}}} W_1^+|||_0^2 + \gamma |||T^\gamma_{\sigma_l} T^\gamma_{\chi_{\mathbf{c}}} W_1^-|||_0^2
$$

in the left-hand side of (5.48). We thus control the L^2 norm of the vector $T^\gamma_{\chi_{\mathbf{c}}} W$, and not only the noncharacteristic part of the vector. We also control the $L^2(H^1)$ norm far from the bicharacteristic curves that originate from the critical set.

5.6. Derivation of energy estimates: the good frequencies. To estimate the trace of $T^\gamma_{\chi_{\mathbf{u}}} W^{\mathrm{nc}}$, one first computes the equation satisfied (in Ω) by the vector $T^\gamma_{\chi_{\mathbf{u}}} W$; then one eliminates the components that belong to the kernel of $\mathbf{I}_2$, and obtains an equation involving only $T^\gamma_{\chi_{\mathbf{u}}} W^{\mathrm{nc}}$. The equation is similar to (5.38). For this reduced equation, one can construct Kreiss' type symmetrizers, because the *uniform* Lopatinskii condition is satisfied in a neighborhood of the support of $\chi_{\mathbf{u}}$. The construction of the symmetrizer is achieved as in the constant coefficients case (see Section 4). Once again, we refer to [11] for a detailed derivation of energy estimates, and we only give the result here. The estimate obtained by this method reads:

$$
\begin{aligned}
(5.49)\quad & \gamma |||T^\gamma_{\chi_{\mathbf{u}}} W|||_{1,\gamma}^2 + \|T^\gamma_{\chi_{\mathbf{u}}} W^{\mathrm{nc}}(0)\|_{1,\gamma}^2 \\
& \leq C(\|G\|_{1,\gamma}^2 + \|W^{\mathrm{nc}}(0)\|_0^2) + \frac{C}{\gamma}(|||F|||_{1,\gamma}^2 + |||W|||_0^2 + |||T^\gamma_r W|||_{1,\gamma}^2).
\end{aligned}
$$

As in (5.48), the symbol r vanishes in a neighborhood of the bicharacteristic curves (the symbol r in (5.49) is a linear combination of the derivatives of $\chi_{\mathbf{u}}$, and the cut-off function $\chi_{\mathbf{u}}$ is identically zero near the bicharacteristic curves).

5.7. Derivation of energy estimates: the poles. To derive an estimate for $T^{\gamma}_{\chi_{\mathbf{p}}} W^{\pm}$, one starts from (5.35a)–(5.35b) and computes an equation similar to (5.36). Then one changes basis, as was done in the constant coefficients case. In the end, one derives a maximal $L^2(H^1)$ estimate because the uniform Lopatinskii condition is satisfied near the poles. The energy estimate is thus similar to the one corresponding to the *good frequencies*:

$$
\begin{aligned}
(5.50)\quad & \gamma \, |||\, T^{\gamma}_{\chi_{\mathbf{p}}} W \,|||^{2}_{1,\gamma} + ||T^{\gamma}_{\chi_{\mathbf{p}}} W^{\mathrm{nc}}(0)||^{2}_{1,\gamma} \\
&\leq C(||G||^{2}_{1,\gamma} + ||W^{\mathrm{nc}}(0)||^{2}_{0}) + \frac{C}{\gamma}(\,|||\, F \,|||^{2}_{1,\gamma} + |||\, W \,|||^{2}_{0} + |||\, T^{\gamma}_{r} W \,|||^{2}_{1,\gamma}).
\end{aligned}
$$

5.8. Proof of Theorem 5.1. We now patch together the microlocalized energy estimates, and show that the estimate (5.27) holds. We first note that, adding (5.48), (5.49), (5.50), we are able to control the norms $\gamma^3 |||W|||^2_0$ and $\gamma^2 \|W^{\mathrm{nc}}{}_{|x_2=0}\|^2_0$. Namely, we first obtain (up to choosing γ large enough):

$$
\begin{aligned}
(5.51)\quad & \text{Left-hand side of (5.48)–(5.49)–(5.50)} \\
&\leq C||G||^{2}_{1,\gamma} + \frac{C}{\gamma}(\,|||\, F \,|||^{2}_{1,\gamma} + |||\, T^{\gamma}_{r} W \,|||^{2}_{1,\gamma}).
\end{aligned}
$$

