Entropy-based moment closure for kinetic equations: Riemann problem and invariant regions

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Abstract

We study a nonlinear hyperbolic system of balance laws that arises from an entropy-based moment closure of a kinetic equation. We show that the corresponding homogeneous Riemann problem can be solved without smallness assumption, and we exhibit invariant regions.

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1 Introduction

This paper is devoted to the analysis of the following PDEs system

$$\begin{cases} \partial_t \rho + \partial_x J = 0\,, \\ \varepsilon^2\, \partial_t J + \partial_x \left(\rho\, \psi\Big(\dfrac{\varepsilon J}{\rho}\Big) \right) = -J\,, \end{cases} \tag{1}$$

where the unknown are the density ρ, and the current J, while ε is a positive scaling parameter. The function ψ that appears in (1) is defined in the following way:

$$\begin{array}{rccl} \psi: & (-1,+1) & \longrightarrow &]0,+\infty[\\ & u & \longmapsto & u^2 + \mathbb{G}'\Big(\mathbb{G}^{-1}(u)\Big) = \dfrac{\mathbb{F}''}{\mathbb{F}}\Big(\mathbb{G}^{-1}(u)\Big)\,, \end{array} \tag{2}$$

where we have let

$$\forall\, \beta \in \mathbb{R}\,, \quad \mathbb{F}(\beta) := \frac{\sinh(\beta)}{\beta}\,, \qquad \mathbb{G}(\beta) := \coth(\beta) - \frac{1}{\beta} = \frac{\mathbb{F}'(\beta)}{\mathbb{F}(\beta)}\,. \tag{3}$$

We note that $\mathbb{G}$ is a $\mathcal{C}^\infty$ diffeomorphism from $\mathbb{R}$ onto $(-1,1)$, so the use of the inverse $\mathbb{G}^{-1}$ is legitimate. For future purposes, it is convenient to remark that

$$\mathbb{F}(\beta) = \int_{-1}^{+1} e^{\beta v}\, d\mu(v)\,,$$

where, here and below, $d\mu$ stands for the normalized Lebesgue measure on $(-1,+1)$. It is also worth noting that $\mathbb{F}$, and ψ are even functions, while $\mathbb{G}$ is an odd function. We will show below that ψ is strictly convex. The following relations will be often used throughout the paper:

$$\mathbb{F}(0) = 1\,, \qquad \mathbb{G}(0) = 0\,, \qquad \psi(0) = \mathbb{G}'(0) = \frac{1}{3}\,, \qquad \psi'(0) = 0\,.$$

Our motivation for studying (1) comes from kinetic theory. The system (1) is indeed derived as an intermediate model between a fully microscopic description and its diffusion asymptotics $\varepsilon \to 0$. Let $f(t,x,v) \geq 0$ stand for a (microscopic) density of particles having at time t a position $x \in \mathbb{R}$ and a velocity $v \in (-1,+1)$, and let us assume that the evolution of f is governed by the linear kinetic equation:

$$\varepsilon\, \partial_t f + v\, \partial_x f = \frac{1}{\varepsilon}\, Q(f)\,, \quad Q(f)(v) = \int_{-1}^{+1} f(v')\, d\mu(v') - f(v)\,. \tag{4}$$

The parameter ε is related to the notion of mean free path: $0 < \varepsilon \ll 1$ means that particles suffer more and more collision events per time unit. As a consequence of the penalization of the collision term in (4), we guess that for small values of ε, the microscopic density f tends to an element of $\mathrm{Ker}(Q)$, which is reduced to constants with respect to the velocity variable:

$$f(t,x,v) \simeq \varrho(t,x)\,. \tag{5}$$

Let us introduce the moments of the density f:

$$\begin{pmatrix} \rho \\ J \\ p \end{pmatrix} (t,x) := \int_{-1}^{+1} \begin{pmatrix} 1 \\ v/\varepsilon \\ v^2 \end{pmatrix} f(t,x,v)\, d\mu(v)\,.$$

Integration of (4) yields the following system

$$\begin{cases} \partial_t \rho + \partial_x J = 0\,, \\ \varepsilon^2 \partial_t J + \partial_x p = -J\,. \end{cases} \tag{6}$$

The formal ansatz (5) leads to $p(t,x) \simeq \varrho(t,x) \int v^2\, d\mu(v) = \varrho(t,x)/3$, so that formally, as ε tends to 0, (6) becomes

$$\partial_t \varrho + \partial_x J = 0\,, \qquad \frac{1}{3}\, \partial_x \varrho = -J\,.$$

Hence, the limit density ϱ should satisfy the heat equation

$$\partial_t \varrho - \frac{1}{3}\, \partial^2_{xx} \varrho = 0\,. \tag{7}$$

The convergence of the solutions of (4) to the solution ϱ of the heat equation (7) can be rigorously established, for this simple model as well as for much more complicated (linear and nonlinear) kinetic models, see e.g. [2].

An important issue, for instance in nuclear engineering or in radiative transfer where this question naturally arises, consists in describing, with enough accuracy, intermediate regimes for small, but nonzero ε. Clearly the solution of the heat equation does not provide such a nice approximation, first of all because it gets rid of the velocity variable. Note also that (7) propagates information with infinite speed while in (4), the speed of propagation does not exceed $1/\varepsilon$ (that is large but finite!). To examine intermediate regimes, a possible strategy consists in closing the moment system (6), by expressing in a suitable way the second moment p as a function of the zero and first moments ρ, and J. If the resulting system is found to be hyperbolic, we shall keep a finite speed of propagation. There exists a huge variety of such closure methods. The system (1) is derived by an entropy minimization principle, as described by Levermore in [12]. Let us set

$$H(f) = \int_{-1}^{+1} f(v)\, \ln f(v)\, d\mu(v)\,,$$

and observe that H is dissipated by (4). Then, for (ρ, J) given in $\mathbb{R}^+ \times \mathbb{R}$, let us define the function $v \mapsto F(v)$ such that:

$$H(F) = \min \left\{ H(g)\,,\ g : (-1,+1) \to \mathbb{R}^+\,,\ \int_{-1}^{+1} \begin{pmatrix} 1 \\ v/\varepsilon \end{pmatrix} g \, d\mu(v) = \begin{pmatrix} \rho \\ J \end{pmatrix} \right\} .$$

The minimizer F is well-defined provided that $\varepsilon|J| < \rho$, and is given by the formula:

$$F(v) = \frac{\rho}{\mathbb{F} \circ \mathbb{G}^{-1}(\varepsilon J/\rho)} \exp\left(v\, \mathbb{G}^{-1}\big(\frac{\varepsilon J}{\rho}\big)\right) .$$

Eventually, we close (6) by requiring p to be the second moment of the minimizer F:

$$p = \int_{-1}^{+1} v^2 \, F(v) \, d\mu(v) = \rho\, \psi\Big(\frac{\varepsilon J}{\rho}\Big) ,$$

and we thus get the system (1). Of course, the closure of the moments system highly depends on the set in which the velocity variable lies. For instance, if one replaces the set $(-1,+1)$, equipped with the normalized Lebesgue measure, by the whole line $\mathbb{R}$, equipped with the Gaussian measure, then the entropy minimization procedure leads to the isothermal Euler system with relaxation:

$$\begin{cases} \partial_t \rho + \partial_x J = 0\,, \\ \varepsilon^2 \Big(\partial_t J + \partial_x \dfrac{J^2}{\rho} \Big) + \partial_x \rho = -J\,. \end{cases} \tag{8}$$

In [5], we have shown that (1) admits global smooth solutions that are bounded away from vacuum, and that the density converges towards the solution to the heat equation (7) as ε tends to zero. The intermediate system (1) is thus consistent with the diffusion limit (7). However, this result is not fully satisfactory because both the kinetic equation (4), and the heat equation (7) are perfectly defined up to $\rho = 0$, and it is therefore natural to try to justify a similar asymptotic result for solutions that may contain vacuum regions. More precisely, we want to show the existence of global weak solutions of (1) for bounded initial data that may contain vacuum, and derive uniform L^∞ bounds with respect to ε. As we shall see later on, vacuum corresponds to the boundary of the domain of hyperbolicity of (1). In view of earlier results, see e.g. [7, 14, 13, 15, 11], it seems natural to show that the vanishing viscosity method for (1) is convergent. We have not been able so far to prove such convergence. However, to prove that this method converges, one needs L^∞ bounds for the viscous approximations, and such bounds can be derived by showing the positive invariance of some domains, using the result of [4]. In this paper, we compute such positive invariant domains. There is a similar question for the homogeneous Riemann problem, that is for the system (1) without relaxation term. We shall show that this homogeneous problem can be solved without smallness assumption, and we shall compute invariant regions. We postpone the convergence of the vanishing viscosity method to a future work.

