When is $L^r(\mathbf{R})$ contained in $L^p(\mathbf{R}) + L^q(\mathbf{R})$?

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Abstract

We prove a necessary and sufficient condition on the exponents $p, q, r \geq 1$ such that $L^r(\mathbf{R}) \subset L^p(\mathbf{R}) + L^q(\mathbf{R})$. In doing so, we explore the structure of $L^p(\mathbf{R}) + L^q(\mathbf{R})$ as a normed vector space.

1 Introduction.

In a recent mathematical note aimed at undergraduate students and their teachers ([3]), J.-B. Hiriart-Urruty and M. Pradel proposed a way to extend the Fourier transformation to all the spaces $L^r(\mathbf{R})$ with $1 \leq r \leq 2$ in the following manner.

– First, they classically define the Fourier transformation on $L^1(\mathbf{R})$. Then they define it on $L^2(\mathbf{R})$ using in that case the much less known Wiener's approach which relies on a specific Hilbertian basis made of the so-called Bernstein functions.

– After having checked the coherence of both definitions on $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, they extend the definition of the Fourier transformation to the space $L^1(\mathbf{R}) + L^2(\mathbf{R})$.

– Finally, and this is the key-point, they prove the inclusion $L^r(\mathbf{R}) \subset L^1(\mathbf{R}) + L^2(\mathbf{R})$ for all $1 \leq r \leq 2$, so that the Fourier transformation can be extended to all the Lebesgue spaces $L^r(\mathbf{R})$.

An obvious question which arises is, what happens if $r \notin [1, 2]$? For example, is the inclusion $L^3(\mathbf{R}) \subset L^1(\mathbf{R}) + L^2(\mathbf{R})$ true or not ? The objective of the present mathematical note is to answer this question. We provide a necessary and sufficient condition for the inclusion $L^r(\mathbf{R}) \subset L^p(\mathbf{R}) + L^q(\mathbf{R})$ to hold true. For that purpose, concentrate on the sum $L^p(\mathbf{R}) + L^q(\mathbf{R})$ and see how it is structured as a normed vector space.

2 How to norm a sum of normed vector spaces ?

Let V and W be two vector subspaces of a "holdall" vector space, E. If V is equipped with a norm $\| \cdot \|_1$ and W with a norm $\| \cdot \|_2$, is there a natural way to define a norm on the vector subspace $V + W$? Of course, we do not assume that $V \cap W = \{O_E\}$. If this was the case, a natural way to define a norm N on $V + W$ would be

$$N(u) = \|v\|_1 + \|w\|_2,$$

whenever $u \in V + W$ is (uniquely) decomposed as $u = v + w$, with $v \in V$ and $w \in W$.

What we have in mind is indeed $V = L^p(\mathbf{R})$, $W = L^q(\mathbf{R})$ and $E = L(\mathbf{R})$ as the "holdall" vector space ($L(\mathbf{R})$ stands for the set of all Lebesgue classes of mesurable functions on $\mathbf{R}$).

The theorem below answers the question posed above. It does not seem to be well-known, except by people who have to deal with the interpolation of functional spaces (like in [1]).

Theorem 1 *Let N be defined on $V + W$ as follows*

$$N(u) := \inf\{\|v\|_1 + \|w\|_2 \ ; \ u = v + w \ \textit{with} \ v \in V \ \textit{and} \ w \in W\}. \tag{1}$$

Then, N is a semi-norm on $V + W$. It is a norm under the following "compatibility" assumption :

$$(\mathcal{T}) \qquad \left.\begin{array}{l} z_k \in V \cap W, \\ z_k \longrightarrow a \ \textit{in} \quad (V, ||.||_1) \\ z_k \longrightarrow b \ \textit{in} \quad (W, ||.||_2) \end{array}\right\} \Longrightarrow a = b.$$

Proof : To check that $N(O_E) = 0$, $N(\lambda u) = |\lambda| N(u)$ for all $\lambda \in \mathbf{R}$, $u \in V + W$, and $N(u_1 + u_2) \leq N(u_1) + N(u_2)$ for all u_1 and u_2 in $V + W$, does not raise any difficulty. It suffices to use the definition of the lower bound (or infimum) of a set of real numbers.

