On the stability of the optimal value and the optimal set in optimization problems
Joint work with N. Dinh and M.A. Goberna

Marco A. López
Alicante University

Colloque JBHU 2010
Consider the optimization problem

\[
\begin{align*}
(P) \quad & \inf f(x) \\
\text{s.t.} \quad & f_t(x) \leq 0, \forall t \in T; \\
\quad & x \in C,
\end{align*}
\]

where:

- \( T \) is an arbitrary (possibly infinite, possibly empty) index set
- \( \emptyset \neq C \subset X \) is an abstract constraint set, \( X \) is a Banach space
- \( f, f_t : X \to \mathbb{R} \cup \{+\infty\} \) for all \( t \in T \)

**MAIN GOAL:** To analyze the stability of the optimal value function and the optimal set mapping of \((P)\), \( \emptyset \) and say \( \mathcal{F}^{opt} \), under different possible types of perturbations of the data preserving the decision space \( X \) and the index set \( T \).
In [5] we studied the effect on the set of feasible solutions, i.e. the set of solutions of the *constraint system*

\[ \sigma := \{ f_t(x) \leq 0, t \in T; \ x \in C \}, \]

also represented \( \sigma = \{ f_t, t \in T; \ C \} \), of perturbing any function \( f_t, \ t \in T \), and possibly the set \( C \), under the condition that these perturbations maintain certain properties of the constraints.
In [5] we studied the effect on the set of feasible solutions, i.e. the set of solutions of the *constraint system*

\[ \sigma := \{ f_t(x) \leq 0, t \in T; \ x \in C \}, \]

also represented \( \sigma = \{ f_t, t \in T; \ C \} \), of perturbing any function \( f_t, t \in T \), and possibly the set \( C \), under the condition that these perturbations maintain certain properties of the constraints.

Different parametric spaces were considered in [5]. Each one, denoted by \( \Theta \) (for certain subindex) is a given family of systems in the same space \( X \) and index set \( T \).
In [5] we studied the effect on the set of feasible solutions, i.e. the set of solutions of the constraint system

$$\sigma := \{f_t(x) \leq 0, t \in T; x \in C\},$$

also represented \(\sigma = \{f_t, t \in T; C\}\), of perturbing any function \(f_t, t \in T\), and possibly the set \(C\), under the condition that these perturbations maintain certain properties of the constraints.

- Different parametric spaces were considered in [5]. Each one, denoted by \(\Theta\) (for certain subindex) is a given family of systems in the same space \(X\) and index set \(T\).

- The main goal of [5] was to study the stability of the feasible set mapping \(\mathcal{F} : \Theta \Rightarrow X\) such that

$$\mathcal{F}(\sigma) = \{x \in X : f_t(x) \leq 0, \forall t \in T; x \in C\}.$$
In [5] we studied the effect on the set of feasible solutions, i.e. the set of solutions of the constraint system

$$\sigma := \{ f_t(x) \leq 0, t \in T; \ x \in C \},$$

also represented $\sigma = \{ f_t, t \in T; \ C \}$, of perturbing any function $f_t$, $t \in T$, and possibly the set $C$, under the condition that these perturbations maintain certain properties of the constraints.

Different parametric spaces were considered in [5]. Each one, denoted by $\Theta$ (for certain subindex) is a given family of systems in the same space $X$ and index set $T$.

The main goal of [5] was to study the stability of the feasible set mapping $F : \Theta \Rightarrow X$ such that

$$F(\sigma) = \{ x \in X : f_t(x) \leq 0, \forall t \in T; \ x \in C \}.$$

If $T \neq \emptyset$, we shall use the function

$$g := \sup \{ f_t, t \in T \}.$$
We consider *parametric spaces* of the form

\[ \Pi = \mathcal{V} \times \Theta, \]

where \( \mathcal{V} \) is a particular family of functions \( f : X \rightarrow \mathbb{R} \cup \{+\infty\} \) and \( \Theta \) is a particular family of systems \( \sigma \).