The main result of this paper is the following:

Theorem 1. *Let $\rho_r, \rho_l > 0$, and let $J_r, J_l \in \mathbb{R}$ satisfy $\varepsilon|J_r| < \rho_r$, and $\varepsilon|J_l| < \rho_l$. Then the homogeneous Riemann problem:*

$$\begin{cases} \partial_t \rho + \partial_x J = 0\,, \\ \varepsilon^2 \, \partial_t J + \partial_x \left(\rho\, \psi\Big(\dfrac{\varepsilon J}{\rho}\Big)\right) = 0\,, \end{cases}$$

with initial data

$$(\rho, J)_{|t=0} = \begin{cases} (\rho_r, J_r) & \text{if } x > 0, \\ (\rho_l, J_l) & \text{if } x < 0, \end{cases}$$

is uniquely solvable, and its solution does not contain vacuum. Moreover, for all $\rho_0 > 0$, and for all J_0 that satisfies $\varepsilon|J_0| < \rho_0$, the region $\mathcal{I}_{\rho_0,J_0}$ defined in (36) *is invariant for the Riemann problem.*

For all $\rho_0 > 0$, and all $\nu > 0$, the region $\mathcal{I}_{\rho_0,0}$ defined in (36) *is positively invariant for the viscous approximation:*

$$\begin{cases} \partial_t \rho + \partial_x J = \nu\, \partial_{xx}\rho\,, \\ \varepsilon^2\, \partial_t J + \partial_x \left(\rho\, \psi(\dfrac{\varepsilon J}{\rho})\right) = -J + \nu\, \partial_{xx} J\,. \end{cases} \tag{9}$$

The definition of the invariant regions requires some notations, so we prefer to give it later on. The paper is organized as follows. In Section 2, we prove some general properties of the relaxation free system corresponding to (1). We show that this system is strictly hyperbolic, and that its characteristic fields are genuinely nonlinear. We also exhibit some Riemann invariants. In Section 3, we investigate the Riemann problem, and show that it is uniquely solvable with invariant regions. We also prove the positive invariance of the region $\mathcal{I}_{\rho_0,0}$ for the parabolic regularization (9).

2 Some general facts on Levermore's system

In this section, we focus on the system (1) with no relaxation term:

$$\begin{cases} \partial_t \rho + \partial_x J = 0\,, \\ \varepsilon^2\, \partial_t J + \partial_x \left(\rho\, \psi(\dfrac{\varepsilon J}{\rho})\right) = 0\,. \end{cases} \tag{10}$$

The functions ψ, $\mathbb{F}$, and $\mathbb{G}$ are defined by (2)-(3). In all what follows, we use the notation $u := \varepsilon J/\rho \in (-1, 1)$ to denote the (rescaled) velocity. System (10) meets the classical hyperbolicity properties, as shown in the following:

Proposition 1. *The system* (10) *is stricly hyperbolic in the convex open set $\{(\rho, J)/\rho > 0\,, \varepsilon\,|J| < \rho\}$. Its characteristic speeds $\lambda_1^\varepsilon, \lambda_2^\varepsilon$ are given by*

$$\lambda_i^\varepsilon(\rho, J) := \frac{1}{\varepsilon}\,\boldsymbol{\lambda}_i(u)\,, \quad i = 1, 2\,,$$
$$\boldsymbol{\lambda}_1(u) := \frac{\psi'(u) - \sqrt{\psi'(u)^2 - 4u\psi'(u) + 4\psi(u)}}{2}\,,\ \boldsymbol{\lambda}_2(u) := \frac{\psi'(u) + \sqrt{\psi'(u)^2 - 4u\psi'(u) + 4\psi(u)}}{2}\,, \tag{11}$$

and the corresponding eigenvectors can be defined as follows:

$$r_i^\varepsilon(\rho, J) = \begin{pmatrix} 1 \\ \lambda_i^\varepsilon(\rho, J) \end{pmatrix}\,, \quad i = 1, 2\,. \tag{12}$$

Moreover, the function

$$\mathbb{H}(\rho, J) := \rho\, \ln \rho - \rho\, \ln\left[\mathbb{F} \circ \mathbb{G}^{-1}(u)\right] + \varepsilon J\, \mathbb{G}^{-1}(u)\,, \tag{13}$$

is a strictly convex entropy for (10).

Proof. If we write system (10) under the compact form

$$\partial_t \begin{pmatrix} \rho \\ J \end{pmatrix} + \partial_x \mathcal{F}_\varepsilon(\rho, J) = 0\,,$$

we can compute the Jacobian matrix

$$D\mathcal{F}_\varepsilon(\rho, J) = \begin{pmatrix} 0 & 1 \\ \dfrac{1}{\varepsilon^2}(\psi(u) - u\psi'(u)) & \dfrac{1}{\varepsilon}\psi'(u) \end{pmatrix}.$$

It is now easy to check that the two eigenvalues of $D\mathcal{F}_\varepsilon(\rho, J)$ are real, distinct, and given by (11). The discriminant of the characteristic polynomial is positive since

$$\psi'(u)^2 + 4(\psi(u) - u\psi'(u)) = (\psi'(u) - 2u)^2 + 4(\psi(u) - u^2) \geq 4(\psi(u) - u^2) > 0\,.$$

Thus, the system (10) is strictly hyperbolic. The reader will easily check that, up to a multiplicative constant, the eigenvectors of $D\mathcal{F}_\varepsilon(\rho, J)$ are given by (12).

That $\mathbb{H}$ is a strictly convex entropy for (10) was already checked in [5], where the corresponding flux is also computed. □

Observe that the characteristic speeds of (10) only depend on the velocity u. This property is shared by the isothermal Euler system. Another analogy with (8) is the following:

Lemma 1. *Let $\boldsymbol{\lambda}_{1,2}$ be defined by (11). Then we have:*

$$\forall\, u \in (-1, 1)\,, \quad \boldsymbol{\lambda}_1(u) < u < \boldsymbol{\lambda}_2(u)\,.$$

The proof is an easy consequence of the definition (11). Thanks to Lemma 1, we are able to define two functions

$$\Lambda_i(u) := \int_0^u \frac{dv}{\boldsymbol{\lambda}_i(v) - v}\,, \quad i = 1, 2\,, \quad u \in (-1, 1)\,. \tag{14}$$

It is clear that Λ_1 is decreasing, while Λ_2 is increasing. These functions are useful to construct Riemann invariants of (10):

Lemma 2. *Let Λ_1, Λ_2 be defined by (14), and let*

$$Z_i(\rho, J) := -\ln \rho + \Lambda_i\Big(\frac{\varepsilon J}{\rho}\Big)\,, \quad i = 1, 2\,.$$

Then Z_1, Z_2 are Riemann invariants for (10):

$$\nabla Z_1 \cdot r_1^\varepsilon = 0\,, \quad \nabla Z_2 \cdot r_2^\varepsilon = 0\,,$$

where the eigenvectors $r_1^\varepsilon, r_2^\varepsilon$ are defined by (12).

The proof is again a basic application of the chain rule, and we omit it. We end this section with a result on the genuine nonlinearity of the characteristic fields:

Proposition 2. *The function ψ is strictly convex, and it satisfies*

$$\lim_{u \to \pm 1} \psi(u) = 1\,, \quad \lim_{u \to \pm 1} \psi'(u) = \pm 2\,.$$

Furthermore $\boldsymbol{\lambda}_1$, and $\boldsymbol{\lambda}_2$ satisfy

$$\forall\, u \in \,(-1, 1)\,, \quad \boldsymbol{\lambda}_i'(u) > 0\,, \quad i = 1, 2\,,$$

and both characteristic fields of (10) are genuinely nonlinear.