To prove that $N(u) = 0$ implies that $u = O_E$ is a bit more tricky. Our experience with that question with undergraduate students shows that they usually fail to answer it correctly. Their common mistake is to deduce that a sequence $(v_k + w_k)_k$ converges to O_E using the fact that $(v_k)_k$ converges to O_E in V and $(w_k)_k$ converges to O_E in W. We therefore provide a proof here.

We first begin by observing that

$$\nu \ : \ V \cap W \ni u \mapsto \nu(u) := \max(\|u\|_1, \|u\|_2) \tag{2}$$

is a norm on $V \cap W$; this is an easy result to prove.

Consider therefore, $u \in V + W$ such that $N(u) = 0$. We take for example

$$u = v + w, \quad \text{with } v \in V \ \text{ and } w \in W. \tag{3}$$

Due to the definition (1) of $N(u)$, for all positive integers k, there exists $v_k \in V$ and $w_k \in W$ such that

$$u = v_k + w_k, \ \text{ and} \tag{4}$$

$$\|v_k\|_1 + \|w_k\|_2 \leq N(u) + \frac{1}{k} = \frac{1}{k}. \tag{5}$$

Thus, $(v_k)_k$ converges to O_E in V and $(w_k)_k$ converges to O_E in w. But what about $(v_k + w_k)_k$? Recall that no norm, hence, no topology, has yet been defined on $V + W$. We infer from (3) and (4) that

$$v + w = v_k + w_k,$$

and thus

$$v - v_k = w_k - w. \tag{6}$$

As a consequence, this common vector $z_k := v - v_k = w_k - w$ lies in $V \cap W$ and, since $\nu(z_k) = \|v - v_k\|_1$ or $\|w_k - w\|_2$,

$$z_k \to v \text{ in } (V, \|\cdot\|_1) \text{ and } z_k \to -w \text{ in } (W, \|\cdot\|_2).$$

The assumption $(\mathcal{T})$ then ensures that

$$v = -w, \text{ that is } u = v + w = O_E.$$

■

Note that the technical "compatibility" assumption $(\mathcal{T})$ is satisfied

– trivially if $V \cap W = \{O_E\}$ (in that case it amounts to $0 = 0$).

– in the cases where $V = L^p(\mathbb{R}),\ W = L^q(\mathbb{R})$ (indeed, convergence of $(f_k)_k$ towards f in $L^p(\mathbb{R})$ implies convergence almost everywhere of a subsequence of $(f_k)_k$ towards f).

– when the "holdall" vector space E is a Hausdorff topological vector space in which V and W are continuously imbedded.

We suppose that $(\mathcal{T})$ is in force for the rest of the section.

The vector space $V+W$, equipped with the norm N as defined in (1), inherits some properties of $(V, \|\cdot\|_1)$ and $(W, \|\cdot\|_2)$. Here is one.

Theorem 2 *If $(V, \|\cdot\|_1)$ and $(W, \|\cdot\|_2)$ are Banach spaces, then so is $(V+W, N)$.*

Proof : The proof of this theorem offers the opportunity to use a caracterization of completeness of normed vector spaces which is not well-known. Let $(X, \|\cdot\|)$ be a normed space. We have indeed

$$\left((X, \|\cdot\|) \text{ is complete }\right) \iff \left(\begin{array}{c}\text{Every series in } X, \text{ of general term } a_k, \text{for} \\ \text{which } \sum_{k=0}^{\infty} \|a_k\| < +\infty \text{ does converges in } X \\ \text{(toward a sum denoted as } \sum_{k=0}^{\infty} a_k)\end{array}\right). \tag{7}$$

The implication $(\Rightarrow)$ is classical, it is the most often used. The converse implication $(\Leftarrow)$ is not often used, but we have an opportunity to do that here. For a proof of the equivalence (7), see for example ([6], Theorems 2-XIV-2.1

and 2.2, pages 164-165), ([4], page 20) or ([7], pages 262 and 270); it is also sketched in ([1], page 24). The proof is not very difficult however, but readers are encouraged to work through it themselves. We now proceed in this manner to a proof of Theorem 2.