The 1st object analyzed in the present paper is the *optimal value function* \( \vartheta : \Pi \rightarrow \mathbb{R} \cup \{-\infty, +\infty\} \) defined as follows

\[
\pi = (f, \sigma) \in \Pi \quad \Rightarrow \quad \vartheta(\pi) := \inf \{ f(x) : x \in \mathcal{F}(\sigma) \} = \inf f(\mathcal{F}(\sigma)).
\]

Conventions: \( \vartheta(\pi) = +\infty \) if \( \mathcal{F}(\sigma) = \emptyset \) (i.e. if \( \sigma \notin \text{dom} \mathcal{F} \)).

If \( \vartheta(\pi) = -\infty \) we say that \( \pi \) is *unbounded*. 
The 2nd object of this work is the \textit{optimal set mapping} 
\[ F^{opt} : \Pi_{\Diamond} \Rightarrow X \]

\[ \pi = (f, \sigma) \in \Pi_{\Diamond} \Rightarrow F^{opt}(\pi) := \{ x \in \mathcal{F}(\sigma): f(x) = \vartheta(\pi) \} \].

If \( \pi \in \text{dom } F^{opt} \) (i.e. \( F^{opt}(\pi) \neq \emptyset \)) we say that \( \pi \) is (optimally) \textit{solvable}.

It is obvious that the stability of \( \vartheta \) and if \( F^{opt} \) will be greatly influenced by the stability of \( \mathcal{F} \), and this why many results in this presentation deal with stability properties of \( \mathcal{F} \).
In this presentation we consider only two parameter spaces, namely:

\[ \Pi_1 := \left\{ \pi \in \Pi : \begin{array}{c} f \text{ and } f_t, t \in T, \text{ are lsc} \\ C \text{ is closed} \end{array} \right\}, \]

\[ \Pi_2 := \left\{ \pi \in \Pi_1 : \begin{array}{c} f \text{ and } f_t, t \in T, \text{ are convex} \\ C \text{ is convex} \end{array} \right\}, \]

where lsc stands for lower semicontinuous.
In this presentation we consider only two parameter spaces, namely:

\[ \Pi_1 := \left\{ \pi \in \Pi : \begin{array}{l} f \text { and } f_t, t \in T, \text { are lsc } \\ C \text { is closed} \end{array} \right\}, \]

\[ \Pi_2 := \left\{ \pi \in \Pi_1 : \begin{array}{l} f \text { and } f_t, t \in T, \text { are convex } \\ C \text { is convex} \end{array} \right\}, \]

where lsc stands for lower semicontinuous.

Obviously, if \( \pi = (f, \sigma) \in \Pi_1 \) both sets (possibly empty) \( \mathcal{F}(\pi) \) and \( \mathcal{F}^{opt}(\pi) \) are closed sets in \( X \).
In this presentation we consider only two parameter spaces, namely:

\[ \Pi_1 := \left\{ \pi \in \Pi : f \text{ and } f_t, t \in T, \text{ are lsc} \right\}, \]

\[ \Pi_2 := \left\{ \pi \in \Pi_1 : f \text{ and } f_t, t \in T, \text{ are convex} \right\}, \]

where lsc stands for lower semicontinuous.

- Obviously, if \( \pi = (f, \sigma) \in \Pi_1 \) both sets (possibly empty) \( \mathcal{F}(\pi) \) and \( \mathcal{F}^{opt}(\pi) \) are closed sets in \( X \).
- If \( \pi = (f, \sigma) \in \Pi_2 \) both sets are also convex.
Let \( A_1, A_2, \ldots, A_n, \ldots \) be a sequence of nonempty subsets of a first countable Hausdorff space \( Y \). We consider the set of \textit{limit points} of this sequence

\[
y \in \lim_{n \to \infty} A_n \iff \text{there exist } y_n \in A_n, \ n = 1, 2, \ldots, \text{ such that } (y_n)_{n \in \mathbb{N}} \text{ converges to } y,
\]

and the set of \textit{cluster points}

\[
y \in \lim_{n \to \infty} A_n \iff \text{there exist } n_1 < n_2 < \ldots < n_k \ldots, \text{ and } y_{n_k} \in A_{n_k}, \text{ such that } (y_{n_k})_{k \in \mathbb{N}} \text{ converges to } y.
\]
Limit sets