In view of (5.51), the only thing to show is how to absorb the term $|||T^{\gamma}_{r} W|||_{1,\gamma}$. Recall that the symbol r is identically zero in the regions where $\chi_{\mathbf{c}}$, $\chi_{\mathbf{u}}$ or $\chi_{\mathbf{p}}$ are equal to 1. We may thus decompose r as a linear combination of the form

$$
\begin{aligned}
r &= \alpha_{\mathbf{u}} \chi_{\mathbf{u}} + \alpha_{\mathbf{p}} \chi_{\mathbf{p}} \\
&+ \alpha_{\mathbf{c}} \begin{pmatrix} \sigma_r & 0 & 0 & & & \\ 0 & \sigma_r & 0 & & \mathbf{O} & \\ 0 & 0 & \lambda^{1,\gamma} & & & \\ & & & \sigma_l & 0 & 0 \\ & \mathbf{O} & & 0 & \sigma_l & 0 \\ & & & 0 & 0 & \lambda^{1,\gamma} \end{pmatrix} \begin{pmatrix} 1 & 0 & & \\ 0 & \chi_1 Q^{r}_{0} & & \\ & & 1 & 0 \\ & & 0 & \chi_1 Q^{l}_{0} \end{pmatrix} \chi_{\mathbf{c}}.
\end{aligned}
$$

The matrices $\alpha_{\mathbf{c},\mathbf{u},\mathbf{p}}$ have a block diagonal structure. We are thus able to absorb the term $|||T^{\gamma}_{r} W|||_{1,\gamma}$, thanks to the left-hand sides of (5.48)–(5.49)–(5.50). We thus obtain (5.27), since the left-hand sides of (5.48)–(5.49)–(5.50) are bounded from below by

$$
c(\gamma^3 \,|||\, W \,|||^{2}_{0} + \gamma^2 ||W^{\mathrm{nc}}{}_{|x_2=0}||^{2}_{0}), \qquad c > 0.
$$

Thanks to Proposition 5.3, the estimate (5.12) for the variable coefficients linearized operators also holds. This completes the proof of Theorem 5.1.

6. CONCLUDING REMARKS

In this paper, we have proved a linear stability result for a wide class of rectilinear compressible vortex sheets. To summarize, once we are given a suitable perturbation of a rectilinear supersonic vortex sheet, the linearized coefficients around this perturbation satisfies an a priori estimate with loss of one derivative (in the tangential variables). We have also proved that the paralinearized version of the linearized equations satisfies the same a priori estimate. The constants appearing in the energy estimates are uniform with respect to the $W^{2,\infty}$ norm of the coefficients.

To prove the local in time existence of nonconstant vortex sheets, the next step will be to build an iteration scheme that takes into account this loss of regularity. In view of [1, 13], there is a strong hope that a Nash-Moser type iteration scheme might answer the problem. However, special attention should be paid, at each step, to the relations (5.3), (5.4), and (5.5), that are crucial in the proof of Theorem 5.1. The verification of the local existence of (supersonic) vortex sheets is postponed to a future work.

The one-dimensional stability of contact discontinuities has received a general treatment in [8] and [9]. Unfortunately, the isentropic Euler equations do not admit contact discontinuities in one space dimension. However, it would be interesting to determine whether the present analysis extends to some contact discontinuities for the general Euler equations, and see the connections with the one-dimensional analysis.

Acknowledgements. The authors thank Sylvie Benzoni-Gavage and Denis Serre for providing them with early versions of [4]. This has been very useful to understand the symmetrizer construction near the poles. J.-F. C. thanks the Mathematics Department of Brescia University for its hospitality during his postdoctoral stay. Research of the authors was supported by the EU financed network HYKE, HPRN-CT-2002-00282.

APPENDIX A. PROOF OF INTERMEDIATE RESULTS

1.1. The proof of Proposition 4.3. Using (4.9) and (4.14), we obtain

$$\beta \begin{pmatrix} E_r & E_l \end{pmatrix} = \begin{pmatrix} (\tau + iv_r\eta)(c^{-1}(\tau + iv_r\eta) - \omega_r) & (\tau + iv_l\eta)(c^{-1}(\tau + iv_l\eta) - \omega_l) \\ -c\omega_r(\tau + iv_l\eta)(c\omega_r - (\tau + iv_r\eta)) & c\omega_l(\tau + iv_r\eta)(c\omega_l - (\tau + iv_l\eta)) \end{pmatrix}$$

for all $(\tau, \eta) \in \Sigma$. This gives the following expression for the Lopatinskii determinant:

$$\Delta(\tau, \eta) = -c^2(\tau + iv_r\eta - c\omega_r)(\tau + iv_l\eta - c\omega_l)(\omega_r\omega_l - \eta^2)(\omega_r + \omega_l).$$

Recall that ω_r and ω_l have negative real part when τ has positive real part, and satisfy the dispersion relations (4.13a)–(4.13b). Because $v_r = -v_l$, we have the identity

$$\omega_r(\tau, \eta) = \omega_l(\tau, -\eta).$$

With the above expression for Δ, it is easy to check that $\Delta(\tau, \eta) = \Delta(\tau, -\eta)$. We shall thus only consider nonnegative values of η in all this section: $\eta \geq 0$.

One first checks that both expressions

$$(\tau + i v_r \eta - c\omega_r) \quad \text{and} \quad (\tau + i v_l \eta - c\omega_l)$$

do not vanish for any $(\tau, \eta) \in \Sigma$, because of (4.13a)–(4.13b).

