Relaxation approximation of the Euler equations

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Abstract

The aim of this paper is to show how solutions to the one-dimensional compressible Euler equations can be approximated by solutions to an enlarged hyperbolic system with a strong relaxation term. The enlarged hyperbolic system is linearly degenerate and is therefore suitable to build an efficient approximate Riemann solver. From a theoretical point of view, the convergence of solutions to the enlarged system towards solutions to the Euler equations is proved for local in time smooth solutions. We also show that arbitrarily large shock waves for the Euler equations admit smooth shock profiles for the enlarged relaxation system. In the end, we illustrate these results of convergence by proposing a numerical procedure to solve the enlarged hyperbolic system. We test it on various cases.

1 Introduction

The introduction of relaxation approximations for hyperbolic systems of conservation laws goes back to the seminal work [7]. In the spirit of [7], we study here a relaxation approximation for the 2×2 and 3×3 compressible Euler equations in one space dimension by considering an enlarged system with only one additional scalar unknown quantity, and a stiff relaxation term. The relaxation systems under consideration in this paper are motivated by the works of Suliciu [11], in the 2×2 case and of Coquel and al. [5], Chalons and Coquel [3] in the 3×3 setting. The idea is to modify only the pressure law in the original compressible Euler equations, which concentrates all the genuine nonlinearities, and to keep the other ones. This approach allows to obtain in both cases an extended first order system with relaxation which is consistent with both the original system and its entropy inequality in the regime of an infinite relaxation parameter. See Liu [8] and Chen, Levermore and Liu [4]. Opposite to [7], the enlarged system is only quasilinear, but it is hyperbolic with the property that all its characteristic fields are linearly degenerate. Then, the Riemann problem can be solved explicitly and as a consequence, the proposed enlarged relaxation system is suitable to construct an efficient approximate Riemann solver for the compressible Euler equations. This approximate Riemann solver is based on a splitting strategy where in a first step one solves a Riemann problem for the convective part of the linearly degenerate enlarged system, and in a second step one makes a projection on the so-called equilibrium manifold, which formally corresponds to an infinite relaxation coefficient. For more details, we refer for instance the reader to [3], [2], [1] and to the now large literature on this numerical issue. This numerical procedure is based on the idea that solutions to the Euler equations are obtained as the limit, when the relaxation coefficient tends to infinity, of solutions to the enlarged system with a stiff relaxation. The aim of this paper is to justify this convergence on a rigorous basis. We first verify the convergence for local in time smooth solutions by applying the main result of [12]. The main problem here is to determine for which initial data the assumptions of [12] are satisfied. Then we show that shock waves of arbitrary strength for the Euler equations admit smooth shock profiles that are traveling waves solutions to the relaxation system. We recall that for shock waves of small amplitude, a general existence result of such shock profiles can be found in [13]. The goal here is to get rid of the smallness assumption of [13], which is made possible by a detailed analysis of the resulting dynamical system. In the 3×3 case, we shall also make use of an explicit conserved quantity for this dynamical system, namely the total energy.

The plan of the paper is as follows: in section 2 we consider the barotropic Euler equations and define the relaxation system. We show that smooth solutions of the relaxation system converge towards smooth solutions of the barotropic Euler equations as the relaxation coefficient tends to infinity. Then we show the existence of arbitrarily large shock profiles. In the end of section 2, we propose a numerical procedure for the relaxation system and verify on various cases that this numerical procedure converges to an approximate Riemann solver for the barotropic Euler equations as the relaxation coefficient tends to infinity. The analysis is done for general pressure laws that only satisfy some standard convexity assumptions. In section 3, we follow the same approach for the full Euler equations. Again, our analysis is performed for general equations of state that only satisfy the so-called Bethe-Weyl conditions.

In all this paper, $H^s(\mathbb{T})$ denotes the Sobolev space of 1-periodic functions with s derivatives in $L^2(\mathbb{T})$.

2 Relaxation of the barotropic Euler equations

In one space dimension, the barotropic Euler equations read:

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p(\tau)) = 0\,, \end{cases} \tag{1}$$

where ρ is the density, u is the velocity, $\tau = 1/\rho$ is the specific volume, and p is the pressure law. We make the following assumption on the pressure:

(H1) p is a $\mathcal{C}^\infty$ function on $]0, +\infty[$ that satisfies $p'(\tau) < 0$ and $p''(\tau) > 0$ for all $\tau > 0$.

In that case, (1) is a strictly hyperbolic system with two genuinely nonlinear characteristic fields, see [6]. The speed of sound c is given by $c(\tau) = \tau\, \sqrt{-p'(\tau)}$. Moreover, the function:

$$\eta = \rho\, \frac{u^2}{2} + \rho\, \varepsilon(\tau)\,, \quad \varepsilon'(\tau) = -p(\tau)\,,$$

is a strictly convex entropy for (1). We will focus on solutions of (1) that satisfy the following classical entropy inequality :

$$\partial_t \eta + \partial_x(\eta\, u + p\, u) \le 0. \tag{2}$$

We are going to show that solutions of (1) can be approximated by solutions to the following system of balance laws:

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \pi) = 0\,, \\ \partial_t(\rho\, \mathcal{T}) + \partial_x(\rho\, \mathcal{T}\, u) = \lambda\, \rho\, (\tau - \mathcal{T})\,, \end{cases} \tag{3}$$

where the so-called relaxed pressure π is given by:

$$\pi = p(\mathcal{T}) + a^2\, (\mathcal{T} - \tau)\,, \tag{4}$$

and a, λ are positive constants. We keep the notation $\tau = 1/\rho$. This definition of π can be understood as a linearization of the pressure p around the relaxation specific volume $\mathcal{T}$.
To be more precise, we are going to show that in some suitable cases, the solution $(\rho^\lambda, u^\lambda, \mathcal{T}^\lambda)$ of (3) converges as λ tends to $+\infty$ towards some function (ρ, u, τ), where $\tau = 1/\rho$ and (ρ, u) satisfies the barotropic Euler equations (1). The choice of the parameter a is crucial, and is determined by the so-called subcharacteristic condition, see e.g. [4, 10]. One of the problems here is to choose a independently of the relaxation parameter λ. We first study the case of smooth solutions by applying the main result of [12]. The verification of the assumptions of [12] is the main issue of this study. Then we discuss the existence of smooth shock profiles. Eventually, we show how to numerically approximate the solutions of (1) by using the relaxation system (3). The efficiency of this numerical procedure is discussed on various cases that will illustrate our theoretical results.
Let us mention to conclude the presentation of the relaxation model that (3) can be endowed with a relaxation entropy defined by:

$$\rho\Sigma = \rho\frac{u^2}{2} + \rho\varepsilon(\mathcal{T}) + \rho\frac{\pi^2 - p^2(\mathcal{T})}{2\,a^2}\,, \tag{5}$$

which coincides with the entropy η at equlibrium $\mathcal{T} = \tau$. By the chain rule and for smooth solutions, we easily get

$$\partial_t(\rho\,\Sigma) + \partial_x(\rho\,\Sigma\,u + \pi\,u) = -\lambda\rho(a^2 + p'(\mathcal{T}))(\mathcal{T} - \tau^2)\,, \tag{6}$$

the right-hand side being negative under the subcharacteristic condition (the relaxation entropy is dissipated by the relaxation procedure). Then, the proposed relaxation process is entropy consistent in the sense of [4].

2.1 Convergence for smooth solutions

Our aim is to apply the convergence result of [12], so we first rewrite the system (3) as a quasilinear system in the variables $(\tau, u, \mathcal{T})$. For smooth solutions, the system (3)-(4) equivalently reads:

$$\begin{cases} \partial_t\tau + u\,\partial_x\tau - \tau\,\partial_x u = 0\,, \\ \partial_t u + u\,\partial_x u - a^2\,\tau\,\partial_x\tau + (a^2 + p'(\mathcal{T}))\,\tau\,\partial_x\mathcal{T} = 0\,, \\ \partial_t\mathcal{T} + u\,\partial_x\mathcal{T} = \lambda\,(\tau - \mathcal{T})\,. \end{cases} \tag{7}$$

We define:

$$U = \begin{pmatrix} \tau \\ u \\ \mathcal{T} \end{pmatrix}\,, \quad A(U) = \begin{pmatrix} u & -\tau & 0 \\ -a^2\,\tau & u & (a^2 + p'(\mathcal{T}))\,\tau \\ 0 & 0 & u \end{pmatrix}\,, \quad Q(U) = \begin{pmatrix} 0 \\ 0 \\ \tau - \mathcal{T} \end{pmatrix}\,,$$

so the quasilinear system (7) can be written in the compact form:

$$\partial_t U + A(U)\,\partial_x U = \lambda\,Q(U)\,. \tag{8}$$

If we let formally λ tend to $+\infty$, we get $\mathcal{T} = \tau$ in the third equation of (7), and the limits τ, u satisfy the quasilinear form of the barotropic Euler equations:

$$\begin{cases} \partial_t\tau + u\,\partial_x\tau - \tau\,\partial_x u = 0\,, \\ \partial_t u + u\,\partial_x u + \tau\,p'(\tau)\,\partial_x\tau = 0\,. \end{cases} \tag{9}$$

The aim of this section is to justify rigorously this convergence.
The following lemma gathers the main structural properties of the relaxation system (8):

Lemma 1. *Let $\mathcal{O}$ be an open subset of* $]0,+\infty[\times\mathbb{R}\times]0,+\infty[$, *and assume that a satisfies:*

$$\forall\,(\tau,u,\mathcal{T})\in\mathcal{O}\,,\quad a^2+p'(\mathcal{T})>0\,. \tag{10}$$

Let $\mathcal{E}=\{(\tau,u,\mathcal{T})\in\mathcal{O}\,|\,\tau=\mathcal{T}\}$. Then there exists a constant invertible matrix P, and there exists a matrix $A_0(U)$ such that the following properties hold:

- *for all $U\in\mathcal{E}$, one has:*
$$P\,DQ(U)\,P^{-1}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-1\end{pmatrix}\,,$$
- *A_0 is a $\mathcal{C}^\infty$ function of $U\in\mathcal{O}$; moreover for all $U\in\mathcal{O}$, the matrix $A_0(U)$ is symmetric definite positive, and the matrix $A_0(U)\,A(U)$ is symmetric,*
- *for all $U\in\mathcal{E}$, one has:*
$$A_0(U)\,DQ(U)+DQ(U)^T\,A_0(U)=-P^T\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}P\,.$$

The set $\mathcal{E}$ is the equilibrium manifold. It is exactly the set of points in $\mathcal{O}$ for which the source term $Q(U)$ in (8) vanishes.

Proof. The first point of lemma 1 is obtained by defining:

$$P=\begin{pmatrix}1&0&0\\0&1&0\\-1&0&1\end{pmatrix}\,,$$

and by observing that for all $U\in\mathcal{E}$, we have:

$$DQ(U)=\begin{pmatrix}0&0&0\\0&0&0\\1&0&-1\end{pmatrix}\,.$$

As a matter of fact, the jacobian matrix $DQ(U)$ is constant, and the above equality holds not only for $U\in\mathcal{E}$ but for all $U\in\mathcal{O}$. We now turn to the definition of the symmetrizer A_0, and let:

$$A_0(U)=\frac{1}{2\,(a^2+p'(\mathcal{T}))}\begin{pmatrix}a^2&0&-(a^2+p'(\mathcal{T}))\\0&1&0\\-(a^2+p'(\mathcal{T}))&0&a^2+p'(\mathcal{T})\end{pmatrix}\,.$$

The end of lemma 1 follows from a straightforward computation. In particular, thanks to assumption **(H1)**, the matrix $A_0(U)$ is symmetric positive definite. $\square$

The structural properties of (8) are the main ingredient to prove the following result:

Theorem 1. *Let $s\geq 2$, and consider initial data $(\tau_0,u_0,\mathcal{T}_0)\in H^{s+2}(\mathbb{T})$ that take values in a compact subset of* $]0,+\infty[\times\mathbb{R}\times]0,+\infty[$. *Then there exists a constant $a>0$, and there exists a time $T>0$ such that:*

- *for all $\lambda\geq 1$, there exists a unique solution $U^\lambda=(\tau^\lambda,u^\lambda,\mathcal{T}^\lambda)\in C([0,T];H^s(\mathbb{T}))$ of (8) with initial data $(\tau_0,u_0,\mathcal{T}_0)$,*
- *the barotropic Euler equations (9) admits a unique solution $(\overline{\tau},\overline{u})\in C([0,T];H^{s+2}(\mathbb{T}))$ with initial data (τ_0,u_0),*

- $(\tau^\lambda, u^\lambda)$ *converges towards* $(\overline{\tau}, \overline{u})$ *in* $C([0,T]; H^s(\mathbb{T}))$ *as* λ *tends to* $+\infty$*, and* $\mathcal{T}^\lambda$ *converges to* $\overline{\tau}$ *in* $L^1([0,T]; H^s(\mathbb{T}))$ *as* λ *tends to* $+\infty$*.*

Proof. We are going to check that all the assumptions of [12] are satisfied. First of all, we consider a compact subset K_0 of $]0,+\infty[\times\mathbb{R}\times]0,+\infty[$ such that $(\tau_0, u_0, \mathcal{T}_0)$ takes its values in K_0. There is no loss of generality in assuming that K_0 is convex. We now consider a second compact subset K_1 of $]0,+\infty[\times\mathbb{R}\times]0,+\infty[$ such that K_1 is convex, and K_0 is contained in the interior of K_1. We also fix the constant $a > 0$ such that:

$$\forall\, (\tau, u, \mathcal{T}) \in K_1\,, \quad a^2 + p'(\mathcal{T}) > 0\,.$$

