On the generalized Jacobian of the inverse of a Lipschitzian mapping

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Abstract. The objective of this short note is to provide an estimate of the generalized Jacobian of the inverse of a Lipschitzian mapping when Clarke's inverse function theorem applies. Contrary to the classical $\mathcal{C}^1$ case, inverting matrices of the generalized Jacobian is not enough. Simple counterexamples show that our results are sharp.

Keywords. Inverse function theorem. Lipschitzian mapping. Generalized Jacobian.

Introduction

In a paper published in 1976 ([3]), F. H. Clarke proposed a generalization of the classical $\mathcal{C}^1$ inverse function theorem for Lipschitzian mappings when all the matrices of what he called generalized Jacobian are invertible. Let us briefly recall what it is about.

When a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitzian (*i.e.*, satisfies a Lipschitz property) in a neigborhood of $x \in \mathbb{R}^n$, one firstly defines the *limiting generalized Jacobian*

$$\underrightarrow{\mathcal{J}F}(x) = \{\lim JF(x_k) : x_k \to x\} \tag{1}$$

(that is to say, we consider all sequences (x_k) converging to x such that F is differentiable at x_k and such that the sequence $(JF(x_k))$ of Jacobian matrices converges). The notation $JF(x_k)$ is used for the (usual) Jacobian matrix of F at x_k ; the notation $\underrightarrow{\mathcal{J}F}(x)$ itself suggests that we collect all the possible *limits* of Jacobian matrices. Indeed, $\underrightarrow{\mathcal{J}F}(x)$ is a (nonempty) compact set of $(n \times n)$ matrices. Taking the convex hull of $\underrightarrow{\mathcal{J}F}(x)$ gives rise to the so-called Clarke *generalized Jacobian* of F at x, denoted here as $\mathcal{J}F(x)$ (other notations are $\partial F(x)$ (the original one) and $\partial_C F(x)$):

$$\mathcal{J}F(x) = \mathrm{co}\left(\underrightarrow{\mathcal{J}F}(x)\right). \tag{2}$$

(Here and below, co(S) means the convex hull of a set S). Therefore, $\mathcal{J}F(x)$ is a convex compact set of $(n \times n)$ matrices.

In the series of questions that we have decided to pose like in a quiz, here is the first one:

Q1. *Does $\mathcal{J}F(x)$ enjoy any property as a convex compact set of matrices?*

The answer is No. It has recently been proved in [1] that any convex compact set of matrices is the CLARKE generalized Jacobian of some Lipschitzian mapping.

An interesting property of $\mathcal{J}F(x)$, proved in [7] and [4], is that its construction (see (1) and (2)) is insensitive or "blind" to sets of null (Lebesgue) measure. That means that in (1) we could impose to x_k staying out of a set of null measure without affecting the final result $\mathcal{J}F(x)$. This is useful in some proofs when one likes to avoid some "nasty" sets; that will happen once in our note (Step 2 in the proof of the main theorem).

CLARKE's inverse function theorem for Lipschitzian mappings is as follows:

Theorem ([3]). *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitzian around $x_0 \in \mathbb{R}^n$. Suppose that all the matrices in $\mathcal{J}F(x_0)$ are invertible. There then exist neighborhoods $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ of x_0 and $F(x_0)$ respectively, and a Lipschitzian mapping $G : V \to \mathbb{R}^n$ such that*

(a) $(G \circ F)(x) = x$ for all $x \in U$

(b) $(F \circ G)(y) = y$ for all $y \in V$.

Like in the classical case (*i.e.*, when F is $\mathcal{C}^1$ in a neighborhood of x_0), by abuse of language and writing, we say that G is the (local) inverse of F, and we denote it by F^{-1} as well. Just as in the classical case, CLARKE's inverse function theorem yields an implicit function theorem, as has been noted by HIRIART-URRUTY ([5, Theorem 11]).

In the classical case, with $y_0 = F(x_0)$, we have $JF^{-1}(y_0) = [JF(x_0)]^{-1}$, a perfectly symmetrical situation. Thus a natural question arises: *What about $\mathcal{J}F^{-1}(y_0)$ when CLARKE's inverse theorem applies?* This is the kind of question we tackle in this note. To the best of our knowledge, the subsequent results have not been published, although they are very natural. Moreover, recent applications feature inversion of CLARKE generalized Jacobians, formally used as classical Jacobians, for implicitely defined input and output relations (see [8, Theorem 2)] for example). The present work sheds light on the accuracy of this type of procedure from a Nonsmooth Analysis point of view.