Clearly, the sum $\omega_r + \omega_l$ can not vanish when τ has positive real part, since both numbers have negative real part. When τ is purely imaginary, one extends $\omega_{r,l}$ by continuity. If $\tau = i\delta \in i\mathbb{R}$ satisfies $(\delta + v_r\eta)^2 \leq c^2\eta^2$, one has

$$\omega_r = -\sqrt{\eta^2 - \frac{1}{c^2}(\delta + v_r\eta)^2} \in \mathbb{R}.$$

If $(\delta + v_r\eta)^2 > c^2\eta^2$, we use Cauchy-Riemann relations to derive

$$\omega_r = -i \operatorname{sgn}(\delta + v_r\eta)\sqrt{\frac{1}{c^2}(\delta + v_r\eta)^2 - \eta^2} \in i\mathbb{R}.$$

The calculations are almost the same as those done in [3]. For ω_l, one just changes v_r into $v_l = -v_r$ and derives similar formulas. Then using the dispersion relations (4.13a)–(4.13b), we easily check that $\omega_r + \omega_l$ vanishes if and only if $\tau = 0$ (and therefore $\eta \neq 0$). For $\eta > 0$, this gives the following values for the eigenmodes:

$$\omega_r = -i\eta\sqrt{\frac{v_r^2}{c^2} - 1} = -\omega_l \in i\mathbb{R}.$$

Recall that $v_r > c\sqrt{2}$.

It now remains to determine whether the expression $(\omega_r\omega_l - \eta^2)$ may vanish. If $\eta = 0$, one has $\omega_r = \omega_l = -\tau/c$, and, therefore, $\omega_r\omega_l \neq 0$. We thus assume $\eta \neq 0$ (that is, $\eta > 0$) and introduce the reduced expressions

$$V := \frac{\tau}{i\eta}, \quad \Omega_{r,l} := \frac{\omega_{r,l}}{i\eta}.$$

Assume that $\Omega_r\Omega_l = -1$. Using (4.13) and $(\Omega_r\Omega_l)^2 = 1$, we obtain the following polynomial equation for V:

$$V^4 - 2(c^2 + v_r^2)V^2 + v_r^2(v_r^2 - 2c^2) = 0.$$

This is a polynomial equation of degree 2 for the unknown V^2, whose roots are real and distinct. If (3.11) holds, both roots are positive. Let us denote these roots by V_1^2 and V_2^2, where $0 < V_1 < V_2$. One has

$$V_1^2 = c^2 + v_r^2 - c\sqrt{c^2 + 4v_r^2},$$
$$V_2^2 = c^2 + v_r^2 + c\sqrt{c^2 + 4v_r^2}.$$

We first show that the root V_2 does not yield any instability. If $V = V_2$, one has $V \pm v_r \geq c$. Because $\eta > 0$, we obtain

$$\Omega_r = -\sqrt{\frac{1}{c^2}(V_2 + v_r)^2 - 1} \quad \text{and} \quad \Omega_l = -\sqrt{\frac{1}{c^2}(V_2 - v_r)^2 - 1},$$

so $\Omega_r \Omega_l \neq -1$. The Lopatinskii determinant Δ does not vanish when $V = V_2$. A similar argument shows that Δ does not vanish either for $V = -V_2$.

Now we show that $V = V_1$ is a root of the Lopatinskii determinant. One first checks that $V_1 + v_r > c$, and $V_1 - v_r < -c$. Hence, for $\eta > 0$ and $\tau = iV_1\eta$, we find

$$\Omega_r = -\sqrt{\frac{1}{c^2}(V_1 + v_r)^2 - 1} \quad \text{and} \quad \Omega_l = \sqrt{\frac{1}{c^2}(V_1 - v_r)^2 - 1}.$$

These relations yield $\Omega_r \Omega_l < 0$ and $(\Omega_r \Omega_l)^2 = 1$, that is, we have $\Omega_r \Omega_l = -1$. For $V = V_1$, the Lopatinskii determinant vanishes. The same argument holds for $V = -V_1$.

This completes the first part of the proof of Proposition 4.3. What remains to show is that the roots of the Lopatinskii determinant are simple. We first show that near $\tau = 0$ and $\eta = 1/v_r$ (this is the only point of Σ such that $\tau = 0$ and $\eta > 0$), we have

$$\omega_r + \omega_l = \tau h(\tau, \eta),$$

for an appropriate C^∞ function h. Since $\eta \neq 0$ near $\tau = 0$, we have

$$\omega_r + \omega_l = i\eta(\Omega_r + \Omega_l),$$

where the notations are those introduced earlier. Near $V = 0$, both functions Ω_r and Ω_l are analytic with respect to V and satisfy

$$\Omega_r^2 = \frac{1}{c^2}(V + v_r)^2 - 1, \quad \Omega_l^2 = \frac{1}{c^2}(V - v_r)^2 - 1.$$

We thus obtain

$$\left(\frac{d\Omega_r}{dV} + \frac{d\Omega_l}{dV}\right)\Big|_{V=0} = \frac{2v_r}{c\Omega_r(0)} \neq 0.$$

Using a classical factorization property of holomorphic functions, we obtain

$$\Omega_r + \Omega_l = VH(V),$$

where H is holomorphic near 0 and $H(0) \neq 0$. This yields

$$\omega_r + \omega_l = \tau H\left(\frac{\tau}{i\eta}\right).$$

The factorization result for Δ is proved near the root $\tau = 0$.

