Weak stability of nonuniformly stable multidimensional shocks

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Abstract

The aim of this paper is to investigate the linear stability of multidimensional shock waves that violate the uniform stability condition derived by A. Majda. Two examples of such shock waves are studied: (1) planar Lax shocks in isentropic gas dynamics (2) phase transitions in an isothermal van der Waals fluid. In both cases we prove an energy estimate on the resulting linearized system. Special attention is paid to the losses of derivatives arising from the failure of the uniform stability condition.

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1 Introduction

The stability of multidimensional shock waves in gas dynamics has been an active field of mathematical research since the late 1940's, see e.g. [9, 12, 13, 19, 30]. The first results proved on this subject were giving some necessary conditions of stability by means of a normal modes analysis. In [21], Lax formulated the definition of a shock wave for an arbitrary sytem of conservation laws, also dictated by some kind of "stability"argument.

More precisely, the number of characteristics impinging on the shock front curve is imposed by the size of the system in order to avoid under- (or over-)determinacy of the resulting free boundary problem. As regards ideal gas dynamics, this definition is known to be equivalent to the requirement that the physical entropy increases upon crossing the shock front curve, see [9].

Using the extensive study of mixed initial boundary value problems for hyperbolic systems (see e.g. [16, 17, 20]), Majda succeeded in the early 1980's in deriving a necessary and sufficient strong stability condition for multidimensional shock waves [24]. The resulting estimates on the linearized problem enabled him to prove a nonlinear existence theorem [23]. We also refer to [25, 34] for a general overview of the method and its application to isentropic gas dynamics. It is worth noting that a different approach developed at the same time by Blokhin [5, 6] gave rise to similar results. However Majda's approach, which has been slightly improved in [26, 28] by using the new ideas of paradifferential calculus introduced by Bony, seems appropriate to our purpose and we shall adopt it for our analysis.

In the study of initial boundary value problems for hyperbolic systems, many physically relevant boundary data are found to violate the uniform stability condition, namely the so-called Kreiss-Lopatinskii condition. Nevertheless many authors have overcome this difficulty in various cases by using particular properties of the involved system, see e.g. [2, 10, 15, 31, 32]. Although Majda's result has the great advantage of dealing with any system of conservation laws, examples of multidimensional shocks are not that numerous and the verification of the uniform stability condition often gives rise to very tedious computations. However such verification can be carried out for the system of gas dynamics. Two cases of non uniformly stable shocks arise and motivate the present study. The first example, which is briefly addressed in [24], is the one of planar Lax shocks in isentropic gas dynamics that violate Majda's inequality (see [24], page 10). This inequality is recalled in section 2. The second example comes from the theory of phase transitions in isothermal van der Waals fluids. These planar discontinuities are undercompressive hocks. They require an additional jump relation to select the relevant ones. Various admissibility criteria have been proposed over the last two decades, see [36] for phase transitions in the context of gas dynamics or [35, 37] and references therein for phase transitions in the context of elastodynamics. We base our analysis on the viscosity-capillarity criterion proposed in [36] under the assumption that the viscosity coefficient is neglected and taken to be zero. In other words, the additional jump relation is written as a generalized equal area rule. It has been shown in [3] that the uniform stability condition is violated because of surface waves (taking viscosity into account would yield uniform stability, see [4]). It is worth noting that the failure of the uniform stability condition in isentropic gas dynamics can only rise from the appearance of boundary waves (but we shall get back to this in the next sections); for a precise statement of the distinction between these two types of waves, we refer the reader to the very nice survey [11].

The purpose of the paper is the derivation of a complete energy estimate on the linearized system resulting from the study of these two problems. Since the *classical* energy estimate is known to be equivalent to the uniform stability condition, as proved in [24], losses of derivatives are to be expected. As shown in theorems 1 and 2 and this is no real surprise, losses of derivatives are more severe when boundary waves occur than when surface waves occur. We point out that this kind of phenomenon had already been mentioned in previous works [11, 31]. Despite the impossibility of using some "dissipativeness"arguments on the boundary conditions in our context, we shall see that the derivation of an energy estimate can be carried out by a suitable modification in the ordinary construction of a Kreiss' symmetrizer. This point will be emphasized in both problems we shall detail.

This paper is divided as follows. In section 2, we recall Majda's method for multidimensional shock waves and introduce some notations. Note that Lax shocks for isentropic Euler equations are uniformly stable in one space dimension and we shall therefore deal with two or three dimensional problems (the one dimensional case is treated in [22]). We warn the reader that many calculations can not be reproduced here to avoid overloading the paper and we shall often refer to previous works on this subject where some details are available. However, special attention will be paid to detail the normal modes analysis on which relies the entire construction of the symbolic symmetrizer. In section 3, we treat the first example, i.e. non uniformly stable Lax shocks for isentropic Euler equations. We show in section 4 how the method developed in section 3 applies in the study of phase transitions in a van der Waals fluid and even gives slightly better results. Once again, we shall focus on two or three dimensional problems since phase transitions are known to be uniformly stable in one space dimension and their existence has already been studied in [14]. Section 5 is devoted to the proof of several technical lemmas used in the construction of Kreiss' symmetrizers. Eventually, we make in section 6 some general remarks on the possible advances for these two problems.

2 General considerations

We study the Euler equations governing the motion of an inviscid isentropic fluid in $\mathbb{R}^d$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \,, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0 \,. \end{cases} \tag{1}$$

We have adopted the following standard notations, that will be used throughout this paper: ρ denotes the density, $\mathbf{u}$ the velocity field, c the sound speed given by the pressure law $p(\rho)$ that the fluid is assumed to obey

$$c(\rho) = \sqrt{p'(\rho)} \,.$$

Since smooth solutions generally develop singularities in finite time, we look for particular weak solutions of the form of functions which are smooth on both sides of a (variable) hypersurface of $\mathbb{R}^d$. A first step in the proof of the existence of such solutions is the study of the linear stability of piecewise constant solutions defined by a relation of the form

$$\overline{U} = \begin{cases} U_l = (\rho_l, \mathbf{u}_l) & \text{if } x \cdot \nu < \sigma t, \\ U_r = (\rho_r, \mathbf{u}_r) & \text{if } x \cdot \nu > \sigma t. \end{cases}$$

Such a function $\overline{U}$ is a weak solution of the Euler equations (1) if and only if it satisfies the Rankine-Hugoniot jump relations which can be written in the following way

$$\begin{cases} \rho_r (\mathbf{u}_r \cdot \nu - \sigma) = \rho_l (\mathbf{u}_l \cdot \nu - \sigma) =: j \,, \\ j[\mathbf{u}] + [p]\nu = 0 \,. \end{cases} \tag{2}$$

We consider dynamical discontinuities and thus assume that the mass transfer j across the hyperplane $\{x \cdot \nu = \sigma t\}$ is not zero. By symmetry arguments, one can therefore assume $j > 0$. We first assume that $\overline{U}$ defines a compressive 1-Lax shock or in other words that the following inequalities hold:

$$M_r = \frac{\mathbf{u}_r \cdot \nu - \sigma}{c(\rho_r)} < 1 \quad , \quad M_l = \frac{\mathbf{u}_l \cdot \nu - \sigma}{c(\rho_l)} > 1 \quad \text{and} \quad \rho_r > \rho_l \, .$$

Note that the above assumptions immediatly imply that the shock is noncharacteristic: the propagation speed of the interface σ is different from the characteristic speeds of system (1) on both sides of the interface. With the above notations, we have the following statement

Proposition 1 (Majda). [24]. *The shock $\overline{U}$ is uniformly stable if and only if*

$$M_r^2 \left(\frac{\rho_r}{\rho_l} - 1 \right) < 1 \, . \tag{3}$$

If inequality (3) *does not hold, then the shock $\overline{U}$ is only weakly stable.*

Inequality (3) holds as long as p is a convex function of the density ρ which is the case for the classical gamma-law but not for more complicated laws (like for instance an isothermal van der Waals pressure law). We shall investigate in section 3 the case where the opposite **strict** inequality holds. We shall also detail why the equality case can not be treated by the techniques used in this paper.

If we now assume that p is a nonmonotone function of ρ (this hypothesis can be viewed as a model of isothermal liquid-vapor phase transitions, see [18]), it is known that subsonic discontinuities can appear, for which we have

$$M_r = \frac{\mathbf{u}_r \cdot \nu - \sigma}{c(\rho_r)} < 1 \quad , \quad M_l = \frac{\mathbf{u}_l \cdot \nu - \sigma}{c(\rho_l)} < 1 \quad \text{and} \quad \rho_r > \rho_l \, .$$

Such inequalities occur if p is for instance given by an isothermal van der Waals pressure law with a temperature below the so-called critical temperature (see [3, 4, 36]). To avoid the natural instability of $\overline{U}$ with respect to small perturbations, one needs to specify an additional jump relation to the Rankine-Hugoniot conditions. The analysis developed in section 4 is based on the capillarity criterion proposed in [36] (the admissibility criterion proposed in [37] is the analogue for elastodynamics, the main idea governing both criteria is that there is no entropy dissipation upon crossing the shock). Together with the Rankine-Hugoniot conditions, this criterion requires that $\overline{U}$ satisfies the generalized equal area rule

$$\int_{v_r}^{v_l} p(v) \, dv \; = \; (v_l - v_r) \, \frac{p(v_r) + p(v_l)}{2} \, , \tag{4}$$

where $v = 1/\rho$ is the specific volume of the fluid. Such phase transitions which differ from Maxwell equilibirum states are noncharacteristic.

We are now able to develop Majda's method to study the linear stability of such multidimensional shocks. Note first that by a change of observer, one can always assume that the unit vector ν is the last vector of the canonical basis of $\mathbb{R}^d$. Since the mass transfer

j is not zero, equations (2) show that the tangential components of the velocity are the same on both sides of the shock front curve. Performing another change of observer one can assume from now on

$$(\mathbf{u}_1^r, \dots, \mathbf{u}_{d-1}^r) = (\mathbf{u}_1^l, \dots, \mathbf{u}_{d-1}^l) = \mathbf{0} \quad \text{and} \quad \sigma = 0\,,$$

which is of no consequence on the stability of the particular solution $\overline{U}$. Note that these operations yield a simplified expression of the mass transfer j across the interface (defined by system (2)): $j = \rho_r u_r = \rho_l u_l$.

We adopt in all the sequel the following notations: all space vectors x in $\mathbb{R}^d$ are decomposed as $x = (y, x_d)$ where y is a vector in $\mathbb{R}^{d-1}$ and x_d is a scalar. Similarly, all velocity vectors $\mathbf{u}$ are decomposed as $\mathbf{u} = (\check{u}, u)$ where $\check{u} \in \mathbb{R}^{d-1}$ is the tangential part of the velocity and $u \in \mathbb{R}$ is the normal velocity.

We are now led to search a weak solution U of (1) defining a compressive 1-Lax shock (or an admissible phase transition) across a smooth hypersurface $\Sigma(t) = \{x_d = \varphi(t, y)\}$ close to the hyperplane $\{x_d = 0\}$. Since $\Sigma(t)$ is part of the unknowns of the problem, one first fixes the front by the well-known transformation in free boundary problems:

$$\big(U:\ (t, y, x_d) \longrightarrow \mathbb{R}^N\ \big) \longrightarrow (U_\pm:\ (t, y, z) \longmapsto U(t, y, \varphi(t, y) \pm z)\ )\ ,$$

both applications $U_+ = (\rho_+, \mathbf{u}_+)$ and $U_- = (\rho_-, \mathbf{u}_-)$ being defined on the same half-space $\{z > 0\}$. The quasilinear form of Euler equations is linearized on both sides of $\Sigma(t)$ around the piecewise constant solution $\overline{U}$ (see [24, 34]). The resulting linear system reads

$$\begin{cases} \partial_t U_+ + \displaystyle\sum_{j=1}^{d-1} A_j(U_r)\, \partial_{x_j} U_+ + A_d(U_r)\, \partial_z U_+ = f_+\,, \\ \partial_t U_- + \displaystyle\sum_{j=1}^{d-1} A_j(U_l)\, \partial_{x_j} U_- - A_d(U_l)\, \partial_z U_- = f_-\,, \end{cases} \tag{5}$$

where $A_j(U_{r,l})$ are $(d+1) \times (d+1)$ matrices corresponding to the quasilinear form of isentropic Euler equations, see [8, 9, 33].