The main results

Consider the following example proposed by CLARKE in his seminal paper [3] from 1976. It will serve as a "red thread" in our note.

Let $F:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $F(x,y)=\begin{pmatrix} u=f(x,y)=|x|+y \\ v=g(x,y)=2x+|y| \end{pmatrix}$. We then have:

$$\underrightarrow{\mathcal{J}F}(0,0)=\left\{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix},\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix},\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix},\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\right\}, \qquad (3)$$

$$\mathcal{J}F(0,0)=\mathrm{co}\left\{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix},\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix},\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix},\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\right\}. \qquad (4)$$

In a more "compact" form,

$$\mathcal{J}F(0,0)=\left\{\begin{bmatrix} \alpha & 1 \\ 2 & \beta \end{bmatrix}:\alpha\in[-1,1],\,\beta\in[-1,1]\right\}. \qquad (5)$$

Clearly, $\mathcal{J}F(0,0)$ is a "flat" convex set in the vector space $\mathcal{M}_2(\mathbb{R})$ of (2×2) matrices; it is of dimension 2 (The dimension of a convex set is, by definition, the dimension of its affine hull). Here, we are lucky enough to have a compact form for $\mathcal{J}F(0,0)$. That helps to infer that all the matrices in $\mathcal{J}F(0,0)$ are invertible.

A question that could cross our minds: *If $\mathcal{S}\subset\mathcal{M}_n(\mathbb{R})$ contains only invertible matrices, is it the same for* $\mathrm{co}(\mathcal{S})$*?* The answer is clearly No: a line-segment with endpoints invertible matrices may contain singular (*i.e.* not invertible) matrices. The answer however is Yes if all the matrices in $\mathcal{S}$ are positive definite (or negative definite). That brings us to the table the difficulty that may exist in verifying the hypotheses in Clarke's inverse function theorem.

In our specific example,

$$[\mathcal{J}F(0,0)]^{-1} :=\ \left\{M^{-1}:M\in\mathcal{J}F(0,0)\right\} \qquad (6)$$

$$=\left\{\frac{1}{\alpha\beta-2}\begin{bmatrix} \beta & -1 \\ -2 & \alpha \end{bmatrix}:\alpha\in[-1,1],\,\beta\in[-1,1]\right\}. \qquad (7)$$

Mimicing what is known in the classical case, we may ask the following question:

Q2. *Under the assumptions of Clarke's inverse function theorem, does $\mathcal{J}F^{-1}(y_0)$ equal $[\mathcal{J}F(x_0)]^{-1}$ (the latter being, by definition, $\{M^{-1}:M\in\mathcal{J}F(x_0)\}$)?*

The answer is No. The reason is that $[\mathcal{J}F(x_0)]^{-1}$ is not, as a general rule, a convex set. It is just a connected compact set (as the image of a compact convex set by the continuous mapping $(.)^{-1}$). To see this possible nonconvexity, consider the specific example above (see (6) and (7)). Let us choose in $[\mathcal{J}F(0,0)]^{-1}$:

$$M = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \text{ corresponding to } \alpha = 0 \text{ and } \beta = 1 \ ;$$

$$N = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} \text{ corresponding to } \alpha = 1 \text{ and } \beta = -1.$$

As a consequence,

$$\frac{1}{3}M + \frac{2}{3}N = \begin{bmatrix} 1/18 & 7/18 \\ 7/9 & -2/9 \end{bmatrix} = -\frac{7}{18}\begin{bmatrix} -1/7 & -1 \\ -2 & 4/7 \end{bmatrix}. \tag{8}$$

For having this matrix of the form of those in the family in (2), one should have

$$-\frac{7}{18}\begin{bmatrix} -1/7 & -1 \\ -2 & 4/7 \end{bmatrix} = \frac{1}{\alpha\beta - 2}\begin{bmatrix} \beta & -1 \\ -2 & \alpha \end{bmatrix} \tag{9}$$

for some $\alpha \in [-1,1]$, $\beta \in [-1,1]$. This induces *three* constraints (the fourth is redundant):

- first, $\frac{1}{\alpha\beta\ -2} = -\frac{7}{18}$, hence $\alpha\beta = -\frac{4}{7}$ (resulting from equality of antidiagonal elements in (9))

- second, $\beta = -1/7$, $\alpha = 4/7$, resulting, afterwards, from equality of diagonal elements in (9).