As regards the situation near those roots of the form $(\pm iV_1\eta, \eta) \in \Sigma$, it is entirely similar and we shall not detail the proof. (The proof is similar because Ω_r and Ω_l are still holomorphic with respect to V in a neighborhood of V_1 and $-V_1$). The result is that the Lopatinskii determinant Δ admits a factorization that reads

$$\Delta(\tau,\eta) = (\tau - iV_1\eta)h(\tau,\eta) \quad \text{or} \quad \Delta(\tau,\eta) = (\tau + iV_1\eta)h(\tau,\eta),$$

where h is C^∞ and does not vanish near the roots of Δ. This completes the proof.

1.2. The proof of Lemma 4.5. In the proof of Proposition 4.3, we have seen that the matrix $\beta(E_r E_l)$ has the following expression:

$$\beta\begin{pmatrix} E_r & E_l \end{pmatrix} = \begin{pmatrix} (\tau + iv_r\eta)(c^{-1}(\tau + iv_r\eta) - \omega_r) & (\tau + iv_l\eta)(c^{-1}(\tau + iv_l\eta) - \omega_l) \\ -c\omega_r(\tau + iv_l\eta)(c\omega_r - (\tau + iv_r\eta)) & c\omega_l(\tau + iv_r\eta)(c\omega_l - (\tau + iv_l\eta)) \end{pmatrix},$$

for all $(\tau,\eta) \in \Sigma$. We have also seen that the quantity $(\tau + iv_r\eta - c\omega_r)$ does not vanish for $(\tau,\eta) \in \Sigma$. Let us now consider a neighborhood $\mathcal{V}$ of a point $(\tau_0,\eta_0) \in \Sigma$ such that $\tau_0 = 0$. Up to shrinking $\mathcal{V}$, the quantity $(\tau + iv_r\eta)$ does not vanish in $\mathcal{V}$. As a consequence, the upper left corner coefficient of $\beta(E_r E_l)$ does not vanish in $\mathcal{V}$. We write

$$\beta\begin{pmatrix} E_r & E_l \end{pmatrix} = \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{pmatrix}.$$

Then, the relation (4.15) can be rewritten as $\Delta = \zeta_1\zeta_4 - \zeta_2\zeta_3$, and ζ_1 does not vanish in $\mathcal{V}$. The identity

$$\begin{pmatrix} 1/\zeta_1 & 0 \\ -\zeta_3/\zeta_1 & 1 \end{pmatrix} \beta\begin{pmatrix} E_r & E_l \end{pmatrix} \begin{pmatrix} 1 & -\zeta_2 \\ 0 & \zeta_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}$$

is a straightforward verification. In particular, this identity yields the estimate

$$|\beta\begin{pmatrix} E_r & E_l \end{pmatrix} Z^-|^2 \geq \kappa \min(1, |\Delta|^2)|Z^-|^2,$$

for all $Z^- \in \mathbb{C}^2$. Using Proposition 4.3, the Lopatinskii determinant Δ can be factorized near (τ_0, η_0):

$$\Delta(\tau, \eta) = \tau h(\tau, \eta), \quad h(\tau_0, \eta_0) \neq 0.$$

Since γ is the real part of τ, we obtain $|\Delta(\tau, \eta)| \geq \kappa\gamma$, for a suitable constant $\kappa > 0$ (still up to shrinking the neighborhood $\mathcal{V}$). This last inequality yields

$$|\beta \left(E_r \quad E_l\right) Z^-|^2 \geq \kappa\gamma^2 |Z^-|^2,$$

for all $Z^- \in \mathbb{C}^2$ and all $(\tau, \eta) \in \mathcal{V}$.

Lemma 4.5 is thus proved when (τ_0, η_0) satisfies $\tau_0 = 0$. The other points where Δ vanishes are those points (τ_0, η_0) such that $\tau_0 = \pm i V_1 \eta_0$. Near those points, the upper left corner coefficient of $\beta(E_r E_l)$ still does not vanish. We can again conclude that in an appropriate neighborhood $\mathcal{V}$ of (τ_0, η_0) (with, for instance, $\tau_0 = i V_1 \eta_0$), one has

$$|\beta \left(E_r \quad E_l\right) Z^-|^2 \geq \kappa \min(1, |\Delta|^2) |Z^-|^2.$$

Now we use the factorization

$$\Delta(\tau, \eta) = (\tau - i V_1 \eta) h(\tau, \eta), \quad h(\tau_0, \eta_0) \neq 0,$$

to conclude. This completes the proof of Lemma 4.5.

