MATHEMATICAL TAPAS

Volume 2

(From Undergraduate to Graduate level)

Foreword

Mathematical tapas... but what are tapas? *Tapas* is a Spanish word (in the Basque country, one would also say *pintxos*) for small savory dishes typically served in bars, with drinks, shared with friends in a relaxed ambience. The offer is varied: it may be meat, fish, vegetables,... Each of the guests of the party selects the tapas he likes most at the moment. This is the spirit of the mathematical tapas that we present to the reader here.

The tapas that we offer are of the same character as in our previous volume, published with the same title, but which was aimed at the undergraduate level.[1] However, the tapas in this volume are more substantial; they therefore could be called *raciones* instead. Consisting of mathematical questions to answer – exercises (more than long problems, in their spirit) of various types – these raciones concern mathematics ranging from the undergraduate to the graduate level (roughly speaking, this corresponds to the end of the third year and the fourth year at university);[2] they do not cover the whole spectrum of mathematics, of course. Clearly, they reflect the mathematical interests and teaching experience of the author:

Metric, normed, BANACH, inner-product and HILBERT spaces:
basic calculus with distances and norms, convergent or CAUCHY sequences, oddities in the infinite-dimensional setting;
Differential calculus:
calculus rules, applications to unconstrained optimization problems in various settings;
Integration:
examples of effective calculations of integrals, nothing on the theoretical aspects;
Matrices:
especially symmetric and positive semidefinite, links with quadratic functions, interplays with geometry, convexity and optimization;
Convexity:
convex sets, convex functions, their use in optimization;
Optimization or "variational" problems:
arising in geometry (triangles or polyhedrons), unconstrained or constrained (mainly with equality constraints).

To reflect the variety of mathematics, there is no specific ordering of topics: the tapas are more or less "randomly" presented, even if some gatherings have been carried out (for example, on mean value theorems, on weakly converging sequences in HILBERT spaces, etc.)

[1] J.-B. HIRIART-URRUTY, *Mathematical tapas.* Vol. 1 (for Undergraduates). Springer Undergraduate Mathematics Series (September 2016).

[2] Called "Licence 3" and "Master 1" in the European Higher Education system.

How have they been chosen?

- Firstly, because "I like them" and have tested them. In other words, each tapa *reveals something*: it could be an interesting inequality among integrals, a useful or surprising property of some mathematical objects (especially in the infinite-dimensional setting), or simply an elegant formula... I am just sensitive to the aesthetics of mathematics.

- Secondly, because they illustrate the following motto: *"if you solve it, you learn something"*. During my career, I have taught hundreds of students and, therefore, posed thousands of exercises (in directed sessions of problem solving, for exams, etc.); but I have not included here (standard) questions whose objective is just to test the ability to calculate a gradient, sums of series or integrals, eigenvalues, etc. I have therefore limited my choice of tapas for this second volume to the (symbolic) number of **222**.

Where have they been chosen from?

I have observed that, year after year, some questions give rise to interest or surprise among students. These mathematical tapas are chosen from them, and also from my favorite journals posing such challenges: the *American Mathematical Monthly*, and the French mathematical journals *Revue de la Filière Mathématique* (formerly *Revue de Mathématiques Spéciales*) and *Quadrature*. From time to time, I have posed or solved questions posted in these journals. Some longer or more substantial tapas have been reconstituted from those already present, in a similar form, in the books that I have written in French in the past (see the references at the end). However, for many tapas, I must confess that I could not remember their origin or history.

How are they classified?

As in restaurant guides, each tapa has one, two or three stars (★):

- One star (★). Tapas of the first level, for students at the end of their undergraduate studies.

- Two stars (★★). Tapas of a more advanced level. That does not mean that solving them necessarily requires more expertise or wit than for one-starred tapas, but sometimes just more maturity (or prerequisite definitions) in mathematics.

- Three stars (★★★). The upper level in the proposed tapas, typically for students in the first years of their graduate studies. Some may be tough and need more chewing.

We admit that this classification is somewhat arbitrary for some of the problems, as it depends on the reader's background.

How are they presented?

Each tapa begins with a *statement*, of course. The statement may contain the answers to the posed questions; this is the case when the questions or proposals are formulated as "Show that... " or "Prove that".

There are no detailed solutions to all questions, as that would have inflated this booklet by a factor of three or four. Moreover, in mathematics, there is no uniform and unique way to write down answers. But, to help solve the posed challenges, I have proposed *hints...* They suggest a path to follow. A question without any indication could be labelled "can't be done" or too time-consuming... ; the same question with "spoon-fed" steps could be considered too easy. We have tried to strike a balance between the two postures which reflects the variety of the tapas. Of course, an interested reader is asked to try to chew and swallow the tapas without having recourse to the hints.

When they are not integrated into the statements, we provide *answers* to questions, numerical results for example.

From time to time, we add some *comments*: on the context of the question, on its origin, on a possible extension.