The linearization of the jump conditions across the interface $\Sigma(t)$ yields the boundary conditions on $\{z = 0\}$. When one deals with a compressive Lax shock, the jump conditions are nothing but the Rankine-Hugoniot relations, and their linearized form read

$$\begin{aligned} u_r\rho_+ + \rho_r u_+ - u_l\rho_- - \rho_l u_- - [\rho]\partial_t\varphi &= g_1\,, \\ \rho_r u_r \check{u}_+ - \rho_l u_l \check{u}_- - [p]\nabla_y\varphi &= \check{g}\,, \\ (u_r^2 + c_r^2)\rho_+ + 2\rho_r u_r u_+ - (u_l^2 + c_l^2)\rho_- - 2\rho_l u_l u_- &= g_{d+1}\,. \end{aligned} \tag{6}$$

When one deals with a subsonic phase transition in a van der Waals fluid, the complete boundary conditions for the linearized problem are obtained by linearizing equation (4) and adding this new relation to the linearized Rankine-Hugoniot relations (6). The complete set of boundary conditions reads in this case

$$\begin{aligned} u_r\rho_+ + \rho_r u_+ - u_l\rho_- - \rho_l u_- - [\rho]\partial_t\varphi &= g_1\,, \\ \rho_r u_r \check{u}_+ - \rho_l u_l \check{u}_- - [p]\nabla_y\varphi &= \check{g}\,, \\ (u_r^2 + c_r^2)\rho_+ + 2\rho_r u_r u_+ - (u_l^2 + c_l^2)\rho_- - 2\rho_l u_l u_- &= g_{d+1}\,, \\ c_r^2\frac{\rho_+}{\rho_r} + u_r u_+ - c_l^2\frac{\rho_-}{\rho_l} - u_l u_- - [u]\partial_t\varphi &= g_{d+2}\,. \end{aligned} \tag{7}$$

It is now clear that, even though both examples rise from two different research areas, they are exactly of the same kind. In both cases, we are led to study a non standard mixed initial boundary value problem

$$\begin{cases} \partial_t U + \sum_{j=1}^{d-1} \mathcal{A}_j \, \partial_{x_j} U + \mathcal{A}_d \, \partial_z U = f & \text{for } z > 0, \\ \partial_t \varphi \, b_0 + \sum_{j=1}^{d-1} \partial_{x_j} \varphi \, b_j + M \, U = g & \text{for } z = 0. \end{cases} \tag{8}$$

The boundary conditions for the study of compressive Lax shocks are given by (6) and the boundary conditions for the study of subsonic phase transitions are given by (7). To write system (8), we have let

$$U = \begin{pmatrix} U_+ \\ U_- \end{pmatrix}, \quad f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ \check{g} \\ g_{d+1} \end{pmatrix} \text{ or } g = \begin{pmatrix} g_1 \\ \check{g} \\ g_{d+1} \\ g_{d+2} \end{pmatrix}$$

$$\mathcal{A}_j = \begin{pmatrix} A_j(U_r) & \mathbf{0} \\ \mathbf{0} & A_j(U_l) \end{pmatrix} \text{ for } 1 \leq j \leq d-1, \quad \mathcal{A}_d = \begin{pmatrix} A_d(U_r) & \mathbf{0} \\ \mathbf{0} & -A_d(U_l) \end{pmatrix}.$$

In both examples, M represents the matrix of the linearized jump conditions (Rankine-Hugoniot relations and the generalized equal area rule in the case of phase transitions). The vectors $b_0, \dots, b_{d-1}$ come from equations (6) and (7). They belong to $\mathbb{R}^{d+1}$ in the study of Lax shocks, while they belong to $\mathbb{R}^{d+2}$ in the study of subsonic phase transitions.

The derivation of an energy estimate for system (8) relies on the introduction of a positive weight γ (see [20, 24]). More precisely, we perform a change of unknown functions

$$v(t, y, z) = e^{-\gamma t} \, U(t, y, z) \quad \text{and} \quad \psi(t, y) = e^{-\gamma t} \, \varphi(t, y),$$

where γ is a nonnegative parameter. We now perform a Fourier transform in the variables t and y (the corresponding dual variables will be respectively denoted δ and η). These operations yield the system of ordinary differential equations

$$\begin{cases} \dfrac{dV}{dz} = \mathcal{A}(\delta, \eta, \gamma) \, V(z) + F & \text{for } z > 0, \\ \chi \, b(\delta, \eta, \gamma) + M \, V(0) = G & \text{for } z = 0, \end{cases} \tag{9}$$

with

$$\mathcal{A}(\delta, \eta, \gamma) = -\mathcal{A}_d^{-1} \left(\tau + i \sum_{j=1}^{d-1} \eta_j \mathcal{A}_j \right) \quad \text{and} \quad b(\delta, \eta, \gamma) = \tau b_0 + i \sum_{j=1}^{d-1} \eta_j b_j \, .$$

For convenience we have let $\tau = \gamma + i\delta$. Note that inverting $\mathcal{A}_d$ is legitimate since the shock is in both examples noncharacteristic. We now turn to the description of the method: in both examples, we show that the boundary conditions in problem (9) can be rewritten so that χ appears only in the last scalar boundary condition. The remaining part of the work consists in deriving an a priori estimate on the resulting initial boundary value problem for U where the boundary conditions take the form of a pseudodifferential operator.

Because of the decoupled nature of system (5), it is clear that matrix $\mathcal{A}(\delta, \eta, \gamma)$ has a block diagonal structure: its first block corresponds to the linearized system ahead of the shock and its second block to the linearized system before the shock (see [24, 25, 34]). The eigenmodes of the first block are $\omega_2^r = -\tau/u_r$ and the roots of the second order polynomial equation

$$(\tau + u_r\,\omega)^2 = c_r^2(\omega^2 - |\eta|^2)\,. \tag{10}$$

In a similar way, the eigenmodes of the second block are $\omega_2^l = \tau/u_l$ and the roots of the second order polynomial equation

$$(\tau - u_l\,\omega)^2 = c_l^2(\omega^2 - |\eta|^2)\,. \tag{11}$$

We briefly analyse the eigenmodes of $\mathcal{A}$ and begin with the eigenmodes of the first block. In both problems analysed in section 3 and section 4, the shock $\overline{U}$ is subsonic with respect to the right state ($M_r < 1$). It is clear that ω_2^r is of negative real part when τ has positive real part (that is when γ is positive). Moreover equation (10) has one root ω_3^r of negative real part when τ has positive real. The other root of (10) is denoted ω_1^r and has positive real part when τ has positive real part. The parametrization of the corresponding eigenspaces, which we use in sections 3 and 4, can be found in [3, 34]. One crucial property of the eigenmodes $\omega_{1,3}^r$ is that they can be extended up to imaginary values of τ. Note that ω_3^r has negative real part if $|\tau| < |\eta|\sqrt{c_r^2 - u_r^2}$ and is purely imaginary if $|\tau| \geq |\eta|\sqrt{c_r^2 - u_r^2}$.

In the case of a compressive 1-Lax shock, that is when the shock is supersonic with respect to the left state, then the second dynamical system does not give any contribution to the stable subspace $\mathcal{E}^-$ of $\mathcal{A}$. Indeed ω_2^l is of positive real part when τ has positive real part. Furthermore equation (11) has two roots ω_1^l and ω_3^l of positive real part when τ has positive real part. One easily checks that the continuous extension of ω_1^l and ω_3^l for purely imaginary values of τ are always distinct.

In the case of a subsonic phase transition, equation (11) has the same behavior as equation (10).More precisely, equation (11) has exactly one root ω_1^l of negative real part when τ has positive real part. The other root of (11) is denoted ω_3^l. It has positive real part when τ has positive real part. When τ is a purely imaginary number, ω_1^l has negative real part if $|\tau| < |\eta|\sqrt{c_l^2 - u_l^2}$ and is purely imaginary if $|\tau| \geq |\eta|\sqrt{c_l^2 - u_l^2}$.

3 Non uniformly stable shocks in gas dynamics

We begin by describing the failure of the uniform stability condition for compressive Lax shocks in isentropic gas dynamics. Let $\overline{U}$ define a compressive 1-Lax shock for isentropic Euler equations (1) as described in the previous section. We study the non standard initial boundary value problem (8) with boundary conditions given by (6). We assume that $\overline{U}$ violates Majda's inequality (3) in the following way

$$M_r^2\left(\frac{\rho_r}{\rho_l} - 1\right) > 1\,.$$

Note that the previous simplifications imply that this inequality is equivalent to

$$u_r u_l > c_r^2 + u_r^2\,. \tag{12}$$

This remark will be useful to complete the proof of lemma 3. Under the assumptions made on $\overline{U}$, the normal modes analysis of problem (9) is summarized in the following result:

Lemma 1. *There exists a positive number V_1 such that for all $(\delta, \eta, \gamma) \in \mathbb{R}^{d+1}$ satisfying $\gamma \geq 0$ and $(\delta, \gamma) \neq (\pm i V_1 |\eta|, 0)$, one has*

$$\big\{(Z, \chi) \in \mathcal{E}^-(\delta, \eta, \gamma) \times \mathbb{C} \ \text{ s.t. } \ \chi\, b(\delta, \eta, \gamma) + M\, Z = 0\big\} = \{0\}\,,$$

and for $\eta \neq 0$, the set

$$\big\{(Z, \chi) \in \mathcal{E}^-(\pm V_1 |\eta|, \eta, 0) \times \mathbb{C} \ \text{ s.t. } \ \chi\, b(\pm V_1 |\eta|, \eta, 0) + M\, Z = 0\big\}$$

is a one dimensional subspace of $\mathbb{C}^{2d+3}$.

By definition, V_1^2 is the smallest root of the polynomial

$$P_1(X) = (c_r^2 - u_r^2)(X^2 + u_r^2 u_l^2) + \left[4 u_r^2 c_r^2 - 2 u_r u_l (c_r^2 + u_r^2)\right] X\,,$$

which has two real positive roots under assumption (12) *(the greatest is denoted V_2^2). Furthermore we have*

$$c_r^2 - u_r^2 < V_1^2 < u_r u_l \frac{c_r^2 - u_r^2}{c_r^2 + u_r^2} < V_2^2\,.$$

Proof. This is a basic extension of the calculations already done in [24] (which can also be found in [34]). First of all, we note that the stable subspace of the dynamical system

$$\frac{dV}{dz} = \mathcal{A}(\delta, \eta, \gamma)\, V$$

consists of all vectors $Z = (Z_r, Z_l)$ such that

$$(u_r \tau - (c_r^2 - u_r^2)\omega_3^r, \rho_r u_r i \eta^T, -\rho_r \tau) \cdot Z_r = 0 \quad \text{and} \quad Z_l = 0\,.$$

With this parametrization of the stable subspace, one easily computes the Lopatinskii determinant associated to (9):

$$\Delta(\delta, \eta, \gamma) = \rho_r^d u_r^{d-1} \left[(c_r^2 - u_r^2)[p] |\eta|^2 + (c_r^2 + u_r^2)[\rho] \tau^2 + 2 u_r [\rho] \tau a_3^r\right]\,,$$

where we have let $a_3^r = u_r \tau - (c_r^2 - u_r^2)\omega_3^r$.It is clear that $\Delta(\delta, 0, \gamma)$ does not vanish for any $(\delta, \gamma) \neq (0, 0)$. One can therefore factor the expression of $\Delta(\delta, \eta, \gamma)$ by $|\eta|^2$ and use the reduced variables

$$V = \frac{\tau}{i|\eta|} \quad , \quad A_3^r = \frac{a_3^r}{i|\eta|}\,.$$

Some simplifications using the Rankine-Hugoniot relations lead to the expression

$$\Delta(\delta, \eta, \gamma) = \rho_r^d u_r^{d-1} |\eta|^2 [\rho] \left[(c_r^2 - u_r^2) u_r u_l - (c_r^2 + u_r^2) V^2 - 2 u_r V A_3^r\right]\,.$$

Let $\mathcal{R}$ denote the complex square root mapping defined by

$$\begin{aligned} \mathcal{R}\ : \mathbb{C} \setminus \mathbb{R}_+ &\longrightarrow \{\zeta \in \mathbb{C} \ \text{s.t. Im}\ \zeta > 0\} \\ w &\longmapsto \mathcal{R}(w) \quad \text{with} \quad \mathcal{R}(w)^2 = w\,. \end{aligned}$$

Then analysing equation (10) shows that for $\gamma > 0$ (or equivalently for V of negative imaginary part) we have

$$A_3^r = -c_r \mathcal{R}(V^2 - (c_r^2 - u_r^2))$$

and therefore if the Lopatinskii determinant vanishes at some point (τ, η), V^2 has to be a root of the polynomial P_1 defined in the lemma. Note that the assumption (12) made on the shock $\overline{U}$ implies that P_1 has two distinct positive roots V_1^2 and V_2^2 that satisfy the properties given in the lemma. This already proves that the possible zeroes of $\Delta(\delta, \eta, \gamma)$ have to satisfy

$$\eta \neq 0 \, , \; \gamma = 0 \; \text{ and } \; \delta^2 > (c_r^2 - u_r^2)|\eta|^2 \, ,$$

and those requirements imply that V is a real number such that $V^2 > c_r^2 - u_r^2$. One has therefore to extend the previous definition of A_3^r to such values of V. This is achieved by using the Cauchy-Riemann relations on holomorphic functions (see [3, 34] for the details):

$$\begin{cases} A_3^r = c_r \sqrt{V^2 - (c_r^2 - u_r^2)} & \text{if } V > \sqrt{c_r^2 - u_r^2}, \\ A_3^r = -c_r \sqrt{V^2 - (c_r^2 - u_r^2)} & \text{if } V < -\sqrt{c_r^2 - u_r^2}. \end{cases}$$

Furthermore, the previous analysis shows that $\Delta(\delta, \eta, 0)$ vanishes if and only if

$$\begin{cases} 2u_r c_r V \sqrt{V^2 - (c_r^2 - u_r^2)} = -(c_r^2 + u_r^2)V^2 + u_r u_l (c_r^2 - u_r^2) & \text{if } V > \sqrt{c_r^2 - u_r^2}, \\ 2u_r c_r V \sqrt{V^2 - (c_r^2 - u_r^2)} = (c_r^2 + u_r^2)V^2 - u_r u_l (c_r^2 - u_r^2) & \text{if } V < -\sqrt{c_r^2 - u_r^2}, \end{cases}$$

and these relations imply $P_1(V^2) = 0$.