Consider a series in $V+W$, of general term u_k for which $\sum_{k=0}^{\infty} N(u_k) < \infty$. We have to prove that $\sum_{k=0}^{n} u_k$ converges as $n \to +\infty$, to some element $u \in V+W$. In view of the definition (1) of N, for all positive integer k, there exist $v_k \in V$ and $w_k \in W$ satisfying

$$u_k = v_k + w_k, \text{ and} \tag{8}$$

$$\|v_k\|_1 + \|w_k\|_2 \leq N(u_k) + \frac{1}{k^2}. \tag{9}$$

Thus, $\sum_{k=0}^{\infty} \|v_k\|_1 < +\infty$ and $\sum_{k=0}^{\infty} \|w_k\|_2 < +\infty$. Since both $(V, \|\cdot\|_1)$ and $(W, \|\cdot\|_2)$ have been assumed to be complete, there exist $v \in V$ and $w \in W$ such that

$$\sum_{k=0}^{n} v_k \underset{n\to+\infty}{\longrightarrow} v \text{ in } V, \text{ and } \quad \sum_{k=0}^{n} w_k \underset{n\to+\infty}{\longrightarrow} w \text{ in } W. \tag{10}$$

Let $u := v + w \in V + W$. Let us check that, as expected, $\sum_{k=0}^{n} u_k$ converges to u in $(V+W, N)$. Indeed,

$$N\left(u - \sum_{k=0}^{n} u_k\right) = N\left(u + v - \sum_{k=0}^{n}(v_k + w_k)\right) \leq \left\|v - \sum_{k=0}^{n} v_k\right\|_1 + \left\|w - \sum_{k=0}^{n} w_k\right\|_2.$$

It remains to apply (10) to get the desired result, $N\left(u - \sum_{k=0}^{n} u_k\right) \underset{n\to+\infty}{\longrightarrow} 0$. ■

Comments : The construction of the norm ν on $V \cap W$ and N on $V + W$ deserves some geometrical interpretation. Even if $\|\cdot\|_1$ (resp. $\|\cdot\|_2$) is only defined on $V \subset E$ (resp. on $W \subset E$), we can extend it to the whole of E by setting $\|u\|_1 = +\infty$ if $u \notin V$ (resp. $\|u\|_2 = +\infty$ if $u \notin W$). We still denote by $\|\cdot\|_1$ and $\|\cdot\|_2$ the extended functions.

Clearly, $\|\cdot\|_1$ and $\|\cdot\|_2$ are convex positively homogeneous functions on E. Modern convex analysis accepts and can handle convex functions possibly taking the value $+\infty$ ([5]). An important geometrical object associated with a convex function $f : E \to \mathbf{R} \cup \{+\infty\}$ is its so-called strict epigraph

$$\text{epi}_s f := \{(x, r) \in E \times \mathbf{R} \ : \ f(x) < r\}$$

(literally, what is strictly above the graph of f). In our situation, $K_1 := \text{epi}_s \|\cdot\|_1$ and $K_2 := \text{epi}_s \|\cdot\|_2$ are open convex cones of E. So, what are the strict epigraphs of the norm functions ν and N ? One easily checks the following

$$\begin{aligned} \text{epi}_s \nu &= (\text{epi}_s \|\cdot\|_1) \cap (\text{epi}_s \|\cdot\|_2), \text{ and} \\ \text{epi}_s N &= (\text{epi}_s \|\cdot\|_1) + (\text{epi}_s \|\cdot\|_2). \end{aligned} \tag{11}$$

The sets where ν (resp. N) is finite, called the domain of ν (resp. of N) in convex analysis, is just $V \cap W$ (resp. $V + W$).