Then according to the notations of lemma 1, we let $\mathcal{O}$ denote an open neighborhood of K_1 in $]0,+\infty[\times\mathbb{R}\times]0,+\infty[$ that satisfies:

$$\forall\, (\tau, u, \mathcal{T}) \in \mathcal{O}\,, \quad a^2 + p'(\mathcal{T}) > 0\,,$$

and we let $\mathcal{E}$ denote the equilibrium manifold $\{(\tau, u, \mathcal{T}) \in \mathcal{O} \,|\, \tau = \mathcal{T}\}$. Lemma 1 shows that the structural assumptions of [12] are satisfied in the open set $\mathcal{O}$. Moreover, the limit system (9), that is obtained by taking formally the limit $\lambda \to +\infty$ in (8), is symmetrizable and is therefore locally well-posed in $H^{s+2}(\mathbb{T})$. In our particular case, this limit system is nothing but the barotropic Euler equations (9). Consequently, if we want to apply the main result of [12], the last point to check is that the Ordinary Differential Equation:

$$\frac{dI}{ds}(s,x) = Q(I(s,x))\,, \quad I(0,x) = (\tau_0, u_0, \mathcal{T}_0)(x)\,, \tag{11}$$

has a global solution that converges exponentially to some limit state that belongs to $\mathcal{E}$. The ODE (11) can be solved explicitly and we obtain:

$$I(s,x) = \big(\tau_0(x), u_0(x), \exp(-s)\, \mathcal{T}_0(x) + (1 - \exp(-s))\, \tau_0(x)\big)\,.$$

Thanks to the convexity of K_0, we have $I(s,x) \in K_0$ for all $(s,x) \in [0,+\infty[\times\mathbb{T}$, and $I(s,x)$ converges exponentially towards $(\tau_0, u_0, \tau_0)(x) \in \mathcal{E} \cap K_0$ as s tends to $+\infty$. This last point shows that we can apply the main result of [12] and obtain the conclusion of the theorem. $\square$

It is worth noting that theorem 1 can be obtained for ill-prepared initial data, that is for initial data U_0 that do not necessarily satisfy $Q(U_0) = 0$. As a matter of fact, this is made possible because the ODE (11) is rather simple to solve, so we can show easily that its solution has the appropriate asymptotic behavior for large times. The price to pay is an initial layer for the function $\mathcal{T}$ that precludes convergence in $\mathcal{C}([0,T]; H^s(\mathbb{T}))$. The convergence can only be obtained in a space $L^p([0,T]; H^s(\mathbb{T}))$, with $1 \le p < +\infty$. For the nonbarotropic system that we shall study in the following section, we shall have to restrict to well-prepared initial data because the corresponding ODE will not be anymore simple enough to be solved explicitly.

2.2 Shock profiles

We consider a shock wave:

$$(\rho, u) = \begin{cases} (\rho_r, u_r)\,, & \text{if } x > \sigma\, t, \\ (\rho_\ell, u_\ell)\,, & \text{if } x < \sigma\, t, \end{cases} \tag{12}$$

solution to the Euler equations (1)-(2). In other words (see [6] for more details), (12) satisfies the Rankine-Hugoniot jump conditions:

$$\rho_r\,(u_r - \sigma) = \rho_\ell\,(u_\ell - \sigma) = j\,, \quad j^2\,(\tau_r - \tau_\ell) = p(\tau_\ell) - p(\tau_r)\,, \tag{13}$$

together with Lax shock inequalities:

$$\begin{aligned} 0 < \frac{u_r - \sigma}{c_r} < 1 < \frac{u_\ell - \sigma}{c_\ell}\,, \quad \text{if } j > 0,\\ 0 < \frac{\sigma - u_\ell}{c_\ell} < 1 < \frac{\sigma - u_r}{c_r}\,, \quad \text{if } j < 0. \end{aligned} \tag{14}$$

In (14), c_r (resp. c_ℓ) denotes the speed of sound in the state r (resp. ℓ). Observe that the case $j = 0$ is ruled out since it corresponds to $u_r = u_\ell$ and $\rho_r = \rho_\ell$, that is to the case of a constant solution.

A shock profile is a traveling wave $(\rho, u, \mathcal{T})(\lambda(x - \sigma\, t))$ solution to the enlarged system (3), that satisfies the asymptotic conditions:

$$\lim_{\xi \to +\infty} (\rho, u, \mathcal{T})(\xi) = (\rho_r, u_r, \tau_r)\,, \quad \lim_{\xi \to -\infty} (\rho, u, \mathcal{T})(\xi) = (\rho_\ell, u_\ell, \tau_\ell)\,. \tag{15}$$

The existence of shock profiles is summarized in the following result:

Theorem 2. *Assume that* **(H1)** *holds, and that* (12) *satisfies* (13), (14). *Let a satisfy:*

$$a^2 > \max(-p'(\tau_r), -p'(\tau_\ell))\,. \tag{16}$$

Then there exists a unique smooth shock profile $(\rho, u, \mathcal{T})(\lambda(x - \sigma\, t))$ solution to (3), (4) *and* (15). *Moreover, all functions $\rho, u, \mathcal{T}$ are monotone.*

Proof. For simplicity, we deal with the case $j > 0$, which corresponds to $\tau_r < \tau_\ell$. The case $j < 0$ is entirely similar so we omit it.

Assume that $(\rho, u, \mathcal{T})(\lambda(x - \sigma\, t))$ is a smooth shock profile. Then for all $\xi \in \mathbb{R}$ we have:

$$\begin{cases} (\rho\,(u - \sigma))'(\xi) = 0\,, \\ (\rho\, u\,(u - \sigma) + \pi)'(\xi) = 0\,, \\ (\rho\, \mathcal{T}\,(u - \sigma))'(\xi) = \rho(\xi)\,(\tau - \mathcal{T})(\xi)\,. \end{cases}$$

Integrating the first two equations, and using the asymptotic conditions (15) as well as (13), we obtain the equivalent system:

$$\begin{cases} \rho(\xi)\,(u(\xi) - \sigma)) = j\,, \\ j\, u(\xi) + p(\mathcal{T}(\xi)) + a^2\,(\mathcal{T}(\xi) - \tau(\xi)) = j\, u_r + p(\tau_r)\,, \\ j\, \mathcal{T}'(\xi) = 1 - \dfrac{\mathcal{T}(\xi)}{\tau(\xi)}\,. \end{cases}$$

Eliminating $u(\xi)$ in the second equation leads to:

$$\begin{cases} u(\xi) = j\, \tau(\xi) + \sigma\,, \\ (a^2 - j^2)\, \tau(\xi) = a^2\, \mathcal{T}(\xi) - j^2 \tau_r + p(\mathcal{T}(\xi)) - p(\tau_r)\,, \\ j\, \mathcal{T}'(\xi) = 1 - \dfrac{\mathcal{T}(\xi)}{\tau(\xi)}\,. \end{cases}$$

Using the strict convexity of p (assumption **(H1)**), we have:

$$-p'(\tau_\ell) < j^2 = \frac{p(\tau_\ell) - p(\tau_r)}{\tau_r - \tau_\ell} < -p'(\tau_r)\,,$$

so a satisfies $a^2 - j^2 > 0$. If we denote:

$$g(\mathcal{T}) = \frac{1}{a^2 - j^2}\big(a^2\, \mathcal{T} - j^2\, \tau_r + p(\mathcal{T}) - p(\tau_r)\big)\,, \tag{17}$$

a shock profile must satisfy the system:

$$\begin{cases} u(\xi) = j\,\tau(\xi) + \sigma\,, \\ \tau(\xi) = g(\mathcal{T}(\xi))\,, \\ \mathcal{T}'(\xi) = \dfrac{1}{j}\left(1 - \dfrac{\mathcal{T}(\xi)}{g(\mathcal{T}(\xi))}\right) = h(\mathcal{T}(\xi))\,. \end{cases} \tag{18}$$

Conversely, if $(\tau, u, \mathcal{T})$ is a solution to (18) that is defined on $\mathbb{R}$, such that $\lim_{+\infty} \mathcal{T} = \tau_r$ and $\lim_{-\infty} \mathcal{T} = \tau_\ell$, then $(\tau, u, \mathcal{T})$ is a shock profile. (It is indeed easy to check that τ and u have the right asymptotic behavior at $\pm\infty$ thanks to the Rankine-Hugoniot conditions (13).)
From the definition (17), we easily check that $g(\tau_r) = \tau_r$, $g(\tau_\ell) = \tau_\ell$, and g is increasing on $[\tau_r, \tau_\ell]$ thanks to the convexity of p and the inequality $a^2 + p'(\tau_r) > 0$. Moreover, thanks to the strict convexity of p, we have:

$$\forall \mathcal{T} \in \,]\tau_r, \tau_\ell[\,, \quad h(\mathcal{T}) = \frac{1}{j}\left(1 - \frac{\mathcal{T}}{g(\mathcal{T})}\right) < 0\,,$$
$$h'(\tau_r) < 0\,, \quad h'(\tau_\ell) > 0\,.$$

Consequently, there exists a smooth function $\mathcal{T}$ that is defined on $\mathbb{R}$, that is a solution to the ordinary differential equation $\mathcal{T}' = h(\mathcal{T})$, and such that $\lim_{+\infty} \mathcal{T} = \tau_r$, $\lim_{-\infty} \mathcal{T} = \tau_\ell$. The function $\mathcal{T}$ is unique up to a shift, and is decreasing. Then the functions τ and u given by the first two equations in (18) are monotone and have the appropriate asymptotic behavior at $\pm\infty$. This completes the proof of the theorem. □

2.3 Numerical approach

In this section, we first propose to illustrate numerically the *convergence* of the solutions of the relaxation system (3)-(4) towards the solutions of the barotropic Euler equations system (1)-(2) when λ goes to infinity. For that, we are going to consider a natural discretization of (3) and test several values of λ. Then, our objective will be to formally set $\lambda = +\infty$ in this natural discretization in order to recover a consistant method for approximating the solution of (1) which does not depend on the source term in (3). Here, the *convergence* has to be understood in the sense of the previous two sections, namely a smooth solution of (3)-(4) converges to a smooth solution of (1), see theorem 1, and an admissible discontinuity of (1)-(2) can be obtained by a shock profile of (3), see theorem 2. The validity of these two theorems relies on some (more or less technical) assumptions at the continuous level. Among them, the so-called Whitham, or sub-characteristic, condition (16) (see also (10)) plays an important part at the discrete level for the stability of the method.

2.3.1 Numerical procedure

For simplicity in the forthcoming notations, we first propose to introduce the following condensed forms for (1) and (3). We set

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0\,, \tag{19}$$

with $\mathbf{u} = (\rho, \rho u)^T$ and $\mathbf{f}(\mathbf{u}) = (\rho u, \rho u^2 + p(\tau))^T$ for (1), and

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \lambda \mathcal{R}(\mathbf{U})\,, \tag{20}$$

with $\mathbf{U} = (\rho, \rho u, \rho\mathcal{T})^T$ and $\mathbf{F}(\mathbf{U}) = (\rho u, \rho u^2 + \pi, \rho\mathcal{T} u)^T$ for (3).
Then, the proposed numerical procedure for (3) is based on a splitting strategy and turns out to be very classical in the context of relaxation systems, see [7]. It is made of two steps : the first step makes the solution evolve in time according to (20) with $\lambda = 0$, which amounts to account for the convective part only, and the second step deals with the source term. Before going into

detail, we first set some notations.
Let Δx and Δt be two constant steps for space and time discretizations. Let $(x_j)_{j\in\mathbb{Z}}$ be a sequence of equidistributed points of $\mathbb{R}$: $x_{j+1} - x_j = \Delta x$. For all $j \in \mathbb{Z}$ and all $n \in \mathbb{N}$, we introduce the notations:

$$x_{j+1/2} = x_j + \frac{\Delta x}{2}, \quad t^n = n\Delta t,$$

and consider the following discretization of the computational domain $\mathbb{R}_x \times \mathbb{R}_t^+$:

$$\mathbb{R}_x \times \mathbb{R}_t^+ = \bigcup_{j\in\mathbb{Z}} \bigcup_{n\geq 0} C_j^n, \quad C_j^n = [x_{j-1/2}, x_{j+1/2}[\times[t^n, t^{n+1}[.$$

As usual in the context of finite volumes methods, the approximate solution $\mathbf{u}_\Delta(x,t)$ of (1)-(2) with initial data $\mathbf{u}(x,0) = \mathbf{u}_0(x)$ is sought as a piecewise constant function on each slab C_j^n. We set

$$\mathbf{u}_\Delta(x,t) = \mathbf{u}_j^n \text{ for } (x,t) \in C_j^n,$$

and for the sake of completeness

$$\mathbf{u}_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}_0(x) dx\,, \quad j \in \mathbb{Z}. \tag{21}$$

Let us assume as given the piecewise constant approximate solution $\mathbf{u}_\Delta(x,t^n)$ at time t^n. In order to advance it to the next time level t^{n+1}, we first define another piecewise constant function $\mathbf{U}_\Delta(x,t^n)$ associated with $\mathbf{u}_\Delta(x,t^n)$ when setting

$$\mathbf{U}_\Delta(x,t^n) = \mathbf{U}_j^n = \begin{pmatrix} \mathbf{u}_j^n \\ (\rho\mathcal{T})_j^n \end{pmatrix} \text{ for } (x,t) \in C_j^n.$$

Actually, we are going to show how to advance $\mathbf{U}_\Delta(x,t^n)$ to the next time level t^{n+1} and $\mathbf{u}_\Delta(x,t^{n+1})$ will coincide with the first two components of $\mathbf{U}_\Delta(x,t^{n+1})$. Note that $\mathbf{U}_\Delta(x,t^n)$ represents a piecewise constant approximate solution of (20) at time t^n.
At time $t = 0$, the function $\mathbf{U}_\Delta(x,t^0)$ is set to be at equilibrium, that is

$$(\rho\mathcal{T})_j^0 := \rho_j^0 \tau_j^0 = 1.$$

We are now in position to precise the two steps of the algorithm.