If the Lopatinskii determinant vanishes at $V = V_2$, then we must have

$$2u_r c_r V_2 \sqrt{V_2^2 - (c_r^2 - u_r^2)} = -(c_r^2 + u_r^2)V_2^2 + u_r u_l (c_r^2 - u_r^2) \, .$$

But the left-hand term of the equality is positive and the right-hand term is negative. Therefore the Lopatinskii determinant can not vanish at $V = V_2$ (and neither at $V = -V_2$ by a similar argument). Since $P_1(V_1^2) = 0$ we have

$$2u_r c_r V_1 \sqrt{V_1^2 - (c_r^2 - u_r^2)} = -(c_r^2 + u_r^2)V_1^2 + u_r u_l (c_r^2 - u_r^2) \, ,$$

because both terms in the equality are positive. Therefore the Lopatinskii determinant vanishes at $V = V_1$ (and similarly at $V = -V_1$).This completes the proof of the existence and the characterization of points where the uniform stability condition fails. The last assertion on the dimension of the correponding kernel follows directly from the shape of the boundary conditions (6). □

Note that in the special case $u_r u_l = c_r^2 + u_r^2$, then $P_1(c_r^2 - u_r^2) = 0$. In other words, the uniform stability condition fails exactly at the points where equation (10) has a double root. At such points, the symbol $\mathcal{A}$ is not diagonalizable and a 2×2 Jordan block arises in the reduction of $\mathcal{A}$ which is used to construct a Kreiss' symmetrizer (see the proof of proposition 2). At the present time, we have not been able to overcome this difficulty. This case is left to a future work.

3.1 Elimination of the front

The first step in the derivation of an energy estimate for the mixed problem (8) is to work in the Fourier space and to isolate the front χ in the last boundary condition for problem (9). This operation can be summarized in the following terms:

Lemma 2. *There exists a C^∞ mapping Q defined on the half-space $\mathbb{R}^d \times \mathbb{R}^+ \setminus \{0\}$, homogeneous of degree* 0 *with values in the set of square $(d+1)\times(d+1)$ invertible matrices such that for all $X \in \mathbb{R}^d \times \mathbb{R}^+ \setminus \{0\}$ the first d components of the vector $Q(X)\, b(X)$ vanish.*

Proof. The Rankine-Hugoniot jump relations together with (6) yield the relations

$$b(\delta,\eta,\gamma) = \begin{pmatrix} -\tau[\rho] \\ -iu_r u_l[\rho]\eta \\ 0 \end{pmatrix} \text{ if } d=2 \quad \text{and} \quad b(\delta,\eta,\gamma) = \begin{pmatrix} -[\rho]\tau \\ -iu_r u_l[\rho]\eta_1 \\ -iu_r u_l[\rho]\eta_2 \\ 0 \end{pmatrix} \text{ if } d=3.$$

To preserve the homogeneity of the physical quantities we handle in the calculations, we fix a reference velocity $\tilde{V}$ and a reference frequency $\tilde{\gamma}$ and we define Σ_+ as the hemisphere

$$\Sigma_+ = \{(\delta,\eta,\gamma) \in \mathbb{R}^d \times \mathbb{R}_+ \text{ s.t. } \gamma^2 + \delta^2 + \tilde{V}^2|\eta|^2 = \tilde{\gamma}^2\}\,.$$

We first define the mapping Q on the hemisphere Σ_+ and then extend it as a homogeneous mapping of degree 0. One easily checks that for $d=2$, the matrix

$$Q(\delta,\eta,\gamma) = \begin{pmatrix} 0 & 0 & 1 \\ iu_r u_l\eta & -\tau & 0 \\ u_r u_l\overline{\tau} & -i\tilde{V}^2\eta & 0 \end{pmatrix}$$

satisfies all required properties. For $d=3$, one can choose for instance

$$Q(\delta,\eta,\gamma) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ iu_r u_l\eta_1 & -\tau & 0 & 0 \\ iu_r u_l\eta_2 & 0 & -\tau & 0 \\ u_r u_l\overline{\tau} & -i\tilde{V}^2\eta_1 & -i\tilde{V}^2\eta_2 & 0 \end{pmatrix}$$

which also satisfies all required properties. This completes the proof. □

We can therefore write boundary conditions for the linearized problem (8) in the equivalent way

$$\begin{pmatrix} B(\delta,\eta,\gamma) \\ \ell(\delta,\eta,\gamma) \end{pmatrix} V(0) + \chi \begin{pmatrix} \mathbf{0}_d \\ \alpha(\delta,\eta,\gamma) \end{pmatrix} = Q(\delta,\eta,\gamma)\, G\,,$$

where $\alpha(\delta,\eta,\gamma)$ is given by

$$\alpha(\delta,\eta,\gamma) = -u_r u_l[\rho]\tilde{\gamma}\sqrt{\gamma^2+\delta^2+\tilde{V}^2|\eta|^2} \neq 0\,,$$

and this relation holds for $d=2$ and $d=3$.

Lemma 1 ensures that the restriction of $B(\delta,\eta,\gamma)$ to the stable subspace $\mathcal{E}^-(\delta,\eta,\gamma)$ is invertible except at the points where the uniform stability condition fails. We thus have to study the behavior of the restriction of $B(\delta,\eta,\gamma)$ to the stable subspace $\mathcal{E}^-$ in the neighbourhood of those points. Next lemma asserts that the Lopatinskii determinant vanishes *at order* 1 or in other words that the roots exhibited in lemma 1 are simple.

For all vector Z belonging to the stable subspace $\mathcal{E}^-$ we denote by Z_3^r and Z_2^r the components of Z on the eigenspaces associated to the eigenmodes ω_3^r and ω_2^r. Then we have the following microlocal estimate:

Lemma 3. *There exists a neighbourhood $\mathcal{V}$ of $(V_1|\eta|,\eta,0)$ in Σ_+ and a constant $c>0$ such that for all $X\in\mathcal{V}$ and for all $Z\in\mathcal{E}^-(X)$, one has*

$$|B(X)\,Z|^2 \geq c\,\gamma^2\left(|Z_3^r|^2+|Z_2^r|^2\right)\,.$$

An analogous estimate holds in a neighbourhood of points $(-V_1|\eta|,\eta,0)$.

Proof. According to lemma 1 we know that the kernel of the restriction of B to the stable subspace $\mathcal{E}^-$ at the point $(V_1|\eta|,\eta,0)$ is a one dimensional space. Therefore in order to proof lemma 3, we only need to show that 0 is a *simple root* of the determinant of the restriction of B to $\mathcal{E}^-$ or more precisely that the partial derivative of this determinant with respect to γ calculated at $\gamma=0$ is not zero.

We deal first with the case $d=2$ and we keep the notation a_3^r introduced in the proof of lemma 1. After a few simplifications, for $Z\in\mathcal{E}^-$, we get

$$B(\delta,\eta,\gamma)\,Z=\begin{pmatrix}\rho_r(c_r^2\tau+u_ra_3^r) & 2ij\eta\\[2ex] \dfrac{ij\eta\tilde{\gamma}(c_r^2\tau+u_la_3^r)}{\sqrt{\gamma^2+\delta^2+\tilde{V}^2\eta^2}} & \dfrac{-\rho_r\tilde{\gamma}(\tau^2+u_ru_l\eta^2)}{\sqrt{\gamma^2+\delta^2+\tilde{V}^2\eta^2}}\end{pmatrix}\begin{pmatrix}Z_3^r\\ Z_2^r\end{pmatrix}.$$

Note that this expression involves $\tilde{\gamma}$ and some square roots because of the homogeneity property of the mapping Q. The determinant of the restriction of B to the stable subspace $\mathcal{E}^-$ is therefore given by

$$\det B^-=\frac{i\tilde{\gamma}\rho_r^2|\eta|^3}{\sqrt{\gamma^2+\delta^2+\tilde{V}^2\eta^2}}\underbrace{\left[c_r^2V(V^2+2u_r^2-u_ru_l)+u_rA_3^r(V^2+u_ru_l)\right]}_{f(V)},$$

where V and A_3^r denote the same reduced quantities as those defined in the proof of lemma 1. One can check that $f(V)$ vanishes at the points where the uniform stability condition fails (thanks to the expression of the Lopatinskii determinant). The final step consists in calculating the partial derivative of $\det B^-$ with respect to γ at $\gamma=0$. Proving that this derivative is not zero is equivalent to proving that the derivative (with respect to V) of the function $f(V)$ calculated at $V=\pm V_1$ is not zero. We have

$$f'(V)=c_r(3V^2+2u_r^2-u_ru_l)+\frac{c_ru_rV\left[3V^2+u_ru_l-2(c_r^2-u_r^2)\right]}{u_rVA_3^r},$$

and thus, using the expression of VA_3^r at $(V_1|\eta|,\eta,0)$, we find the expression

$$f'(V_1)=c_r^2(3V_1^2+2u_r^2-u_ru_l)-\frac{2c_r^2u_r^2V_1^2\left[3V_1^2+u_ru_l-2(c_r^2-u_r^2)\right]}{(c_r^2+u_r^2)V_1^2-(c_r^2-u_r^2)u_ru_l}\,.$$

Eventually $f'(V_1) = 0$ if and only if V_1^2 is a root of the polynomial

$$Q_1(X) = 3(c_r^2 - u_r^2)X^2 + 2\left[u_r^2(3c_r^2 - u_r^2) - 2u_r u_l c_r^2\right] X + u_r u_l (c_r^2 - u_r^2)(u_r u_l - 2u_r^2)\,.$$

Assume that $Q_1(V_1^2) = 0$. Since V_1^2 is also a root of the polynomial P_1 defined in lemma 1, we get the relation

$$\left[u_r u_l (c_r^2 + 3u_r^2) - u_r^2(3c_r^2 + u_r^2)\right] V_1^2 - u_r u_l (c_r^2 - u_r^2)(u_r u_l + u_r^2) = 0\,,$$

and one easily checks that the previous term between brackets is positive since $u_r u_l > c_r^2 + u_r^2$. Plugging this explicit expression of V_1^2 into the definition of P_1 yields the relation

$$(1 - M_r^2)S^3 + (2M_r^4 + 3M_r^2 - 1)S^2 - M_r^2(M_r^4 + 5M_r^2 + 2)S + M_r^4(3 + M_r^2) = 0\,,$$

where we have let $S = u_r u_l / c_r^2$. One easily checks that this polynomial (in S) vanishes for $S = 1$ and this value can not be reached by S since $S > 1 + M_r^2$. We thus obtain the relation

$$(1 - M_r^2)S^2 + 2M_r^2(1 + M_r^2)S - M_r^4(3 + M_r^2) = 0\,.$$

But the value of this polynomial is greater than 1 for $S = 1 + M_r^2$ so S is always larger than the greatest root of this last polynomial. We are thus led to a contradiction. Therefore V_1^2 can not be a root of the polynomial Q_1 which means exactly that $f'(V_1) \neq 0$.

If $d = 3$ and Z is a vector in the stable subspace $\mathcal{E}^-$, we have the relation

$$B(\delta, \eta, \gamma)\, Z = \begin{pmatrix} \rho_r(c_r^2\tau + u_r a_3^r) & 2ij\eta_1 & 2ij\eta_2 \\ \dfrac{ij\eta_1\tilde{\gamma}(c_r^2\tau + u_l a_3^r)}{\sqrt{\gamma^2 + \delta^2 + \tilde{V}^2|\eta|^2}} & \dfrac{-\rho_r\tilde{\gamma}(\tau^2 + u_r u_l \eta_1^2)}{\sqrt{\gamma^2 + \delta^2 + \tilde{V}^2|\eta|^2}} & \dfrac{-\rho_r\tilde{\gamma} u_r u_l \eta_1\eta_2}{\sqrt{\gamma^2 + \delta^2 + \tilde{V}^2|\eta|^2}} \\ \dfrac{ij\eta_2\tilde{\gamma}(c_r^2\tau + u_l a_3^r)}{\sqrt{\gamma^2 + \delta^2 + \tilde{V}^2|\eta|^2}} & \dfrac{-\rho_r\tilde{\gamma} u_r u_l \eta_1\eta_2}{\sqrt{\gamma^2 + \delta^2 + \tilde{V}^2|\eta|^2}} & \dfrac{-\rho_r\tilde{\gamma}(\tau^2 + u_r u_l \eta_1^2)}{\sqrt{\gamma^2 + \delta^2 + \tilde{V}^2|\eta|^2}} \end{pmatrix} \begin{pmatrix} Z_3^r \\ Z_2^r \end{pmatrix}$$

from which we get the expression

$$\det B^- = \frac{i\tilde{\gamma}^2\rho_r^3 V^2|\eta|^5}{\gamma^2 + \delta^2 + \tilde{V}^2\eta^2}\, f(V)\,.$$

Therefore the previous analysis made in the case $d = 2$ applies and the conclusion of the lemma follows. □

In order to simplify the sequel of the analysis, we assume that the reference speed $\tilde{V}$ and the reference frequency $\tilde{\gamma}$ are **normalized** and taken equal to 1. This is of pure convenience and does not affect the following results but it will clarify the introduction of weighted Sobolev spaces.