Proof. We first show the strict convexity of ψ. Using the definition (2), we compute:

$$\psi''(u) = 2 + \frac{\mathbb{G}'''}{(\mathbb{G}')^2}\big(\mathbb{G}^{-1}(u)\big) - \frac{(\mathbb{G}'')^2}{(\mathbb{G}')^3}\big(\mathbb{G}^{-1}(u)\big)\,.$$

Therefore, ψ is strictly convex if, and only if the following condition holds:

$$\forall\,\beta \in \mathbb{R}\,, \quad 2\,\mathbb{G}'(\beta)^3 + \mathbb{G}'(\beta)\,\mathbb{G}'''(\beta) - \mathbb{G}''(\beta)^2 > 0\,.$$

Using the relation $\mathbb{G}(\beta) = \mathbb{F}'(\beta)/\mathbb{F}(\beta)$, and recalling that $\mathbb{F}(\beta)$ is positive, see (3), we obtain that ψ is strictly convex if, and only if, we have

$$\forall\,\beta \in \mathbb{R}\,, \quad -\mathbb{F}''(\beta)^3 + \mathbb{F}(\beta)\,\mathbb{F}''(\beta)\,\mathbb{F}^{(4)}(\beta) + 2\mathbb{F}'(\beta)\,\mathbb{F}''(\beta)\,\mathbb{F}'''(\beta) - \mathbb{F}'(\beta)^2\,\mathbb{F}^{(4)}(\beta) - \mathbb{F}(\beta)\,\mathbb{F}'''(\beta)^2 > 0\,.$$

We now use the expression (3) of $\mathbb{F}$, and are thus reduced to proving the following inequality:

$$\forall\,\beta \in \mathbb{R}\,, \quad \Upsilon(\beta) := \frac{1}{\beta^9}\,\big\{(\sinh\beta)^3 + 2\,\beta^3\cosh\beta - 3\,\beta^2\sinh\beta - \beta^4\sinh\beta\big\} > 0\,. \tag{15}$$

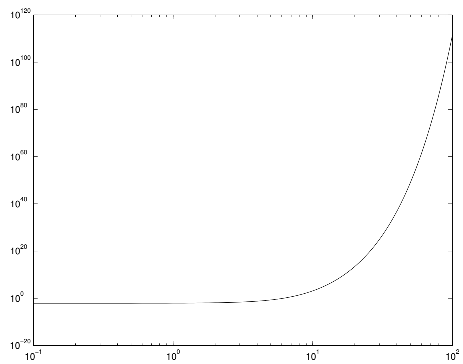

Figure 1: The function Υ on $[0,100]$ (with a log-log scale).

A numerical evidence of this inequality is shown in Figure 1, where we have depicted the function Υ on the interval $[0,100]$. For the sake of completeness, we are going to prove that $\Upsilon(\beta)$ is positive for all β, by expanding Υ in power series. (Observe that Υ can indeed be expanded in power series, and the radius of convergence is $+\infty$). After some simplifications, we are led to

$$\Upsilon(\beta) = \sum_{k=0}^{+\infty} \upsilon_k\,\beta^{2k}\,, \qquad \upsilon_k := \frac{19683 * 9^k - 8931 - 10552\,k - 4640\,k^2 - 896\,k^3 - 64\,k^4}{4\,(2k+9)!}\,.$$

The reader will easily check that υ_0, and υ_1 are positive. When $k \geq 2$, one has

$$\begin{aligned}\frac{19683 * 9^k}{k^4} \geq \frac{19683 * 9^2}{2^4} &\geq 64 + \frac{896}{2} + \frac{4640}{4} + \frac{10552}{8} + \frac{8931}{16} \\ &\geq 64 + \frac{896}{k} + \frac{4640}{k^2} + \frac{10552}{k^3} + \frac{8931}{k^4}\,.\end{aligned}$$

Consequently, the υ_k's are nonnegative for all $k \geq 1$, and $\Upsilon(\beta)$ is bounded from below by $\upsilon_0 > 0$. This proves that ψ is strictly convex.

From the definition (3) of the function $\mathbb{G}$, we have

$$\lim_{|\beta|\to+\infty} \mathbb{G}'(\beta) = 0\,, \qquad \lim_{u\to\pm1} \mathbb{G}^{-1}(u) = \pm\infty\,.$$

Since we have $\psi(u) = u^2 + \mathbb{G}' \circ \mathbb{G}^{-1}(u)$, we get

$$\lim_{|u|\to 1} \psi(u) = 1\,.$$

To compute the limit of $\psi'(u)$ at ± 1, we use the expression

$$\psi'(u) = 2u + \frac{\mathbb{G}''}{\mathbb{G}'} \circ \mathbb{G}^{-1}(u)\,. \tag{16}$$

The function $\mathbb{G}$ satisfies

$$\mathbb{G}'(\beta) = \frac{1}{\beta^2} - \frac{1}{(\sinh\beta)^2}\,, \quad \mathbb{G}''(\beta) = \frac{-2}{\beta^3} + \frac{2\cosh\beta}{(\sinh\beta)^3}\,,$$

which implies

$$\lim_{|\beta|\to+\infty} \frac{\mathbb{G}''}{\mathbb{G}'}(\beta) = 0\,.$$

This yields the desired limit for $\psi'(u)$ at ± 1.

We are now going to show that the characteristic fields are genuinely nonlinear. Using the definition (11), the $\boldsymbol{\lambda}_i$'s satisfy the second order polynomial equation

$$\forall\, u \in (-1,1)\,, \quad \boldsymbol{\lambda}_i(u)(\boldsymbol{\lambda}_i(u) - \psi'(u)) - (\psi(u) - u\psi'(u)) = 0\,.$$

Differentiating this equation with respect to u, and using the relations

$$-[2\boldsymbol{\lambda}_1(u) - \psi'(u)] = 2\boldsymbol{\lambda}_2(u) - \psi'(u) = \sqrt{\Delta(u)}\,,$$

where

$$\forall\, u \in (-1,1)\,, \quad \Delta(u) := \psi'(u)^2 + 4(\psi(u) - u\psi'(u)) > 0\,, \tag{17}$$

we end up with

$$\boldsymbol{\lambda}_1'(u)\,\sqrt{\Delta(u)} = \psi''(u)\,(u - \boldsymbol{\lambda}_1(u))\,, \quad \boldsymbol{\lambda}_2'(u)\,\sqrt{\Delta(u)} = \psi''(u)\,(\boldsymbol{\lambda}_2(u) - u)\,.$$

Using Lemma 1 and the strict convexity of ψ, we get $\boldsymbol{\lambda}_1'(u) > 0$, and $\boldsymbol{\lambda}_2'(u) > 0$.

Using (11) and (12), we compute

$$\nabla\lambda_i^\varepsilon \cdot r_i^\varepsilon = \frac{1}{\varepsilon\rho}\,\boldsymbol{\lambda}_i'(u)\,(\boldsymbol{\lambda}_i(u) - u) \neq 0\,,$$

so both characteristic fields are genuinely nonlinear. $\square$

Because the characteristic fields are genuinely nonlinear, we can expect to solve the Riemann problem for (10) with a juxtaposition of shock waves and rarefaction waves (see e.g. [3, 10, 16] for a complete description of the theory). In the next section, we are going to detail the Rankine-Hugoniot relations, and exhibit the 1-Lax shocks and the 2-Lax shocks (meaning the discontinuities satisfying Lax' shock inequalities). The rarefaction curves will be computed by using the Riemann invariants of Lemma 2. Then we shall show that the Riemann problem for (10) is uniquely solvable, without any restriction on the data.

3 Global solvability of the Riemann problem

3.1 The shock curves

In this paragraph, we are going to compute the shock curves for the hyperbolic system (10). We fix, once and for all, a state (ρ_-, J_-) in the domain of hyperbolicity. We are going to compute the states (ρ_+, J_+) such that there exists a speed $s \in \mathbb{R}$ verifying the Rankine-Hugoniot relations for the system (10). The Rankine-Hugoniot relations read

$$\begin{cases} J_+ - J_- = s\,(\rho_+ - \rho_-)\,, \\ \dfrac{1}{\varepsilon^2}\,[\rho_+\,\psi(u_+) - \rho_-\,\psi(u_-)] = s\,(J_+ - J_-)\,. \end{cases} \tag{18}$$

In (18), we have used the notations $u_\pm = \varepsilon J_\pm/\rho_\pm$.

If $\rho_+ = \rho_-$, we get the obvious solution $J_+ = J_-$. We thus assume from now on $\rho_+ \neq \rho_-$, and we rewrite the Rankine-Hugoniot relations (18) in the following way:

$$s = \frac{1}{\varepsilon}\,\frac{\rho_+ u_+ - \rho_- u_-}{\rho_+ - \rho_-}\,, \quad (\rho_+ - \rho_-)(\rho_+\psi(u_+) - \rho_-\psi(u_-)) = (\rho_+ u_+ - \rho_- u_-)^2\,. \tag{19}$$

We have the following result:

