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When only global optimization matters

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Abstract In this short note, the objective of which is essentially pedagogical, we show that in the well-known problem which consists of minimizing the rank of a matrix, every admissible point is a local minimizer. Hence, in this problem like in various other ones, only global minimization matters.

Keywords Matrix rank · Lower-semicontinuous function · Local minimizer

1 Introduction

Readers of the Journal of Global Optimization, as other users of results and techniques from optimization, know that there are minimization problems (theoretical or practical) where only global minimizers have the desired properties or are useful for the pursued goals. Such instances arise for example in nonlinear least squares problems, energy minimization problems, robust statistics [1], etc. We consider here another problem, that of minimizing the rank of a matrix:

$$(\mathcal{P}) \quad \begin{cases} \text{Minimize } f(A) := \text{rank of } A \\ \text{subject to } A \in C, \end{cases}$$

where C is a subset of $\mathcal{M}_{m,n}(\mathbb{R})$ (the vector space of m by n real matrices). A related (or cousin) problem, actually equivalent in terms of difficulty, stated in $\mathbb{R}^n$ this time, consists in minimizing the so-called counting function $x = (x_1, \ldots, x_n) \mapsto c(x) :=$ number of nonzero components x_i in x:

$$(\mathcal{Q}) \quad \begin{cases} \text{Minimize } c(x) \\ \text{subject to } x \in S, \end{cases}$$

where S is a subset of $\mathbb{R}^n$.

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There are dozens of papers devoted to these optimization problems: modelling practical problems giving rise to such formulations, convex relaxations thereof, adapted algorithms to solve these relaxed forms, etc. In order not to be immediately drowned, we advise the interested reader to peruse the two survey papers [2,3]. Therefore, problems $(\mathcal{P})$ and $(\mathcal{Q})$ are not "pathological"; they however share the following bizarre property: every admissible point is a local minimizer.

2 The result

We state the result only for the problem $(\mathcal{P})$.

Theorem 1 *Every admissible point in $(\mathcal{P})$ is a local minimizer.*

Proof The two ingredients of the proof are the lower-semicontinuity of the function $f : A \in \mathcal{M}_{m,n}(\mathbb{R}) \mapsto f(A) :=$ rank of A and the fact that f takes finitely many values.

Let therefore any A be in the constraint set C. Since the rank function f is lower-semicontinuous,

$$\liminf_{B \to A} f(B) \geqslant f(A).$$

In a detailed way, for any $\epsilon > 0$, there exists a neighborhood $\mathcal{N}$ of A such that

$$f(B) \geqslant f(A) - \epsilon \text{ for all } B \in \mathcal{N}. \tag{1}$$

We now choose $\epsilon < 1$, say $\epsilon = 1/2$. Since f can only take the integer values $0, 1, .., p := \min(m, n)$, we infer from (1):

$$f(B) \geqslant f(A) \text{ for all } B \in \mathcal{N} \cap C.$$

Hence we have proved that A is a local minimizer of f on C. □

Comments

1. To have a constraint set C in $(\mathcal{P})$ or not does not change anything; the two above-mentioned properties of the objective function f suffice.
2. A natural question arising from the consideration of nonsmooth (even discontinuous here) functions in optimization, is: For this particular function $f(A) =$ rank of A, what is its generalized subdifferential $\partial f(A)$? Again the result is somehow strange: the various generalized subdifferentials, including CLARKE's one, all coincide and, of course, contain the zero element [4]. In the simpler context where the counting function c is at hand, $\partial c(x)$ is always a vector space V, ranging from $\{0\}$ to the whole space $\mathbb{R}^n$.

3 Conclusion

When studying calculus or, more generally, analysis, students are confronted with functions enjoying properties like semicontinuity, continuity or differentiability. They maybe know or are able to prove the following result: A continuous function $f : \mathbb{R}^n \to \mathbb{R}$, for which any point is a local minimizer, is necessarily constant on $\mathbb{R}^n$ (This is even true if one restricts the question to some arcwise-connected subset S of $\mathbb{R}^n$). For lower-semicontinuous functions

f, this result is no more true. With the rank function, or, in a simpler context, the counting function c, they have such examples, moreover arising from truly interesting optimization problems. As a result, as far as optimization of these functions is concerned, only global optimization matters.

References

1. Flores, S.: Global Optimization Problems in Robust Statistics. Ph D Thesis of the University of Toulouse (February 2011)
2. Recht, B., Fazel, M., Parrilo, P.A.: Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. SIAM Rev. **52**(3), 471–501 (2010)
3. Tseng, P.: Approximation accuracy, gradient methods, and error bound for structured convex optimization. Math. Program. Ser. B 125(2), 263–295 (2010)
4. Le, H.Y.: Ph D Thesis of the University of Toulouse (in progress)