3.2 A priori estimate on the linearized equations

We begin with a result of existence of a microlocal Kreiss' symmetrizer for system (8). The proof of this result is detailed in the next subsection. Except at the particular points where the uniform stability condition fails, the method is the one developed in [20] (see also [7]) whose first purpose was the resolution of mixed initial boundary value problems for strictly hyperbolic systems when the boundary conditions do not have any "dissipativeness" property. We point out that this method was later used in [24] (see also [26, 28]) to deal with multidimensional shock waves where no "dissipativeness" argument holds since the boundary conditions B take the form of a pseudodifferential operator of order 0. In our case, since we have limited the study to constant coefficients systems, these boundary conditions take the simpler form of a Fourier multiplicator.

We shall see in the proof of theorem 1 that the failure of the uniform stability condition in the so-called hyperbolic region gives rise to some poor energy estimates compared to the maximal L^2 estimates obtained under the uniform stability condition. In fact, we can state the following result:

Proposition 2. *For all $X_0 \in \Sigma_+$, there exists an open neighbourhood $\mathcal{V}$ of X_0 and matrices $r(X)$, $T(X)$ of class C^∞ with respect to $X \in \mathcal{V}$ which satisfy*

$r(X)$ is hermitian,

$T(X)$ is invertible and defining $a(X) = T(X)^{-1}\mathcal{A}(X)\,T(X)$, $\tilde{B}(X) = B(X)\,T(X)$, there exist two positive constants C et $c > 0$ such that

$$\begin{aligned} \mathrm{Re}\ (r(X)\,a(X)) &\geq c\gamma I\,, \\ r(X) + C\tilde{B}(X)^*\tilde{B}(X) &\geq cI\,, \end{aligned}$$

if the Lopatinskii determinant does not vanish at X_0, and

$$\begin{aligned} \mathrm{Re}\ (r(X)\,a(X)) &\geq c\gamma^3 I\,, \\ r(X) + C\tilde{B}(X)^*\tilde{B}(X) &\geq c\gamma^2 I\,, \end{aligned}$$

if X_0 is a root of the Lopatinskii determinant. In this later case, $r(X)$ can be chosen under the following diagonal form

$$r(X) = \begin{pmatrix} -\gamma^2 I_d & 0 \\ 0 & \lambda I_{d+2} \end{pmatrix}\,,$$

where λ is a real number greater than 1.

We make a few comments on proposition 2. Recall first of all that under the uniform stability condition, one can construct a Kreiss' symmetrizer R which satisfies

$$\begin{aligned} \mathrm{Re}\ (R(X)\mathcal{A}(X)) &\geq c\gamma I\,, \\ R(X) + CB(X)^*B(X) &\geq cI\,. \end{aligned}$$

Comparing to the result of proposition 2, we see that losses of derivatives appear, both in the interior domain and on the boundary. This is a quite remarkable difference between our study and previous works as [10, 11, 31] where derivatives were only lost on the boundary.

Proposition 2 enables to derive an energy estimate on system (8) in some appropriate weighted spaces. We define two domains Ω and ω as

$$\Omega = \mathbb{R} \times \mathbb{R}^d_+ = \{(t, y, z) \in \mathbb{R}^{d+1} \text{ s.t. } z > 0\} \quad \text{and} \quad \omega = \mathbb{R} \times \mathbb{R}^{d-1} = \partial\overline{\Omega}\,.$$

For $\gamma > 0$ and $s \in \mathbb{R}$ we define the following symbols

$$\forall \xi \in \mathbb{R}^d, \quad \lambda^{s,\gamma}(\xi) = (\gamma^2 + |\xi|^2)^{s/2}.$$

The usual Sobolev spaces $H^s(\omega)$ are equipped with the weighted norms (depending on the positive parameter γ):

$$\|u\|_{s,\gamma}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \lambda^{2s,\gamma}(\xi)\, |\hat{u}(\xi)|^2 \, d\xi \, .$$

These weighted norms enable to construct a parameter version of the classical pseudo-differential calculus which is of constant use in the study of mixed initial boundary value problems for hyperbolic systems, see [1, 20, 24].

For all integer k, we equip the usual Sobolev space $H^k(\Omega)$ with the following norm

$$|\!|\!| U |\!|\!|_{k,\gamma}^2 = \sum_{j=0}^{k} \int_0^{+\infty} \|\partial_z^j U(.,z)\|_{k-j,\gamma}^2 \, dz \, .$$

We now define two operators $\mathcal{L}$ and $\mathcal{B}$ by

$$\mathcal{L}(U) = \partial_t U + \sum_{j=1}^{d-1} \mathcal{A}_j \, \partial_{x_j} U + \mathcal{A}_d \, \partial_z U \quad \text{for } z > 0,$$

$$\mathcal{B}(\varphi, U) = \partial_t \varphi \, b_0 + \sum_{j=1}^{d-1} \partial_{x_j} \varphi \, b_j + M\, U \quad \text{for } z = 0.$$

The change of unknown functions described in section 2 leads to the introduction of the "weighted" operators

$$\mathcal{L}^\gamma(U) = \mathcal{L}(U) + \gamma U \quad \text{and} \quad \mathcal{B}^\gamma(\varphi, U) = \mathcal{B}(\varphi, U) + \gamma \varphi \, b_0 \, .$$

These notations enable to state our first weak stability theorem:

Theorem 1. *There exists a constant $C > 0$ such that for all $U \in H^2(\Omega)$, for all $\varphi \in H^2(\omega)$ and for all $\gamma \geq 1$, the following estimate holds:*

$$\gamma |\!|\!| U |\!|\!|_{0,\gamma}^2 + \|U\|_{0,\gamma}^2 + \|\varphi\|_{1,\gamma}^2 \leq C \left(\frac{1}{\gamma^3} |\!|\!| \mathcal{L}^\gamma U |\!|\!|_{1,\gamma}^2 + \frac{1}{\gamma^2} \|\mathcal{B}^\gamma(\varphi, U)\|_{1,\gamma}^2 \right).$$

Proof. The result is a consequence of the existence of a symbolic symmetrizer r given by proposition 2. Since Σ_+ is a compact set, we can fix a finite covering $(\mathcal{V}_i)_{1\leq i\leq I}$ of Σ_+ by open sets defined in proposition 2. Let $(\psi_i)_{1\leq i\leq I}$ be a partition of unity associated to this covering. More precisely, the functions ψ_i are nonnegative, C^∞ and satisfy

$$\forall i = 1, \ldots, I, \quad \text{Supp } \psi_i \subset \mathcal{V}_i \quad \text{and} \quad \sum_{i=1}^{I} \phi_i^2 \equiv 1 \, .$$

Let now $U \in H^2(\Omega)$ and $\varphi \in H^2(\omega)$. We denote $\hat{U}(\xi, z)$ the Fourier transform of $U(t, y, z)$ with respect to the d first variables (t, y). We also define

$$F(t, y, z) = \mathcal{L}^\gamma U(t, y, z) \in H^1(\Omega) \, ,$$
$$G(t, y) = \mathcal{B}^\gamma(\varphi, U) \in H^1(\omega) \, .$$

Lemma 2 ensures that there exists a constant $C > 0$ such that

$$\lambda^{2,\gamma}(\xi)\,|\hat{\varphi}(\xi)|^2 \leq C\left(|\hat{U}(\xi,0)|^2 + |\hat{G}(\xi)|^2\right),$$

with $\xi = (\delta, \eta)$. Integrating with respect to ξ and using Plancherel's theorem yield the inequalities

$$\begin{aligned}\|\varphi\|_{1,\gamma}^2 &\leq C\left(\|U\|_{0,\gamma}^2 + \|\mathcal{B}^\gamma(\varphi,U)\|_{0,\gamma}^2\right)\\ &\leq C\left(\|U\|_{0,\gamma}^2 + \gamma^{-2}\|\mathcal{B}^\gamma(\varphi,U)\|_{1,\gamma}^2\right).\end{aligned}$$

We now need to estimate the norms $\|U\|_{0,\gamma}^2$ and $|||U|||_{0,\gamma}^2$ in terms of $\|G\|_{1,\gamma}^2$ and $|||F|||_{1,\gamma}^2$. We define

$$V_i(X,z) = \psi_i(X)\,T_i(X)^{-1}\,\hat{U}(\xi,z).$$

Since ψ_i has compact support in $\mathcal{V}_i$, we extend the mappings r_i and T_i on all Σ_+ assuming them to be constant outside of $\mathcal{V}_i$ (this is of pure convenience since only the value of these mappings on Supp ψ_i will be involved in the sequel). Then we extend r_i and T_i (and thus a) as homogeneous functions of degree 0 in $X = (\xi, \gamma)$ (this is the method developed in [7, 20, 28]).

Using the definition of the matrix $a(X)$, we know that $V_i(X,z)$ satisfies the ordinary differential equation

$$\frac{dV_i}{dz} = a(X)\,T_i(X)^{-1}\,V_i + \psi_i(X)\,T_i(X)^{-1}\,\mathcal{A}_d^{-1}\,\hat{F}.$$

We deal first with the case where $\mathcal{V}_i$ is a neighbourhood of a root of the Lopatinskii determinant. We take the scalar product of the previous equation by $\lambda^{2,\gamma}(\xi)\,r_i(X)V_i$ and integrate with respect to $\xi = (\delta,\eta) \in \mathbb{R}^d$. Then we integrate with respect to z from 0 to $+\infty$. Using the properties of the symmetrizer r_i, we get

$$\begin{aligned}&-2\ \mathrm{Re}\ \langle\langle r(X)\,V_i, \psi_i(X)\,\lambda^{2,\gamma}(\xi)\,T_i(X)^{-1}\,\mathcal{A}_d^{-1}\hat{F}\rangle\rangle\\ &\qquad\geq 2c\gamma^2\|\psi_i\hat{U}\|_{0,\gamma}^2 - C\|\psi_i B\,\hat{U}\|_{1,\gamma}^2 + 2\ \mathrm{Re}\ \langle\langle V_i, \lambda^{2,\gamma}\,r_i(X)\,a(X)\,V_i\rangle\rangle.\end{aligned}$$

Define a matrix Σ as

$$\Sigma = \begin{pmatrix}\dfrac{\gamma}{\sqrt{\gamma^2+|\xi|^2}} & 0\\ 0 & \sqrt{\lambda}\end{pmatrix},$$

where λ is a real number greater than 1 as stated in proposition 2. We clearly have Re $r_i(X)\,a(X) \geq C\gamma\Sigma^2$ for X in the support of ψ_i. Since a and r_i are diagonal matrices on $\mathcal{V}_i$, we have

$$2\ \mathrm{Re}\ \langle\langle V_i, \lambda^{2,\gamma}\,r_i(X)\,a(X)\,V_i\rangle\rangle \geq c\gamma|||\lambda^{1,\gamma}\,\Sigma V_i|||_{0,\gamma}^2,$$

and the Cauchy-Schwarz inequality yields the estimate

$$\begin{aligned}-2\ \mathrm{Re}\ \langle\langle r(X)\,V_i, \psi_i(X)\,\lambda^{2,\gamma}(\xi)\,T_i(X)^{-1}\,\mathcal{A}_d^{-1}\hat{F}\rangle\rangle &\leq c\gamma|||\lambda^{1,\gamma}\,\Sigma V_i|||_{0,\gamma}^2 + \frac{C}{\gamma}|||\lambda^{1,\gamma}\,\Sigma\hat{F}|||_{0,\gamma}^2\\ &\leq c\gamma|||\lambda^{1,\gamma}\,\Sigma V_i|||_{0,\gamma}^2 + +\frac{C}{\gamma}|||F|||_{1,\gamma}^2.\end{aligned}$$

Eventually we get the following estimate

$$2c\gamma^2\|\psi_i\hat{U}\|^2_{0,\gamma} + c\gamma\|\!|\lambda^{1,\gamma}\,\Sigma V_i\|\!|^2_{0,\gamma} \leq \frac{C}{\gamma}\|\!|F\|\!|^2_{1,\gamma} + C\|\psi_i B\,\hat{U}\|^2_{1,\gamma}\,,$$

from which we finally obtain

$$\gamma^2\|\psi_i\hat{U}\|^2_{0,\gamma} + \gamma^3\|\!|\psi_i\hat{U}\|\!|^2_{0,\gamma} \leq \frac{C}{\gamma}\|\!|F\|\!|^2_{1,\gamma} + C\|\psi_i B\,\hat{U}\|^2_{1,\gamma}\,.$$

When $\mathcal{V}_i$ is a neighbourhood of a point X_0 where the Lopatinskii determinant does not vanish, the result is directly obtained by the analysis made by Kreiss [20] (see also [7, 28]) which gives the maximal L^2 estimate. All these inequalities give an estimate on U in terms of $\mathcal{L}^\gamma(U)$ and $\mathcal{B}^\gamma(\varphi, U)$. The previous estimate on the front φ added to this first estimate on U gives the result. $\square$

Note that when $\mathcal{L}^\gamma(U) = 0$, we recover Majda's statement on weakly stable shocks (see [24], page 10). However theorem 1 is a little more precise since it indicates two types of loss of derivatives arising in this problem. Some regularity is lost on the boundary, as pointed out in Majda's work. But in addition, a very sever loss of regularity occurs in the domain Ω.