Not all questions in mathematics have known answers... To illustrate this, at the end we have added 8 open problems or conjectures. This is our *open bar* section... the reader helps himself. Of course, there are neither hints nor answers for them since they are unknown... I have chosen these open problems to match the level and topics considered in this volume (numbers, real analysis, matrices, optimization,...): easy to understand, concerning various areas of mathematics, original for some of them. They are marked with the symbol (♣♣♣).

In spite of my efforts, some misprints or even mistakes may have slipped in; I just hope that they are not irreparable.

An essential characteristic of mathematics is to be universal and thus international. So, imagine a student or someone who has some knowledge in mathematics (say, undergraduate level) in seclusion for some time on an isolated island, or just put into jail... With a book like the one containing these tapas, he might even enjoy his time and savour some of them.

Bon appétit !

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Notations

All the notations, abbreviations and appellations we use are standard; however, here we make some of them more precise.

vs, abbreviation of *versus:* in opposition with, or faced with.
i.e., abbreviation of *id est:* that is to say.

$\mathbb{N}$: the set of natural integers, that is, $\{0, 1, 2...\}$.

x positive means $x > 0$; x nonnegative means $x \geqslant 0$. This varies from country to country, sometimes positive is used for nonnegative and strictly positive for positive. Here, we stand by the first appellations.

$f : I \to \mathbb{R}$ is increasing on the interval I means that $f(x) \leqslant f(y)$ whenever $x < y$ in I; to call such functions nondecreasing is a mistake from the logical viewpoint.

$f : I \to \mathbb{R}$ is strictly increasing on the interval I means that $f(x) < f(y)$ whenever $x < y$ in I.

$[a, b]$ denotes the closed interval of $\mathbb{R}$ with end-points a and b.

(a, b) is a somewhat ambiguous notation used to denote the open interval with end-points a and b; in some countries the (better) notation $]a, b[$ is used instead.

log or ln: used indifferently for the natural (or Napierian) logarithm.

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$; also denoted C_n^k in some countries.

For two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, $u \leqslant v$ means componentwise inequality: $u_i \leqslant v_i$ for all $i = 1, \ldots, n$.

$\|\cdot\|$: unless otherwise specified, this denotes the usual Euclidean norm in $\mathbb{R}^n$.

tr$A$ or trace(A) stands for the trace of A.

det$A$ stands for the determinant of A.

$\mathcal{S}_n(\mathbb{R})$ is the set of $n \times n$ real symmetric matrices. Semidefinite and positive definite matrices are always symmetric; this is often recalled, and assumed if not.

$A \succcurlyeq 0$ (resp. $A \succ 0$) means that the (symmetric) matrix A is positive semidefinite (resp. positive definite).

$\langle \cdot, \cdot \rangle$ is the generic notation for an inner-product (or scalar product). However, for the usual inner product of two vectors u and v in $\mathbb{R}^d$, we also use the notation $u^T v$ for $\langle u, v \rangle$; for example, the quadratic form on $\mathbb{R}^d$ associated with $A \in \mathcal{S}_d(\mathbb{R})$ is denoted $\langle Ax, x \rangle$ or $x^T Ax$. Moreover, to make a distinction with the usual Euclidean space $(\mathbb{R}^d, \langle \cdot, \cdot \rangle)$, we make use of the notation $\langle\langle U, V \rangle\rangle = \text{tr}(U^T V)$ for the standard scalar product in $\mathcal{M}_{m,n}(\mathbb{R})$.

$f : H \to \mathbb{R}$ is a primitive or an anti-gradient of the mapping $g : H \to H$ if f is differentiable and $\nabla f = g$.

A point x is called critical (or stationary) for the differentiable function $f : H \to \mathbb{R}$ whenever $\nabla f(x) = 0$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable at x, $\nabla^2 f(x)$ or $Hf(x)$ stands for the Hessian matrix of f at x, *i.e.*, the $n \times n$ symmetric matrix whose entries are the second-order partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$.

$\mathrm{co}(A)$: the convex hull of A, *i.e.*, the smallest convex set containing A.

$\overline{\mathrm{co}}(A)$: the closed convex hull of A, *i.e.*, the smallest closed convex set containing A, which is also the closure of $\mathrm{co}(A)$.

$\mathbf{1}_S$: the indicator function of $S \subset E$. By definition, $\mathbf{1}_S(x) = 1$ if $x \in S$, $\mathbf{1}_S(x) = 0$ if $x \notin S$.

Integrals: The notations $\int_a^b f(x)\, \mathrm{d}x$ (the RIEMANN integral of the continuous function f on $[a, b]$) and $\int_{[a,b]} f(x)\, \mathrm{d}\lambda(x)$, or even $\int_{[a,b]} f(x)\, \mathrm{d}x$ (the LEBESGUE integral of the measurable function f on $[a, b]$) cohabit in the text.

Classification by levels of difficulty

1–103: ★
104–189: ★★
190–222: ★★★
223–230: ♣♣♣

Classification by topics