APPENDIX B. PARADIFFERENTIAL CALCULUS WITH A PARAMETER

In this appendix, we collect the main results of the paradifferential calculus of Bony and Meyer [5, 26] that we use in this paper, see [25] for the introduction of the parameter. We refer to these papers for the proofs of the results stated below. We first recall the classification of paradifferential symbols.

Definition B.1. A paradifferential symbol of degree $m \in \mathbb{R}$ and regularity k ($k \in \mathbb{N}$) is a function $a(x, \xi, \gamma) : \mathbb{R}^2 \times \mathbb{R}^2 \times [0, +\infty[\to \mathbb{C}^{N \times N}$ such that a is C^∞ with respect to ξ and for all $\alpha \in \mathbb{N}^2$, there exists a constant C_α verifying

$$\forall\, (\xi, \gamma), \quad \|\partial_\xi^\alpha a(\cdot, \xi, \gamma)\|_{W^{k,\infty}(\mathbb{R}^2)} \leq C_\alpha \lambda^{m-|\alpha|,\gamma}(\xi) = C_\alpha (\gamma^2 + |\xi|^2)^{(m-|\alpha|)/2}.$$

The set of paradifferential symbols of degree m and regularity k is denoted by Γ_k^m. We denote by Σ_k^m the subset of paradifferential symbols $a \in \Gamma_k^m$ such that for a suitable $\varepsilon \in]0, 1[$ one has

$$\forall\, (\xi, \gamma), \quad \operatorname{Supp} \mathcal{F}_x a(\cdot, \xi, \gamma) \subset \{\zeta \in \mathbb{R}^2 / |\zeta| \leq \varepsilon (\gamma^2 + |\xi|^2)^{1/2}\}.$$

Of course, the symbols in Σ_k^m are C^∞ functions with respect to both variables x and ξ, and for all $a \in \Sigma_k^m$, we have the estimates

$$\forall\ (x,\xi,\gamma), \quad |\partial_x^\beta \partial_\xi^\alpha a(x,\xi,\gamma)| \leq C_{\alpha,\beta} \lambda^{m-|\alpha|+|\beta|,\gamma}(\xi).$$

Thus any symbol $a \in \Sigma_k^m$ belongs to Hörmander's class $S_{1,1}^m$ [16] and defines an operator $\mathrm{Op}^\gamma(a)$ on the Schwartz' class S by the usual formula

$$\forall\ u \in S, \quad \mathrm{Op}^\gamma(a)u(x) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\cdot\xi}\, a(x,\xi,\gamma)\hat{u}(\xi)\, d\xi.$$

We shall use the following terminology:

Definition B.2. A family of operators $\{P^\gamma\}$ defined for $\gamma \geq 1$ will be said of order $\leq m$ ($m \in \mathbb{R}$) if the operators P^γ are uniformly bounded from H^{s+m} to H^s:

$$\forall\ \gamma \geq 1,\ \forall\ u \in H^{s+m}, \quad \|P^\gamma u\|_{s,\gamma} \leq C(s,m)\|u\|_{s+m,\gamma}.$$

The following theorem is crucial for the sequel of the analysis.

Theorem B.3. *If $a \in \Sigma_k^m$, the family $\{\mathrm{Op}^\gamma(a)\}$ is of order $\leq m$.*

The regularization of symbols in the class Γ_k^m is achieved by a convolution with admissible cut-off functions.

Definition B.4. Let $\psi : \mathbb{R}^2 \times \mathbb{R}^2 \times [1,+\infty[\ \to [0,+\infty[$ be a C^∞ function such that the following estimates hold for all α, $\beta \in \mathbb{N}^2$:

$$\forall\ (\zeta,\xi,\gamma), \quad |\partial_\zeta^\alpha \partial_\xi^\beta \psi(\zeta,\xi,\gamma)| \leq C_{\alpha,\beta} \lambda^{-|\alpha|-|\beta|,\gamma}(\xi).$$

We shall say that ψ is an admissible cut-off function if there exist real numbers $0 < \varepsilon_1 < \varepsilon_2 < 1$ satisfying

$$\begin{aligned} \psi(\zeta,\xi,\gamma) &= 1 \quad \text{if } |\zeta| \leq \varepsilon_1(\gamma^2+|\xi|^2)^{1/2}, \\ \psi(\zeta,\xi,\gamma) &= 0 \quad \text{if } |\zeta| \geq \varepsilon_2(\gamma^2+|\xi|^2)^{1/2}. \end{aligned}$$