- Let $A_1, A_2, ..., A_n, \ldots$ be a sequence of nonempty subsets of a first countable Hausdorff space $Y$. We consider the set of limit points of this sequence

$$y \in \lim_{n \to \infty} A_n \iff \left\{ \begin{array}{l} \text{there exist } y_n \in A_n, \ n = 1, 2, \ldots, \\ \text{such that } (y_n)_{n \in \mathbb{N}} \text{ converges to } y \end{array} \right.;$$

and the set of cluster points

$$y \in \limsup_{n \to \infty} A_n \iff \left\{ \begin{array}{l} \text{there exist } n_1 < n_2 < \ldots < n_k \ldots, \text{ and } y_{n_k} \in A_{n_k} \\ \text{such that } (y_{n_k})_{k \in \mathbb{N}} \text{ converges to } y \end{array} \right..$$

- Clearly $\lim_{n \to \infty} A_n \subset \limsup_{n \to \infty} A_n$ and both sets are closed.
Let $A_1, A_2, ..., A_n, ..$ be a sequence of nonempty subsets of a first countable Hausdorff space $Y$. We consider the set of *limit points* of this sequence

$$y \in \lim_{n \to \infty} A_n \iff \begin{cases} \text{there exist } y_n \in A_n, \; n = 1, 2, ..., \text{such that } (y_n)_{n \in \mathbb{N}} \text{ converges to } y \end{cases};$$

and the set of *cluster points*

$$y \in \limsup_{n \to \infty} A_n \iff \begin{cases} \text{there exist } n_1 < n_2 < ... < n_k..., \text{ and } y_{n_k} \in A_{n_k} \text{ such that } (y_{n_k})_{k \in \mathbb{N}} \text{ converges to } y \end{cases}.$$

Clearly $\lim_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n$ and both sets are closed.

We say that $A_1, A_2, ..., A_n, ..$ is *Kuratowski-Painlevé* convergent to the closed set $A$ if $\lim_{n \to \infty} A_n = \limsup_{n \to \infty} A_n = A$, and we write then $A = K - \lim_{n \to \infty} A_n$. 
Multivalued mappings

Let $Y$ and $Z$ be two topological spaces, and consider a set-valued mapping $S: Y \rightrightarrows Z$. $S$ is lower semicontinuous (in the Berge sense) at $y \in Y$ (lsc, in brief) if, for each open set $W \subseteq Z$ such that $W \cap S(y) \neq \emptyset$, there exists an open set $V \subseteq Y$ containing $y$, such that $W \cap S(y_0) = \emptyset$ for each $y_0 \in V$.

$S$ is upper semicontinuous (in the Berge sense) at $y \in Y$ (usc, in brief) if, for each open set $W \subseteq Z$ such that $S(y) \cap W \neq \emptyset$, there exists an open set $V \subseteq Y$ containing $y$, such that $S(y_0) \subseteq W$ for each $y_0 \in V$.

If both $Y$ and $Z$ are first countable Hausdorff spaces, $S$ is closed at $y \in Y$ if for every pair of sequences $(y_n)_{n \in \mathbb{N}} \subseteq Y$ and $(z_n)_{n \in \mathbb{N}} \subseteq Z$ satisfying $z_n \in S(y_n)$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} y_n = y$ and $\lim_{n \to \infty} z_n = z$, one has $z \in S(y)$.

$S$ is said to be closed if it is closed at every $y \in Y$. Obviously, $S$ is closed if and only if $\text{gph} S = \{ (y, z) \in Y \times Z : z \in S(y) \}$ is closed.

Marco A. López (Alicante University)
Multivalued mappings

- Let $Y$ and $Z$ be two topological spaces, and consider a set-valued mapping $S : Y \rightharpoonup Z$.
- $S$ is lower semicontinuous (in the Berge sense) at $y \in Y$ (lsc, in brief) if, for each open set $W \subset Z$ such that $W \cap S(y) \neq \emptyset$, there exists an open set $V \subset Y$ containing $y$, such that $W \cap S(y') \neq \emptyset$ for each $y' \in V$. 