One cannot have, at the same time, $\alpha\beta = -4/7$, $\beta = -1/7$, $\alpha = 4/7$. So, $\frac{1}{3}M + \frac{2}{3}N$ belongs to $\mathrm{co}\,[\mathcal{J}F(0,0)]^{-1}$ but not to $[\mathcal{J}F(0,0)]^{-1}$.

The question posed in the previous page can be answered more generally as follows. Let $\mathcal{M}$ be any compact convex set of non-singular matrices in $\mathcal{M}_n(\mathbb{R})$ such that $\mathcal{M}^{-1}$ is not convex. By [1] find a Lipschitzian mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(0) = 0$ and $\mathcal{J}F(0) = \mathcal{M}$. This gives the answer. Such $\mathcal{M}$ do exist, even very simple ones like line-segments. For example, with $n = 2$, consider the line-segment $\mathcal{M}$ with endpoints $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Easy calculations lead to:

$$\mathcal{M}^{-1} = \left\{ \frac{1}{1-\lambda\mu}\begin{bmatrix} 1 & -\mu \\ -\lambda & 1 \end{bmatrix} : \lambda \geqslant 0, \mu \geqslant 0, \lambda + \mu = 1 \right\},$$

$$\frac{1}{2}A^{-1} + \frac{1}{2}B^{-1} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} \notin \mathcal{M}^{-1}.$$

In short: Taking the convex hull (the "co" operation) and inverting matrices (the $(.)^{-1}$ operation) are two operations that do not go well together. The "co" operation goes well with linear (or affine) operations on matrices, while the inversion operation of matrices is quite nasty from the geometrical point of view.

Q3. *Under the assumptions of Clarke's inverse function theorem, does* $\mathcal{J}F^{-1}(y_0)$ *equal* $\mathrm{co}\,[\mathcal{J}F(x_0)]^{-1}$*?*

The answer is again No. We will see a counterexample later on. Nevertheless, with $\mathrm{co}\,[\mathcal{J}F(x_0)]^{-1}$, we have an outer estimate of $\mathcal{J}F^{-1}(y_0)$. Here is our main theorem.

Theorem 1. *Under the assumptions of Clarke's inverse function theorem, we have*:

$$\left\{M^{-1} : M \in \underrightarrow{\mathcal{J}F}(x_0)\right\} \subseteq \underrightarrow{\mathcal{J}F^{-1}}(y_0), \tag{10}$$

$$\mathcal{J}F^{-1}(y_0) = \mathrm{co}\left\{M^{-1} : M \in \underrightarrow{\mathcal{J}F}(x_0)\right\}, \tag{11}$$

$$\mathcal{J}F^{-1}(y_0) \subseteq \mathrm{co}\,[\mathcal{J}F(x_0)]^{-1}. \tag{12}$$

Before going into the proof, note the following:

- Contrary to what one may think at the first glance, the inclusion (10) cannot be immediately "symmetrized": nothing guarantees that the assumption made on F holds true for F^{-1}, especially that all the matrices in $\mathcal{J}F^{-1}(y_0)$ (or even in $\underrightarrow{\mathcal{J}F^{-1}}(y_0)$) are invertible.

- According to the relation between $\underrightarrow{\mathcal{J}H}(.)$ and $\mathcal{J}H(.)$ for a Lipschitzian mapping H (see (2)), a consequence of (10) is that

$$\mathrm{co}\left\{M^{-1} : M \in \underrightarrow{\mathcal{J}F}(x_0)\right\} \subseteq \mathcal{J}F^{-1}(y_0). \tag{13}$$

Therefore, what has to be proved to get at (11) is the converse inclusion.

- Due again to the relation between $\underrightarrow{\mathcal{J}H}(.)$ and $\mathcal{J}H(.)$ for a Lipschitzian mapping H, (12) is a direct consequence of (11).

Proof of Theorem 1.

Step 1. Proof of the inclusion (10).

There is no loss of generality in assuming $x_0 = 0$ and $F(x_0) = 0$.