An example of cut-off function is the following: let χ be a nonnegative C^∞ function on $\mathbb{R}^2 \times \mathbb{R}$ such that

$$\gamma_1^2 + |\xi_1|^2 \geq \gamma_2^2 + |\xi_2|^2 \Rightarrow \chi(\xi_1,\gamma_1) \leq \chi(\xi_2,\gamma_2),$$

$$\begin{cases} \chi(\xi,\gamma) = 1 & \text{if } (\gamma^2+|\xi|^2)^{1/2} \leq \frac{1}{2}, \\ \chi(\xi,\gamma) = 0 & \text{if } (\gamma^2+|\xi|^2)^{1/2} \geq 1. \end{cases}$$

We define a function $\varphi(\xi,\gamma) := \chi(\xi/2,\gamma/2) - \chi(\xi,\gamma)$. Then the function ψ_0 defined by

$$\psi_0(\zeta,\xi,\gamma) := \sum_{p\geq 0} \chi(2^{2-p}\zeta,0)\varphi(2^{-p}\xi,2^{-p}\gamma)$$

is an admissible cut-off function (one can take $\varepsilon_1 = \frac{1}{16}$ and $\varepsilon_2 = \frac{1}{2}$).

If ψ is an admissible cut-off function, the inverse Fourier transform K^ψ of $\psi(\cdot,\xi,\gamma)$ satisfies

$$\forall\,(\xi,\gamma), \quad \|\partial_\xi^\alpha K^\psi(\cdot,\xi,\gamma)\|_{L^1(\mathbb{R}^2)} \leq C_\alpha \lambda^{-|\alpha|,\gamma}(\xi).$$

These L^1 bounds for the derivatives $\partial_\xi^\alpha K^\psi$ enable us to establish the following proposition.

Proposition B.5. *Let ψ be an admissible cut-off function. The mapping*

$$a \longmapsto \sigma_a^\psi(x,\xi,\gamma) := \int_{\mathbb{R}^2} K^\psi(x-y,\xi,\gamma)a(y,\xi,\gamma)\,dy$$

is continuous from Γ_k^m to Σ_k^m for all m.

If $a \in \Gamma_1^m$, then $a - \sigma_a^\psi \in \Gamma_0^{m-1}$. In particular, if ψ_1 and ψ_2 are two admissible cut-off functions and $a \in \Gamma_1^m$, then $\sigma_a^{\psi_1} - \sigma_a^{\psi_2} \in \Sigma_0^{m-1}$.

Fixing an admissible cut-off function ψ, we define the paradifferential operator $T_a^{\psi,\gamma}$ by the formula

$$T_a^{\psi,\gamma} := \mathrm{Op}^\gamma(\sigma_a^\psi).$$

If ψ_1 and ψ_2 are two admissible cut-off functions and $a \in \Gamma_1^m$, then Proposition B.5 and Theorem B.3 show that the family $\{T_a^{\psi_1,\gamma} - T_a^{\psi_2,\gamma}\}$ is of order $\leq (m-1)$.

The symbolic calculus is based on the following theorem.

Theorem B.6.

– *Let $a \in \Gamma_1^m$ and $b \in \Gamma_1^{m'}$. Then $ab \in \Gamma_1^{m+m'}$ and the family*

$$\{T_a^{\psi,\gamma} \circ T_b^{\psi,\gamma} - T_{ab}^{\psi,\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m + m' - 1$ for all admissible cut-off function ψ.

– *Let $a \in \Gamma_1^m$. Then for all admissible cut-off function ψ, the family*

$$\{(T_a^{\psi,\gamma})^* - T_{a^*}^{\psi,\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m - 1$.

– *Let $a \in \Gamma_2^m$ and $b \in \Gamma_2^{m'}$. Then $ab \in \Gamma_2^{m+m'}$ and the family*

$$\{T_a^{\psi,\gamma} \circ T_b^{\psi,\gamma} - T_{ab}^{\psi,\gamma} - T_{-i\sum_j \partial_{\xi_j} a\,\partial_{x_j} b}^{\psi,\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m + m' - 2$ for all admissible cut-off function ψ.

– *Let $a \in \Gamma_2^m$. Then the family*

$$\{(T_a^{\psi,\gamma})^* - T_{a^*}^{\psi,\gamma} - T_{-i\sum_j \partial_{\xi_j}\partial_{x_j} a^*}^{\psi,\gamma}\}_{\gamma \geq 1}$$

is of order $\leq m-2$ for all admissible cut-off function ψ.

The next theorem is the parameter version of Gårding's inequality.

Theorem B.7. *Let $a \in \Gamma_1^{2m}$ and let ψ be and admissible cut-off function. Assume that there exists a constant $c > 0$ such that*

$$\forall\, (x, \xi, \gamma), \quad \Re a(x, \xi, \gamma) \geq c\lambda^{2m,\gamma}(\xi) I.$$

Then there exists $\gamma_0 \geq 1$ such that

$$\forall\, \gamma \geq \gamma_0,\ \forall\, u \in H^m, \quad \Re \langle T_a^{\psi,\gamma} u, u\rangle_{H^{-m},H^m} \geq \frac{c}{2}\|u\|_{m,\gamma}^2.$$

We also have a microlocalized version of Gårding's inequality.