Marco A. López (Alicante University)
Multivalued mappings

- Let $Y$ and $Z$ be two topological spaces, and consider a set-valued mapping $S : Y \rightrightarrows Z$.
- $S$ is *lower semicontinuous (in the Berge sense)* at $y \in Y$ (lsc, in brief) if, for each open set $W \subseteq Z$ such that $W \cap S(y) \neq \emptyset$, there exists an open set $V \subseteq Y$ containing $y$, such that $W \cap S(y') \neq \emptyset$ for each $y' \in V$.
- $S$ is *upper semicontinuous (in the Berge sense)* at $y \in Y$ (usc, in brief) if, for each open set $W \subseteq Z$ such that $S(y) \subseteq W$, there exists an open set $V \subseteq Y$ containing $y$, such that $S(y') \subseteq W$ for each $y' \in V$.
Multivalued mappings

- Let $Y$ and $Z$ be two topological spaces, and consider a set-valued mapping $S : Y \nrightarrow Z$.

- $S$ is *lower semicontinuous (in the Berge sense)* at $y \in Y$ (lsc, in brief) if, for each open set $W \subset Z$ such that $W \cap S(y) \neq \emptyset$, there exists an open set $V \subset Y$ containing $y$, such that $W \cap S(y') \neq \emptyset$ for each $y' \in V$.

- $S$ is *upper semicontinuous (in the Berge sense)* at $y \in Y$ (usc, in brief) if, for each open set $W \subset Z$ such that $S(y) \subset W$, there exists an open set $V \subset Y$ containing $y$, such that $S(y') \subset W$ for each $y' \in V$.

- If both $Y$ and $Z$ are first countable Hausdorff spaces, $S$ is *closed* at $y \in Y$ if for every pair of sequences $(y_n)_{n \in \mathbb{N}} \subset Y$ and $(z_n)_{n \in \mathbb{N}} \subset Z$ satisfying $z_n \in S(y_n)$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} y_n = y$ and $\lim_{n \to \infty} z_n = z$, one has $z \in S(y)$.
Multivalued mappings

- Let $Y$ and $Z$ be two topological spaces, and consider a set-valued mapping $S : Y \rightrightarrows Z$.
- $S$ is **lower semicontinuous (in the Berge sense)** at $y \in Y$ (lsc, in brief) if, for each open set $W \subset Z$ such that $W \cap S(y) \neq \emptyset$, there exists an open set $V \subset Y$ containing $y$, such that $W \cap S(y') \neq \emptyset$ for each $y' \in V$.
- $S$ is **upper semicontinuous (in the Berge sense)** at $y \in Y$ (usc, in brief) if, for each open set $W \subset Z$ such that $S(y) \subset W$, there exists an open set $V \subset Y$ containing $y$, such that $S(y') \subset W$ for each $y' \in V$.
- If both $Y$ and $Z$ are first countable Hausdorff spaces, $S$ is **closed** at $y \in Y$ if for every pair of sequences $(y_n)_{n \in \mathbb{N}} \subset Y$ and $(z_n)_{n \in \mathbb{N}} \subset Z$ satisfying $z_n \in S(y_n)$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} y_n = y$ and $\lim_{n \to \infty} z_n = z$, one has $z \in S(y)$.
- $S$ is said to be **closed** if it is closed at every $y \in Y$. Obviously, $S$ is closed if and only if $\text{gph} \ S := \{(y, z) \in Y \times Z : z \in S(y)\}$ is closed.
We say that \( \pi = (f, \sigma) \) (or, equivalently, \( \sigma \)) satisfies the strong Slater condition if there exists some \( \bar{x} \in \text{int} \ C \) and some \( \rho > 0 \) such that \( f_t(\bar{x}) < -\rho \) for all \( t \in T \) (i.e., \( g(\bar{x}) \leq -\rho \)).
Strong Slater constraint qualification

- We say that $\pi = (f, \sigma)$ (or, equivalently, $\sigma$) satisfies the *strong Slater condition* if there exists some $\bar{x} \in \text{int}\ C$ and some $\rho > 0$ such that $f_t(\bar{x}) < -\rho$ for all $t \in T$ (i.e., $g(\bar{x}) \leq -\rho$).

- In such a case, $\bar{x}$ is called *strong Slater (SS) point of $\pi$ (or $\sigma$) with associated constant $\rho$*. 

Marco A. López (Alicante University)  
Stability in optimization  
Colloque JBHU 2010
Metrics for functions and sets

In order to define a suitable topology on the parameter spaces $\Pi \diamond$, we proceed in two steps. Let us start with the 1st step.