Proposition 3. *When $u_+ \neq u_-$, the solutions of* (19) *consist in the two following curves, that are parametrized by u_+:*

$$\begin{aligned} (\rho_+, J_+, s) &= \left(\rho_-\,X_1(u_-,u_+), \frac{\rho_-\,X_1(u_-,u_+)\,u_+}{\varepsilon}, \frac{1}{\varepsilon}\,\frac{X_1(u_-,u_+)\,u_+ - u_-}{X_1(u_-,u_+) - 1}\right), \\ \text{and } (\rho_+, J_+, s) &= \left(\rho_-\,X_2(u_-,u_+), \frac{\rho_-\,X_2(u_-,u_+)\,u_+}{\varepsilon}, \frac{1}{\varepsilon}\,\frac{X_2(u_-,u_+)\,u_+ - u_-}{X_2(u_-,u_+) - 1}\right), \end{aligned}$$

where $u_+ \in (-1, u_-) \cup (u_-, 1)$, and where $X_1(u_-,u_+)$, $X_2(u_-,u_+)$ are the roots of the second order polynomial equation

$$\mathcal{P}(X) := X^2\,(\psi(u_+) - u_+^2) - X\,(\psi(u_+) + \psi(u_-) - 2u_-u_+) + (\psi(u_-) - u_-^2) = 0\,, \tag{20}$$

and satisfy

$$0 < X_2(u_-,u_+) < 1 < X_1(u_-,u_+)\,.$$

Proof. Defining $X := \rho_+/\rho_-$, the second equation of (19) is found to be equivalent to (20). Recall that $\psi(u) - u^2 > 0$, see (2). We thus only need to show that $\mathcal{P}$ has two positive real roots when $u_+ \neq u_-$. The discriminant $\widetilde{\delta}(u_-,u_+)$ of $\mathcal{P}$ is given by the relation:

$$\begin{aligned} &\widetilde{\delta}(u_-,u_+) = (u_+ - u_-)^2\,\delta(u_-,u_+)\,, \\ &\delta(u_-,u_+) := z^2 - 4u_+z + 4\psi(u_+) = z^2 - 4u_-z + 4\psi(u_-)\,, \quad \text{with} \quad z := \frac{\psi(u_+) - \psi(u_-)}{u_+ - u_-}\,. \end{aligned} \tag{21}$$

Since we have

$$\widetilde{\delta}(u_-,u_+) = (u_+ - u_-)^2\,[(z - 2u_+)^2 + 4(\psi(u_+) - u_+^2)] > 0\,,$$

$\mathcal{P}$ has two distinct real roots $X_2(u_-,u_+) < X_1(u_-,u_+)$. (The reader will understand the choice of our notations when we shall prove Lax' shock inequalities). Note that these roots are simple when $u_+ \neq u_-$, therefore X_1, X_2 are $\mathcal{C}^\infty$ functions of (u_-,u_+) by the implicit functions Theorem. Moreover, we have

$$\begin{gathered} X_1X_2 > 0\,, \quad (\psi(u_+) - u_+^2)(X_1 + X_2) = \psi(u_+) + \psi(u_-) - 2u_-u_+ > u_+^2 + u_-^2 - 2u_-u_+ > 0\,, \\ \mathcal{P}(1) = -(u_+ - u_-)^2 < 0\,, \end{gathered}$$

and we thus have the inequalities

$$0 < X_2 < 1 < X_1 \,,$$

as long as $u_+ \neq u_-$. The proof is complete. $\square$

We have thus constructed all the nontrivial solutions to (10) of the form

$$(\rho, J) = \begin{cases} (\rho_-, J_-) & \text{if } x < s\,t\,, \\ (\rho_+, J_+) & \text{if } x > s\,t\,. \end{cases}$$

We now need to characterize those discontinuous solutions that are admissible, in the sense that they satisfy Lax' shock inequalities, see [10]. As expected from the general theory, only half of each curve defined above is admissible:

Proposition 4. *When $u_+ < u_-$, Lax' shock inequalities hold, that is, we have:*

$$\frac{1}{\varepsilon}\,\boldsymbol{\lambda}_1(u_+) < \frac{1}{\varepsilon}\,\frac{X_1(u_-,u_+)\,u_+ - u_-}{X_1(u_-,u_+) - 1} < \frac{1}{\varepsilon}\,\boldsymbol{\lambda}_2(u_+)\,, \quad \frac{1}{\varepsilon}\,\frac{X_1(u_-,u_+)\,u_+ - u_-}{X_1(u_-,u_+) - 1} < \frac{1}{\varepsilon}\,\boldsymbol{\lambda}_1(u_-)\,,$$
$$\frac{1}{\varepsilon}\,\boldsymbol{\lambda}_1(u_-) < \frac{1}{\varepsilon}\,\frac{X_2(u_-,u_+)\,u_+ - u_-}{X_2(u_-,u_+) - 1} < \frac{1}{\varepsilon}\,\boldsymbol{\lambda}_2(u_-)\,, \quad \frac{1}{\varepsilon}\,\boldsymbol{\lambda}_2(u_+) < \frac{1}{\varepsilon}\,\frac{X_2(u_-,u_+)\,u_+ - u_-}{X_2(u_-,u_+) - 1}\,.$$

When $u_+ > u_-$, Lax' shock inequalities do not hold.

Proof. We first assume $u_+ > u_-$. In this case, Proposition 2 yields $\boldsymbol{\lambda}_i(u_+) > \boldsymbol{\lambda}_i(u_-)$, for $i = 1, 2$. Therefore, Lax' shock inequalities cannot hold when $u_+ > u_-$, and the discontinuity is not admissible.

From now on, we assume $u_+ \in (-1, u_-)$, and we want to prove Lax' shock inequalities, that is:

$$\boldsymbol{\lambda}_1(u_+) < \frac{X_1(u_-,u_+)\,u_+ - u_-}{X_1(u_-,u_+) - 1} < \boldsymbol{\lambda}_2(u_+)\,, \quad \frac{X_1(u_-,u_+)\,u_+ - u_-}{X_1(u_-,u_+) - 1} < \boldsymbol{\lambda}_1(u_-)\,.$$

Recall that the velocities $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ are defined in Proposition 1, and that $X_1(u_-,u_+) > 1$.

a) We first show the inequality

$$\big[X_1(u_-,u_+) - 1\big]\,\boldsymbol{\lambda}_1(u_+) < X_1(u_-,u_+)u_+ - u_-\,. \tag{22}$$

From the definition of $X_1(u_-,u_+)$, see (20) and (21) for the definition of δ and z, we obtain:

$$\frac{X_1(u_-,u_+) - 1}{u_- - u_+} = \frac{z - 2u_+ + \sqrt{\delta(u_-,u_+)}}{2(\psi(u_+) - u_+^2)}\,,$$
$$\frac{X_1(u_-,u_+)u_+ - u_-}{u_- - u_+} = \frac{u_+ z - 2\psi(u_+) + u_+\sqrt{\delta(u_-,u_+)}}{2(\psi(u_+) - u_+^2)}\,.$$

Simplifying (22) with the help of these two relations, it turns out that (22) is equivalent to

$$(\boldsymbol{\lambda}_1(u_+) - u_+)z + 2(\psi(u_+) - u_+\boldsymbol{\lambda}_1(u_+)) < (u_+ - \boldsymbol{\lambda}_1(u_+))\sqrt{\delta(u_-,u_+)}\,. \tag{23}$$

Because $u_+ > \boldsymbol{\lambda}_1(u_+)$ (see Lemma 1), (23) will be satisfied provided that we have the weaker inequality

$$\Big[(\boldsymbol{\lambda}_1(u_+) - u_+)z + 2(\psi(u_+) - u_+\boldsymbol{\lambda}_1(u_+))\Big]^2 < \Big[u_+ - \boldsymbol{\lambda}_1(u_+)\Big]^2\,\delta(u_-,u_+)\,. \tag{24}$$

Expanding (24), and using (21), (24) can be rewritten as

$$z\,(u_+ - \boldsymbol{\lambda}_1(u_+)) - \psi(u_+) + \boldsymbol{\lambda}_1(u_+)^2 > 0\,. \tag{25}$$

Because ψ is strictly convex, and $u_+ < u_-$, we obtain (we use (11) for the last equality):

$$z\,(u_+ - \boldsymbol{\lambda}_1(u_+)) - \psi(u_+) + \boldsymbol{\lambda}_1(u_+)^2 > \psi'(u_+)\,(u_+ - \boldsymbol{\lambda}_1(u_+)) - \psi(u_+) + \boldsymbol{\lambda}_1(u_+)^2 = 0\,.$$

As a consequence, (25) holds, and therefore (22) is proved.

b) We now prove the inequality

$$X_1(u_-, u_+)\,u_+ - u_- < [X_1(u_-, u_+) - 1]\,\boldsymbol{\lambda}_2(u_+)\,.$$

This inequality is seen to be equivalent to

$$(u_+ - \boldsymbol{\lambda}_2(u_+))z - 2(\psi(u_+) - u_+\boldsymbol{\lambda}_2(u_+)) < (\boldsymbol{\lambda}_2(u_+) - u_+)\sqrt{\delta(u_-, u_+)}\,. \tag{26}$$

Because $\boldsymbol{\lambda}_2(u_+) > u_+$, the right-hand side of (26) is positive, and (26) will hold provided that the left-hand side is nonpositive, that is,

$$(\boldsymbol{\lambda}_2(u_+) - u_+)z + 2(\psi(u_+) - u_+\boldsymbol{\lambda}_2(u_+)) \geq 0\,. \tag{27}$$

Observe that we have

$$(\boldsymbol{\lambda}_2(u_+) - u_+)\,z > (\boldsymbol{\lambda}_2(u_+) - u_+)\,\psi'(u_+)\,.$$