First step : evolution in time $(t^n \to t^{n+1-})$
In this step, we take $\lambda = 0$ and solve (20) with $\mathbf{U}_\Delta(x,t^n)$ as initial data. It is easily seen that provided $a > 0$ and the density ρ remains positive, this system is strictly hyperbolic with the following eigenvalues : $\lambda_1(\mathbf{U}) = u - a\tau$, $\lambda_2(\mathbf{U}) = u$ and $\lambda_3(\mathbf{U}) = u + a\tau$. Moreover, all these eigenvalues are associated with a linearly degenerate field. The consequence of the latter property that the solution of the corresponding Riemann problem is explicitly known (see below theorem 3) is going to be used and justifies by itself the use of the relaxation system (20), in the regime $\lambda \to +\infty$, for approximating the solutions of (19).
Let us assume that Δt obeys the usual CFL condition

$$\frac{\Delta t}{\Delta x} \max_{\mathbf{U}}(|\lambda_i(\mathbf{U})|, i = 1,2,3) < \frac{1}{2}. \tag{22}$$

Then, the solution of (20) with $\lambda = 0$ and $\mathbf{U}_\Delta(x,t^n)$ as initial data is obtained by solving a sequence of non interacting Riemann problems set at each cell interface $x_{j+1/2}$. More precisely we have :

$$\mathbf{U}(x,t) = \mathbf{U}(\frac{x - x_{j+1/2}}{t}; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n), \text{ for } (x,t) \in [x_j, x_{j+1}]\times]0, \Delta t], \quad j \in \mathbb{Z},$$

where $(x,t) \to \mathbf{U}(\frac{x}{t};\mathbf{U}_L,\mathbf{U}_R)$ denotes the self similar solution of the following Riemann problem

$$\left\{\begin{array}{l} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0,\ x \in \mathbb{R},\ t>0, \\ \mathbf{U}(x,0) = \left\{\begin{array}{lll} \mathbf{U}_L & \text{if} & x<0, \\ \mathbf{U}_R & \text{if} & x>0. \end{array}\right. \end{array}\right. \tag{23}$$

Recall that this solution is actually known thanks to the brief discussion above and theorem 3 below. We are thus tempted to define the new values $\mathbf{U}_j^{n+1-}$, $j \in \mathbb{Z}$ by means of the celebrated Godunov method. It writes :

$$\mathbf{U}_j^{n+1-} = \mathbf{U}_j^n - \frac{\Delta t}{\Delta x}(g(\mathbf{U}_j^n,\mathbf{U}_{j+1}^n) - g(\mathbf{U}_{j-1}^n,\mathbf{U}_j^n)), \quad j \in \mathbb{Z}, \quad n \geq 0, \tag{24}$$

with

$$g(\mathbf{U}_j^n,\mathbf{U}_{j+1}^n) = \mathbf{F}(\mathbf{U}(0;\mathbf{U}_j^n,\mathbf{U}_{j+1}^n)). \tag{25}$$

Let us now briefly discuss the definition of the parameter a. In order to prove the convergence results in theorems 1 and 2, it is already known that a must fulfill the conditions (10) and (16). From a numerical point of view, we propose to take into account these stability conditions when defining a at each intermediate time t^n according to the following constraint :

$$a^2 > \max_{j\in\mathbb{Z}}(-p'(\tau_j^n)). \tag{26}$$

The corresponding value of a is used in (24)-(25) for defining $\mathbf{U}_j^{n+1-}$. Actually, a deeper analysis of the relaxation system (3), carried out on the associated rate of entropy dissipation, would highlight that this rate increases with the parameter a. In order to lower the numerical diffusion of the scheme, it would be preferable to define a locally at each interface $x_{j+1/2}$ (this value would be used for the definition of $g(\mathbf{U}_j^n,\mathbf{U}_{j+1}^n)$) and as small as possible according to a local version of (26). This point is not addressed here and we refer for instance the reader to [3], [2]. Note that according to (26) the value of a is updated at each time iteration.

We end up this first step when giving the Riemann solution of (23).

Theorem 3.
Let $\mathbf{U}_L$ and $\mathbf{U}_R$ two constant states such that $\rho_L > 0$ and $\rho_R > 0$. Assume that $a > 0$ satisfies the condition

$$\begin{array}{c} \lambda_1(\mathbf{U}_L) = u_L - a\tau_L < u^\star < \lambda_3(\mathbf{U}_R) = u_R + a\tau_R, \\ \\ u^* = \frac{1}{2}(u_L+u_R) + \frac{1}{2a}(\pi_L - \pi_R). \end{array} \tag{27}$$

Then, the self-similar solution $(x,t) \to \mathbf{U}(x/t;\mathbf{U}_L,\mathbf{U}_R)$ of the Riemann problem (23) is made of four constant states separated by three contact discontinuities :

$$\mathbf{U}(x/t;\mathbf{U}_L,\mathbf{U}_R) = \left\{\begin{array}{lll} \mathbf{U}_L & \textit{if} & \frac{x}{t} < \lambda_1(\mathbf{U}_L), \\ \mathbf{U}_L^* & \textit{if} & \lambda_1(\mathbf{U}_L) < \frac{x}{t} < \lambda_2(\mathbf{U}_L^*), \\ \mathbf{U}_R^* & \textit{if} & \lambda_2(\mathbf{U}_R^*) < \frac{x}{t} < \lambda_3(\mathbf{U}_R), \\ \mathbf{U}_R & \textit{if} & \lambda_3(\mathbf{U}_R) < \frac{x}{t}, \end{array}\right.$$

with $\lambda_2(\mathbf{U}_L^*) = \lambda_2(\mathbf{U}_R^*) = u^*$. The intermediate states $\mathbf{U}_L^*$ and $\mathbf{U}_R^*$ are obtained from the following relations :

$$\begin{array}{l} \tau_L^* = \tau_L + (u^* - u_L)/a, \quad \tau_R^* = \tau_R - (u^* - u_R)/a, \\ u_L^* = u_R^* = u^*, \\ \mathcal{T}_L^* = \mathcal{T}_L, \quad \mathcal{T}_R^* = \mathcal{T}_R. \end{array}$$

In addition, we have $\rho_L^* = 1/\tau_L^* > 0$ and $\rho_R^* = 1/\tau_R^* > 0$.

Proof. We already know that the three characteristic fields of (20) (when λ is taken to be 0) are linearly degenerate. Then, the solution is made of four constant states, let us say $\mathbf{U}_L$, $\mathbf{U}_L^*$, $\mathbf{U}_R^*$ and $\mathbf{U}_R$, separated by three contact discontinuities respectively propagating with the corresponding characteristic speeds $\lambda_1(\mathbf{U}_L) = \lambda_1(\mathbf{U}_L^*)$, $\lambda_2(\mathbf{U}_L^*) = \lambda_2(\mathbf{U}_R^*)$ and $\lambda_3(\mathbf{U}_R^*) = \lambda_3(\mathbf{U}_R)$. Using the Rankine-Hugoniot jump relations across these discontinuities easily leads to the expected intermediate states $\mathbf{U}_L^*$ and $\mathbf{U}_R^*$. $\square$

Second step : source term $(t^{n+1-} \to t^{n+1})$
In this step, we propose to take into account the source term when solving

$$\partial_t \mathbf{U} = \lambda \mathcal{R}(\mathbf{U}),$$

with $\mathbf{U}_\Delta(x, t^{n+1-})$ as initial data. By the form of $\mathcal{R}$, it amounts to keep ρ and ρu unchanged, and to make evolve $\rho\mathcal{T}$ according to the ordinary differential equation :

$$\partial_t(\rho\mathcal{T}) = \lambda(1 - \rho\mathcal{T}) \tag{28}$$

which can be exactly solved. Then we simply set for all $j \in \mathbb{Z}$:

$$\left\{ \begin{array}{l} \rho_j^{n+1} = \rho_j^{n+1-}, \\ (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \\ (\rho\mathcal{T})_j^{n+1} = 1 - (1 - (\rho\mathcal{T})_j^{n+1-}) \exp(-\lambda \Delta t), \end{array} \right. \tag{29}$$

and define

$$\mathbf{U}_j^{n+1} = \left(\begin{array}{c} \rho_j^{n+1} \\ (\rho u)_j^{n+1} \\ (\rho\mathcal{T})_j^{n+1} \end{array} \right) = \left(\begin{array}{c} \mathbf{u}_j^{n+1} \\ (\rho\mathcal{T})_j^{n+1} \end{array} \right).$$

This completes the proposed algorithm.

Our objective is now to see how this scheme behaves for various values of λ. It is expected from the previous section that the more λ is large, the more the numerical solution is close to the solution of (1). This would prove numerically the convergence of the solutions of (3) towards the solutions of (1) when λ goes to infinity. Let us notice that instead of considering a finite value for λ, expected to be large in order to get a good approximation of the solution of (1)-(2), one could simply choose formally $\lambda = +\infty$ so that the second step would consist in setting

$$\left\{ \begin{array}{l} \rho_j^{n+1} = \rho_j^{n+1-}, \\ (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \\ (\rho\mathcal{T})_j^{n+1} = 1, \end{array} \right.$$

instead of (29). In other words, the numerical solution obtained at the end of the first step is projected on equilibrium at each intermediate time and a discretization of (28) is no longer necessary. This case will be considered in the numerical experiments. It provides a numerical strategy for approximating the solution of (19) which is free of the relaxation term $\mathcal{R}(\mathbf{u})$ in (20).

2.3.2 Numerical experiments

We consider three Riemann initial data

$$\mathbf{u}_0(x) = \left\{ \begin{array}{ll} \mathbf{u}_L & \text{if} \quad x < 0, \\ \mathbf{u}_R & \text{if} \quad x > 0, \end{array} \right. \tag{30}$$

where the initial states $\mathbf{u}_L$ and $\mathbf{u}_R$ are chosen as follows :

Test 1 (shock-shock)		
$\mathbf{u}_L$:	$\rho_L = 1$	$u_L = 1$
$\mathbf{u}_R$:	$\rho_R = 2$	$u_R = 0.5$

Test 2 (rarefaction-rarefaction)		
$\mathbf{u}_L$:	$\rho_L = 0.5$	$u_L = -0.5$
$\mathbf{u}_R$:	$\rho_R = 1$	$u_R = -0.2$

Test 3 (rarefaction-shock)		
$\mathbf{u}_L$:	$\rho_L = 1$	$u_L = -0.5$
$\mathbf{u}_R$:	$\rho_R = 0.5$	$u_R = -0.5$

The corresponding Riemann solutions of (19)-(30) respectively develop two shocks, two rarefaction waves and a rarefaction wave followed by a shock wave. In this way, the numerical *convergence* will be observed for (piecewise) smooth solutions as well as for shock discontinuities. Without restriction, the pressure p is taken to be

$$p(\tau) = K\tau^{-\gamma} \text{ with } K = \frac{(\gamma-1)^2}{4\gamma} \text{ and } \gamma = 1.6.$$

On figures 1, 2 and 3 are plotted the profiles of ρ, u and $\rho\mathcal{T}$ for several values of λ, namely $\lambda = 1, 10, 100$ and $\lambda = +\infty$. The mesh is made of 300 points. We observe that the more λ is large, the more the density and the velocity of the numerical solution of (20) correctly approach the solution of (19)-(30). At the same time, $\rho\mathcal{T}$ becomes closer to 1, as it is expected.