3.3 Construction of a Kreiss' symmetrizer: proof of proposition 2

In this subsection, we prove proposition 2 and construct a microlocal symmetrizer. This construction relies on the so-called block structure of the symbol $\mathcal{A}$ which was introduced by Kreiss in the case of strictly hyperbolic systems [20]. In [24], Majda extended this property in a general definition and proved that isentropic Euler equations (1) met all the requirements. We point out that in a recent paper [27], Métivier succeeded in proving that Majda's definition of the block structure condition was a property satisfied by all hyperbolic systems of conservation laws with constant multiplicity eigenvalues.

We need to distinguish four cases corresponding to the different behaviors of the eigenmodes $\omega_k^{l,r}$. We recall that when $\gamma = 0$, the eigenmodes ω_1^l and ω_3^l are always distinct (see section 2).

Construction of r in the elliptic region

Let $X_0 \in \Sigma_+$ suvh that $\gamma > 0$. The symbol $\mathcal{A}(X_0)$ has no purely imaginary eigenvalue and one can therefore choose two closed curves C^- (resp. C^+) lying in the half-plane $\{\text{ Re } z < 0\}$ (resp. $\{\text{ Re } z > 0\}$), such that the eigenvalues of negative (resp. positive) real part of $\mathcal{A}(X_0)$ stand in the domain delimited by C^- (resp. C^+). Using the generalized eigenprojectors associated to $C^\pm$, one gets the existence of a C^∞ mapping $T(X)$ with values in the set of $2(d+1)\times 2(d+1)$ invertible matrices, defined on a neighbourhood of X_0 such that

$$\forall X \in \mathcal{V}\,, \quad T(X)^{-1}\mathcal{A}(X)\,T(X) = \begin{pmatrix} a^-(X) & \mathbf{0} \\ \mathbf{0} & a^+(X) \end{pmatrix}\,,$$

and the spectrum of $a^-(X)$ (resp. $a^+(X)$) is contained in the half-space $\{ \text{ Re } z < 0\}$ (resp. $\{ \text{ Re } z > 0\}$).

Define now the positive definite hermitian matrices

$$H^- = 2 \int_0^{+\infty} \exp(ta^-(X_0))^* \exp(ta^-(X_0))\, dt\,,$$

and

$$H^+ = 2 \int_0^{+\infty} \exp(-ta^+(X_0))^* \exp(-ta^+(X_0))\, dt\,.$$

One easily checks that

$$\begin{aligned} \text{Re } (H^+ a^+(X_0)) &:= (H^+ a^+(X_0) + a^+(X_0)^* H^+)/2 = I\,,\\ \text{Re } (H^- a^-(X_0)) &:= (H^- a^-(X_0) + a^-(X_0)^* H^-)/2 = -I\,, \end{aligned}$$

so that in a neighbourhood $\mathcal{V}$ of X_0, one has

$$\forall\, X \in \mathcal{V}\,,\ \text{Re } H^- a^-(X) \leq -\frac{1}{2} I \quad \text{and} \quad \text{Re } H^+ a^+(X) \geq -\frac{1}{2} I\,.$$

We now define

$$r = \begin{pmatrix} -H^- & 0 \\ 0 & \lambda H^+ \end{pmatrix}\,,$$

where λ will be a real number fixed greater than 1 in the sequel. It is clear that r satisfies the first property of the lemma. Moreover if Z denotes any vector of $\mathbb{C}^{2(d+1)}$, we can write

$$\tilde{B}(X_0)\, Z = \tilde{B}(X_0) \begin{pmatrix} Z^- \\ 0 \end{pmatrix} + \tilde{B}(X_0) \begin{pmatrix} 0 \\ Z^+ \end{pmatrix}\,.$$

Since the Lopatinski determinant does not vanish at any point of $\mathcal{V}$, there exists a constant $C > 0$ such that

$$|Z^-|^2 \leq C \left(|Z^+|^2 + |\tilde{B}(X_0)\, Z|^2 \right)\,.$$

Following [7, 20], one can check that for sufficiently large λ, we have

$$r + C \tilde{B}(X_0)^* \tilde{B}(X_0) \geq cI\,,$$

for some constant $c > 0$, and this estimate holds in all $\mathcal{V}$ by a continuity argument (replacing c by $c/2$).

Construction of r at an hyperbolic diagonalization point

Let $X_0 \in \Sigma_+$ such that $\gamma = 0$, $\eta \neq 0$ and $\delta \neq \pm|\eta|\sqrt{c_{r,l}^2 - u_{r,l}^2}$. We also assume that the Lopatinski determinant does not vanish at X_0 and therefore does not vanish in a suitable neighbourhood of X_0. Using the parametrization of the eigenspaces associated to the eigenmodes $\omega^{l,r}$, it is clear that one can construct a C^∞ mapping T such that for all X in a neighbourhood $\mathcal{V}$ of X_0, one has

$$\forall X \in \mathcal{V}\,, \quad T(X)^{-1} \mathcal{A}(X)\, T(X) = \begin{pmatrix} \omega_3^r & & & & & \\ & \omega_2^r I_{d-1} & & & \mathbf{0} & \\ & & \omega_1^r & & & \\ & & & \omega_1^l & & \\ & \mathbf{0} & & & \omega_2^l I_{d-1} & \\ & & & & & \omega_3^l \end{pmatrix}\,.$$

To achieve the construction of the symmetrizer in this case, we first need to study the behavior of ω_1^r and ω_3^r near X_0. We shall prove in section 5 that there exists a constant $c > 0$ such that

$$\forall X \in \mathcal{V}\,, \quad \begin{cases} -\mathrm{Re}\ \omega_3^r \geq c\gamma\,, \\ \ \ \mathrm{Re}\ \omega_1^r \geq c\gamma\,. \end{cases}$$

Similar results hold for the behavior of the eigenmodes ω_1^l and ω_3^l. Then it is sufficient to choose r under diagonal form

$$r = \begin{pmatrix} -1 & & & & & \\ & -I_{d-1} & & & \mathbf{0} & \\ & & \lambda & & & \\ & & & \lambda & & \\ & \mathbf{0} & & & \lambda I_{d-1} & \\ & & & & & \lambda \end{pmatrix},$$

and performing the same analysis as in the elliptic region yield the required properties on the symmetrizer r.

Construction of r in the neighbourhood of Jordan points

Let $X_0 \in \Sigma_+$ such that $\gamma = 0$ and $\delta = \pm|\eta|\sqrt{c_r^2 - u_r^2}$. Using the same type of arguments as in the case $\gamma > 0$, one can prove that there exists a C^∞ mapping $T(X)$ with values in the set of $2(d+1) \times 2(d+1)$ invertible matrices, defined on a neighbourhood of X_0, such that

$$\forall X \in \mathcal{V}\,, \quad T(X)^{-1}\mathcal{A}(X)\,T(X) = \begin{pmatrix} \omega_2^r I_{d-1} & & & & \\ & a_r(X) & & \mathbf{0} & \\ & & \omega_1^l & & \\ & \mathbf{0} & & \omega_2^l I_{d-1} & \\ & & & & \omega_3^l \end{pmatrix},$$

with $a_r(X)$ some 2×2 matrix satsifying

$$a_r(X_0) = \begin{pmatrix} \lambda_r & i \\ 0 & \lambda_r \end{pmatrix},$$

$\lambda_r = i\kappa_r$ being the double (purely imaginary) root of the polynomial

$$(c_r^2 - u_r^2)X^2 \pm 2i|\eta|u_r\sqrt{c_r^2 - u_r^2}X - u_r^2|\eta|^2\,,$$

which is nothing but (10) at point X_0. We shall show in section 5 that T can be chosen such that for $X \in \mathcal{V} \cap \{\gamma = 0\}$, then $a_r(X)$ has purely imaginary coefficients. Furthermore if $D_r(X)$ denotes the partial derivative of $a_r(X)$ with respect to γ, the lower left corner coefficient α_r of $D_r(X_0)$ is a non zero real number.

We define $r(X)$ in the following way

$$r(X) = \begin{pmatrix} -1 & & & & \\ & -I_{d-1} & & \mathbf{0} & \\ & & h_r(X) & & \\ & \mathbf{0} & & \lambda I_{d-1} & \\ & & & & \lambda \end{pmatrix},$$

λ being once again some real number greater than 1 fixed in the sequel. Following the analysis of Kreiss [7, 20], we choose h_r of the form

$$h_r(X) = \underbrace{\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}}_{E} + \underbrace{\begin{pmatrix} f(X) & 0 \\ 0 & 0 \end{pmatrix}}_{F(X)} \underbrace{-i\gamma \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix}}_{G},$$

where e_1, e_2 and g are real numbers and f is a C^∞ real valued function that we shall fix in the sequel. The Taylor expansion of $a_r(X)$ reads

$$a_r(X) = i(\kappa_r I + N - iB_r(\tilde{X})) + \gamma D_r(\tilde{X}) + \gamma^2 M(X),$$

where $\tilde{X} = (\delta, \eta, 0)$ if $X = (\delta, \eta, \gamma)$, and $B_r(\tilde{X}) = a_r(\tilde{X}) - a_r(X_0)$; in the previous relation, N denotes the nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We know that B_r reads

$$B_r(\tilde{X}) = i \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

with real valued C^∞ functions b_{ij} vanishing at X_0. We fix f by the following formula

$$f(X) = \frac{e_1(b_{11} - b_{22}) + e_2 b_{21}}{1 + b_{12}}$$

so that f has the required property. Moreover, this choice of f implies that

$$(E + F(X)) \left(N - iB_r(\tilde{X})\right)$$

is a real symmetric matrix. As a consequence, one gets

$$\mathrm{Re}\ (h_r(X)\, a_r(X)) = \gamma\ \mathrm{Re}\ (G\, N \ +\ E\, D_r(\tilde{X})) + \gamma\, L(X),$$

where L is a C^∞ hermitian matrix which vanishes at X_0. The shape of E and G yields

$$\mathrm{Re}\ (G\, N \ +\ E\, D_r(X_0)) = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix} + \begin{pmatrix} e_1\alpha_r & * \\ * & * \end{pmatrix},$$

where quantities denoted by $*$ depend only on e_1 and e_2. We fix $e_1 = 1/\alpha_r$ and g sufficiently large so that

$$\mathrm{Re}\ (h_r(X)\, a_r(X)) \geq c\gamma I.$$

This is possible as long as the choice of e_2 does not depend on g. In fact, e_2 will be fixed in order to give the estimate with respect to the boundary conditions $\tilde{B}$ and the choice will not involve g. Indeed, the choice of h_r implies that

$$r(X_0) = \begin{pmatrix} -I_{d-1} & & & & & \\ & 0 & e_1 & & \mathbf{0} & \\ & e_1 & e_2 & & & \\ & & & \lambda & & \\ & \mathbf{0} & & & \lambda I_{d-1} & \\ & & & & & \lambda \end{pmatrix},$$

and a rather tedious analysis (essentially based on Cayley-Hamilton theorem) shows that the stable subspace $\mathcal{E}^-(X_0)$ is spanned by the d first vectors of our new basis. Since the Lopatinski determinant does not vanish in $\mathcal{V}$, we can therefore fix sufficiently large e_2 and λ (independantly of g) to get an estimate of the type

$$r + C\tilde{B}(X_0)^*\tilde{B}(X_0) \geq cI\,.$$

An appropriate choice of g achieves the construction.

We now turn to the last case of points where the uniform stability condition fails. Note that the previous result on the behavior of the eigenmodes still hold because of the properties of V_1^2. Indeed, one can diagonalize the symbol $\mathcal{A}$ in a neighbourhood of $(V_1|\eta|,\eta,0)$; in other words we still have the existence of a C^∞ mapping T satisfying

$$\forall X \in \mathcal{V}\,,\quad T(X)^{-1}\mathcal{A}(X)\,T(X) = \begin{pmatrix} \omega_3^r & & & & & \\ & \omega_2^r I_{d-1} & & & \mathbf{0} & \\ & & \omega_1^r & & & \\ & & & \omega_1^l & & \\ & \mathbf{0} & & & \omega_2^l I_{d-1} & \\ & & & & & \omega_3^l \end{pmatrix}.$$

To recover the estimate of r with respect to the boundary conditions B, one has to choose r of the form

$$r = \begin{pmatrix} -\gamma^2 & & & & & \\ & -\gamma^2 I_{d-1} & & & \mathbf{0} & \\ & & \lambda & & & \\ & & & \lambda & & \\ & \mathbf{0} & & & \lambda I_{d-1} & \\ & & & & & \lambda \end{pmatrix}.$$

Using lemma 3 and performing the same analysis as in the elliptic region yields the estimate

$$r(X) + C\tilde{B}(X)^*\tilde{B}(X) \geq c\gamma^2 I\,,$$

for sufficiently large λ. Since r is diagonal, we have immediately the estimate

$$\mathrm{Re}\;(r\,a(X)) \geq c\gamma^3 I\,,$$

and this completes the proof of proposition 2.