Using the definition (11) of $\boldsymbol{\lambda}_2(u_+)$, and the definition (17) of Δ, we can write

$$\boldsymbol{\lambda}_2(u_+) = \frac{\psi'(u_+) + \sqrt{\Delta(u_+)}}{2}\,.$$

Then, one checks that

$$(\boldsymbol{\lambda}_2(u_+) - u_+)\,\psi'(u_+) + 2(\psi(u_+) - u_+\boldsymbol{\lambda}_2(u_+)) = \Delta(u_+) + (\psi'(u_+) - 2u_+)\sqrt{\Delta(u_+)} \geq 0\,.$$

Hence (27) holds, and the second Lax inequality is proved.

c) Eventually, we prove the inequality

$$\frac{X_1(u_-, u_+)u_+ - u_-}{X_1(u_-, u_+) - 1} < \boldsymbol{\lambda}_1(u_-)\,. \tag{28}$$

Differentiating with respect to $u \in (-1, u_-)$, one checks that the function

$$u \longmapsto \frac{X_1(u_-, u)u - u_-}{X_1(u_-, u) - 1}$$

is strictly increasing. To prove (28), it is therefore sufficient to show the inequality

$$\lim_{u_+ \to u_-} \frac{X_1(u_-, u_+)u_+ - u_-}{X_1(u_-, u_+) - 1} = \boldsymbol{\lambda}_1(u_-)\,. \tag{29}$$

From the equality

$$\frac{X_1(u_-, u_+)u_+ - u_-}{X_1(u_-, u_+) - 1} = \frac{u_+ z - 2\psi(u_+) + u_+\sqrt{\delta(u_-, u_+)}}{z - 2u_+ + \sqrt{\delta(u_-, u_+)}}\,, \quad z = \frac{\psi(u_+) - \psi(u_-)}{u_+ - u_-}\,,$$

we compute

$$\lim_{u_+ \to u_-} \frac{X_1(u_-, u_+)u_+ - u_-}{X_1(u_-, u_+) - 1} = \frac{u_-\psi'(u_-) - 2\psi(u_-) + u_-\sqrt{\Delta(u_-)}}{\psi'(u_-) - 2u_- + \sqrt{\Delta(u_-)}}$$
$$= \frac{\psi'(u_-) - \sqrt{\Delta(u_-)}}{2} = \boldsymbol{\lambda}_1(u_-)\,,$$

and (29) holds. We have thus proved (28), and this concludes the proof of the 1-Lax shock inequalities. The 2-Lax shock inequalities are proved in a completely similar way, and we shall not reproduce the calculations. □

Remark 1. *It would be interesting to know whether the admissible shocks also satisfy the entropy inequality*

$$s\Big(\mathbb{H}(\rho_+, J_+) - \mathbb{H}(\rho_-, J_-)\Big) - \big(q(\rho_+, J_+) - q(\rho_-, J_-)\big) \geq 0\,,$$

where $\mathbb{H}$ is the strictly convex entropy defined by (13), and q is the corresponding flux. From the general theory, we know that this inequality holds for small amplitude shocks, that is, when $u_+ < u_-$, and u_+ close to u_-, see e.g. [3]. Unfortunately, we have not been able to prove this inequality in the general case.

The following Lemma will be useful when solving the Riemann problem:

Lemma 3. *The functions X_1 and X_2 satisfy*

$$\forall\, u_- \in (-1,1)\,, \qquad \lim_{u_+ \to -1} X_1(u_-, u_+) = +\infty\,,$$
$$\forall\, u_+ \in (-1,1)\,, \qquad \lim_{u_- \to 1} X_2(u_-, u_+) = 0\,.$$

Proof. When $u_+ < u_-$, the definition of X_1 is

$$X_1(u_-, u_+) = \frac{\psi(u_+) + \psi(u_-) - 2u_- u_+ + (u_- - u_+)\sqrt{\delta(u_-, u_+)}}{2(\psi(u_+) - u_+^2)}\,,$$

see Proposition 3, and (21). We have

$$\begin{aligned}\psi(u_+) + \psi(u_-) - 2u_- u_+ + (u_- - u_+)\sqrt{\delta(u_-, u_+)} &\geq \psi(u_+) + \psi(u_-) - 2u_- u_+ \\ &\geq \psi(u_+) - u_+^2 + \psi(u_-) - u_-^2 \geq \psi(u_-) - u_-^2 > 0\,,\end{aligned}$$

and we also have

$$\lim_{u_+ \to -1} [\psi(u_+) - u_+^2] = 0^+\,.$$

The first part of the Lemma follows.

For $u_+ < u_-$, we also have

$$X_2(u_-, u_+) = \frac{\psi(u_+) + \psi(u_-) - 2u_- u_+ - \sqrt{\widetilde{\delta}(u_-, u_+)}}{2(\psi(u_+) - u_+^2)}\,,$$

with

$$\widetilde{\delta}(u_-, u_+) = [\psi(u_+) + \psi(u_-) - 2u_- u_+]^2 - 4(\psi(u_+) - u_+^2)(\psi(u_-) - u_-^2)\,.$$

We thus compute

$$\psi(u_+) + \psi(u_-) - 2u_- u_+ - \sqrt{\widetilde{\delta}(u_-, u_+)} \to 0\,, \quad \text{when} \quad u_- \to 1\,,$$

and the proof is complete. □

3.2 The rarefaction curves

In this paragraph, we construct the rarefaction curves. Recall that a i-rarefaction wave ($i = 1, 2$), connecting two states (ρ_-, J_-) and (ρ_+, J_+), is a solution to (10) of the form

$$(\rho, J)(t, x) = \begin{cases} (\rho_-, J_-) & \text{if } x \leq \lambda_i^\varepsilon(\rho_-, J_-)t\,, \\ W(x/t) & \text{if } \lambda_i^\varepsilon(\rho_-, J_-)t \leq x \leq \lambda_i^\varepsilon(\rho_+, J_+)t\,, \\ (\rho_+, J_+) & \text{if } \lambda_i^\varepsilon(\rho_+, J_+)t \leq x\,, \end{cases}$$

with $\lambda_i^\varepsilon(\rho_-, J_-) \le \lambda_i^\varepsilon(\rho_+, J_+)$, and where $W(\xi)$ is a solution to the ODE:

$$W'(\xi) = \frac{r_i^\varepsilon(W(\xi))}{\nabla\lambda_i^\varepsilon(W(\xi)) \cdot r_i^\varepsilon(W(\xi))}, \quad \xi \in [\lambda_i^\varepsilon(\rho_-, J_-), \lambda_i^\varepsilon(\rho_+, J_+)].$$

Thanks to Lemma 2, we know that the integral curves of the vector field r_1^ε (resp. r_2^ε) are nothing but the level sets of the Riemann invariant Z_1 (resp. Z_2). Moreover, the constraint $\lambda_i^\varepsilon(\rho_-, J_-) \le \lambda_i^\varepsilon(\rho_+, J_+)$ is equivalent to $u_- \le u_+$, because the functions $\boldsymbol{\lambda}_i$ are increasing (see Lemma 2). We thus have:

Proposition 5. *Two states (ρ_-, J_-) and (ρ_+, J_+) are connected by a 1-rarefaction wave if and only if*

$$u_- \le u_+, \quad \text{and} \quad \rho_+ = \rho_- \exp \int_{u_-}^{u_+} \frac{dw}{\boldsymbol{\lambda}_1(w) - w}.$$

Two states (ρ_-, J_-) and (ρ_+, J_+) are connected by a 2-rarefaction wave if and only if

$$u_- \le u_+, \quad \text{and} \quad \rho_+ = \rho_- \exp \int_{u_-}^{u_+} \frac{dw}{\boldsymbol{\lambda}_2(w) - w}.$$

Moreover, we have the following asymptotic behavior for the rarefaction curves:

Lemma 4. *We have*

$$\int_{-1}^{0} \frac{dw}{w - \boldsymbol{\lambda}_1(w)} = \int_{0}^{1} \frac{dw}{w - \boldsymbol{\lambda}_1(w)} = \int_{-1}^{0} \frac{dw}{\boldsymbol{\lambda}_2(w) - w} = \int_{0}^{1} \frac{dw}{\boldsymbol{\lambda}_2(w) - w} = +\infty.$$

Proof. From (11) and (2), we have

$$2(u - \boldsymbol{\lambda}_1(u)) = 2u - \psi'(u) + \sqrt{\Delta(u)} = \sqrt{\Delta(u)} - \frac{\mathbb{G}''}{\mathbb{G}'}(\mathbb{G}^{-1}(u)),$$

$$\text{and} \quad \Delta(u) = 4\,\mathbb{G}'(\mathbb{G}^{-1}(u)) + \left(\frac{\mathbb{G}''}{\mathbb{G}'}(\mathbb{G}^{-1}(u))\right)^2.$$