To get at (10), the reasoning is rather similar to the one used in the classical case. It is indeed a consequence of Lemma 6.1 and the proof of it in [6, page 104]; it says the following.

Lemma 1. *If* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is a Lipschitzian homeomorphism near* 0 *and* $x \in \mathbb{R}^n$ *is a point where* F *is differentiable, with* $JF(x)$ *invertible, then the* *inverse mapping $G := F^{-1}$ is differentiable at the image-point $y = F(x)$ with $JG(y) = [JF(x)]^{-1}$.*

For the convenience of the reader, we provide here a self-contained proof of this statement.

Let

$$\begin{aligned}\Delta &:= F^{-1}\ [F(x)+h] - F^{-1}\ [F(x)] - [JF(x)]^{-1} h \\ &= F^{-1}\ [F(x)+h] - x - [JF(x)]^{-1} h.\end{aligned}$$

We intend to prove that $\Delta = o(h)$.

By assumption, F is differentiable at x, hence

$$F(x+d) = F(x) + JF(x)d + o(d).$$

We rewrite this first-order development with $d := [JF(x)]^{-1} h$; we get at

$$F\left\{x + [JF(x)]^{-1} h\right\} = F(x) + h + o(h).$$

Now, because F^{-1} is Lipschitzian around y, we infer from the above

$$x + [JF(x)]^{-1} h = F^{-1}\ [F(x)+h] + o(h).$$

Thus, $\Delta = o(h)$. The lemma above is proved.

Now pick any $M \in \underrightarrow{\mathcal{J}F}(0) \subset \mathcal{J}F(0)$. By assumption in the Theorem 1, M is invertible. Due to the definition of $\underrightarrow{\mathcal{J}F}(0)$, there exists a sequence (x_k) converging to 0 such that F is differentiable at x_k and such that the sequence $(JF(x_k))$ of Jacobian matrices converges to M. Consider $y_k = F(x_k)$. According to the lemma 1 above, for every k, the inverse mapping $G := F^{-1}$ is differentiable at $y_k = F(x_k)$ and $JG(y_k) = [JF(x_k)]^{-1}$. Clearly, due to the continuity of F and that of the $(.)^{-1}$operation on the open set of invertible matrices, when $k \to \infty$, we have:

$$\begin{array}{ccc} y_k & \longrightarrow & F(0) = 0, \\ JG(y_k) & = & [JF(x_k)]^{-1} \longrightarrow M^{-1.} \end{array}$$

Hence, $M^{-1} \in \underrightarrow{\mathcal{J}G}(0) = \underrightarrow{\mathcal{J}F^{-1}}(0)$. We thus have proved that

$$\left\{M^{-1} : M \in \underrightarrow{\mathcal{J}F}(0)\right\} \subset \underrightarrow{\mathcal{J}F^{-1}}(0).$$

Step 2. Here, we intend to prove

$$\mathcal{J}F^{-1}(0) \subseteq \text{co}\left\{M^{-1} : M \in \underrightarrow{\mathcal{J}F}(0)\right\}, \qquad (14)$$

that is the converse inclusion to (13).

Let Λ_0 and Λ_1 denote the sets of points where F and F^{-1} fail to be differentiable, respectively. These sets are of null (Lebesgue) measure. Because F is Lipschitzian, a simple reflexion reveals that $F(\Lambda_0)$ is also of null measure. Thus $\Lambda := \Lambda_0 \cup \Lambda_1 \cup F(\Lambda_0)$ is of null measure. As recalled at the beginning (after the answer to **Q1**), the construction of $\mathcal{J}F^{-1}(0)$ is insensitive to sets of null measure, so

$$\mathcal{J}F^{-1}(0) = \text{co}\left\{\lim JF^{-1}(y_k) : y_k \to 0, y_k \notin \Lambda\right\}. \tag{15}$$

The trick in our proof is to work only with points $y_k \notin \Lambda$. Fix any $y_k \notin \Lambda$. Realizing that F^{-1} is differentiable at y_k and F is differentiable at $x_k := F^{-1}(y_k)$, chain rule for differentiable mappings yields

$$I_n = J(Id_{\mathbb{R}^n})(y_k) = JF(x_k)JF^{-1}(y_k).$$

Recalling that $\mathcal{J}F(0)$ consists of invertible matrices only, we conclude that for k large enough (because $x_k \to 0$), $JF(x_k)$ is invertible; so $JF^{-1}(y_k) = [JF(x_k)]^{-1}$. Moreover, $JF(x_k)$ converges to an invertible matrix (in $\underrightarrow{\mathcal{J}F}(0)$); thus, $JF^{-1}(y_k)$ converges to the inverse of a matrix in $\underrightarrow{\mathcal{J}F}(0)$.