4 Subsonic phase transitions in a van der Waals fluid

In this section, we consider the non standard initial boundary value problem (8) with boundary conditions given by (7). We follow the method adopted in section 3 and begin by recalling the main result of [3].

Lemma 4 (Benzoni). [3]. *There exists a positive number V_0 such that for all $(\delta,\eta,\gamma) \in \mathbb{R}^{d+1}$ satisfying $\gamma \geq 0$ and $(\delta,\gamma) \neq (\pm iV_0|\eta|,0)$, one has*

$$\big\{(Z,\chi) \in \mathcal{E}^-(\delta,\eta,\gamma) \times \mathbb{C} \;\text{ s.t. }\; \chi\, b(\delta,\eta,\gamma) + M\,Z = 0\big\} = \{0\}\,,$$

and for $\eta \neq 0$, *the set*

$$\Big\{(Z,\chi) \in \mathcal{E}^-(\pm V_0|\eta|,\eta,0) \times \mathbb{C} \ \text{ s.t. } \ \chi\, b(\pm V_0|\eta|,\eta,0) + M\, Z = 0\Big\}$$

is a one dimensional subspace of $\mathbb{C}^{2d+3}$. *If* (Z,χ) *belongs to this subspace, then*

$$Z_r \in \mathbb{C} \begin{pmatrix} \rho_r(\tau + u_r\, \omega_3^r) \\ -c_r^2\, i\eta \\ -c_r^2\, \omega_3^r \end{pmatrix} \quad \textit{and} \ \ Z_l \in \mathbb{C} \begin{pmatrix} \rho_l(\tau - u_l\, \omega_1^l) \\ -c_l^2\, i\eta \\ c_l^2\, \omega_1^l \end{pmatrix} ,$$

that is Z_r *has no component on the eigenspace associated to the eigenvalue* ω_2^r. *At all points of the form* $(\pm V_0|\eta|,\eta,0)$, *both eigenmodes* ω_3^r *and* ω_1^l *have negative real part (which explains the designation "surface waves").*

By definition, V_0^2 *is the positive root of the polynomial*

$$P_2(X) = \frac{c_r^2 c_l^2 - u_r^2 u_l^2}{u_r^2 u_l^2} X^2 + (c_r^2 - u_r^2 + c_l^2 - u_l^2) X - (c_r^2 - u_r^2)(c_l^2 - u_l^2)\,,$$

and the following inequalities hold:

$$V_0^2 < \min(c_r^2 - u_r^2, c_l^2 - u_l^2) \ \ \textit{and} \ \ V_0^2 < u_r u_l\,.$$

4.1 Elimination of the front

As we did in section 3 we begin by isolating the shock front in the last boundary condition of (9). This is stated as follows:

Lemma 5. *There exists a* C^∞ *mapping* Q *defined on the half-space* $\mathbb{R}^d \times \mathbb{R}^+ \setminus \{0\}$, *homogeneous of degree* 0 *with values in the set of square* $(d+2) \times (d+2)$ *invertible matrices such that for all* $X \in \mathbb{R}^d \times \mathbb{R}^+ \setminus \{0\}$ *the first* $d+1$ *components of the vector* $Q(X)\, b(X)$ *vanish.*

Proof. The Rankine-Hugoniot jump relations together with (7) yield the relations

$$b(\delta,\eta,\gamma) = \begin{pmatrix} -\tau[\rho] \\ ij[u]\eta \\ 0 \\ -\tau[u] \end{pmatrix} \ \text{if } d=2 \quad \text{and} \quad b(\delta,\eta,\gamma) = \begin{pmatrix} -\tau[\rho] \\ ij[u]\eta_1 \\ ij[u]\eta_2 \\ 0 \\ -\tau[u] \end{pmatrix} \ \text{if } d=3.$$

The mapping Q is first defined on the hemisphere Σ_+ and then extended by homogeneity. Note that we go back to the first definition of Σ_+ with a reference velocity $\tilde{V}$ and a reference frequency $\tilde{\gamma}$ to take the physical dimension of the quantities into account.

One easily checks that for $d=2$, the matrix

$$Q(\delta,\eta,\gamma) = \begin{pmatrix} [u] & 0 & 0 & -[\rho] \\ 0 & \tau & 0 & ij\eta \\ 0 & 0 & 1 & 0 \\ 0 & i\tilde{V}^2\eta & 0 & j\overline{\tau} \end{pmatrix}$$

satisfies all required properties. For $d=3$, one can choose for instance

$$Q(\delta,\eta,\gamma)=\begin{pmatrix} [u] & 0 & 0 & 0 & -[\rho] \\ 0 & \tau & 0 & -i\tilde{V}\eta_2 & ij\eta_1 \\ 0 & 0 & \tau & i\tilde{V}\eta_1 & ij\eta_2 \\ 0 & -i\tilde{V}\eta_2 & i\tilde{V}\eta_1 & \overline{\tau} & 0 \\ 0 & i\tilde{V}^2\eta_1 & i\tilde{V}^2\eta_2 & 0 & j\overline{\tau} \end{pmatrix}$$

which also satisfies all required properties. This completes the proof. □

We can therefore write boundary conditions for the linearized problem (8) in the equivalent way

$$\begin{pmatrix} B(\delta,\eta,\gamma) \\ \ell(\delta,\eta,\gamma) \end{pmatrix} V(0)+\chi\begin{pmatrix} \mathbf{0}_{d+1} \\ \beta(\delta,\eta,\gamma) \end{pmatrix}=Q(\delta,\eta,\gamma)\,G\,,$$

where $\beta(\delta,\eta,\gamma)$ is given by

$$\beta(\delta,\eta,\gamma)=-j[u]\tilde{\gamma}\sqrt{\gamma^2+\delta^2+\tilde{V}^2|\eta|^2}\neq 0\,,$$

and this relation holds for $d=2$ and $d=3$. We now turn to the study of the behavior of the restriction of $B(\delta,\eta,\gamma)$ to the stable subspace $\mathcal{E}^-$ in the neighbourhood of the points where the uniform stability condition fails. According to lemma 4, the symbol $\mathcal{A}$ is diagonalizable in the neighbourhood of such points.

We decompose all vector Z belonging to the stable subspace $\mathcal{E}^-$ on the three different eigenspaces denoting by Z_3^r, Z_2^r and Z_1^- the components of Z on the eigenspaces associated to the eigenmodes ω_3^r, ω_2^r and ω_1^-. Then we have:

Lemma 6. *There exists a neighbourhood $\mathcal{V}$ of $(V_0|\eta|,\eta,0)$ and a constant $c>0$ such that for all $X\in\mathcal{V}$ and for all $Z\in\mathcal{E}^-(X)$, one has*

$$|B(X)\,Z|^2\geq c\,\gamma^2\left(|Z_3^r|^2+|Z_1^l|^2\right)+c|Z_2^r|^2\,.$$

An analogous estimate holds in a neighbourhood of points of the form $(-V_0|\eta|,\eta,0)$.

Proof. According to lemma 4 we know that the kernel of the restriction of B to the stable subspace $\mathcal{E}^-$ is a one-dimensional space whose vectors have no Z_2^r component. Therefore in order to proof the stated result, it is again sufficient to prove that 0 is a *simple root* of the determinant of the restriction of B to $\mathcal{E}^-$.

To avoid overloading this paper, we shall only detail the different steps of the proof in the two dimensional case. The three dimensional case is carried out by similar arguments but the calculations are much more complicated due to the expression of the mapping Q defined at the the previous lemma which involves the complex conjugate $\overline{\tau}$ (which was not the case in section 3).

Let $d=2$ and define (as in the proof of lemma 1) the following quantities

$$a_3^r=\tau u_r-(c_r^2-u_r^2)\omega_3^+\quad,\quad a_1^l=\tau u_l+(c_l^2-u_l^2)\omega_1^-\,.$$

Keeping the definition of the complex square root $\mathcal{R}$ introduced in the proof of lemma 1, we also define two quantities $W_{r,l}(V)$ as

$$W_{r,l}(V)=\mathcal{R}(V^2-(c_{r,l}^2-u_{r,l}^2))\,.$$

Because of the properties of V_0 (see lemma 4), both expressions W_r and W_l depend analytically on V in a neighbourhood of V_0 and it is shown in [3] that V_0 also satisfies

$$c_r^2 c_l^2 V_0^2 + u_r u_l W_l(V_0) W_r(V_0) = 0\,.$$

A direct calculation shows that V_0 is a simple root of the above analytical function (as mentionned in [4]).

Let now Z be any vector in the stable subspace $\mathcal{E}^-(\delta,\eta,\gamma)$ with components Z_3^r, Z_2^r and Z_1^- on the eigenspaces associated to the eigenmodes ω_3^r, ω_2^r and ω_1^-. We have

$$B(\delta,\eta,\gamma)\,Z = \begin{pmatrix} \rho_r[u]a_3^+ - c_r^2[\rho]\tau & i(\rho_l+\rho_r)[u]\eta & c_l^2[\rho]\tau - \rho_l[u]a_1^- \\ 0 & j\tilde{\gamma}\dfrac{\tau^2/u_r - u_r\eta^2}{\sqrt{\gamma^2+\delta^2+\tilde{V}^2\eta^2}} & 0 \\ \rho_r(c_r^2\tau + u_r a_3^+) & 2ij\eta & -\rho_l(c_l^2\tau + u_l a_1^-) \end{pmatrix} \begin{pmatrix} Z_3^r \\ Z_2^r \\ Z_1^- \end{pmatrix}$$

from which we get the expression of the restriction of B to the stable subspace $\mathcal{E}^-$. Letting $X = (\delta,\eta,\gamma)$, one gets the expression of the determinant of the above matrix:

$$\det B^-(X) = h_2(\gamma)\left[c_r^2 c_l^2 V^2 + u_r u_l W_l(V) W_r(V)\right]\,,$$

where h_2 is given by

$$h_2(\gamma) = \frac{-j\tilde{\gamma}c_r c_l[\rho]^2|\eta|^4(V^2+u_r^2)}{u_r\sqrt{\gamma^2+V_0^2|\eta|^2+\tilde{V}^2|\eta|^2}}\,.$$

With the preceding remarks, it is now a straightforward verification that the partial derivative of this determinant with respect to γ calculated at $\gamma=0$ is not zero, simply because $h_2(0)\neq 0$.

For the three dimensional case ($d=3$), one proceeds in the same way. The expression of the determinant of the restriction B^- is

$$\det B^-(X) = h_3(\gamma)\left[c_r^2 c_l^2 V^2 + u_r u_l W_l(V) W_r(V)\right]\,,$$

where h_3 is given by

$$h_3(\gamma) = \frac{j^2\tilde{\gamma}^3 c_r c_l[\rho]^2|\eta|^6\,\overline{\tau}}{(\gamma^2+V_0^2|\eta|^2+\tilde{V}^2|\eta|^2)^{3/2}}\left[\frac{V^4}{u_r^2} + V^2 - \tilde{V}^2\left(\frac{V^2}{u_r^2}+1\right)\frac{\tau}{\overline{\tau}}\right]\,.$$

Once again (since $h_3(0)\neq 0$) the partial derivative of the determinant with respect to γ calculated at $\gamma=0$ is not zero. $\square$

4.2 A priori estimate on the linearized equations

We begin with a result of existence of a global Kreiss' symmetrizer for system (8):

Proposition 3. *There exist a C^∞ mapping R defined on the half-space $\mathbb{R}^d \times \mathbb{R}_+ \setminus \{0\}$, homogeneous of degree* 0*, and two positive constants c and C such that*

$$\mathrm{Re}\ (R(X)\mathcal{A}(X)) \geq \frac{c\gamma^2}{\sqrt{\gamma^2+\delta^2+|\eta|^2}}\,,$$
$$R(X)+CB(X)^*B(X) \geq \frac{c\gamma^2}{\gamma^2+\delta^2+|\eta|^2}\,,$$

for all $X=(\delta,\eta,\gamma) \in \mathbb{R}^d \times \mathbb{R}_+ \setminus \{0\}$.

This result will be directly derived from the microlocal analysis developed in the next subsection. We simply make the following remark: as in the study of non uniformly stable Lax shocks for isentropic Euler equations, the failure of the uniform stability condition yields two types of losses of derivatives. Some regularity is lost in the interior domain and some is lost on the boundary.

The previous result enables to derive the second main result of this paper, namely the complete energy estimate on the linearized problem (8) in the case of subsonic phase transitions. We keep the notations introduced in subsection 3.2 for the domains Ω and its boundary ω and for the linearized operators $\mathcal{L}^\gamma$ and $\mathcal{B}^\gamma$.