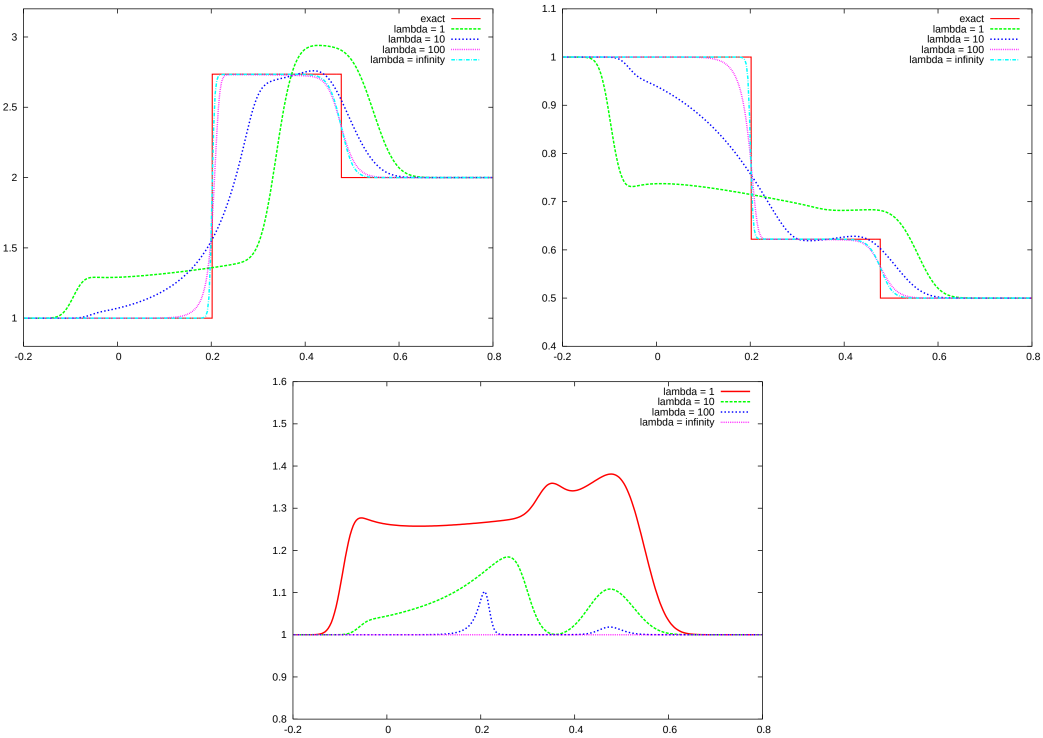

Figure 1: **Test 1** : ρ (Left), u (middle) and $\rho\mathcal{T}$ (Right) at time 0.5

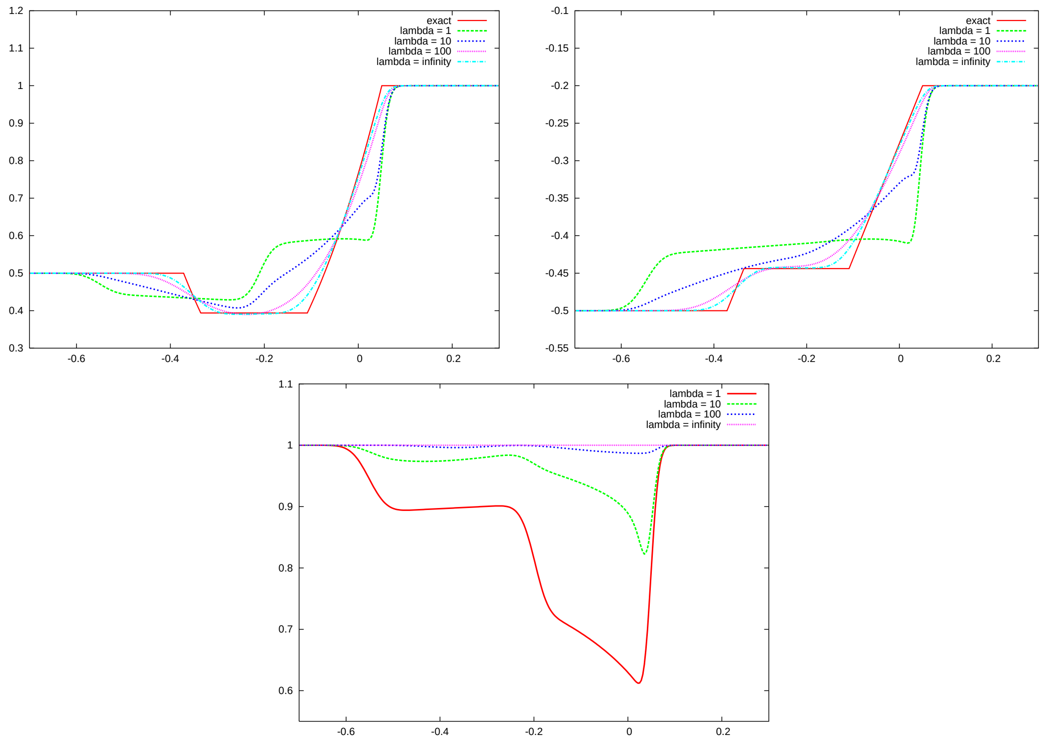

Figure 2: **Test 2** : ρ (Left), u (middle) and $\rho\mathcal{T}$ (Right) at time 0.5

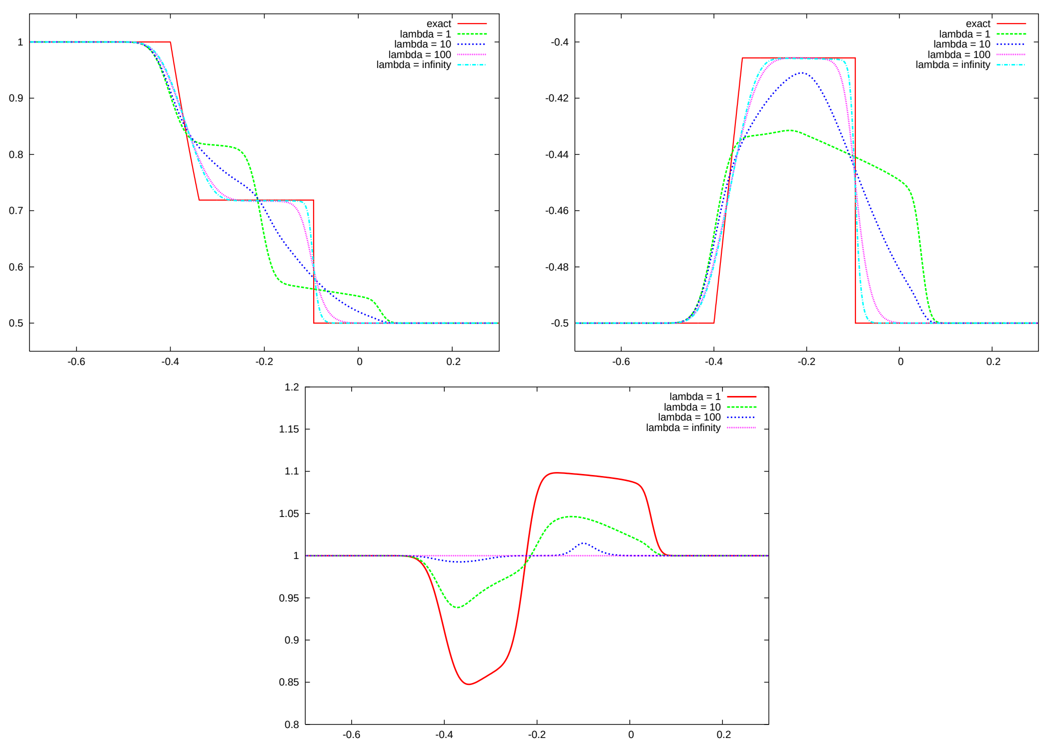

Figure 3: **Test 3** : ρ (Left), u (middle) and $\rho\mathcal{T}$ (Right) at time 0.5

3 Relaxation of the Euler equations

In one space dimension, the Euler equations read:

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p) = 0\,, \\ \partial_t(\rho\, E) + \partial_x(\rho\, E\, u + p\, u) = 0\,, \end{cases} \tag{31}$$

where ρ is the density, u is the velocity, p is the pressure, $E = u^2/2 + \varepsilon$ stands for the specific total energy, and ε denotes the specific internal energy. We assume that the fluid is endowed with a complete equation of state $\varepsilon = \varepsilon(\tau, S)$, where $\tau = 1/\rho$ is the specific volume while S is the specific entropy, and that this equation of state satisfies the classical thermodynamical requirements (see e.g. [9] and the references therein):

(H2) ε is a $\mathcal{C}^\infty$ function on $]0, +\infty[\times\mathbb{R}$ such that $p = -\partial_\tau \varepsilon > 0$ and $\theta = \partial_S \varepsilon > 0$. Moreover, the derivatives of ε satisfy:

$$\frac{\partial^2 \varepsilon}{\partial \tau^2} > 0\,, \quad \frac{\partial^2 \varepsilon}{\partial \tau\, \partial S} < 0\,, \quad \frac{\partial^2 \varepsilon}{\partial \tau^2}\, \frac{\partial^2 \varepsilon}{\partial S^2} > \left(\frac{\partial^2 \varepsilon}{\partial \tau\, \partial S} \right)^2\,,$$
$$-\frac{\partial^2 \varepsilon}{\partial \tau\, \partial S} < \frac{2\,\theta}{p}\, \frac{\partial^2 \varepsilon}{\partial \tau^2}\,, \quad \frac{\partial^3 \varepsilon}{\partial \tau^3} < 0\,.$$

The function θ is the temperature of the fluid. It is given as a function of the specific volume and the specific entropy. Using assumption **(H2)**, we can define the sound speed $c = \tau\sqrt{-\partial_\tau p}$. Moreover, it is shown in [9] that under assumption **(H2)**, (31) is a strictly hyperbolic system with two extreme genuinely nonlinear fields and one intermediate linearly degenerate field. The function $-\rho\, S$ is a strictly convex entropy for (31). As usual, we shall focus on weak solutions of (31) that satisfy the classical entropy inequality :

$$\partial_t \rho S + \partial_x(\rho S u) \geq 0. \tag{32}$$

We also refer to [9] for results on the global solvability of the Riemann problem.

The aim of this section is to follow the analysis of the barotropic case, that is to show that smooth solutions of (31) can be approximated by solutions to an enlarged system with a strong relaxation, and that shock waves of (31) admit smooth shock profiles solutions to this enlarged system. The enlarged system reads as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0\,, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \pi) = 0\,, \\ \partial_t(\rho\, S) + \partial_x(\rho\, S\, u) = \lambda\, \rho\, (\tau - \mathcal{T})^2 \left(a^2 + \partial_\tau p(\mathcal{T}, S)\right)\,, \\ \partial_t(\rho\, \mathcal{T}) + \partial_x(\rho\, \mathcal{T}\, u) = \lambda\, \rho\, (\tau - \mathcal{T}) \left(\theta(\mathcal{T}, S) + (\mathcal{T} - \tau)\, \partial_S p(\mathcal{T}, S)\right)\,, \end{cases} \tag{33}$$

where $\tau = 1/\rho$, and the new pressure π is defined by:

$$\pi = p(\mathcal{T}, S) + a^2\, (\mathcal{T} - \tau)\,. \tag{34}$$

Again, a and λ are positive constants and π can be understood as a linearization of p with respect to the first variable and around $\mathcal{T}$. We recall that in (33), θ denotes the temperature (that is the partial derivative of the internal energy with respect to the specific entropy). We also highlight the fact that in (33), all quantities θ, p etc. are evaluated at $(\mathcal{T}, S)$ and not at (τ, S).

An important quantity for the enlarged system (33)-(34) is the so-called relaxation specific total energy Σ, that we define as:

$$\Sigma = \frac{u^2}{2} + \varepsilon(\mathcal{T}, S) + \frac{\pi^2 - p^2(\mathcal{T}, S)}{2\, a^2}\,. \tag{35}$$

and that coincides with E when $\mathcal{T} = \tau$. Repeated applications of the chain rule show that for smooth solutions of (33)-(34), the relaxed total energy is conserved:

$$\partial_t(\rho\,\Sigma) + \partial_x(\rho\,\Sigma\, u + \pi\, u) = 0\,. \tag{36}$$

Observe at the same time and in view of (33) that the relaxation entropy ρS is dissipated by the relaxation procedure under the subcharacteristic condition $a^2 + \partial_\tau p(\mathcal{T}, S) > 0$. Then, the proposed system is consistent in the sense of [4].