We thus have

$$2(u - \boldsymbol{\lambda}_1(u))\,\mathbb{G}'(\mathbb{G}^{-1}(u)) = \left[\sqrt{4(\mathbb{G}')^3 + (\mathbb{G}'')^2} - \mathbb{G}''\right](\mathbb{G}^{-1}(u)).$$

Recall the expression

$$\mathbb{G}(\beta) = \frac{1}{\tanh\beta} - \frac{1}{\beta},$$

so the function $\mathbb{G}$ satisfies

$$\lim_{\beta\to+\infty} \beta^2\,\mathbb{G}'(\beta) = 1,$$

$$\lim_{\beta\to+\infty} \beta^3 \left[\sqrt{4\mathbb{G}'(\beta)^3 + \mathbb{G}''(\beta)^2} - \mathbb{G}''(\beta)\right] = 2\sqrt{2} + 2.$$

We also compute

$$\lim_{\beta\to+\infty} \beta\,(1 - \mathbb{G}(\beta)) = 1, \quad \text{that is,} \quad \lim_{u\to1} \mathbb{G}^{-1}(u)\,(1-u) = 1.$$

Eventually, we obtain the following asymptotic behavior:

$$\frac{1}{u - \boldsymbol{\lambda}_1(u)} \sim \frac{1}{(1+\sqrt{2})(1-u)}, \quad \text{when} \quad u \to 1^-,$$

and the result for the first integral follows. One shows that the three other integrals diverge in a similar way. ∎

3.3 Global solvability of the Riemann problem

In this section, we show how to solve the Riemann problem for (10), that is, we solve (10) with an initial datum of the form

$$(\rho_0, J_0)(x) = \begin{cases} (\rho_l, J_l) & \text{if } x < 0, \\ (\rho_r, J_r) & \text{if } x > 0, \end{cases}$$

with $\rho_r, \rho_l > 0$, $\varepsilon|J_r| < \rho_r$, and $\varepsilon|J_l| < \rho_l$. From the general theory, see e.g. [3, 16], we know that the solution is a 1-wave (either a shock or a rarefaction) connecting (ρ_l, J_l) to an intermediate state (ρ_m, J_m), followed by a 2-wave connecting (ρ_m, J_m) to (ρ_r, J_r). In order to determine the intermediate state (ρ_m, J_m), we define the following functions for $i = 1, 2$:

$$\forall\,(u_-, u_+) \in (-1,1) \times (-1,1)\,, \quad \mathcal{L}_i(u_-, u_+) := \begin{cases} X_i(u_-, u_+) & \text{if } u_+ < u_-, \\ \exp \displaystyle\int_{u_-}^{u_+} \frac{dw}{\boldsymbol{\lambda}_i(w) - w} & \text{if } u_+ \geq u_-. \end{cases} \tag{30}$$

The $\mathcal{L}_i$'s satisfy the following regularity and monotonicity properties:

Proposition 6. *The functions $\mathcal{L}_1$ and $\mathcal{L}_2$ are positive, and within $\mathcal{C}^2((-1,1) \times (-1,1))$. Moreover, one has*

$$\frac{\partial \mathcal{L}_1}{\partial u_+}(u_-, u_+) < 0\,, \quad \frac{\partial \mathcal{L}_2}{\partial u_-}(u_-, u_+) < 0\,,$$

and

$$\lim_{u_+ \to -1} \mathcal{L}_1(u_-, u_+) = +\infty\,, \quad \lim_{u_+ \to 1} \mathcal{L}_1(u_-, u_+) = 0\,,$$
$$\lim_{u_- \to -1} \mathcal{L}_2(u_-, u_+) = +\infty\,, \quad \lim_{u_- \to 1} \mathcal{L}_2(u_-, u_+) = 0\,.$$

Proof. The $\mathcal{L}_i$'s are clearly positive, and they are also $\mathcal{C}^\infty$ on either side of the diagonal $\{u_+ = u_-\}$. To prove the $\mathcal{C}^2$ smoothness, one only needs to show that the partial derivatives, up to order 2, can be extended by continuity on either side of the diagonal, and do not experience a jump. This can be checked directly from the definitions (30). First, it is clear that the $\mathcal{L}_i$'s are continuous. Now we compute the partial derivative of $\mathcal{L}_1$ with respect to u_+:

$$\frac{\partial \mathcal{L}_1}{\partial u_+} = \frac{1}{\boldsymbol{\lambda}_1(u_+) - u_+}\,\mathcal{L}_1\,, \quad \text{if } u_+ \geq u_-\,,$$

while

$$2(\psi(u_+) - u_+^2)\,\frac{\partial \mathcal{L}_1}{\partial u_+} + 2(\psi'(u_+) - 2u_+)\,\mathcal{L}_1 = \psi'(u_+) - 2u_- - \sqrt{\delta} + (u_- - u_+)\frac{\partial \sqrt{\delta}}{\partial u_+}\,, \quad \text{if } u_+ < u_-\,.$$

The partial derivatives thus have finite limits on either side of the diagonal, and we compute

$$\frac{\partial \mathcal{L}_1}{\partial u_+}(u_-, u_- + 0) = \frac{2}{\psi'(u_-) - 2u_- - \sqrt{\Delta(u_-)}}\,,$$
$$\frac{\partial \mathcal{L}_1}{\partial u_+}(u_-, u_- - 0) = \frac{\psi'(u_-) - 2u_- + \sqrt{\Delta(u_-)}}{-2(\psi(u_-) - u_-^2)}\,.$$

These two quantities are equal thanks to (17). Therefore $\mathcal{L}_1$ has a continuous partial derivative with respect to u_+. For the partial derivative with respect to u_-, the calculations are similar, as for the second order partial derivatives. We shall leave the end of the calculations to the interested reader.

The regularity property of $\mathcal{L}_1, \mathcal{L}_2$ is no suprise, since it is predicted by the general theory, see [3, 10, 16]. Moreover, the limits stated in the Proposition are easily deduced from Lemmas 3 and 4. The only thing left to prove is the monotonicity property of $\mathcal{L}_1, \mathcal{L}_2$. We only detail the proof for $\mathcal{L}_1$. For $u_+ \geq u_-$, one has

$$\frac{\partial \mathcal{L}_1}{\partial u_+}(u_-, u_+) = \frac{1}{\boldsymbol{\lambda}_1(u_+) - u_+}\, \mathcal{L}_1(u_-, u_+) < 0\,,$$

see Lemma 1. From now on, we thus restrict to the case $u_+ < u_-$, and we wish to prove

$$\forall\, u_+ < u_-\,, \quad \frac{\partial X_1}{\partial u_+}(u_-, u_+) < 0\,.$$

Using (20), and differentiating with respect to u_+, we obtain

$$\big[2(\psi(u_+) - u_+^2)X_1 - (\psi(u_+) + \psi(u_-) - 2u_- u_+)\big]\frac{\partial X_1}{\partial u_+} = (\psi'(u_+) - 2u_-)X_1 - (\psi'(u_+) - 2u_+)X_1^2\,. \tag{31}$$

Using the discriminant (21) of the polynomial $\mathcal{P}$, the equality (31) also reads

$$\frac{(u_- - u_+)\sqrt{\delta}}{X_1}\,\frac{\partial X_1}{\partial u_+} = (1 - X_1)(\psi'(u_+) - 2u_+) + 2(u_+ - u_-)\,. \tag{32}$$

We thus want to prove that the right-hand side of (32) is negative when $u_+ < u_-$.

Before going on, we state an intermediate result:

Lemma 5.