Theorem 2. *There exists a constant $C>0$ such that for all $U \in H^2(\Omega)$, for all $\varphi \in H^{3/2}(\omega)$ and for all $\gamma \geq 1$, the following estimate holds:*

$$\gamma^2 \left(\|\!|U|\!\|^2_{0,\gamma} + \|U\|^2_{-1/2,\gamma} + \|\varphi\|^2_{1/2,\gamma} \right) \leq C \left(\frac{1}{\gamma^2} \|\!|\mathcal{L}^\gamma U|\!\|^2_{1,\gamma} + \|\mathcal{B}^\gamma(\varphi,U)\|^2_{1/2,\gamma} \right).$$

Proof. The result is a direct consequence of the existence of a symbolic symmetrizer R given by proposition 3. Let $U \in H^2(\Omega)$ and $\varphi \in H^{3/2}(\omega)$. We denote $\hat{U}(\xi,z)$ the Fourier transform of $U(t,y,z)$ with respect to the d first variables (t,y). We also define

$$F(t,y,z) = \mathcal{L}^\gamma U(t,y,z) \in H^1(\Omega)\,,$$
$$G(t,y) = \mathcal{B}^\gamma(\varphi,U) \in H^{1/2}(\omega)\,.$$

Then lemma 4 ensures that there exists a constant $C_1>0$ such that

$$\lambda^{1,\gamma}(\xi)\,|\hat{\varphi}(\xi)|^2 \leq C_1 \lambda^{-1,\gamma}(\xi) \left(|\hat{U}(\xi,0)|^2 + |\hat{G}(\xi)|^2 \right)\,,$$

with $\xi=(\delta,\eta)$. Integrating with respect to ξ and using Plancherel's theorem yield the estimates

$$\begin{aligned} \|\varphi\|^2_{1/2,\gamma} &\leq C_1 \left(\|U\|^2_{-1/2,\gamma} + \|\mathcal{B}^\gamma(\varphi,U)\|^2_{-1/2,\gamma} \right) \\ &\leq C_1 \left(\|U\|^2_{-1/2,\gamma} + \gamma^{-2} \|\mathcal{B}^\gamma(\varphi,U)\|^2_{1/2,\gamma} \right)\,. \end{aligned}$$

Furthermore, $\hat{U}$ satisfies the ordinary differential equation

$$\frac{d\hat{U}}{dz} = \mathcal{A}(\xi,\gamma)\,\hat{U} + \mathcal{A}_d^{-1}\,\hat{F}\,.$$

We take the scalar product of this equation by $\lambda^{1,\gamma}(\xi)\, R(\xi,\gamma)\, \hat{U}$ and integrate with respect to $\xi = (\delta, \eta) \in \mathbb{R}^d$. Then we integrate with respect to z from 0 to $+\infty$ and take the real part of the corresponding equality. Using the properties of the symmetrizer R, we get

$$-2\ \mathrm{Re}\ \langle\!\langle \hat{U}, \lambda^{1,\gamma}(\xi)\, \mathcal{A}_d^{-1} \hat{F} \rangle\!\rangle \geq 2c\gamma^2 \|\!| U |\!\|^2_{0,\gamma} + 2c\gamma^2 \|U\|^2_{-1/2,\gamma} - C_2 \|\mathcal{B}^\gamma(\varphi, U)\|^2_{1/2,\gamma}\,.$$

The Cauchy-Schwartz inequality yields the estimate

$$-2\ \mathrm{Re}\ \langle\!\langle \hat{U}, \lambda^{1,\gamma}(\xi)\, \mathcal{A}_d^{-1} \hat{F} \rangle\!\rangle \leq c\gamma^2 \|\!| U |\!\|^2_{0,\gamma} + \frac{C_3}{\gamma^2} \|\!| \mathcal{L}^\gamma U |\!\|^2_{1,\gamma}\,.$$

This last inequality added to the previous estimate on the front φ enables to conclude. □

4.3 Construction of a Kreiss' symmetrizer

We first construct a microlocal symmetrizer from which we will deduce the result of proposition 3.

Proposition 4. *For all $X_0 \in \Sigma_+$, there exists an open neighbourhood $\mathcal{V}$ of X_0 and matrices $r(X)$, $T(X)$ of class C^∞ with respect to $X \in \mathcal{V}$ which satisfy*

$r(X)$ is hermitian,

$T(X)$ is invertible and defining $a(X) = T(X)^{-1}\mathcal{A}(X)\, T(X)$, $\tilde{B}(X) = B(X)\, T(X)$, there exist two positive constants C and c such that

$$\mathrm{Re}\ (r(X)\, a(X)) \geq c\gamma^2 I\,,$$
$$r(X) + C\tilde{B}(X)^* \tilde{B}(X) \geq c\gamma^2 I\,.$$

Proof. Many steps of the proof are identical to what has been done in the case of Lax shocks and we shall not repeat them: in the so-called elliptic region $\{\gamma > 0\}$ and at Jordan points, the construction is entirely similar. Note that the equality $c_r^2 - u_r^2 = c_l^2 - u_l^2$ is not precluded in the context of phase transitions though it is highly unlikely. In such a case, the reduction of $\mathcal{A}$ would involve two distinct Jordan blocks but the microlocal construction of r would be a direct extension of what has been done in the case of a single

The only difference relies on the properties of the symbol $\mathcal{A}$ in the neighbourhood of the points where the uniform stability condition fails. Let $X_0 = (\pm V_0|\eta|, \eta, 0)$ be a point where the Lopatinskii determinant vanishes. We already know that $\mathcal{A}$ is diagonalizable in a neighbourhood $\mathcal{V}$ of X_0 and that $\mathcal{V}$ may be suitably chosen so that ω_3^r and ω_1^l have negative real part in $\mathcal{V}$. We thus choose r of the form

$$r(X) = \begin{pmatrix} -\gamma^2 & & & & & \\ & -I_{d-1} & & & \mathbf{0} & \\ & & -\gamma^2 & & & \\ & & & \lambda & & \\ & \mathbf{0} & & & \lambda I_{d-1} & \\ & & & & & \lambda \end{pmatrix},$$

where λ is a real number greater than 1 which will be fixed in the sequel. Since there exist a C^∞ invertible matrix $T(X)$ such that

$$\forall X \in \mathcal{V}\,, \quad T(X)^{-1}\mathcal{A}(X)\,T(X) = \begin{pmatrix} \omega_3^r & & & & & \\ & \omega_2^r I_{d-1} & & & \mathbf{0} & \\ & & \omega_1^l & & & \\ & & & \omega_3^l & & \\ & \mathbf{0} & & & \omega_2^l I_{d-1} & \\ & & & & & \omega_1^r \end{pmatrix}\,,$$

we have Re $(r(X)\,a(X)) \geq c\gamma^2 I$ for all X in $\mathcal{V}$. We now have to fix λ in order to get the estimate on the boundary conditions. For this, we let $Z \in \mathbb{C}^{2(d+1)}$ and define Z^- (resp. Z^+) as the vector formed by the $(d+1)$ first (resp. last) components of Z. Writing $Z^- = (Z_1^-, \check{Z}^-, Z_{d+1}^-)$, lemma 6 ensures that there exists a constant $c > 0$ which does not depend on Z such that

$$c\gamma^2 \left(|Z_1^-|^2 + |Z_{d+1}^-|^2\right) + c|\check{Z}^-|^2 \leq C \left(|Z^+|^2 + |\tilde{B}(X)\,Z|^2\right)\,.$$

By the same techniques as used in the construction of the symmetrizer in the elliptic region, it is clear that for a sufficiently large λ, there exists a constant $C > 0$ such that the following estimate holds

$$r(X) + C\tilde{B}(X)^*\tilde{B}(X) \geq c\gamma^2 I\,.$$

This completes the proof of proposition 4.

We can now turn to the proof of proposition 3, using the gluing technique developed in [7, 28]. We fix a finite covering $(\mathcal{V}_i)_{1\leq i\leq I}$ of Σ_+ by open sets defined in proposition 4. Let $(\psi_i)_{1\leq i\leq I}$ be a partition of unity associated to this covering. We define a C^∞ mapping R on Σ_+ by the formula

$$\forall X \in \Sigma_+, \quad R(X) = \sum_{i=1}^{I} \psi_i(X)\left(T_i(X)^{-1}\right)^* r_i(X)\,T_i(X)^{-1}$$

so that R has values in the set of hermitian matrices. Moreover, we have

$$\text{Re }(R(X)\mathcal{A}(X)) \geq c\gamma^2 \sum_{i=1}^{I} \psi_i(X)\left(T_i(X)^{-1}\right)^* T_i(X)^{-1}\,,$$

$$R(X) + CB(X)^*B(X) \geq c\gamma^2 \sum_{i=1}^{I} \psi_i(X)\left(T_i(X)^{-1}\right)^* T_i(X)^{-1}\,,$$

for some positive constants c and C. It is clear that for all X in the compact set Σ_+, the matrix

$$\sum_{i=1}^{I} \psi_i(X)\left(T_i(X)^{-1}\right)^* T_i(X)^{-1}$$

is hermitian positive definite. We can therefore conclude that there exists positive constants c and C such that for all X in Σ_+

$$\text{Re }(R(X)\,\mathcal{A}(X)) \geq c\gamma^2 I\,,$$
$$R(X) + CB(X)^*B(X) \geq c\gamma^2 I\,.$$

The result of proposition 3 follows by extending R in a homogeneous function of degree 0 and using the homogeneity properties of symbols $\mathcal{A}$ and B. $\square$

We point out that the result of theorem 2 is not optimal in the sense that we could define new spaces to get a refined estimate since only 1/2 of derivative is lost in the interior domain (and only in the tangential variables). However, we have not feared stating the result in this way to make the result easier to visualize. Furthermore, the proof of the theorem appears much more simple than the prrof of theorem 1 where attention needs to be paid to get the best result as possible.

5 Some technical lemmas

In this section, we prove three results used in the proof of propositions 2 and 4. Though our proof uses some particular properties of system (1), they are essentially the same as in the general case, see [7, 20, 29].

We first begin by studying the behavior of the eigenmodes ω_1^r and ω_3^r in a neighbourhood of points $X_0 = (\delta, \eta, 0)$.

Lemma 7. *Let $X_0 \in \Sigma_+$ such that $\gamma = 0$, $\eta \neq 0$ and $\delta \neq \pm|\eta|\sqrt{c_r^2 - u_r^2}$. There exists a neighbourhood $\mathcal{V}$ of X_0 in Σ_+ and a positive constant c such that*

$$\forall\, X \in \mathcal{V}\,, \quad \begin{cases} -\mathrm{Re}\ \omega_3^r \geq c\gamma\,, \\ \ \ \mathrm{Re}\ \omega_1^r \geq c\gamma\,. \end{cases}$$

Proof. Let $X_0 = (\delta_0, \eta_0, 0)$ satisfy the assumptions of the lemma. Using the proof of proposition 2, we already know that $\mathcal{A}$ is diagonalizable in a neighbourhood $\mathcal{V}$ of X_0:

$$\forall X \in \mathcal{V}\,, \quad T(X)^{-1}\mathcal{A}(X)\,T(X) = \begin{pmatrix} \omega_3^r & & & & & \\ & \omega_2^r I_{d-1} & & & \mathbf{0} & \\ & & \omega_1^r & & & \\ & & & \omega_1^l & & \\ & \mathbf{0} & & & \omega_2^l I_{d-1} & \\ & & & & & \omega_3^l \end{pmatrix}.$$

If both eigenmodes ω_1^r and ω_3^r are not purely imaginary at X_0, the result comes from a simple continuity argument. We shall therefore assume that both eigenmodes are purely imaginary at X_0. We fix $\eta = \eta_0$ and define

$$Q(\delta, \gamma, \omega) = (\omega + i\omega_1^r(X))(\omega + i\omega_3^r(X))(\omega + i\omega_2^r(X))^{d-1}\,. \tag{13}$$

For $\tau = \gamma + i\delta$ close to $i\delta_0$, the eigenmodes ω_k^r are pairwise distinct and the hyperbolicity of the system (1) shows that for all $\xi \in \mathbb{R}$, Q is given by

$$\begin{aligned} Q(\delta, \gamma, \xi) = \alpha \left[\delta - i\gamma + \left(\xi u_r + c_r\sqrt{|\eta_0|^2 + \xi^2}\right)\right] & \\ \left[\delta - i\gamma + \left(\xi u_r - c_r\sqrt{|\eta_0|^2 + \xi^2}\right)\right] & (\delta - i\gamma + \xi u_r)^{d-1}\,, \end{aligned}$$

for some real constant $\alpha \neq 0$. Thus for all real ξ we have

$$Q(\delta, 0, \xi) \in \mathbb{R} \quad \text{and} \quad \frac{\partial Q}{\partial \gamma}(\delta, 0, \xi) \in i\mathbb{R}\,. \tag{14}$$

Moreover, the definition of Q gives the relation

$$\frac{\partial Q}{\partial \gamma}(\delta_0, 0, -i\omega_1^r(X_0)) = i\frac{\partial \omega_1^r}{\partial \gamma}(X_0)(-i\omega_1^r(X_0) + i\omega_3^r(X_0))(-i\omega_1^r(X_0) + i\omega_2^r(X_0))^{d-1},$$

from which we conclude that the partial derivative $\partial_\gamma \omega_1^r(X_0)$ is a real number. A similar result holds for ω_3^r. We are now going to prove that this partial derivative is not zero. Equation (10) reads

$$(c_r^2 - u_r^2)(\omega_1^r)^2 - 2\tau u_r \omega_1^r - \tau^2 - c_r^2|\eta_0|^2 = 0,$$

and thus, differentiating with respect to γ yields the equality

$$(c_r^2 - u_r^2)2\omega_1^r \frac{\partial \omega_1^r}{\partial \gamma} - 2u_r\omega_1^r - 2\tau u_r \frac{\partial \omega_1^r}{\partial \gamma} - 2\tau = 0.$$