Theorem B.8. *Let $a \in \Gamma_1^{2m}$, $\chi \in \Gamma_1^0$ and ψ be and admissible cut-off function. Assume that there exists $\tilde{\chi} \in \Gamma_1^0$ and a constant $c > 0$ such that $\tilde{\chi} \geq 0$, $\tilde{\chi}\chi = \chi$, and*

$$\forall\, (x, \xi, \gamma), \quad \tilde{\chi}^2(x, \xi, \gamma) \Re a(x, \xi, \gamma) \geq c\tilde{\chi}^2(x, \xi, \gamma)\lambda^{2m,\gamma}(\xi) I.$$

Then there exists $\gamma_0 \geq 1$ and $C > 0$ such that

$$\Re \langle T_a^{\psi,\gamma} T_\chi^{\psi,\gamma} u, T_\chi^{\psi,\gamma} u\rangle_{H^{-m},H^m} \geq \frac{c}{2}\|T_\chi^{\psi,\gamma} u\|_{m,\gamma}^2 - C\|u\|_{m-1,\gamma}^2, \qquad \forall\, \gamma \geq \gamma_0,\ \forall\, u \in H^m.$$

We now study the case of paraproducts: they are defined by the particular choice of ψ_0 as cut-off function. We shall write T_a^γ instead of $T_a^{\psi_0,\gamma}$ for the paradifferential operators obtained after convolution by the function ψ_0. We have the following important result.

Theorem B.9. *Let $a \in W^{1,\infty}(\mathbb{R}^2)$, $u \in L^2(\mathbb{R}^2)$, and $\gamma \geq 1$. Then we have*

$$\|au - T_a^\gamma u\|_0 \leq \frac{C}{\gamma}\|a\|_{W^{1,\infty}(\mathbb{R}^2)}\,\|u\|_0,$$
$$\|a\,\partial_{x_j} u - T_a^\gamma\,\partial_{x_j} u\|_0 \leq C\|a\|_{W^{1,\infty}(\mathbb{R}^2)}\,\|u\|_0,$$

for a suitable constant C that is independent of (a, u, γ).

If $a \in W^{2,\infty}(\mathbb{R}^2)$, we have

$$\|au - T_a^\gamma u\|_{1,\gamma} \leq \frac{C}{\gamma}\|a\|_{W^{2,\infty}(\mathbb{R}^2)}\,\|u\|_0,$$
$$\|a\,\partial_{x_j} u - T_a^\gamma\,\partial_{x_j} u\|_{1,\gamma} \leq C\|a\|_{W^{2,\infty}(\mathbb{R}^2)}\,\|u\|_0,$$

for a suitable constant C that is independent of (a, u, γ).

We can extend the paradifferential calculus to symbols defined on a half-space in the following way: we still denote by Γ_k^m the set of symbols $a(x_0, x_1, x_2, \xi, \gamma)$ defined on $\Omega \times (\mathbb{R}^d \times [0, +\infty[\setminus \{0\})$ such that the mapping $x_2 \mapsto a(\cdot, x_2, \cdot)$ is bounded into Γ_k^m. We define the paradifferential operator T_a^γ by the formula

$$\forall\, u \in C_c^\infty(\bar{\Omega}),\ \forall\, x_2 \geq 0, \quad (T_a^\gamma u)(\cdot, x_2) := T_{a(x_2)}^\gamma u(\cdot, x_2).$$

Using Theorem B.9 and integrating with respect to x_2, we obtain for all symbols $a \in W^{1,\infty}(\Omega)$ and all $u \in L^2(\Omega)$ the estimates:

$$\|\!| au - T_a^\gamma u \|\!|_0 \leq \frac{C}{\gamma} \|a\|_{W^{1,\infty}(\Omega)} \|\!| u \|\!|_0,$$
$$\|\!| a\partial_{x_j} u - T_a^\gamma \partial_{x_j} u \|\!|_0 \leq C \|a\|_{W^{1,\infty}(\Omega)} \|\!| u \|\!|_0, \quad j = 0, 1.$$

REFERENCES

[1] S. ALINHAC, *Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels*, Comm. Partial Differential Equations **14** (1989), 173–230. MR 90h:35147b (French, with English summary)

[2] MIGUEL ARTOLA and ANDREW J. MAJDA, *Nonlinear development of instabilities in supersonic vortex sheets. I. The basic kink modes*, Phys. D **28** (1987), 253–281, http://dx.doi.org/10.1016/0167-2789(87)90019-4. MR 88i:76025

[3] S. BENZONI-GAVAGE, *Stability of multi-dimensional phase transitions in a van der Waals fluid*, Nonlinear Anal. **31** (1998), 243–263. MR 99b:76070

[4] S. BENZONI-GAVAGE and D. SERRE, *First order systems of hyperbolic partial differential equations with applications* (In preparation).