The binary operation which builds a convex function f from two other ones f_1 and f_2, via the geometric construction

$$\mathrm{epi}_s f = \mathrm{epi}_s f_1 + \mathrm{epi}_s f_2$$

is called the *infimal convolution* of f_1 and f_2 ([2],[5]). This operation enjoys properties similar to the usual (integral) convolution in classical analysis.

In brief, the norm N has been designed as an infimal convolution of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$.

Returning to our particular setting $E = L(\mathbf{R}), V = L^p(\mathbf{R})$ and $W = L^q(\mathbf{R})$ with $1 \leq p, q < +\infty$, the vector space $L^p(\mathbf{R}) + L^q(\mathbf{R})$ can be equipped with a norm that we denote $\|\cdot\|_{p,q}$ as follows,

$$\|f\|_{p,q} = \inf\{\|g\|_p + \|h\|_q \ ; \ f = g + h \ \text{ with } g \in L^p(\mathbf{R}) \text{ and } h \in L^q(\mathbf{R})\}. \quad (12)$$

As proved in Theorem 2, $(L^p(\mathbf{R}) + L^q(\mathbf{R}), \|\cdot\|_{p,q})$ is a Banach space.

3 Comparing $L^r(\mathbf{R})$ with $L^p(\mathbf{R}) + L^q(\mathbf{R})$

We know that the Lebesgue spaces $L^r(\mathbf{R})$ and $L^s(\mathbf{R})$ (for $1 \leq r \neq s < +\infty$) cannot be compared. Neither $L^r(\mathbf{R})$ is contained in $L^s(\mathbf{R})$ nor the converse. A direct comparison is however possible if we deal with the sum of these spaces. Here is the main result of this section.

Theorem 3 *If $1 \leq p < q < +\infty$, then we have the following :*

1. *$L^r(\mathbf{R})$ is contained in $L^p(\mathbf{R}) + L^q(\mathbf{R})$ whenever $r \in [p, q]$,*
2. *if $1 \leq r < +\infty$ does not lie in $[p, q]$, then $L^r(\mathbf{R})$ is not contained in $L^p(\mathbf{R}) + L^q(\mathbf{R})$.*

Proof : (1) Let $f \in L^r(\mathbf{R})$ and consider $X := \{x \in \mathbf{R} \ : \ |f(x)| > 1\}$ (a measurable set defined within a set of null measure) as well as $X^c = \mathbf{R} \setminus X$ (the complementary set of X in $\mathbf{R}$). We decompose f as follows :

$$f = f_1 + f_2, \quad \text{with } f_1 = f \cdot \mathbf{1}_X \ \text{ and } f_2 = f \cdot \mathbf{1}_{X^c}. \quad (13)$$

We claim that (13) provides an explicit decomposition of f in $L^p(\mathbf{R}) + L^q(\mathbf{R})$, that is to say $f_1 \in L^p(\mathbf{R})$ and $f_2 \in L^q(\mathbf{R})$. We first prove that,

$$\int_{\mathbf{R}} |f_1(x)|^p d\lambda(x) = \int_X |f(x)|^p d\lambda(x) = \int_X |f(x)|^{p-r} \cdot |f(x)|^r d\lambda(x). \quad (14)$$

For $x \in X$, $|f(x)| > 1$ and, since the exponent $p-r$ is nonpositive, $|f(x)|^{p-r} \leq 1$. Consequently, the last integral in the string of equalities (14) is bounded above by $\int_X |f(x)|^r d\lambda(x)$. Finally,

$$(14') \qquad \int_{\mathbf{R}} |f_1(x)|^p d\lambda(x) \leq \int_X |f(x)|^r d\lambda(x) \leq \int_{\mathbf{R}} |f(x)|^r d\lambda(x) < +\infty.$$

We thus have proved that $f_1 \in L^p(\mathbf{R})$.