Since ω_1^r et ω_2^r are distinct for all $X \in \mathcal{V}$, it is clear that $\partial_\gamma \omega_1^r$ does not vanish at X_0. The end of the proof relies on a simple Taylor expansion of ω_1^r at X_0, using the fact that ω_1^r is of positive real part for $\gamma > 0$. $\square$

We now turn to the study of the reduced symbol in the neighbourhood of Jordan points. Let $X_0 = (\delta_0, \eta_0, 0)$ be such that

$$\delta_0 = |\eta_0|\sqrt{c_r^2 - u_r^2},$$

so that, according to the proof of proposition 2, we have

$$T(X)^{-1}\mathcal{A}(X)\,T(X) = \begin{pmatrix} \omega_2^r I_{d-1} & & & & \\ & a_r(X) & & \mathbf{0} & \\ & & \omega_1^l & & \\ & \mathbf{0} & & \omega_2^l I_{d-1} & \\ & & & & \omega_3^l \end{pmatrix},$$

with $a_r(X)$ some 2×2 matrix satsifying

$$a_r(X_0) = \begin{pmatrix} \lambda_r & i \\ 0 & \lambda_r \end{pmatrix} = \lambda_r I_2 + iN.$$

Recall that $\lambda_r = i\kappa_r$ is the double root of the polynomial

$$(c_r^2 - u_r^2)X^2 \pm 2i|\eta|u_r\sqrt{c_r^2 - u_r^2}X - u_r^2|\eta|^2.$$

With these notations, we have

Lemma 8. *Defining* $D_r(X) = \dfrac{\partial a_r}{\partial \gamma}(X)$ *for* X *close to* X_0*, then the lower left corner coefficient* α_r *of* $D_r(X_0)$ *is a non zero real number.*

Proof. We fix $\eta = \eta_0$ and let $\tau = \gamma + i\delta$ be close to $i\delta_0$. We define a polynomial Q by (13) (see the proof of lemma 7) and two polynomials Q_r and $\tilde{Q}$ by the following formulae:

$$\begin{aligned} Q_r(\delta, \gamma, \omega) &= \det\left[\omega I_2 + ia_r(\delta, \eta_0, \gamma)\right], \\ \tilde{Q}(\delta, \gamma, \omega) &= (\omega + i\omega_2^+)^{d-1} = (\omega - i\tau/u_r)^{d-1}, \end{aligned}$$

so that $Q = Q_r\tilde{Q}$. We already know by relation (14) that for all real ξ

$$Q(\delta, 0, \xi) \in \mathbb{R} \quad \text{and} \quad \frac{\partial Q}{\partial \gamma}(\delta, 0, \xi) \in i\mathbb{R}\,.$$

It is also clear that for $\xi \in \mathbb{R}$, one has $\tilde{Q}(\delta, 0, \xi) \in \mathbb{R}$.

For δ close to δ_0, $Q(\delta, 0, \omega)$ seen as a polynomial in ω has real coefficients and therefore has real roots or conjugate complex roots. Moreover $\tilde{Q}(\delta, 0, \omega)$ has exactly one real root so $Q_r(\delta, 0, \omega)$ has two real roots or two conjugate complex roots. Thus for δ close to δ_0, we have

$$\forall\, \xi \in \mathbb{R}\,, \quad Q_r(\delta, 0, \xi) \in \mathbb{R}\,. \tag{15}$$

The definition of λ_r yields

$$\frac{\partial Q}{\partial \gamma}(\delta_0, 0, -i\lambda_r) = \tilde{Q}(\delta_0, 0, -i\lambda_r)\,\frac{\partial Q_r}{\partial \gamma}(\delta_0, 0, -i\lambda_r)\,,$$

and since $\lambda_r \neq -i\delta_r/u_r$, we can conclude that $\partial_\gamma Q_r(\delta_0, 0, -i\lambda_r)$ is a purely imaginary number. It is clear that 0 is a simple root of the polynomial $Q(\delta_0, \cdot, -i\lambda_r)$ and therefore the partial derivative $\partial_\gamma Q_r(\delta_0, 0, -i\lambda_r)$ is a non zero purely imaginary number.

To complete the proof, we note that

$$ia_r(X_0) = i\lambda_r I_2 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\,,$$

from which we get

$$\frac{\partial Q_r}{\partial \gamma}(\delta_0, 0, -i\lambda_r) = i\alpha_r \in i\mathbb{R} \setminus \{0\}\,.$$

□

The last thing to check is the invertible matrix $T(X)$ may be chosen in such a way that $a_r(X)$ has purely imaginary coefficients for $X \in \mathcal{V} \cap \{\gamma = 0\}$. We base our proof of this result on a technique developed in [29]. Let X_0 be the triple $(|\eta_0|\sqrt{c_r^2 - u_r^2}, \eta_0, 0)$. For $X = (\delta, \eta, \gamma)$ close to X_0, we define $\tilde{X} = (\delta, \eta, 0)$. With these notations, we have

Lemma 9. *There exists a C^∞ change of basis of $\mathbb{C}^2$ such that for all X close to X_0, $a_r(\tilde{X})$ has purely imaginary coefficients.*

Proof. Let (f_1, f_2) be the canonical basis of $\mathbb{C}^2$. For X close to X_0, define

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \quad B_r(X) = a_r(X) - a_r(X_0)\,.$$

Since $B_r(X_0)$ is zero, the couple of vectors $(f_1' = (N - iB_r(\tilde{X}))\, f_2, f_2)$ forms a basis of $\mathbb{C}^2$ for X close to X_0 and f_1' is a C^∞ vector valued function of X. In this new basis, $N - iB_r(\tilde{X}$ reads

$$\begin{pmatrix} b_1 & 1 \\ b_2 & 0 \end{pmatrix}\,,$$

and the characteristic polynomial of $N - iB_r(\tilde{X}$ is therefore

$$P(\xi) = \xi^2 - b_1\xi - b_2\,.$$

We also have the relation $N - iB_r(\tilde{X}) = -ia_r(\tilde{X}) - \kappa_r I_2$, from which we get

$$P(\xi) = \det\left[ia_r(\tilde{X}) + (\kappa_r + \xi)I_2\right]\,,$$

and relation (15) asserts that P has real coefficients. This completes the proof. □

6 Concluding remarks

In both problems detailed in this paper, a weak stability result has been proved. Though the present study is just a constant coefficients analysis, it indicates the way to follow in order to get a nonlinear existence result (we warn the reader that such a result is not guaranteed at the present time).

Since both problems give rise to losses of derivatives on the solution of the corresponding linearized system, special attention should be paid when dealing with a variable coefficients linearized system. The usual linearized system (8) used in [24, 26, 28] is not appropriate in this case since the right-hand side would involve some terms whose Sobolev norm need to be controlled when one wants to construct an iterative scheme. Higher order terms in the Taylor expansion should therefore be taken into account when linearizing equations (1) around a variable coefficients state $\overline{U}$.

However, it appears from theorem 2 that the case of phase transitions in a van der Waals fluid is rather similar to the problem treated in [31]. The study of the variable linearized system should be carried out by using a parameter version of paradifferential calculus which has been developed in [28].

To conclude, it is known since Majda's work that planar discontinuities for a multidimensional scalar conservation law are only weakly stable. Since our method heavily depends on the behavior of the *boundary matrix* B^- in the neighbourhood of the points where the uniform stability condition fails, it cannot directly apply to a general scalar conservation law. We postpone the redaction of the previous results in a general framework to a future work.

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References

[1] M. S. Agranovič. Boundary value problems for systems with a parameter. *Math. USSR-Sb.*, 13:25–64, 1971.

[2] H. Beirão da Veiga. On the existence theorem for the barotropic motion of a compressible inviscid fluid in the half-space. *Ann. Mat. Pura Appl. (4)*, 163:265–289, 1993.

[3] S. Benzoni-Gavage. Stability of multi-dimensional phase transitions in a van der Waals fluid. *Nonlinear Anal.*, 31(1-2):243–263, 1998.

[4] S. Benzoni-Gavage. Stability of subsonic planar phase boundaries in a van der Waals fluid. *Arch. Rational Mech. Anal.*, 150(1):23–55, 1999.

[5] A. M. Blokhin. Estimation of the energy integral of a mixed problem for gas dynamics equations with boundary conditions on the shock wave. *Sibirsk. Mat. Zh.*, 22(4):23–51, 229, 1981 [in Russian].

[6] A. M. Blokhin. Uniqueness of the classical solution of a mixed problem for equations of gas dynamics with boundary conditions on a shock wave. *Sibirsk. Mat. Zh.*, 23(5):17–30, 222, 1982 [in Russian].

[7] J. Chazarain, A. Piriou. *Introduction to the theory of linear partial differential equations.* North-Holland Publishing Co., Amsterdam, 1982. Translated from the French.

[8] R. Courant, D. Hilbert. *Methods of mathematical physics. Vol. II: Partial differential equations.* Interscience Publishers (a division of John Wiley & Sons), New York-London, 1962.

[9] R. Courant, K. O. Friedrichs. *Supersonic Flow and Shock Waves.* Interscience Publishers, Inc., New York, N. Y., 1948.

[10] W. Domański. Admissible boundary conditions for linear symmetric hyperbolic systems with applications to mechanics and electrodynamics. *Ph.D. Thesis*, 1986, [in Polish].

[11] W. Domański. Surface and boundary waves for linear hyperbolic systems: applications to basic equations of electrodynamics and mechanics of continuum. *J. Tech. Phys.*, 30(3-4):283–300, 1989.

[12] S. P. D'yakov. On the stability of shock waves. *Ž. Eksper. Teoret. Fiz.*, 27:288–295, 1954 [in Russian].

[13] J. Erpenbeck. Stability of step shocks. *Phys. Fluids*, 5:1181–1187, 1962.

[14] H. Freistühler. The persistence of ideal shock waves. *Appl. Math. Lett.*, 7(6):7–11, 1994.

[15] O. Guès. Problème mixte hyperbolique quasi-linéaire caractéristique. *Comm. Partial Differential Equations*, 15(5):595–645, 1990 [in French].

[16] R. Hersh. Mixed problems in several variables. *J. Math. Mech.*, 12:317–334, 1963.

[17] R. L. Higdon. Initial-boundary value problems for linear hyperbolic systems. *SIAM Rev.*, 28(2):177–217, 1986.

[18] D. Y. Hsieh, X. P. Wang. Phase transition in van der Waals fluid. *SIAM J. Appl. Math.*, 57(4):871–892, 1997.

[19] V. M. Kontorovič. Stability of shock waves in relativistic hydrodynamics. *Soviet Physics JETP*, 34 (7):127–132, 1958.

[20] H. O. Kreiss. Initial boundary value problems for hyperbolic systems. *Comm. Pure Appl. Math.*, 23:277–298, 1970.

[21] P. D. Lax. Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[22] T. T. Li, W. C. Yu. *Boundary value problems for quasilinear hyperbolic systems.* Duke University Mathematics Department, Durham, NC, 1985.

[23] A. Majda. The existence of multi-dimensional shock fronts. *Memoirs Amer. Math. Soc.*, 281, 1983.

[24] A. Majda. The stability of multi-dimensional shock fronts. *Memoirs Amer. Math. Soc.*, 275, 1983.

[25] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables.* Springer-Verlag, New York, 1984.

[26] G. Métivier. Problèmes mixtes non linéaires et stabilité des chocs multidimensionnels. *Astérisque*, 152-153:37–53, 1987 [in French].

[27] G. Métivier. The block structure condition for symmetric hyperbolic systems. *Bull. London Math. Soc.*, 32(6):689–702, 2000.

[28] A. Mokrane. Problèmes mixtes hyperboliques non-linéaires. *Ph.D. Thesis, Université de Rennes I*, 1987 [in French].

[29] J. V. Ralston. Note on a paper of Kreiss. *Comm. Pure Appl. Math.*, 24(6):759–762, 1971.

[30] R. D. Richtmyer. Taylor instability in shock acceleration of compressible fluids. *Comm. Pure Appl. Math.*, 13:297–319, 1960.

[31] M. Sablé-Tougeron. Existence pour un problème de l'élastodynamique Neumann non linéaire en dimension 2. *Arch. Rational Mech. Anal.*, 101(3):261–292, 1988 [in French].

[32] P. Secchi. Inflow-outflow problems for inviscid compressible fluids. *Commun. Appl. Anal.*, 2(1):81–110, 1998.

[33] D. Serre. *Systems of conservation laws. 1.* Cambridge University Press, Cambridge, 1999.

[34] D. Serre. *Systems of conservation laws. 2.* Cambridge University Press, Cambridge, 2000.

[35] M. Shearer. Admissibility criteria for shock wave solutions of a system of conservation laws of mixed type. *Proc. Roy. Soc. Edinburgh Sect. A*, 93(3-4):233–244, 1982/83.

[36] M. Slemrod. Admissibility criteria for propagating phase boundaries in a van der Waals fluid. *Arch. Rational Mech. Anal.*, 81(4):301–315, 1983.

[37] L. Truskinovsky. About the "normal growth" approximation in the dynamical theory of phase transitions. *Contin. Mech. Thermodyn.*, 6(3):185–208, 1994.