3.1 Convergence for smooth solutions

We proceed as in the previous section, and first rewrite the relaxation system (33) in a quasilinear form. For smooth solutions, (33)-(34) reads:

$$\begin{cases} \partial_t \tau + u\,\partial_x \tau - \tau\,\partial_x u = 0\,, \\ \partial_t u + u\,\partial_x u - a^2\,\tau\,\partial_x\tau + \tau\,\partial_S p(\mathcal{T}, S)\,\partial_x S + \tau\,(a^2 + \partial_\tau p(\mathcal{T}, S))\,\partial_x \mathcal{T} = 0\,, \\ \partial_t S + u\,\partial_x S = \lambda\,(\tau - \mathcal{T})^2 \left(a^2 + \partial_\tau p(\mathcal{T}, S)\right), \\ \partial_t \mathcal{T} + u\,\partial_x \mathcal{T} = \lambda\,(\tau - \mathcal{T}) \left(\theta(\mathcal{T}, S) + (\mathcal{T} - \tau)\,\partial_S p(\mathcal{T}, S)\right). \end{cases} \tag{37}$$

We define:

$$U = \begin{pmatrix} \tau \\ u \\ S \\ \mathcal{T} \end{pmatrix}, \quad A(U) = \begin{pmatrix} u & -\tau & 0 & 0 \\ -a^2\,\tau & u & \tau\,\partial_S p(\mathcal{T}, S) & (a^2 + \partial_\tau p(\mathcal{T}, S))\,\tau \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix},$$

$$Q(U) = \begin{pmatrix} 0 \\ 0 \\ (\tau - \mathcal{T})^2\,(a^2 + \partial_\tau p(\mathcal{T}, S)) \\ (\tau - \mathcal{T})\,(\theta(\mathcal{T}, S) + (\mathcal{T} - \tau)\,\partial_S p(\mathcal{T}, S)) \end{pmatrix},$$

so the quasilinear system (37) can be written in the compact form:

$$\partial_t U + A(U)\,\partial_x U = \lambda\, Q(U)\,. \tag{38}$$

We keep the same notations as in the previous section in order to highlight the similarities in the analysis.
If we let formally λ tend to $+\infty$ in (37) and assume that all quantities are smooth and have a limit, we get $\mathcal{T} - \tau = O(\lambda^{-1})$ in the fourth equation of (37). Consequently the limits τ, u, S satisfy the quasilinear form of the Euler equations:

$$\begin{cases} \partial_t \tau + u\,\partial_x \tau - \tau\,\partial_x u = 0\,, \\ \partial_t u + u\,\partial_x u + \tau\,\partial_\tau p(\tau, S)\,\partial_x \tau + \tau\,\partial_S p(\tau, S)\,\partial_x S = 0\,, \\ \partial_t S + u\,\partial_x S = 0\,. \end{cases} \tag{39}$$

and by (36)

$$\partial_t(\rho\, E) + \partial_x(\rho\, E\, u + p\, u) = 0.$$

A rigorous proof of such convergence is based on some structural properties of the relaxation system (38). Such properties are gathered in the following lemma:

Lemma 2. *Let $\mathcal{O}$ be an open subset of $]0,+\infty[\times\mathbb{R}^2\times]0,+\infty[$, and assume that a satisfies:*

$$\forall\,(\tau,u,S,\mathcal{T})\in\mathcal{O}\,,\quad a^2+\partial_\tau p(\mathcal{T},S)>0\,. \tag{40}$$

Let $\mathcal{E}=\{(\tau,u,S,\mathcal{T})\in\mathcal{O}\,|\,\tau=\mathcal{T}\}$. Then there exists a constant invertible matrix P, and there exists a matrix $A_0(U)$ such that the following properties hold:

- *for all $U\in\mathcal{E}$, one has:*

$$P\,DQ(U)\,P^{-1}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-\theta(\mathcal{T},S)\end{pmatrix},$$

- *A_0 is a $\mathcal{C}^\infty$ function of $U\in\mathcal{O}$; moreover for all $U\in\mathcal{O}$, the matrix $A_0(U)$ is symmetric definite positive, and the matrix $A_0(U)\,A(U)$ is symmetric,*

- *for all $U\in\mathcal{E}$, one has:*

$$A_0(U)\,DQ(U)+DQ(U)^T\,A_0(U)=-P^T\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}P\,.$$

Proof. The first point is obtained by defining:

$$P=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-1&0&0&1\end{pmatrix},$$

and by observing that for all $U\in\mathcal{E}$, we have:

$$DQ(U)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\\theta(\mathcal{T},S)&0&0&-\theta(\mathcal{T},S)\end{pmatrix}.$$

We now turn to the definition of the symmetrizer A_0, and define:

$$A_0(U)=\frac{1}{2\,\theta\,(a^2+\partial_\tau p)}\begin{pmatrix}a^2&0&-\partial_S p&-(a^2+\partial_\tau p)\\0&1&0&0\\-\partial_S p&0&-3(\partial_S p)^2/\partial_\tau p&0\\-(a^2+\partial_\tau p)&0&0&a^2+\partial_\tau p\end{pmatrix},$$

where the partial derivatives $\partial_\tau p$, $\partial_S p$, and the temperature θ are all evaluated at $(\mathcal{T},S)$. Then the second point of the lemma follows from the direct calculation of $A_0(U)\,A(U)$. Moreover, using assumption **(H2)**, we can check that the matrix $A_0(U)$ is symmetric positive definite. The final point of the lemma follows also from a direct calculation. ☐

Using lemma 2, we are ready to prove our main convergence result:

Theorem 4. *Let $s\geq 2$, and consider initial data $(\tau_0,u_0,S_0,\mathcal{T}_0)\in H^{s+2}(\mathbb{T})$ that take values in a compact subset of $]0,+\infty[\times\mathbb{R}^2\times]0,+\infty[$. Assume morerover that $\mathcal{T}_0=\tau_0$. Then there exists a constant $a>0$, and there exists a time $T>0$ such that:*

- *for all $\lambda\geq 1$, there exists a unique solution $U^\lambda=(\tau^\lambda,u^\lambda,S^\lambda,\mathcal{T}^\lambda)\in C([0,T];H^s(\mathbb{T}))$ of* (37) *with initial data $(\tau_0,u_0,S_0,\mathcal{T}_0)$,*

- *the Euler equations* (39) *admits a unique solution* $(\overline{\tau}, \overline{u}, \overline{S}) \in C([0,T]; H^{s+2}(\mathbb{T}))$ *with initial data* (τ_0, u_0, S_0)*,*
- $(\tau^\lambda, u^\lambda, S^\lambda)$ *converges towards* $(\overline{\tau}, \overline{u}, \overline{S})$ *in* $C([0,T]; H^s(\mathbb{T}))$ *as* λ *tends to* $+\infty$*, and* $\mathcal{T}^\lambda$ *converges to* $\overline{\tau}$ *in* $C([0,T]; H^s(\mathbb{T}))$ *as* λ *tends to* $+\infty$*.*

Proof. We follow the arguments of [12]. As in the proof of theorem 1, it is possible to define compact sets K_0 and K_1 such that the initial data take values in K_0, and K_0 is contained in the interior of K_1. Moreover, lemma 2 shows that the structural assumptions of [12] are satisfied in an open neighborhood $\mathcal{O}$ of K_1. Since the limit system (39) is locally well-posed in $H^{s+2}(\mathbb{T})$, the only thing left to check is that the Ordinary Differential Equation:

$$\frac{dI}{ds}(s,x) = Q(I(s,x)) \,, \quad I(0,x) = (\tau_0, u_0, S_0, \mathcal{T}_0)(x) \,, \tag{41}$$

has a global solution that converges exponentially to some limit state that belongs to the equilibrium manifold $\mathcal{E}$. For initial data that already take values in the equilibrium manifold $\mathcal{E}$ (that is when $\mathcal{T}_0 = \tau_0$), the solution of the ODE (41) is the stationary solution $I(s,x) = I(0,x)$, so it is trivial in this case that the solution converges exponentially towards a limit state that belongs to $\mathcal{E}$. This last point shows that we can apply the main result of [12] and obtain the conclusion of the theorem. The convergence of $\mathcal{T}^\lambda$ occurs in $C([0,T]; H^s(\mathbb{T}))$ because there is no initial layer (the data are well-prepared). $\square$

The restriction to well-prepared initial data in Theorem 4 is motivated by the following observation: the aim of the relaxation system (33) is to provide an approximation of the solution to the Euler equations (31). In particular the limit, as λ tends to $+\infty$, of the solutions to (33) should be a solution to the Euler equations (31). However, it appears from the analysis of [12] that for smooth solutions, the limit of the solutions to (33) is a solution to (31) with the initial data $\overline{U}(x) = \lim_{s \to +\infty} I(s,x)$ (and $I(s,x)$ is the solution to the ODE (41)). If the initial data are ill-prepared, that is when $\mathcal{T}_0 \neq \tau_0$, it is not clear that the solution to (41) is defined for all positive times, and that it has a limit at $+\infty$. (Here the source term Q is highly nonlinear and depends on the parameter a). Even if it could be proved that I has a limit as s tends to $+\infty$, it is possible to prove that the asymptotic state depends (in some complicated way) on a, so the initial data for (31) would be some function that is determined by the choice of a. This is not really acceptable because the final goal is to solve the Euler equations with some specific initial data (and not for initial data that are only given by some complicated limiting procedure). This explains why the restriction to well-prepared data is not a drawback of the relaxation system.

3.2 Shock profiles

We consider a shock wave:

$$(\rho, u, S) = \begin{cases} (\rho_r, u_r, S_r) \,, & \text{if } x > \sigma\, t, \\ (\rho_\ell, u_\ell, S_\ell) \,, & \text{if } x < \sigma\, t, \end{cases} \tag{42}$$

solution to the Euler equations (31)-(32). In other words, (42) satisfies the Rankine-Hugoniot conditions:

$$\begin{aligned} \rho_r\,(u_r - \sigma) = \rho_\ell\,(u_\ell - \sigma) = j \,, \\ j^2\,(\tau_r - \tau_\ell) + p(\tau_r, S_r) - p(\tau_\ell, S_\ell) = 0 \,, \\ \varepsilon(\tau_r, S_r) - \varepsilon(\tau_\ell, S_\ell) + \frac{p(\tau_r, S_r) + p(\tau_\ell, S_\ell)}{2}\,(\tau_r - \tau_\ell) = 0 \,, \end{aligned} \tag{43}$$

with $j \neq 0$, and the entropy criterion:

$$j\,(S_r - S_\ell) \geq 0 \,. \tag{44}$$

As shown in [9], under the assumption **(H2)** on the equation of state, the Rankine-Hugoniot relations (43) and the entropy inequality (44) yield the classical Lax' shock inequalities:

$$
\begin{aligned}
&0 < \frac{u_r - \sigma}{c_r} < 1 < \frac{u_\ell - \sigma}{c_\ell}\,, \quad \text{if } j > 0,\\
&0 < \frac{\sigma - u_\ell}{c_\ell} < 1 < \frac{\sigma - u_r}{c_r}\,, \quad \text{if } j < 0.
\end{aligned}
\tag{45}
$$

We also recall that $\tau_r < \tau_\ell$ and $S_r > S_\ell$ if $j > 0$, while $\tau_r > \tau_\ell$ and $S_r < S_\ell$ if $j < 0$ (see [9]).

A shock profile is a smooth traveling wave $(\rho, u, S, \mathcal{T})(\lambda(x - \sigma\, t))$ solution to the enlarged system (33)-(34) (and also (36)), that satisfies the asymptotic conditions:

$$
\lim_{\xi \to +\infty} (\rho, u, S, \mathcal{T})(\xi) = (\rho_r, u_r, S_r, \tau_r)\,, \quad \lim_{\xi \to -\infty} (\rho, u, S, \mathcal{T})(\xi) = (\rho_\ell, u_\ell, S_\ell, \tau_\ell)\,. \tag{46}
$$

The existence of shock profiles is summarized in the following theorem:

Theorem 5. *Assume that* **(H2)** *holds, and that* (42) *satisfies* (43) *and* (45). *Then if a satisfies:*

$$
a^2 + \max_{[\tau_r,\tau_\ell]\times[S_\ell,S_r]} \partial_\tau p > 0\,, \quad a^2 > j^2\,\frac{\max(\tau_r,\tau_\ell)}{\min(\tau_r,\tau_\ell)}\,, \tag{47}
$$

there exists a smooth shock profile $(\rho, u, S, \mathcal{T})(\lambda(x - \sigma\, t))$ *solution to* (33), (34), (36) *and* (46). *Moreover, all functions* $\rho, u, S, \mathcal{T}$ *are monotone.*

Before proving theorem 5, we first prove two lemmas that will be used in the proof of theorem 5. As in the preceeding section, we restrict the proof of theorem 5 to the case $j > 0$. We thus consider a shock wave (42) for which $\tau_r < \tau_\ell$, and $S_r > S_\ell$.

Lemma 3. *If a satisfies* (47), *then the function:*

$$
G\,:\,(\mathcal{T}, S) \longmapsto \frac{1}{a^2 - j^2}\,\Big(a^2\,\mathcal{T} - j^2\,\tau_r + p(\mathcal{T}, S) - p(\tau_r, S_r)\Big)\,,
$$

takes positive values on $[\tau_r, \tau_\ell] \times [S_\ell, S_r]$.

Proof. Observe that $a^2 > j^2$ because of Lax' shock inequalities (45) and (47), the function G is well-defined and it satisfies:

$$
\partial_{\mathcal{T}} G(\mathcal{T}, S) = \frac{1}{a^2 - j^2}\,\Big(a^2 + \partial_\tau p(\mathcal{T}, S)\Big)\,,
$$

so choosing a as in (47), we get $\partial_{\mathcal{T}} G(\mathcal{T}, S) > 0$ for all $(\mathcal{T}, S) \in [\tau_r, \tau_\ell] \times [S_\ell, S_r]$. Then for $(\mathcal{T}, S) \in [\tau_r, \tau_\ell] \times [S_\ell, S_r]$ we have:

$$
\begin{aligned}
G(\mathcal{T}, S) \ge G(\tau_r, S) &= \frac{1}{a^2 - j^2}\,\Big((a^2 - j^2)\,\tau_r + p(\tau_r, S) - p(\tau_r, S_r)\Big)\\
&\ge \frac{1}{a^2 - j^2}\,\Big((a^2 - j^2)\,\tau_r + p(\tau_r, S_\ell) - p(\tau_r, S_r)\Big)\\
&\ge \frac{1}{a^2 - j^2}\,\Big((a^2 - j^2)\,\tau_r + p(\tau_\ell, S_\ell) - p(\tau_r, S_r)\Big) = \frac{1}{a^2 - j^2}\,\Big(a^2\,\tau_r - j^2\,\tau_\ell\Big)\,,
\end{aligned}
$$

where we have used the properties $\partial_S p > 0$, $\partial_\tau p < 0$, and the Rankine-Hugoniot conditions (43). If a satisfies (47), we obtain $G(\mathcal{T}, S) > 0$. $\square$

The next lemma gives a description of the set $\{G(\mathcal{T}, S) = \mathcal{T}\}$:

Lemma 4. *If a satisfies* (47), *then there exists a function* $\mathcal{T}_0$ *that is* $\mathcal{C}^\infty$ *on the interval* $[S_\ell, S_r]$, *that takes its values in* $[\tau_r, \tau_\ell]$, *and such that:*

$$
\{(\mathcal{T}, S) \in [\tau_r, \tau_\ell] \times [S_\ell, S_r] \,|\, G(\mathcal{T}, S) = \mathcal{T}\} = \{(\mathcal{T}_0(S), S)\,, S \in [S_\ell, S_r]\} \cup \{(\tau_r, S_r)\}\,.
$$

Moreover, the function $\mathcal{T}_0$ *is decreasing,* $\mathcal{T}_0(S_\ell) = \tau_\ell$, *and* $\mathcal{T}_0(S_r) \in]\tau_r, \tau_\ell[$.