- *One has $\mathbb{G}''(\beta) > 0$ for $\beta < 0$, and $\mathbb{G}''(\beta) < 0$ for $\beta > 0$. Moreover, one has $\psi'(u) > 2u$ for $u < 0$, and $\psi'(u) < 2u$ for $u > 0$.*
- *For all $\beta > 0$, one has*
$$2\,\mathbb{G}'(\beta)^2 + \mathbb{G}''(\beta)\,(1 - \mathbb{G}(\beta)) > 0\,.$$
Moreover, when $0 < u_+ < u_-$, one has
$$4(\psi(u_+) - u_+^2) + (2u_+ - \psi'(u_+))(2u_+ - z) > 0\,.$$

The proof of this Lemma is given in an appendix at the end of this paper. Using the first result of Lemma 5, we already find that the right-hand side of (32) is negative when u_+ is nonpositive. We can thus assume $0 < u_+ < u_-$, and $(\psi'(u_+) - 2u_+) < 0$. In this case, the right-hand side of (32) is negative if and only if:

$$\frac{X_1 - 1}{u_- - u_+} < \frac{2}{2u_+ - \psi'(u_+)}\,.$$

Using the expression of X_1, this is found to be equivalent to

$$\sqrt{\delta} < \frac{4(\psi(u_+) - u_+^2)}{2u_+ - \psi'(u_+)} - (z - 2u_+)\,. \tag{33}$$

By the second point of Lemma 5, the right-hand side of (33) is positive, and (33) is therefore equivalent to

$$(2u_+ - z)^2 + 4(\psi(u_+) - u_+^2) < \left[\frac{4(\psi(u_+) - u_+^2)}{2u_+ - \psi'(u_+)} - (z - 2u_+)\right]^2\,. \tag{34}$$

Developping and making a few simplifications, we find that (34) is equivalent to

$$4\psi(u_+) - \psi'(u_+)^2 + 2z(\psi'(u_+) - 2u_+) > 0\,.$$

We now use the expression (21) of z, and we thus want to prove that

$$\Theta(u_-) := (4\psi(u_+) - \psi'(u_+)^2)(u_- - u_+) + 2(\psi'(u_+) - 2u_+)(\psi(u_-) - \psi(u_+)) > 0\,, \tag{35}$$

as long as $0 < u_+ < u_- < 1$. We consider that u_+ is fixed, and that u_- is a variable in the open interval $(u_+, 1)$. We compute

$$\Theta''(u_-) = 2(\psi'(u_+) - 2u_+)\psi''(u_-) < 0\,.$$

Hence Θ is a strictly concave function, and we get

$$\forall\, u_- \in (u_+, 1)\,, \quad \Theta(u_-) > \min(\Theta(u_+), \Theta(1)) = \min(0, \Theta(1))\,.$$

Eventually, using the notation $\beta_+ := \mathbb{G}^{-1}(u_+)$, we compute

$$\begin{aligned}\Theta(1) = (2 - \psi'(u_+))\big[2\psi(u_+) + \psi'(u_+) - 2u_+ - u_+\psi'(u_+)\big]& \\ = \frac{2 - \psi'(u_+)}{\mathbb{G}'(\beta_+)}\,\big[2\mathbb{G}'(\beta_+)^2 + \mathbb{G}''(\beta_+)(1 - \mathbb{G}(\beta_+))\big]\,.&\end{aligned}$$

Thanks to Lemma 5, this last term is positive, and we can conclude that (35) holds. Going back to the beginning, this means that the right-hand side of (32) is always negative. This finishes the proof. $\square$

Using the monotonicity properties of the functions $\mathcal{L}_1, \mathcal{L}_2$, we can solve the Riemann problem without any restriction on the data. Indeed, let $\rho_r, \rho_l > 0$, and let J_r, J_l satisfy $\varepsilon|J_r| < \rho_r$, and $\varepsilon|J_l| < \rho_l$. To solve the Riemann problem, we need to find an intermediate state (ρ_m, J_m) such that (ρ_l, J_l) is connected to (ρ_m, J_m) by a 1-wave, and (ρ_m, J_m) is connected to (ρ_r, J_r) by a 2-wave. This means that ρ_m, and $u_m = \varepsilon J_m/\rho_m$ must solve

$$\frac{\rho_m}{\rho_l} = \mathcal{L}_1(u_l, u_m)\,, \quad \text{and} \quad \frac{\rho_r}{\rho_m} = \mathcal{L}_2(u_m, u_r)\,.$$

Thanks to Proposition 6, we know that there exists a unique solution $u_m \in (-1, 1)$ to the equation

$$\frac{\rho_r}{\rho_l} = \mathcal{L}_1(u_l, u_m)\,\mathcal{L}_2(u_m, u_r)\,.$$

Consequently, the Riemann problem admits a unique solution.

We now prove the invariance of some regions for the Riemann problem. Let $\rho_0 > 0$, and let J_0 satisfy $\varepsilon|J_0| < \rho_0$. Define the following set:

$$\mathcal{I}_{\rho_0,J_0} := \Big\{(\rho, J) \in (0, \rho_0) \times \mathbb{R}/\varepsilon|J| < \rho\,,\ Z_i(\rho, J) \geq Z_i(\rho_0, J_0) \quad i = 1, 2\Big\}\,. \tag{36}$$

We want to show that $\mathcal{I}_{\rho_0,J_0}$ is an invariant region for the Riemann problem. (Recall that Z_1, Z_2 are defined in Lemma 2). Thanks to the characterization of [8, Corollary 3.7], it is enough to prove that the set $\mathcal{I}_{\rho_0,J_0}$ is convex (in the (ρ, J) plane). Using (14), and Lemma 4, we first get that Λ_1 is a decreasing diffeomorphism from $(-1, 1)$ onto $\mathbb{R}$, while Λ_2 is an increasing diffeomorphism from $(-1, 1)$ onto $\mathbb{R}$. Moreover, we can rewrite $\mathcal{I}_{\rho_0,J_0}$ as

$$\begin{aligned}\mathcal{I}_{\rho_0,J_0} = \Big\{&(\rho, J) \in (0, \rho_0) \times \mathbb{R}/ \\ &\Lambda_2^{-1}\Big(\Lambda_2(u_0) + \ln\rho - \ln\rho_0\Big) \leq \frac{\varepsilon J}{\rho} \leq \Lambda_1^{-1}\Big(\Lambda_1(u_0) + \ln\rho - \ln\rho_0\Big)\Big\}\,.\end{aligned}$$

Differentiating with the chain rule, one checks that the derivative of the function

$$\mathcal{J}_2 : \quad \rho \in (0, \rho_0) \longmapsto \rho\, \Lambda_2^{-1}\Big(\Lambda_2(u_0) + \ln \rho - \ln \rho_0\Big)\,,$$

is given by

$$\mathcal{J}_2'(\rho) = \boldsymbol{\lambda}_2 \circ \Lambda_2^{-1}\Big(\Lambda_2(u_0) + \ln \rho - \ln \rho_0\Big)\,.$$

Hence $\mathcal{J}_2'$ is increasing, because $\boldsymbol{\lambda}_2$ and Λ_2^{-1} are increasing, and $\mathcal{J}_2$ is convex. In the same way, one shows that the function

$$\mathcal{J}_1 : \quad \rho \in (0, \rho_0) \longmapsto \rho\, \Lambda_1^{-1}\Big(\Lambda_1(u_0) + \ln \rho - \ln \rho_0\Big)$$

is concave. Therefore, $\mathcal{I}_{\rho_0, J_0}$ is convex, and invariant for the Riemann problem.

4 Positively invariant regions for the viscous approximation

To contruct global weak solutions to the system (1) by the vanishing viscosity method, we introduce its regularized version (9), and we can try to pass to the limit $\nu \to 0^+$. For the isentropic gas dynamics system (with no relaxation term), this procedure was first achieved by DiPerna [7], see also [6] for the case with a source term. DiPerna's work was later extended in [14, 13] to all pressure laws of the form ρ^γ, $\gamma > 1$. In the isothermal case $\gamma = 1$, the analysis of [14, 13] does not apply, due to the singular nature of the equation for the entropies (see the next paragraph). However, the existence of global weak solutions (with vacuum) for the isothermal Euler equations was proved recently in [11] (see also [9]). Since the system (1) is "close" to the isothermal Euler system with a relaxation, it is really tempting to believe that the vanishing viscosity method (9) also works for Levermore's system (1).

A preliminary information is given by the existence of positively invariant regions for (9). Let $\rho_0 > 0$, and consider the region $\mathcal{I}_{\rho_0,0}$ defined in (36) (for $J_0 = 0$). If there was no relaxation term $-J$ in the right-hand side of (9), the invariance of $\mathcal{I}_{\rho_0,0}$ would simply follow from the result of [4]. For the system (9), we follow [1, page 66], and we thus only need to check that at each point of the boundary $\partial\mathcal{I}_{\rho_0,0}$, the vector $(0, -J)^T$ points into the interior of $\mathcal{I}_{\rho_0,0}$. The same type of analysis was used in [15] for a closely related problem.

The boundary of $\mathcal{I}_{\rho_0,0}$ is the union of two curves:

$$\partial\mathcal{I}_{\rho_0,0} = \Big\{(\rho, J) \in (0, \rho_0) \times \mathbb{R} / \varepsilon J = \rho\, \Lambda_2^{-1}\Big(\ln \rho - \ln \rho_0\Big)\Big\} \\ \bigcup \Big\{(\rho, J) \in (0, \rho_0) \times \mathbb{R} / \varepsilon J = \rho\, \Lambda_1^{-1}\Big(\ln \rho - \ln \rho_0\Big)\Big\}\,.$$

On the first of these curves, an outgoing normal vector is given by

$$n := \frac{N}{|N|}\,, \quad N := \Big(\frac{1}{\varepsilon}\, \boldsymbol{\lambda}_2 \circ \Lambda_2^{-1}(\ln \rho - \ln \rho_0), -1\Big)^T\,,$$

so that

$$n \cdot (0, -J)^T = \frac{J}{|N|} = \frac{\rho\, \Lambda_2^{-1}(\ln \rho - \ln \rho_0)}{|N|} < 0\,,$$

the last inequality being valid because Λ_2^{-1} is increasing and vanishes at 0.