Finally, the description (15) leads to the announced inclusion (14).

At this stage, it is natural to pose the question of sharpness in the general inclusion (12).

Q4. *Under the assumptions of Clarke's inverse function theorem, could the inclusion (12) be strenghtened to an equality?*

The answer is No. There are examples where equality holds true in (12) and examples where the inclusion in (12) is strict.

Q5. *Under the assumptions of Clarke's inverse function theorem, if equality (resp., a strict inclusion) holds true for F in (12), does equality (resp., a strict inclusion) hold true for $G := F^{-1}$ in (12)?*

The answer is again No.

To illustrate all these answers, let us go back to Clarke's example F given at the beginning. Is it possible to get at an explicit expression of

$$F^{-1}(u,v) = \begin{pmatrix} x = \varphi(u,v) \\ y = \psi(u,v) \end{pmatrix}$$

at least in a neighborhood of $(0,0)$ (where F is known to have an inverse)? Indeed, F is globally invertible (on $\mathbb{R}^2$), its inverse is again a piecewise linear mapping. We are fortunate enough to have an explicit expression of $G := F^{-1}$. Here it is:

$$G(u,v) = \begin{cases} \begin{pmatrix} x = v - u \\ y = 2u - v \end{pmatrix} \text{ if } u \leqslant v \leqslant 2u; \\[2ex] \begin{pmatrix} x = (v-u)/3 \\ y = (2u+v)/3 \end{pmatrix} \text{ if } -2u \leqslant v \leqslant u; \\[2ex] \begin{pmatrix} x = (u+v)/3 \\ y = (2u-v)/3 \end{pmatrix} \text{ if } 2u \leqslant v \text{ and } -u \leqslant v; \\[2ex] \begin{pmatrix} x = u + v \\ y = 2u + v \end{pmatrix} \text{ if } v \leqslant -u \text{ and } v \leqslant -2u. \end{cases} \tag{16}$$

We have:

$$\underrightarrow{\mathcal{J}G}(0,0) = \left\{ \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \right\}; \tag{17}$$

$$\mathcal{J}G(0,0) = \operatorname{co}\left(\underrightarrow{\mathcal{J}G}(0,0)\right). \tag{18}$$

Unfortunately, we have no "compact expression" for $\mathcal{J}G(0,0)$ like for $\mathcal{J}F(0,0)$ in (5).

As expected (see (10) and the proof of (11)) - and easy to check in our particular example - we have the following relationship

$$\underrightarrow{\mathcal{J}G}(0,0) = \left\{ M^{-1} : M \in \underrightarrow{\mathcal{J}F}(0,0) \right\}.$$

Illustration 1. According to (12) applied to G, knowing that $G^{-1} = F$, we have

$$\mathcal{J}G^{-1}(0,0) = \mathcal{J}F(0,0) \subseteq \operatorname{co}\left[\mathcal{J}G(0,0)\right]^{-1}. \tag{19}$$

The inclusion (19) is strict. One can think of two ways of proving it. First, as observed in [2, page 17], the convex set $\operatorname{co}\left[\mathcal{J}G(0,0)\right]^{-1}$ is of dimension 3, while we know that $\mathcal{J}F(0,0)$ is a convex set of dimension 2. Another way, more explicit, is to exhibit a matrix M which lies in $\operatorname{co}\left[\mathcal{J}G(0,0)\right]^{-1}$ (even in the smaller $\left[\mathcal{J}G(0,0)\right]^{-1}$) but which does not belong to $\mathcal{J}F(0,0)$. For that purpose, consider the four matrices $M_1, \ldots, M_4$ in $\underrightarrow{\mathcal{J}G}(0,0)$, see (17), and define $M = \frac{1}{4}(M_1 + M_2 + M_3 + M_4) \in \mathcal{J}G(0,0)$. We have:

$$M = \begin{bmatrix} 0 & 2/3 \\ 4/3 & 0 \end{bmatrix}, \text{ so that } M^{-1} = \begin{bmatrix} 0 & 3/4 \\ 3/2 & 0 \end{bmatrix}.$$

This matrix M^{-1} does not belong to $\mathcal{J}F(0,0)$ (whose general form is given in (5)).