Second, we prove that $f_2 \in L^q(\mathbf{R})$. Indeed,

$$\int_{\mathbf{R}} |f_2(x)|^q d\lambda(x) = \int_{X^c} |f(x)|^q d\lambda(x) = \int_{X^c} |f(x)|^{q-r} \cdot |f(x)|^r d\lambda(x). \qquad (15)$$

For $x \in X^c$, $|f(x)| \leq 1$ and, since the exponent $q-r$ is nonegative, $|f(x)|^{q-r} \leq 1$. Again, the last integral in the string of equalities (15) is bounded above by $\int_{X^c} |f(x)|^r d\lambda(x)$. As a result,

$$\int_{\mathbf{R}} |f_2(x)|^q d\lambda(x) = \int_{X^c} |f(x)|^r d\lambda(x) = \int_{\mathbf{R}} |f(x)|^r d\lambda(x) < +\infty. \qquad (16)$$

We therefore, have proved that $f_2 \in L^q(\mathbf{R})$.

(2) The second part of Theorem 3 is a bit harder to prove (like most of the negative results in mathematics). We actually have to distinguish two cases for r in the segment $[p, q]$: $r < p$ and $r > q$.

Case 1 : $r < p$. Choose α satisfying $1/p < \alpha < 1/r$ and let f be defined on $\mathbf{R}$ by $f(x) = x^{-\alpha}\mathbf{1}_{(0,1]}(x)$.

Since $|f(x)|^r = x^{-\alpha r}$ for $x \in (0, 1]$ and 0 elsewhere, the choice of α implies that $f \in L^r(\mathbf{R})$ (since $\alpha r < 1$). The same argument shows that $f \notin L^p(\mathbf{R})$ (since $\alpha p > 1$).

Suppose now that $L^r(\mathbf{R}) \subset L^p(\mathbf{R}) + L^q(\mathbf{R})$. Then

$$f \in L^r(\mathbf{R}) \subset L^p(\mathbf{R}) + L^q(\mathbf{R}) \subset L^p([0, 1]) + L^q([0, 1]). \qquad (17)$$

But since $[0, 1]$ is of Lebesgue finite measure and $p < q$, $L^q([0, 1])$ is contained in $L^p([0, 1])$, so that (17) yields that $f \in L^p([0, 1])$. This is not the case.

Thus we have proved that $L^r(\mathbf{R})$ is not contained in $L^p(\mathbf{R}) + L^q(\mathbf{R})$.

Case 2 : $r > q$. Our proof in this case relies on a technical lemma that we present separately.

Lemma 1 *Let $1 \leq p < q < +\infty$, let Ω be a mesurable subset of $\mathbf{R}$, and let $f \in L^p(\Omega) + L^q(\Omega)$. Then $f \in L^q(\Omega)$ whenever it is essentially bounded on Ω.*

Proof of the Lemma : Let f be decomposed as $f = f_p + f_q$, with $f_p \in L^p(\Omega)$ and $f_q \in L^q(\Omega)$. So, to prove that $f \in L^q(\Omega)$ amounts to proving that $f_p \in L^q(\Omega)$.

Let $X := \{x \in \mathbf{R} \;:\; |f_p(x)| > 1\}$. To show that $\int_\Omega |f_p(x)|^q d\lambda(x)$ is finite, we cut it into two pieces : $\int_{\Omega\cap X^c} |f_p(x)|^q d\lambda(x)$ and $\int_{\Omega\cap X} |f_p(x)|^q d\lambda(x)$.

Consider the first piece. Since $f_p \in L^p(\mathbf{R})$, the set X is of finite (Lebesgue) measure. Now with the definition of X and the fact that $q - p > 0$, we obtain

$$\begin{aligned}\int_{\Omega\cap X^c} |f_p(x)|^q d\lambda(x) &= \int_{\Omega\cap X^c} |f_p(x)|^{q-p} \cdot |f_p(x)|^p d\lambda(x)\\ &\leq \int_{\Omega\cap X^c} |f_p(x)|^p d\lambda(x) \leq \int_{\mathbf{R}} |f_p(x)|^p d\lambda(x) < +\infty.\end{aligned}$$

This concludes the argument for the first piece.