- We equip the space $\mathcal{V}$ of all functions of the form $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ with the topology of uniform convergence on bounded sets of $X$.
- It is well known that a compatible metric for this topology is given by

$$d(f, h) := \sum_{k=1}^{+\infty} 2^{-k} \min \{1, \sup_{\|x\| \leq k} |f(x) - h(x)|\}.$$ 

Here, by convention, we understand that $(+\infty) - (+\infty) = 0$, $| -\infty | = | +\infty | = +\infty$.

- Let $f, f_n \in \mathcal{V}$, $n = 1, 2, \ldots$. Then $d(f_n, f) \rightarrow 0$ if and only if the sequence $f_1, f_2, \ldots, f_n, \ldots$ converges uniformly to $f$ on the bounded sets of $X$.

- The function spaces $\mathcal{V}_1 := \{f \in \mathcal{V} : f \text{ is lsc}\}$ and $\mathcal{V}_2 := \{f \in \mathcal{V}_1 : f \text{ is convex}\}$, with the metric $d$, are complete metric spaces.
Distances between sets

- In the space of closed sets in $X$ we shall consider the *Attouch-Wets topology*, which is the inherited topology from the one considered in $\mathcal{V}_1$ under the identification

$$C \longleftrightarrow d_C(\cdot),$$

with $d_C(x) = \inf_{c \in C} \|x - c\|$.

- The sequence of nonempty closed sets $(C_n)_{n \in \mathbb{N}}$ *converges in the sense of Attouch-Wets* to the nonempty closed set $C$ if the sequence of functions $(d_{C_n})_{n \in \mathbb{N}}$ converges to $d_C$ uniformly on the bounded sets of $X$.

- This topology is compatible with the distance

$$\tilde{d}(C, D) := \sum_{k=1}^{+\infty} 2^{-k} \min \left\{ 1, \sup_{\|x\| \leq k} |d_C(x) - d_D(x)| \right\},$$

i.e. $\tilde{d}(C, D) = d(d_C, d_D)$. 

Marco A. López (Alicante University)

Stability in optimization

Colloque JBHU 2010
More on convergence of sets

- The space of all closed sets in $X$ equipped with this distance $\tilde{d}$ becomes a complete metric space.

- Because $X$ is Banach, we have that if the sequence $(d_{C_n})_{n \in \mathbb{N}}$ converges uniformly on bounded sets of $X$ to a continuous function $f$, there exists a nonempty closed set $C$ such that $f = d_C$.

- The sequence of nonempty closed sets $(C_n)_{n \in \mathbb{N}}$ converges in Attouch-Wets sense to the nonempty closed $C$ if and only if

$$\forall k \in \mathbb{N} : \lim_{n \to \infty} \max \{ e(C_n \cap k\mathbb{B}, C), e(C \cap k\mathbb{B}, C_n) \} = 0,$$

where

$$e(A, B) := \sup_{a \in A} d_B(a) = \inf \{ \alpha > 0 : B + \alpha\mathbb{B} \supset A \},$$

and $\mathbb{B} := \{ x \in X : \|x\| \leq 1 \}$. 

Our metric

Given \( \pi = (f, \{f_t, t \in T; C\}) \), \( \pi' = (f', \{f'_t, t \in T; C'\}) \) \( \in \Pi \), we define

\[
    d(\pi, \pi') := \max \{d(f, f'), \sup_{t \in T} d(f_t, f'_t), \tilde{d}(C, C')\}. \tag{1}
\]

If \( T = \emptyset \), we take \( \sup_{t \in T} d(f_t, f'_t) = 0 \).

Theorem

\((\Pi_i, d), i = 1, 2, \) are complete metric spaces.
Lemma

Let $C$ be a closed set in $X$, $x_0 \in \text{int} \, C$, and consider $\varepsilon > 0$ such that $x_0 + \varepsilon B \subset C$. Then there is $\rho > 0$ such that

$$\tilde{d}(C, C') < \rho \implies (x_0 + \varepsilon B) \cap C' \neq \emptyset.$$ 

Lemma

Consider $\pi = (f; \{f_t, t \in T; C\}) \in \Pi_1$ and suppose that the marginal function $g = \sup_{t \in T} f_t$ is usc (and so, continuous). If $\hat{x}$ is an SS-point of $\pi$, then there exists $\varepsilon > 0$ such that

$$x \in \hat{x} + \varepsilon B \text{ and } d(\pi, \pi') < \varepsilon \implies g'(x) < 0,$$

with $\pi' = (f'; \{f'_t, t \in T; C'\}) \in \Pi_1$ and $g' := \sup_{t \in T} f'_t$. 