On the second curve, one can also compute

$$n := \frac{N}{|N|}\,, \quad N := \Big(-\frac{1}{\varepsilon}\, \boldsymbol{\lambda}_1 \circ \Lambda_1^{-1}(\ln \rho - \ln \rho_0), 1\Big)^T\,,$$

so that

$$n\cdot(0,-J)^T=\frac{-J}{|N|}=\frac{-\rho\,\Lambda_1^{-1}(\ln\rho-\ln\rho_0)}{|N|}<0\,,$$

the last inequality being valid because Λ_1^{-1} is decreasing and vanishes at 0. Consequently, the argument of [1] shows that $\mathcal{I}_{\rho_0,0}$ is positively invariant for (9). This completes the proof of Theorem 1.

The existence of an invariant region for (9) is a crucial step towards proving the convergence of the sequence of solutions to (9), because it yields L^∞ bounds that are uniform with respect to ν. (Observe that these bounds are also uniform with respect to ε for the density ρ and for the velocity u). However, the region $\mathcal{I}_{\rho_0,0}$ reaches the boundary of the domain of hyperbolicity, which yields many technical difficulties in showing the convergence of the vanishing viscosity method.

Remark 2. *It is rather easy to compute the entropies for the system* (1)*. Indeed, choosing the* (ρ,u) *variables instead of the conservative variables* (ρ,J)*, the equation for the entropies is found to be*

$$\partial_{\rho\rho}\eta=\frac{\psi(u)-u^2}{\rho^2}\,\partial_{uu}\eta+\frac{2u-\psi'(u)}{\rho}\,\partial_{\rho u}\eta+\frac{\psi'(u)-2u}{\rho^2}\,\partial_u\eta\,. \tag{37}$$

Note that when $\psi(u)=1+u^2$*, one recovers the classical equation of the entropies for the isothermal Euler equations. Note also that the coefficients of* (37) *involve the velocity* u*, which is a consequence of the non-Galilean invariance of* (1)*.*

The equation (37) *can be simplified a little if one uses the variable* $\beta=\mathbb{G}^{-1}(u)\in\mathbb{R}$ *instead of* $u\in(-1,1)$*:*

$$\partial_{\rho\rho}\eta=\frac{1}{\rho^2\,\mathbb{G}'(\beta)}\,\partial_{\beta\beta}\eta-\frac{\mathbb{G}''(\beta)}{\rho\,\mathbb{G}'(\beta)}\,\partial_{\rho\beta}\eta\,.$$

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A Proof of Lemma 5

We begin with the first point of Lemma 5. With $\mathbb{G}(\beta)=1/\tanh(\beta)-1/\beta$, we compute

$$\mathbb{G}''(\beta)=\frac{1}{2[\beta\,\sinh(\beta)]^3}\,\left[4\,\beta^3\,\cosh(\beta)+3\sinh(\beta)-\sinh(3\beta)\right]\,.$$

Of course, $\mathbb{G}''$ is odd. Expanding in power series (with infinite radius of convergence), we get

$$4\,\beta^3\,\cosh(\beta)+3\sinh(\beta)-\sinh(3\beta)=\sum_{p=3}^{+\infty}a_p\,\beta^{2p+1}\,,\quad a_p:=\frac{3-8p+32p^3-3*9^p}{(2p+1)!}<0\,.$$

This yields the first part of the Lemma. Using (2), we compute

$$\psi'(u)-2u=\frac{\mathbb{G}''}{\mathbb{G}'}(\mathbb{G}^{(-1)}(u))\,.$$

and $\mathbb{G}^{(-1)}(u)$ has the same sign as u. We have thus proved the first point of Lemma 5.

We now turn to the second point of Lemma 5. Starting from the relation

$$\begin{aligned}&2\,\mathbb{G}'(\beta)^2+\mathbb{G}''(\beta)\,(1-\mathbb{G}(\beta))\\&\quad=\frac{1-\tanh(\beta)}{\beta^3\,\sinh(\beta)^2\,\tanh(\beta)}\,\left[2\beta^3+\beta^2\,(\exp(2\beta)+1)-2\beta\,(\exp(2\beta)-1)+\cosh(2\beta)-1\right]\,,\end{aligned}$$

and expanding in power series, we obtain

$$2\beta^3 + \beta^2\,(\exp(2\beta)+1) - 2\beta\,(\exp(2\beta)-1) + \cosh(2\beta) - 1 = \sum_{p=6}^{+\infty} \mu_p\,\beta^p\,,$$

with

$$\mu_p = \begin{cases} \dfrac{2^{p-2}\,(p-1)(p-4)}{p!}\,, & \text{if } p \text{ is even,} \\ \dfrac{2^{p-1}\,(p-5)}{(p-1)!}\,, & \text{if } p \text{ is odd.} \end{cases}$$

We can deduce

$$\forall\,\beta > 0\,, \quad 2\,\mathbb{G}'(\beta)^2 + \mathbb{G}''(\beta)\,(1-\mathbb{G}(\beta)) > 0\,.$$

Eventually, we define

$$A := 4(\psi(u_+) - u_+^2) + (2u_+ - \psi'(u_+))(2u_+ - z)\,.$$

Then, setting $\beta_+ := \mathbb{G}^{(-1)}(u_+) > 0$, and using (2), we obtain

$$A = 4\,\mathbb{G}'(\beta_+) - 2\,\mathbb{G}(\beta_+)\,\frac{\mathbb{G}''(\beta_+)}{\mathbb{G}'(\beta_+)} + z\,\frac{\mathbb{G}''(\beta_+)}{\mathbb{G}'(\beta_+)}\,.$$

Because ψ is strictly convex, we have $z < \psi'(u_-) < 2$, see Proposition 2, and we also have $\mathbb{G}''(\beta_+) < 0$ thanks to the first point of Lemma 5. Therefore, we have

$$A > 4\,\mathbb{G}'(\beta_+) + 2\,(1-\mathbb{G}(\beta_+))\,\frac{\mathbb{G}''(\beta_+)}{\mathbb{G}'(\beta_+)} > 0\,,$$

and the proof is complete.

References

[1] C. Bardos. Introduction aux problèmes hyperboliques non linéaires. volume 1047 of *Lecture Notes in Math.*, pages 1–74. Springer, 1984.

[2] C. Bardos, R. Santos, and R. Sentis. Diffusion approximation and computation of the critical size. *Trans. Amer. Math. Soc.*, 284(2):617–649, 1984.

[3] A. Bressan. *Hyperbolic systems of conservation laws.* Oxford University Press, 2000.

[4] K. N. Chueh, C. C. Conley, and J. A. Smoller. Positively invariant regions for systems of nonlinear diffusion equations. *Indiana Univ. Math. J.*, 26(2):373–392, 1977.

[5] J.-F. Coulombel, F. Golse, and T. Goudon. Diffusion approximation and entropy-based moment closure for kinetic equations. *Asymptotic Analysis*, 45(1,2):1–39, 2005.

[6] X. Ding, G.-Q. Chen, and P. Luo. Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics. *Comm. Math. Phys.*, 121(1):63–84, 1989.

[7] R. J. DiPerna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.*, 91(1):1–30, 1983.

[8] D. Hoff. Invariant regions for systems of conservation laws. *Trans. Amer. Math. Soc.*, 289(2):591–610, 1985.

[9] H. Huang, Z. Wang. Convergence of viscosity solutions for isothermal gas dynamics. *SIAM J. Math. Anal.*, 34(3):595–610, 2002.

[10] P. D. Lax. Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[11] P. G. LeFloch, V. Shelukhin. Symmetries and global solvability of the isothermal gas dynamics equations. *Arch. Ration. Mech. Anal.*, 175(3):389–430, 2005.

[12] C. D. Levermore. Moment closure hierarchies for kinetic theories. *J. Statist. Phys.*, 83(5-6):1021–1065, 1996.

[13] P.-L. Lions, B. Perthame, and P. E. Souganidis. Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates. *Comm. Pure Appl. Math.*, 49(6):599–638, 1996.

[14] P.-L. Lions, B. Perthame, and E. Tadmor. Kinetic formulation of the isentropic gas dynamics and p-systems. *Comm. Math. Phys.*, 163(2):415–431, 1994.

[15] P. Marcati and A. Milani. The one-dimensional Darcy's law as the limit of a compressible Euler flow. *J. Differential Equations*, 84(1):129–147, 1990.

[16] D. Serre. *Systems of conservation laws. 1.* Cambridge University Press, Cambridge, 1999.