Proof. We define the function:

$$\mathcal{G}(\mathcal{T},S) = G(\mathcal{T},S) - \mathcal{T} = \frac{1}{a^2-j^2}\left(j^2\,(\mathcal{T}-\tau_r) + p(\mathcal{T},S) - p(\tau_r,S_r)\right),$$

that satisfies:

$$\partial_{\mathcal{T}}\mathcal{G} = \frac{1}{a^2-j^2}\left(j^2 + \partial_\tau p(\mathcal{T},S)\right), \qquad \partial^2_{\mathcal{T}\mathcal{T}}\mathcal{G} = \frac{\partial^2_{\tau\tau} p(\mathcal{T},S)}{a^2-j^2} > 0\,, \quad \partial_S\mathcal{G} = \frac{\partial_S p(\mathcal{T},S)}{a^2-j^2} > 0\,.$$

Consequently, for all $S \in]S_\ell, S_r[$, the function $(\mathcal{T} \mapsto \mathcal{G}(\mathcal{T},S))$ is strictly convex on $[\tau_r,\tau_\ell]$ and it satisfies:

$$\mathcal{G}(\tau_r,S) = \frac{1}{a^2-j^2}\left(p(\tau_r,S) - p(\tau_r,S_r)\right) < 0\,, \quad \mathcal{G}(\tau_l,S) = \frac{1}{a^2-j^2}\left(p(\tau_\ell,S) - p(\tau_\ell,S_\ell)\right) > 0\,.$$

Consequently, the function $(\mathcal{T} \mapsto \mathcal{G}(\mathcal{T},S))$ has one and only one zero in the closed interval $[\tau_r,\tau_\ell]$, and this zero belongs to the open interval $]\tau_r,\tau_\ell[$. We let $\mathcal{T}_0(S)$ denote this zero. The strict convexity of $(\mathcal{T} \mapsto \mathcal{G}(\mathcal{T},S))$ yields $\partial_{\mathcal{T}}\mathcal{G}(\mathcal{T}_0(S),S) > 0$.

The same kind of analysis shows that for $S = S_\ell$, the function $(\mathcal{T} \mapsto \mathcal{G}(\mathcal{T},S_\ell))$ vanishes for $\mathcal{T} = \tau_\ell$ and has no other zero in the closed interval $[\tau_r,\tau_\ell]$. Using Lax' shock inequalities (45), the derivative $\partial_{\mathcal{T}}\mathcal{G}(\tau_\ell,S_\ell)$ is positive. We define $\mathcal{T}_0(S_\ell) = \tau_\ell$.

For $S = S_r$, the function $(\mathcal{T} \mapsto \mathcal{G}(\mathcal{T},S_r))$ vanishes for $\mathcal{T} = \tau_r$, and using Lax' shock inequalities (45) we have $\partial_{\mathcal{T}}\mathcal{G}(\tau_r,S_r) < 0$. We also have $\mathcal{G}(\tau_\ell,S_r) > 0$, so $\mathcal{G}(\cdot,S_r)$ has one and only one zero $\mathcal{T}_0(S_r)$ in the interval $]\tau_r,\tau_\ell[$. We also have $\partial_{\mathcal{T}}\mathcal{G}(\mathcal{T}_0(S_r),S_r) > 0$.

We have thus constructed the function $\mathcal{T}_0$ on the closed interval $[S_\ell,S_r]$. The regularity of $\mathcal{T}_0$ follows from the implicit function theorem, because we have seen that for all $S \in [S_\ell,S_r]$, the derivative $\partial_{\mathcal{T}}\mathcal{G}(\mathcal{T}_0(S),S)$ is positive (and in particular nonzero). To show that $\mathcal{T}_0$ is decreasing, we differentiate the relation $\mathcal{G}(\mathcal{T}_0(S),S) = 0$ with respect to S:

$$\underbrace{\partial_{\mathcal{T}}\mathcal{G}(\mathcal{T}_0(S),S)}_{>0}\,\mathcal{T}_0'(S) + \underbrace{\partial_S\mathcal{G}(\mathcal{T}_0(S),S)}_{>0} = 0\,,$$

so the conclusion follows. □

We now turn to the proof of theorem 5.

Proof of theorem 5. We proceed as in the proof of theorem 2, and first reduce the shock profile equation. We wish to solve the ODE:

$$\begin{cases} (\rho\,(u-\sigma))' = 0\,, \\ (\rho\,u\,(u-\sigma) + \pi)' = 0\,, \\ (\rho\,S\,(u-\sigma))' = \rho\,(\tau-\mathcal{T})^2\left(a^2 + \partial_\tau p(\mathcal{T},S)\right), \\ (\rho\,\mathcal{T}\,(u-\sigma))' = \rho\,(\tau-\mathcal{T})\left(\theta(\mathcal{T},S) + (\mathcal{T}-\tau)\,\partial_S p(\mathcal{T},S)\right), \end{cases} \tag{48}$$

with the asymptotic conditions (46). Integrating once with respect to ξ, and using the Rankine-Hugoniot conditions (43), the first two equations of (48) read:

$$\begin{cases} u(\xi) = j\,\tau(\xi) + \sigma\,, \\ \tau(\xi) = G(\mathcal{T}(\xi),S(\xi))\,, \end{cases} \tag{49}$$

where G is given by lemma 3. Recall that π is defined by (34). We can then eliminate τ and u in the third and fourth equations of (48). These manipulations yield the following reduced system of ODEs:

$$\begin{pmatrix} \mathcal{T} \\ S \end{pmatrix}' = \frac{G(\mathcal{T},S)-\mathcal{T}}{j\,G(\mathcal{T},S)} \begin{pmatrix} \theta(\mathcal{T},S) + (\mathcal{T}-G(\mathcal{T},S))\,\partial_S p(\mathcal{T},S) \\ (G(\mathcal{T},S)-\mathcal{T})\,(a^2 + \partial_\tau p(\mathcal{T},S)) \end{pmatrix}. \tag{50}$$

We want to construct a global solution to (50) with the asymptotic conditions $\lim_{+\infty}(\mathcal{T}, S) = (\tau_r, S_r)$ and $\lim_{-\infty}(\mathcal{T}, S) = (\tau_\ell, S_\ell)$.

We let $\mathbb{F}(\mathcal{T}, S)$ denote the vector field of the ODE (50), that is:

$$\mathbb{F}(\mathcal{T}, S) = \frac{G(\mathcal{T}, S) - \mathcal{T}}{j\, G(\mathcal{T}, S)} \begin{pmatrix} \theta(\mathcal{T}, S) + (\mathcal{T} - G(\mathcal{T}, S))\, \partial_S p(\mathcal{T}, S) \\ (G(\mathcal{T}, S) - \mathcal{T})\,(a^2 + \partial_\tau p(\mathcal{T}, S)) \end{pmatrix} .$$

Using lemma 3, $\mathbb{F}$ is well-defined and $\mathcal{C}^\infty$ on an open neighborhood $\mathcal{U}$ of the rectangle $Q = [\tau_r, \tau_\ell] \times [S_\ell, S_r]$, and a has been chosen such that $a^2 + \partial_\tau p$ is positive on Q. Then the critical points of (50) in Q are $\{(\tau_r, S_r)\}$ and the curve $\{(\mathcal{T}_0(S), S)\, ,\, S \in [S_\ell, S_r]\}$, see lemma 4. The curve of critical points divides the square Q in two sub-regions:

$$Q_1 = \{(\mathcal{T}, S) \in [\tau_r, \tau_\ell] \times [S_\ell, S_r] \,|\, \mathcal{T} \le \mathcal{T}_0(S)\}\, , \quad Q_2 = \{(\mathcal{T}, S) \in [\tau_r, \tau_\ell] \times [S_\ell, S_r] \,|\, \mathcal{T} \ge \mathcal{T}_0(S)\}\, ,$$

see figure 4. In Q_1 one has $G(\mathcal{T}, S) \le \mathcal{T}$, and in Q_2 one has $G(\mathcal{T}, S) \ge \mathcal{T}$.

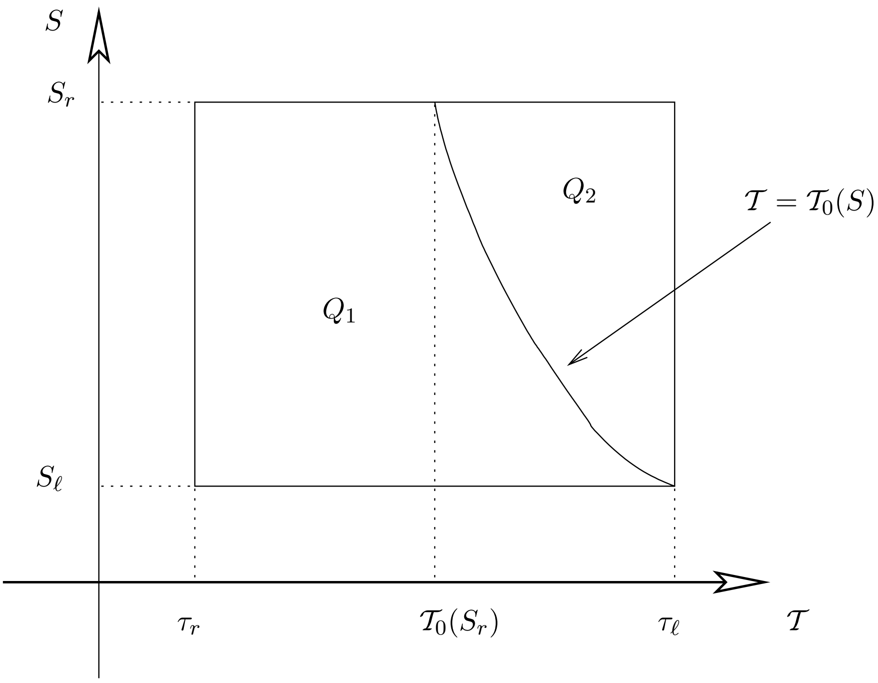

Figure 4: The square $Q = [\tau_r, \tau_\ell] \times [S_\ell, S_r]$, and the subsets Q_1, Q_2.

Let us now prove that the ODE (50) has a heteroclinic orbit that connects the critical points (τ_ℓ, S_ℓ) and (τ_r, S_r), and that takes its values in the compact region Q_1. We first compute the Jacobian matrix of $\mathbb{F}$ at (τ_ℓ, S_ℓ):

$$D\mathbb{F}(\tau_\ell, S_\ell) = \frac{\theta(\tau_\ell, S_\ell)}{j\,(a^2 - j^2)\, \tau_\ell} \begin{pmatrix} j^2 + \partial_\tau p(\tau_\ell, S_\ell) & \partial_S p(\tau_\ell, S_\ell) \\ 0 & 0 \end{pmatrix} .$$

As already mentionned earlier, the quantity $j^2 + \partial_\tau p(\tau_\ell, S_\ell)$ is positive because of Lax' shock inequalities (45). We can therefore apply the unstable manifold theorem: there exists a maximal solution $(\mathcal{T}, S)$ of (50) that is defined on an open interval of the form $]-\infty, \xi_M[$, that is not constant, and whose graph is tangent to the half-line $(\tau_\ell, S_\ell) + \mathbb{R}^-\,(1, 0)$ at $-\infty$. It remains to show that this solution is defined on all $\mathbb{R}$ (that is $\xi_M = +\infty$) and tends to (τ_r, S_r) at $+\infty$. We first observe that $G(\mathcal{T}(\xi), S(\xi)) - \mathcal{T}(\xi)$ does not vanish, and using the asymptotic behavior of $(\mathcal{T}, S)$ at $-\infty$, we get $G(\mathcal{T}(\xi), S(\xi)) < \mathcal{T}(\xi)$ for all ξ. Moreover, for all ξ in a neighborhood of $-\infty$ one has $a^2 + \partial_\tau p(\mathcal{T}(\xi), S(\xi)) > 0$, and therefore $S'(\xi) > 0$. This shows that for all ξ in a neighborhood of $-\infty$, $(\mathcal{T}(\xi), S(\xi))$ belongs to the interior of the compact region Q_1.

Furthermore, the function $\mathcal{T}$ is decreasing and, as long as $(\mathcal{T}(\xi), S(\xi))$ belongs to Q_1, S is increasing.