Illustration 2. The inclusion (12) for our particular F at $(0,0)$ is actually an equality, that is:

$$\mathcal{J}F^{-1}(0,0) = \mathrm{co}\,[\mathcal{J}F(0,0)]^{-1}. \tag{20}$$

For that, it is enough to prove the following:

$$[\mathcal{J}F(0,0)]^{-1} \subset \mathrm{co}\left\{M^{-1} : M \in \underrightarrow{\mathcal{J}F}(0,0)\right\}, \tag{21}$$

that is to say: given any matrix $S_{\alpha,\beta}$ in $[\mathcal{J}F(0,0)]^{-1}$ (we know a parametrization of it, see (7)), one should find a convex combination of the four matrices S_i in $\underrightarrow{\mathcal{J}F}^{-1}(0,0)$ $(= \left\{M^{-1} : M \in \underrightarrow{\mathcal{J}F}(0,0)\right\})$ (see (17)) which equals $S_{\alpha,\beta}$. We therefore have to solve the following linear system (real variables x, y, z, t)

$$xS_1 + yS_2 + zS_3 + tS_4 = S_{\alpha,\beta}, \tag{22}$$

which gives in detail:

$$\begin{aligned}
-x - \frac{y}{3} + \frac{z}{3} + t &= \frac{\beta}{\alpha\beta - 2}\\
x + \frac{y}{3} + \frac{z}{3} + t &= \frac{-1}{\alpha\beta - 2}\\
-x + \frac{y}{3} - \frac{z}{3} + t &= \frac{\alpha}{\alpha\beta - 2}\\
2x + \frac{2y}{3} + \frac{2z}{3} + 2t &= \frac{-2}{\alpha\beta - 2}\\
x + y + z + t &= 1,
\end{aligned}$$

where $\alpha \in [-1,1]$, $\beta \in [-1,1]$ (two independent parameters). In the above list, we note that the second and the fourth equations coming from equality (22) between (2×2) matrices are the same.

The (unique) solution of (22) is:

$$\left\{\begin{aligned}
x &= \tfrac{(\alpha+1)(\beta+1)}{4(2-\alpha\beta)},\\
y &= \tfrac{3(1-\alpha)(\beta+1)}{4(2-\alpha\beta)},\\
z &= \tfrac{3(\alpha+1)(1-\beta)}{4(2-\alpha\beta)},\\
t &= \tfrac{(1-\alpha)(1-\beta)}{4(2-\alpha\beta)}.
\end{aligned}\right. \tag{23}$$

All these coefficients are nonnegative. We therefore have found coefficients x, y, z, t such that the convex combination $xS_1 + yS_2 + zS_3 + tS_4$ equals $S_{\alpha,\beta}$.

To summarize the two illustrations above, (12) is an equality for the considered (piecewise linear) mapping F, while it is a strict inclusion for the (piecewise linear) mapping $G := F^{-1}$. With these two examples, it is also easy to imagine a mapping $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ where the inclusion (12) is strict for both Φ and Φ^{-1}.

Remark. Of course, things are easier and much simplified in the one-dimensional case ($n = 1$), something observed for a long time: if all the elements of the generalized derivative $\partial\sigma(x_0)$ are nonzero, then $\partial\sigma^{-1}(y_0) = \frac{1}{\partial\sigma(x_0)}$.

Conclusion

Under the assumptions of CLARKE's inverse function theorem on F at x_0, the main result that we have proved is a nonsymmetrical relationship on generalized Jacobians of F at x_0 and that of F^{-1}at $y_0 = F(x_0)$:

$$\mathcal{J}F^{-1}(y_0) \subseteq \mathrm{co}\,[\mathcal{J}F(x_0)]^{-1}.$$

Examples above show that one cannot do better than that.

This state of affairs, deemed unsatisfactory for what they aimed at, led the authors in [2] to define and study "conservative" generalized Jacobians $\mathcal{J}^cF(x)$. This was justified by the need for a conservative calculus in a framework of inverse functions, and to provide a variational meaning to $[\mathcal{J}^cF(x_0)]^{-1}$.

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