Marco A. López (Alicante University)  Stability in optimization  Colloque JBHU 2010
Consider a convex set \( C \) with \( 0 \in \text{int} \ C \neq \emptyset \), and the associated *Minkovski gauge function* defined as

\[
p_C(x) := \inf\{\lambda \geq 0 \mid x \in \lambda C\},
\]

and for any positive real number \( \mu \), define a set

\[
C_\mu := \{x \in X \mid p_C(x) \leq \mu\}.
\]

Given \( \varepsilon > 0 \), there exists \( \mu \in ]0, 1[ \) such that

\[
\tilde{d}(C, C_\mu) \leq \varepsilon.
\]

The system \( \sigma \) is said to be *Tuy regular* if there exists \( \varepsilon > 0 \) such that for any \( u \in \mathbb{R}^T \) and for any nonempty convex set \( C' \subset X \) satisfying

\[
\max\{\sup_{t \in T} |u_t|, \tilde{d}(C, C')\} < \varepsilon,
\]

the system

\[
\sigma' = \{f_t(x) - u_t \leq 0, t \in T; x \in C'\} \in \text{dom} \mathcal{F}.
\]

The last definition is inspired in a similar one of H. Tuy ([3]).
Theorem

The feasible set mapping $\mathcal{F}$ is closed on $\Theta_i$, $i = 1, 2$.

Theorem

Let $\sigma = \{f_t, t \in T; C\} \in \Theta_1$ with $T \neq \emptyset$, and consider the following statements:

(i) $\mathcal{F}$ is lsc at $\sigma$;
(ii) $\sigma \in \text{int dom } \mathcal{F}$;
(iii) $\sigma$ is Tuy regular;
(iv) $\sigma$ satisfies the strong Slater condition;
(v) $\mathcal{F}(\sigma)$ is the closure of the set of SS points of $\sigma$.

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (iv). Moreover, if $C$ is convex, and $\text{int } C \neq \emptyset$, then (i) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (iv).

If, in addition, $\sigma \in \Theta_2$ and $g = \sup_{t \in T} f_t$ is usc, then all the statements (i) — (v) are equivalent.
This property has important consequences in the overall stability of a system $\sigma$, as well as in the sensitivity analysis of perturbed systems, affecting even the numerical complexity of the algorithms conceived for finding a solution of the system.

Many authors ([Aubin84], [Ausl84], [Com90], [JuThi90], [KlaHenr98], [KlKu85], [Rob75,76], [ZoKur79], etc.) have investigated this property and explored the relationship of this property with standard constraint qualification as Mangasarian-Fromovitz CQ, Slater CQ, Robinson CQ, etc.

For instance, in [KlaHenr98] the relationships among the metric regularity, the metric regularity with respect to RHS perturbations, and the extended Mangasarian-Fromowitz CQ are established in a non-convex differentiable setting.
Let us remember the definition of metric regularity in our specific setting:

**Definition**

$\mathcal{F}^{-1}$ is said to be *metrically regular at* $(x, \sigma) \in \text{gph} \mathcal{F}^{-1}$ if there exist real numbers $\varepsilon, \delta > 0$ and $\kappa \geq 0$ such that

$$
\begin{align*}
\text{d}(\sigma, \sigma') < \delta \\
\|x - x'\| < \varepsilon
\end{align*}
\right\} \Rightarrow \text{d}(x', \mathcal{F}(\sigma')) \leq \kappa \text{d}(\sigma', \mathcal{F}^{-1}(x')).
$$

This inequality is specially useful if the residual $\text{d}(\sigma', \mathcal{F}^{-1}(x'))$ can be easily computed.