Let us assume that there exists a $\xi_0 \in]-\infty, \xi_M[$ such that $(\mathcal{T}(\xi_0), S(\xi_0))$ belongs to the boundary of Q_1. In that case, there is no loss of generality in assuming that ξ_0 is minimal for this property. We thus have $S(\xi_0) > S_\ell$ and $\mathcal{T}(\xi_0) < \tau_\ell$. Moreover, the orbit $(\mathcal{T}(\xi), S(\xi))$ can not reach the curve $Q_1 \cap Q_2$ because this curve is made of critical points for (50), and for the same reason, it can not reach the point (τ_r, S_r). Consequently, the point $(\mathcal{T}(\xi_0), S(\xi_0))$ belongs either to $\{\tau_r\}\times]S_\ell, S_r[$ or to $]\tau_r, \mathcal{T}_0(S_r)[\times\{S_r\}$, see figure 4. It is time to use the conservation of the total energy (36). More precisely, we observe that the quantity:

$$H(\mathcal{T}, S) = \frac{j^2 - a^2}{2} G^2(\mathcal{T}, S) + \frac{a^2}{2}\mathcal{T} + \varepsilon(\mathcal{T}, S) + \mathcal{T}\, p(\mathcal{T}, S)\,,$$

is conserved for the solutions of (50), as can be shown by repeated applications of the chain rule. In particular, we get:

$$H(\mathcal{T}(\xi), S(\xi)) = H(\tau_\ell, S_\ell) = \frac{j^2}{2}\tau_\ell^2 + \varepsilon(\tau_\ell, S_\ell) + \tau_\ell\, p(\tau_\ell, S_\ell)\,. \tag{51}$$

Using the Rankine-Hugoniot conditions (43), we observe that $H(\tau_\ell, S_\ell) = H(\tau_r, S_r)$. If the point $(\mathcal{T}(\xi_0), S(\xi_0))$ belongs to $\{\tau_r\}\times]S_\ell, S_r[$, we get $H(\tau_r, S(\xi_0)) = H(\tau_r, S_r)$ so by Rolle's theorem, there exists some $S_1 \in]S(\xi_0), S_r[$ such that $\partial_S H(\tau_r, S_1) = 0$. We compute:

$$\partial_S H(\tau_r, S_1) = \underbrace{\theta(\tau_r, S_1)}_{>0} + \underbrace{(\tau_r - G(\tau_r, S_1))}_{>0}\,\underbrace{\partial_S p(\tau_r, S_1)}_{>0}\,,$$

which is a contradiction. The only possibility left is $(\mathcal{T}(\xi_0), S(\xi_0)) \in]\tau_r, \mathcal{T}_0(S_r)[\times\{S_r\}$ which yields $H(\mathcal{T}(\xi_0), S_r) = H(\tau_r, S_r)$ so by Rolle's theorem, there exists some $\mathcal{T}_1 \in]\tau_r, \mathcal{T}(\xi_0)[$ such that $\partial_{\mathcal{T}} H(\mathcal{T}_1, S_r) = 0$. However, we compute:

$$\partial_{\mathcal{T}} H(\mathcal{T}_1, S_r) = \underbrace{(a^2 + \partial_\tau p(\mathcal{T}_1, S_r))}_{>0}\,\underbrace{(\mathcal{T}_1 - G(\mathcal{T}_1, S_r))}_{>0}\,,$$

which is another contradiction. We can conclude that for all ξ, $(\mathcal{T}(\xi), S(\xi))$ belongs to the interior of the compact set Q_1, and the solution is therefore defined on all $\mathbb{R}$. Both $\mathcal{T}$ and S are monotone and bounded so they have a limit at $+\infty$. This asymptotic state (τ_+, S_+) belongs to the compact set Q_1, is a critical point of (50), satisfies $H(\tau_+, S_+) = H(\tau_\ell, S_\ell) = H(\tau_r, S_r)$, $S_+ > S_\ell$, and $\tau_+ < \tau_\ell$. The only possibility is $(\tau_+, S_+) = (\tau_r, S_r)$.

To complete the proof, it remains to show that τ and u, that are given by (49) are monotone. Using $\tau = G(\mathcal{T}, S)$, we compute:

$$\begin{aligned}
\tau' &= \partial_{\mathcal{T}} G(\mathcal{T}, S)\,\mathcal{T}' + \partial_S G(\mathcal{T}, S)\, S' \\
&= \frac{a^2 + \partial_\tau p(\mathcal{T}, S)}{a^2 - j^2}\,\mathcal{T}' + \frac{\partial_S p(\mathcal{T}, S)}{a^2 - j^2}\, S' \\
&= \frac{(G(\mathcal{T}, S) - \mathcal{T})\,(a^2 + \partial_\tau p(\mathcal{T}, S))\,\theta(\mathcal{T}, S)\,\partial_S p(\mathcal{T}, S)}{(a^2 - j^2)\, j\, G(\mathcal{T}, S)} < 0\,.
\end{aligned}$$

It is clear from (49) that u is also decreasing. □

The second condition in (47) might be unnecessary to prove the existence of smooth shock profiles. However, it simplifies the proof because the flux $\mathbb{F}$ of the ODE (41) can be then defined on the whole rectangle Q.

3.3 Numerical approach

The objective of this section is similar to the one of section 2.3, namely to illustrate numerically the convergence of the solutions of the relaxation system towards the solutions of the Euler equations. Here of course, the Euler and relaxation systems to be considered are (31) and (33), while the theoretical convergence has now to be understood in the sense of theorems 4 and 5. For that we will again consider several values of λ in (33). On the basis of this work and as in section 2.3, we will eventually recover a relevant numerical strategy for approximating the solutions of (31) which is free of the source term in (33) by formally taking $\lambda = +\infty$ (see [3]).

3.3.1 Numerical procedure

Here again, we begin by introducing two condensed forms for the relaxation model and the Euler equations. Since no confusion is possible, we use the same notations but with a different meaning. More precisely, we set

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \tag{52}$$

with $\mathbf{u} = (\rho, \rho u, \rho E)^T$ and $\mathbf{f}(\mathbf{u}) = (\rho u, \rho u^2 + p, \rho E u + pu)^T$ for the Euler system (31), and

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \lambda \mathcal{R}(\mathbf{U}), \tag{53}$$

with $\mathbf{U} = (\rho, \rho u, \rho\Sigma, \rho\mathcal{T})^T$ and $\mathbf{F}(\mathbf{U}) = (\rho u, \rho u^2 + \pi, \rho\Sigma u + \pi u, \rho\mathcal{T} u)^T$ for the relaxation system. It is important to notice at this stage that the total energy Σ is prefered to the entropy S for defining $\mathbf{U}$. In other words, Σ is now considered as a main unknown of the relaxation system (33)-(36), and S has to be understood as a function of $\mathbf{U}$. Actually, this choice is not always well defined since easy calculations allow to obtain that

$$\partial_S \Sigma = \theta(\mathcal{T}, S) + \partial_S p(\mathcal{T}, S)(\mathcal{T} - \tau),$$

which means that the sign of $\partial_S \Sigma$ may change so that Σ cannot be inverted with respect to S generally speaking. However, it is expected that this change of variable is admissible close to the equilibrium $\mathcal{T} = \tau$ (recall that the temperature θ is positive). This will be sufficient for our numerical purpose and the proposed definition of $\mathbf{U}$ will allow to ensure that the total energy is conserved at the discrete level.

Let us now describe the numerical strategy, which is actually similar to the one in section 2.3. Only few things are going to change, but we keep in mind that $\mathbf{u}$ and $\mathbf{U}$ got a new definition in the present section. Define $\mathbf{u}_j^0$ from the initial data $\mathbf{u}_0$ by formula (21) and $\mathbf{U}_j^0$ at equilibrium from $\mathbf{u}_j^0$ by

$$\mathbf{U}_j^0 = \begin{pmatrix} \mathbf{u}_j^0 \\ (\rho\mathcal{T})_j^0 = 1 \end{pmatrix}.$$

Assuming as given $\mathbf{u}_j^n$ and $\mathbf{U}_j^n$ naturally defined by

$$\mathbf{U}_j^n = \begin{pmatrix} \mathbf{u}_j^n \\ (\rho\mathcal{T})_j^n \end{pmatrix},$$

the definition of $\mathbf{U}_j^{n+1}$ is now proposed in two steps.

First step : evolution in time $(t^n \to t^{n+1-})$
In this step, we solve (53) with $\lambda = 0$. This system is strictly hyperbolic with the following eigenvalues : $\lambda_1(\mathbf{U}) = u - a\tau$, $\lambda_2(\mathbf{U}) = u$ and $\lambda_3(\mathbf{U}) = u + a\tau$, provided that $a > 0$ and $\rho > 0$. The corresponding fields are linearly degenerate, so that once more the solution $\mathbf{U}(x,t) = \mathbf{U}(\frac{x}{t}; \mathbf{U}_L, \mathbf{U}_R)$ of the Rieman problem

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0, \ x \in \mathbb{R}, \ t > 0, \\ \mathbf{U}(x, 0) = \begin{cases} \mathbf{U}_L & \text{if} \quad x < 0, \\ \mathbf{U}_R & \text{if} \quad x > 0. \end{cases} \end{cases} \tag{54}$$

is explicitly known, and given in theorem 6 below. This justifies the use of the Godunov method for defining the sequence $\{\mathbf{U}_j^{n+1-}\}_{j\in\mathbb{Z}}$. Under the CFL condition (22), we get the same update formula (24) and (25) but with a different definition of $\mathbf{U}$ and $\mathbf{F}$. Concerning the definition of the parameter a, we propose to take into account at the discrete level the stability condition (40) as follows :

$$a^2 > \max_{j\in\mathbb{Z}}(-\partial_\tau p(\tau_j^n, S_j^n)). \tag{55}$$

This value is taken to be constant in space (*i.e.* at each interface) but is recomputed at each intermediate time t^n. Here again, we refer the reader to [3], [2] for a more detailed analysis of this condition.

We now give the solution of the Riemann problem (54) :

Theorem 6.
Let $\mathbf{U}_L$ and $\mathbf{U}_R$ two constant states such that $\rho_L > 0$ and $\rho_R > 0$. Assume that $a > 0$ satisfies the condition

$$\begin{aligned} &\lambda_1(\mathbf{U}_L) = u_L - a\tau_L < u^* < \lambda_3(\mathbf{U}_R) = u_R + a\tau_R, \\ &u^* = \tfrac{1}{2}(u_L + u_R) + \tfrac{1}{2a}(\pi_L - \pi_R). \end{aligned} \tag{56}$$

Then, the self-similar solution $(x,t) \to \mathbf{U}(x/t; \mathbf{U}_L, \mathbf{U}_R)$ of the Riemann problem (54) is made of four constant states separated by three contact discontinuities :

$$\mathbf{U}(x/t; \mathbf{U}_L, \mathbf{U}_R) = \begin{cases} \mathbf{U}_L & \text{if } \frac{x}{t} < \lambda_1(\mathbf{U}_L), \\ \mathbf{U}_L^* & \text{if } \lambda_1(\mathbf{U}_L) < \frac{x}{t} < \lambda_2(\mathbf{U}_L^*), \\ \mathbf{U}_R^* & \text{if } \lambda_2(\mathbf{U}_R^*) < \frac{x}{t} < \lambda_3(\mathbf{U}_R), \\ \mathbf{U}_R & \text{if } \lambda_3(\mathbf{U}_R) < \frac{x}{t}, \end{cases}$$

with $\lambda_2(\mathbf{U}_L^*) = \lambda_2(\mathbf{U}_R^*) = u^*$. The intermediate states $\mathbf{U}_L^*$ and $\mathbf{U}_R^*$ are obtained from the following relations :

$$\begin{aligned} &\tau_L^* = \tau_L + (u^* - u_L)/a, \quad \tau_R^* = \tau_R - (u^* - u_R)/a, \\ &u_L^* = u_R^* = u^*, \\ &\Sigma_L^* = \Sigma_L + (\pi_L u_L - \pi^* u^*)/a, \quad \Sigma_R^* = \Sigma_R - (\pi_R u_R - \pi^* u^*)/a, \\ &\mathcal{T}_L^* = \mathcal{T}_L, \quad \mathcal{T}_R^* = \mathcal{T}_R. \end{aligned}$$

In addition, we have $\rho_L^* = 1/\tau_L^* > 0$ and $\rho_R^* = 1/\tau_R^* > 0$.

The proof of this result is similar to the one of theorem 3.

Second step : source term $(t^{n+1-} \to t^{n+1})$
We now solve

$$\partial_t \mathbf{U} = \lambda \mathcal{R}(\mathbf{U}).$$

By the form of $\mathcal{R}$, ρ, ρu and $\rho\Sigma$ are constant in this step and $\rho\mathcal{T}$ evolves according to

$$\partial_t(\rho\mathcal{T}) = \lambda\,\rho\,(\tau - \mathcal{T})\left(\theta(\mathcal{T}, S) + (\mathcal{T} - \tau)\,\partial_S p(\mathcal{T}, S)\right). \tag{57}$$

In order to solve this ordinary differential equation, we first have to express the right-hand side as a function of $\mathbf{U}$, which may raise some difficulties in the general setting (see the discussion at the beginning of the section). However, we will see below that this can be easily done in the case of a perfect gas equation of state. In order to convince the reader that this is not really a restriction, let us recall our objectives in this section. First, to illustrate the property that when $\lambda \to +\infty$, the solution of (33)-(36) goes to the solution of (31) : for that a perfect gas equation of state is to be considered. Then, to recover an algorithm for approximating the solutions of (31) which is free of the source term in (33) and then of the ordinary differential equation (57).