Consider now the second piece. Since f has been assumed essentially bounded on Ω,

$$|f_p(x)| \leq |f(x) - f_q(x)| \leq \|f\|_\infty + |f_q(x)| \quad \text{for almost all } x \text{ in } \Omega.$$

Consequently,

$$\int_{\Omega\cap X} |f_p(x)|^q d\lambda(x) \leq \int_{\Omega\cap X} \left(\|f\|_\infty + |f_q(x)|\right)^q d\lambda(x). \tag{18}$$

The convexity of the function $t \mapsto t^q$ on $[0, +\infty)$ implies that $\left(\|f\|_\infty + |f_q(x)|\right)^q \leq 2^{q-1}\left(\|f\|_\infty^q + |f_q(x)|^q\right)$. So, we pursue the string of inequalites (18) with

$$\begin{aligned}\int_{\Omega\cap X} |f_p(x)|^q d\lambda(x) &\leq 2^{q-1} \int_{\Omega\cap X} \left(\|f\|_\infty^q + |f_q(x)|^q\right) d\lambda(x)\\ &\leq 2^{q-1}\left[\lambda(X)\|f\|_\infty^q + (\|f_q\|_q)^q\right].\end{aligned}$$

To summarize, we have proved that

$$\int_\Omega |f_p(x)|^q d\lambda(x) = \int_{\Omega\cap X^c} |f_p(x)|^q d\lambda(x) + \int_{\Omega\cap X} |f_p(x)|^q d\lambda(x) < +\infty.$$

Thus, $f_p \in L^p(\Omega)$, which was our objective. That concludes the proof of the technical lemma. □

Let us go back to the second part of the proof of Theorem 3, the case where $r > q$. Choose α satisfying $1/r < \alpha < 1/q$ and let f be defined on $\mathbf{R}$ by $f(x) = x^{-\alpha}\mathbf{1}_{[1,+\infty)}(x)$.

Since $|f(x)|^r = x^{-\alpha r}$ for $x \in [1, +\infty)$ and 0 elsewhere, the choice of α implies that $f \in L^r(\Omega)$ (since $\alpha r > 1$). But also f is essentially bounded on $\mathbf{R}$. If f were in $L^p(\mathbf{R}) + L^q(\mathbf{R})$, the technical lemma would imply that $f \in L^q(\mathbf{R})$. But, this is not the case since $|f(x)|^q = x^{-\alpha q}$ for $x \in [1, +\infty)$ and 0 elsewhere, the choice of α implies that $\alpha q < 1$.

Thus, again in the case where $r > q$, $L^r(\mathbf{R})$ is not contained in $L^p(\mathbf{R}) + L^q(\mathbf{R})$. ■

We end this note with the following observation, which links sections 2 and 3. In the first part of Theorem 3, we have proved that $L^r(\mathbf{R}) \subset L^p(\mathbf{R}) + L^q(\mathbf{R})$

whenever $r \in [p,q]$. In the course of its proof, a simple explicit decomposition of $f \in L^r(\mathbf{R})$ as $f = f_1 + f_2$, with $f_1 \in L^p(\mathbf{R})$ and $f_2 \in L^q(\mathbf{R})$ has been provided (see (13) and the upper bounds (14') and (16)). Indeed, as a consequence of (14) and (16),

$$\|f\|_{p,q} \leq \|f_1\|_p + \|f_2\|_q \leq \|f\|_r^{r/p} + \|f\|_r^{r/q}. \tag{19}$$

Hence, the injection of $L^r(\mathbf{R})$ into $L^p(\mathbf{R}) + L^q(\mathbf{R})$ is continuous ; therefore, there exists $C > 0$ such that

$$\|f\|_{p,q} \leq C\|f\|_r. \tag{20}$$

Indeed, using the inequality (19), we can get an upper bound for the norm of this injection (a somewhat complicated expression in terms of p, q, r). This result complements a more classical one which says that, when $r \in [p,q]$, $L^p(\mathbf{R}) \cap L^q(\mathbf{R})$ is contained in $L^r(\mathbf{R})$, then the injection is continuous.

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