The existence of an abstract constraint set makes the computation of $\text{d}(\sigma', \mathcal{F}^{-1}(x'))$ very difficult. In fact, if $\sigma' = \{f'_t, t \in T, C\}$ we have

$$
\text{d}(\sigma', \mathcal{F}^{-1}(x')) = \max \left\{ \left[ g'(x') \right]_+, \tilde{d}(C', C_{x'}(X)) \right\},
$$

where $C_{x'}(X)$ is the family of all the closed convex sets $C \subset X$ such that $x' \in C$, and

$$
\tilde{d}(C', C_{x'}(X)) = \inf \left\{ \tilde{d}(C', C) : C \in C_{x'}(X) \right\}.
$$
Nevertheless, when we assume that $C$ is the whole space $X$, the property makes sense. In fact, if $C$ is constantly equal to $X$ and $\sigma' = \{f'_t, t \in T\}$, it is straightforward that

$$d(\sigma', \mathcal{F}^{-1}(x')) = \left[ \sup_{t \in T} f'_t(x') \right]_+ \equiv [g'(x')]_+,$$

where $g' = \sup_{t \in T} f'_t$ and $[\alpha]_+ := \max\{\alpha, 0\}.$

**Theorem**

Let $\mathcal{F} : \Theta \otimes X$ and $(x, \sigma) \in \text{gph} \mathcal{F}^{-1}$ with $\sigma = \{f_t, t \in T\}$, where $\Theta \otimes$ is the set of parameters whose constraint set is $X$ and $f_t$ is convex for all $t \in T$. Then the following statements are true:

(i) If $g = \sup_{t \in T} f_t$ is usc at $x$, and $\mathcal{F}^{-1}$ is metrically regular at $(x, \sigma)$, then $\mathcal{F}$ is lsc at $\sigma$.

(ii) If $X$ is a Hilbert space, and $\mathcal{F}$ is lsc at $\sigma$, then $\mathcal{F}^{-1}$ is metrically regular at $(x, \sigma)$.
We now study the upper semicontinuity of the optimal value function $\vartheta$.

**Theorem**

Let $\pi = (f, \sigma) \in \Pi_1$. The following statements hold.

(i) If $\mathcal{F}$ is lsc at $\sigma$ then $\vartheta$ is usc at $\pi$ provided that $f$ is usc.

(ii) If $\vartheta$ is usc at $\pi$ then $\mathcal{F}$ is lsc at $\sigma$ provided that the functions $f_t$, $t \in T$, are convex, $C$ is convex (i.e., if $\sigma \in \Theta_2$), $\text{int} \, C \neq \emptyset$, and the corresponding marginal function $g = \sup_{t \in T} f_t$ is usc.
Consider the sublevel sets mapping $\mathcal{L} : \Pi \times \mathbb{R} \rightarrow X$:

$$\mathcal{L}(\pi, \lambda) := \{x \in \mathcal{F}(\sigma) : f(x) \leq \lambda\}, \text{ with } \pi = (f, \sigma).$$

**Theorem**

The mapping $\mathcal{L}$ is closed at any point $(\pi, \lambda) \in \Pi \times \mathbb{R}$.

**Definition**

Let $Y$ and $Z$ be two top. spaces and $S : Y \rightarrow Z$. We say that $S$ is uniformly compact-bounded at $y_0 \in Y$ if $\exists$ a compact set $K \subset Z$ and a neighborhood $V$ of $y_0$ such that $y \in V \implies S(y) \subset K$.

**Theorem**

(a) If $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$ with $\pi \in \Pi$, then $\vartheta$ is lsc at $\pi$.

(b) Suppose that $X = \mathbb{R}^n$, and $\pi \in \Pi_2$. If $\mathcal{F}^{opt}(\pi)$ is a nonempty compact set, then $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$. 
This section starts with a sufficient condition for the closedness of $\mathcal{F}^{opt}$.

**Theorem**

Consider $\pi = (f, \sigma) \in \Pi_1$ such that $f$ is usc and $\mathcal{F}$ is lsc at $\sigma$. Then $\mathcal{F}^{opt}$ is closed at $\pi$.

**Theorem**

Consider $\pi = (f, \sigma) \in \Pi_1$ such that $f$ is usc, $\mathcal{F}$ is lsc at $\sigma$, and $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$. Then, $\vartheta$ is continuous at $\pi$ and $\mathcal{F}^{opt}$ is usc at $\pi$. 