[5] JEAN-MICHEL BONY, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. (4) **14** (1981), 209–246. MR 84h:35177 (French)

[6] JACQUES CHAZARAIN and ALAIN PIRIOU, *Introduction to the theory of linear partial differential equations*, Studies in Mathematics and its Applications, vol. 14, North-Holland Publishing Co., Amsterdam, 1982, ISBN 0-444-86452-0. MR 83j:35001

[7] JEAN-YVES CHEMIN, *Fluides parfaits incompressibles*, Astérisque, 1995. MR 97d:76007 (French, with French summary)

[8] ANDREA CORLI, *Asymptotic analysis of contact discontinuities*, Ann. Mat. Pura Appl. (4) **173** (1997), 163–202. MR 99f:35128

[9] ANDREA CORLI and MONIQUE SABLÉ-TOUGERON, *Stability of contact discontinuities under perturbations of bounded variation*, Rend. Sem. Mat. Univ. Padova **97** (1997), 35–60. MR 98h:35149

[10] JEAN-FRANÇOIS COULOMBEL, *Weak stability of nonuniformly stable multidimensional shocks*, SIAM J. Math. Anal. **34** (2002), 142–172 (electronic), http://dx.doi.org/10.1137/S0036141001392803. MR 2004c:35274

[11] J.-F. COULOMBEL, *Weakly stable multidimensional shocks*, Annales de l'IHP, Analyse Non Linéaire (to appear).

[12] J. A. FEJER and JOHN W. MILES, *On the stability of a plane vortex sheet with respect to three-dimensional disturbances*, J. Fluid Mech. **15** (1963), 335–336. MR 27 #4455

[13] JACQUES FRANCHETEAU and GUY MÉTIVIER, *Existence de chocs faibles pour des systèmes quasi-linéaires hyperboliques multidimensionnels*, Astérisque, 2000. MR 2002c:35183 (French)

[14] DAVID HATEAU, *Instabilité des feuilles de tourbillon avec conduction de chaleur*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 629–633, http://dx.doi.org/10.1016/S0764-4442(00)00230-5. MR 2001b:76039 (French)

[15] REUBEN HERSH, *Mixed problems in several variables*, J. Math. Mech. **12** (1963), 317–334. MR 26 #5304

[16] LARS HÖRMANDER, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 26, Springer-Verlag, Berlin, 1997, ISBN 3-540-62921-1. MR 98e:35103

[17] HEINZ-OTTO KREISS, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math. **23** (1970), 277–298. MR 55 #10862

[18] P. D. LAX, *Hyperbolic systems of conservation laws. II*, Comm. Pure Appl. Math. **10** (1957), 537–566. MR 20 #176

[19] P. D. LAX and R. S. PHILLIPS, *Local boundary conditions for dissipative symmetric linear differential operators*, Comm. Pure Appl. Math. **13** (1960), 427–455. MR 22 #9718

[20] ANDREW MAJDA, The Stability of Multidimensional Shock Fronts, Mem. Amer. Math. Soc., vol. 41, 1983. MR 84e:35100

[21] ———, The Existence of Multidimensional Shock Fronts, Mem. Amer. Math. Soc., vol. 43, 1983. MR 85f:35139

[22] ANDREW J. MAJDA and ANDREA L. BERTOZZI, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, Cambridge, 2002, ISBN 0-521-63057-6, 0-521-63948-4. MR 2003a:76002

[23] ANDREW MAJDA and STANLEY OSHER, *Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math. **28** (1975), 607–675. MR 53 #13857

[24] GUY MÉTIVIER, *The block structure condition for symmetric hyperbolic systems*, Bull. London Math. Soc. **32** (2000), 689–702, http://dx.doi.org/10.1112/S0024609300007517. MR 2001i:35198

[25] ———, *Stability of multidimensional shocks*, Advances in the Theory of Shock Waves, Progr. Nonlinear Differential Equations Appl., vol. 47, Birkhäuser Boston, Boston, MA, 2001, pp. 25–103. MR 2002m:35148

[26] YVES MEYER, *Remarques sur un théorème de J.-M. Bony*, Rend. Circ. Mat. Palermo (2) (1981), 1–20. MR 83b:35169 (French)

[27] JOHN W. MILES, *On the disturbed motion of a plane vortex sheet*, J. Fluid Mech. **4** (1958), 538–552. MR 20 #4394

[28] JAMES V. RALSTON, *Note on a paper of Kreiss*, Comm. Pure Appl. Math. **24** (1971), 759–762. MR 58 #29326

[29] DENIS SERRE, *Systems of conservation laws. 1*, Cambridge University Press, Cambridge, 1999, ISBN 0-521-58233-4. MR 2000g:35142

[30] ———, *Systems of conservation laws. 1*, Cambridge University Press, Cambridge, 1999, ISBN 0-521-58233-4. MR 2000g:35142

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2000 MATHEMATICS SUBJECT CLASSIFICATION: Author: Please supply.
KEY WORDS AND PHRASES: Author: Please supply.
Received: *September 17th, 2003.*