But since this method no longer needs to invert Σ with respect to S, it can be used for any equation of state.
Let us then consider a perfect gas equation of state $\varepsilon(\tau, S) = \tau^{1-\gamma}\exp(S/C_v)$ with an adiabatic coefficient γ and a specific heat C_v. Easy calculations successively yield

$$p(\mathcal{T}, S) = (\gamma - 1)\mathcal{T}^{-\gamma}\exp(S/C_v), \quad \theta(\mathcal{T}, S) = \frac{\mathcal{T}^{1-\gamma}}{C_v}\exp(S/C_v),$$

$$(\mathcal{T} - \tau)\,\partial_S p(\mathcal{T}, S) = \frac{(\gamma-1)(\mathcal{T}-\tau)}{\mathcal{T}}\theta(\mathcal{T}, S),$$

and

$$\theta(\mathcal{T}, S) + (\mathcal{T} - \tau)\,\partial_S p(\mathcal{T}, S) = \big(1 + (\gamma-1)\frac{(\rho\mathcal{T}-1)}{\rho\mathcal{T}}\big)\theta(\mathcal{T}, S).$$

Then, it remains to express $\theta(\mathcal{T}, S)$ with respect to $\mathbf{U}$. By the definition (35) and the relation $\varepsilon(\tau, S) = C_v\theta(\mathcal{T}, S)$, we easily get

$$\theta(\mathcal{T}, S) = \frac{\Sigma - \frac{u^2}{2} - \frac{a^2}{2}(\mathcal{T}-\tau)^2}{C_v\big(1 + (\gamma-1)\frac{(\rho\mathcal{T}-1)}{\rho\mathcal{T}}\big)}.$$

Finally, the ordinary differential equation (57) reads in this case

$$\partial_t(\rho\mathcal{T}) = \frac{\lambda}{C_v}(1 - \rho\mathcal{T})\big(\Sigma - \frac{u^2}{2} - \frac{a^2}{2\rho^2}(\rho\mathcal{T}-1)^2\big),$$

with initial condition $\rho\mathcal{T}(t^{n+1-}) = (\rho\mathcal{T})_j^{n+1-}$. Recall that ρ, u and Σ are constant in this second step. Then, this ordinary differential equation has "separated variables" and can be solved explicitly between times $t = t^{n+1-}$ and $t = t^{n+1-} + \Delta t$. From a numerical point of view, we are thus able to define $(\rho\mathcal{T})_j^{n+1}$ for all $j \in \mathbb{Z}$ when simply setting $(\rho\mathcal{T})_j^{n+1} = \rho\mathcal{T}(t^{n+1-} + \Delta t)$. Of course, we have also

$$\begin{cases} \rho_j^{n+1} = \rho_j^{n+1-}, \\ (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \\ (\rho\Sigma)_j^{n+1} = (\rho\Sigma)_j^{n+1-}, \end{cases} \tag{58}$$

and

$$\mathbf{U}_j^{n+1} = \begin{pmatrix} \rho_j^{n+1} \\ (\rho u)_j^{n+1} \\ (\rho\Sigma)_j^{n+1} \\ (\rho\mathcal{T})_j^{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_j^{n+1} \\ (\rho\mathcal{T})_j^{n+1} \end{pmatrix}.$$

This completes the proposed algorithm for finite values of λ. As in section 2.3, it is however natural to take $\lambda = +\infty$ in this numerical strategy so that the second step simply reduces to

$$\begin{cases} \rho_j^{n+1} = \rho_j^{n+1-}, \\ (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \\ (\rho\Sigma)_j^{n+1} = (\rho\Sigma)_j^{n+1-}, \\ (\rho\mathcal{T})_j^{n+1} = 1, \end{cases}$$

and no longer depends on $\mathcal{R}(\mathbf{U})$. We refer the reader to [3] and the references therein for more details on this strategy and the stability properties it enjoys.

3.3.2 Numerical experiments

Again, we consider three typical Riemann initial data

$$\mathbf{u}_0(x) = \begin{cases} \mathbf{u}_L & \text{if} \quad x < 0, \\ \mathbf{u}_R & \text{if} \quad x > 0, \end{cases} \tag{59}$$

where the initial states $\mathbf{u}_L$ and $\mathbf{u}_R$ are chosen as follows :

Test 4 (shock-contact-shock)

$\mathbf{u}_L$:	$\rho_L = 0.9$	$u_L = 3$	$p_L = 2$
$\mathbf{u}_R$:	$\rho_R = 0.5$	$u_R = 2$	$p_R = 1$

Test 5 (rarefaction-contact-rarefaction)

$\mathbf{u}_L$:	$\rho_L = 1$	$u_L = 1$	$p_L = 2$
$\mathbf{u}_R$:	$\rho_R = 2$	$u_R = 2$	$p_R = 2$

Test 6 (rarefaction-contact-shock)

$\mathbf{u}_L$:	$\rho_L = 1$	$u_L = 0$	$p_L = 1$
$\mathbf{u}_R$:	$\rho_R = 0.125$	$u_R = 0.$	$p_R = 0.1$

The corresponding solutions develop shocks, contact discontinuities and rarefaction waves. Without restriction, we take $\gamma = 1.4$ and $C_v = 1$. Figures 5, 6 and 7 plot the profiles of ρ, u, p and $\rho\mathcal{T}$ for several values of λ. The mesh is made of 300 points. Here again, we observe that when λ becomes large, $\rho\mathcal{T}$ becomes close to 1 and the density, velocity and pressure go to the exact solution of the Euler system. This illustrates numerically the *convergence* theorems 4 and 5 established in section 3.

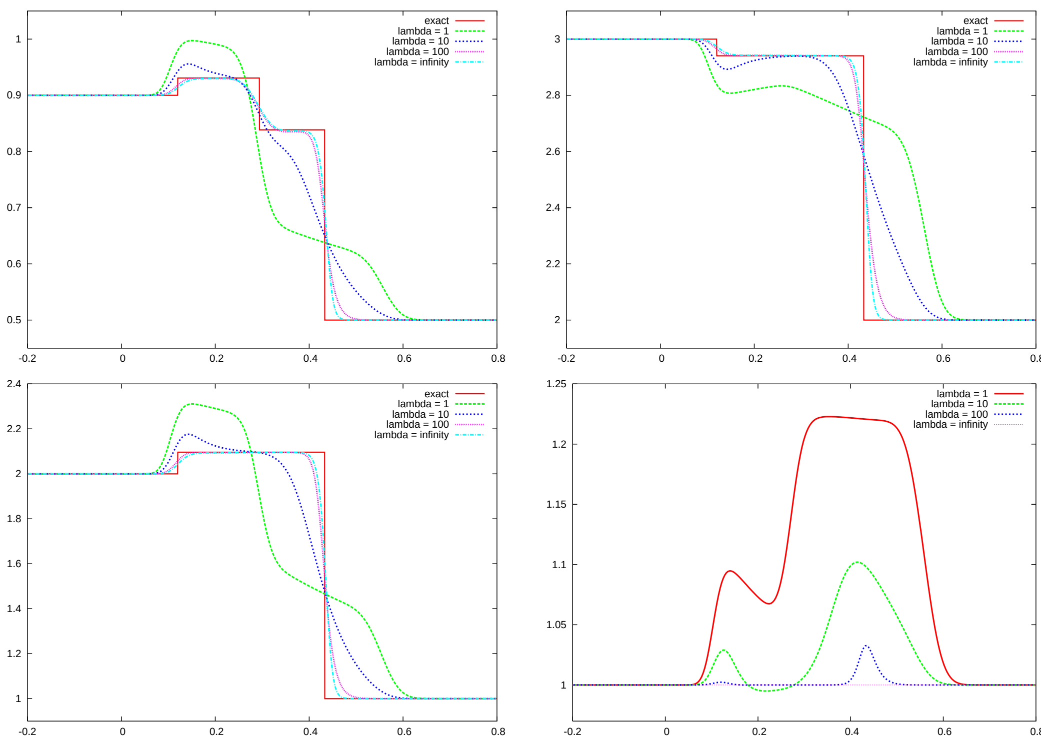

Figure 5: **Test 4** : ρ (left top), u (right top), p (left bottom) and $\rho\mathcal{T}$ (right bottom) at time 0.1

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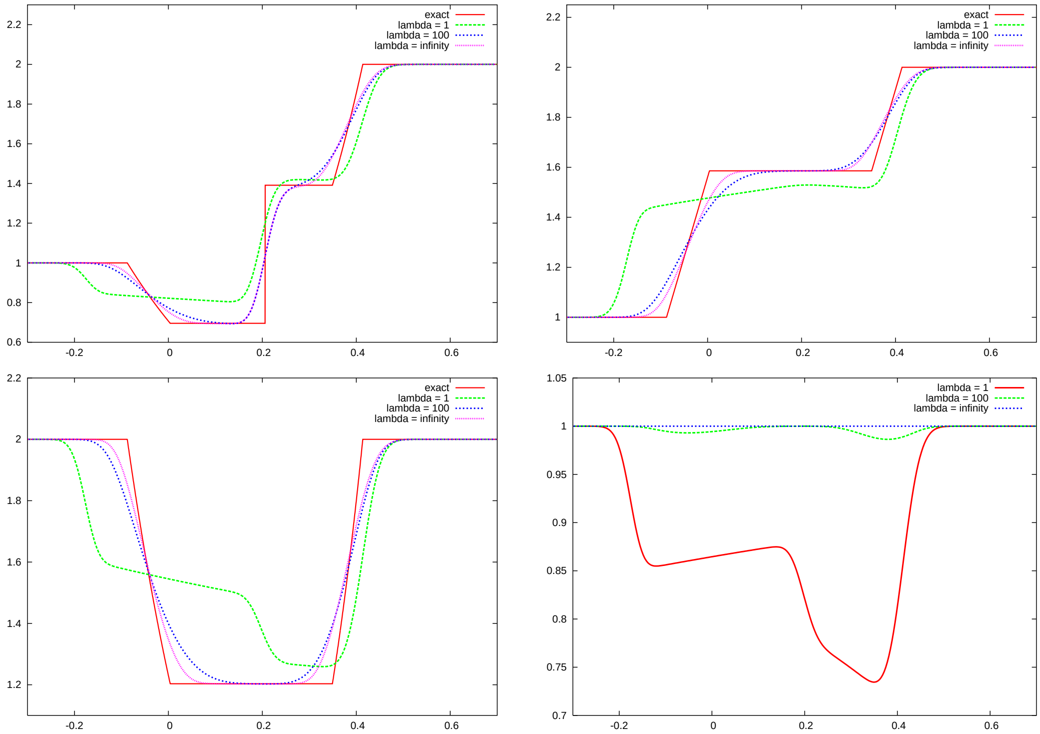

Figure 6: **Test 5** : ρ (left top), u (right top), p (left bottom) and $\rho\mathcal{T}$ (right bottom) at time 0.13

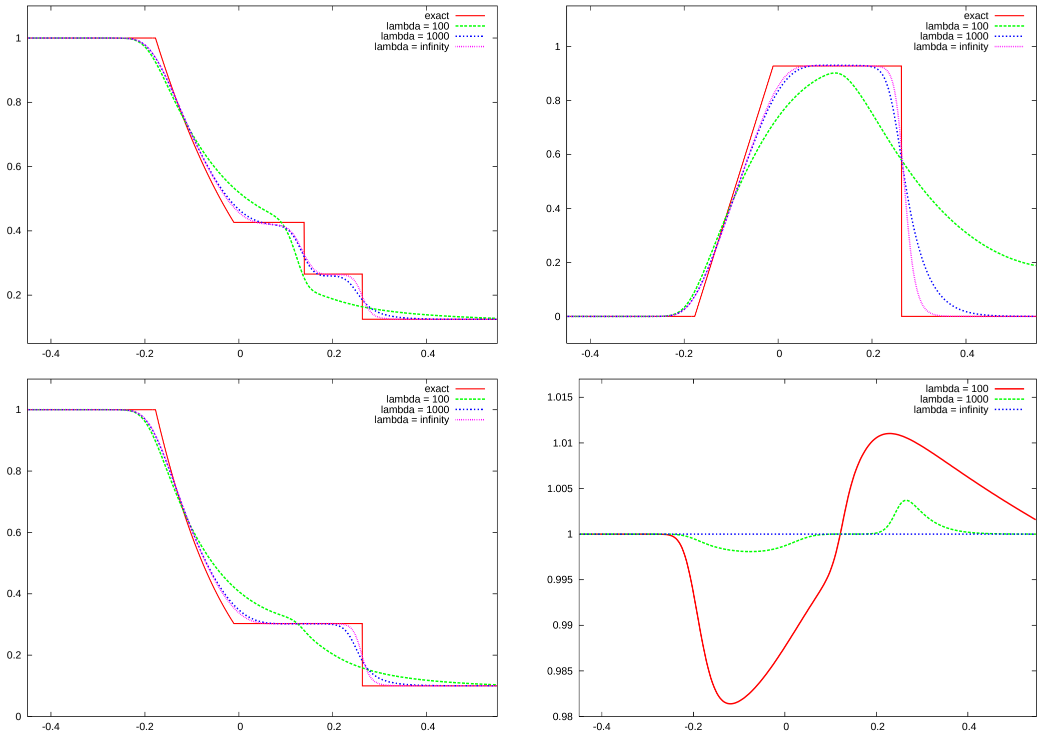

Figure 7: **Test 6** : ρ (left top), u (right top), p (left bottom) and $\rho\mathcal{T}$ (right bottom) at time 0.15

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