THE HEXAGON OF TRIGONOMETRIC FUNCTIONS

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Trigonometric (or circular) functions are basic mathematical objects that all the pupils (in highschools) and beginners in studying sciences (in universities) try to tame... Usually, referring to them brings to mind difficulties of memorizing connections linking them, calculus rules, etc. What we propose here is a synthetic way to show the relationships between the main trigonometric functions, in order to help in grasping them as well as in memorizing the various connections. We do that via a diagram, shaped as an hexagon whose vertices are the six basic trigonometric functions. This way of doing were taught in Chinese highschools some years ago, apparently not so much anymore. We take advantage of this presentation to complete with another diagram of the same vein, concerning this time hyperbolic functions.

1 *Let us begin with definitions : what are we talking about ?*

The four first trigonometric functions are : the sine function (*sin* in short), the cosine function (*cos*), the tangent (*tan*) and the cotangent (*cotan* or *cot*). The prefix "co" is there to express "which goes with", like a roomate : cosine goes with sine, cotangent with tangent. All these mathematical objects are defined and introduced in highschools while studying the triangles. Those triangles could have been named as well *trigons* (the Greek root of which means "with three sides") ; that explains the origin of the qualifying word *trigonometric.*

In addition to these four functions, may be considered two further ones : the secant (*sec* in short) and the cosecant (*cosec* or *csc*) :

$$\sec(\theta) = \frac{1}{\cos(\theta)}; \ \csc(\theta) = \frac{1}{\sin(\theta)}. \tag{1}$$

These two additional functions are widely used in North-America and many other countries, but almost never in France.

2 *The hexagon of the six trigonometric functions*

The six trigonometric functions introduced above are now placed at the vertices of a hexagon, in such a way that some very nice relationships among them can be "seen" on the picture and memorized. We are going to list and comment them. For the sake of completeness, we have put the number 1 at the center of the hexagon.

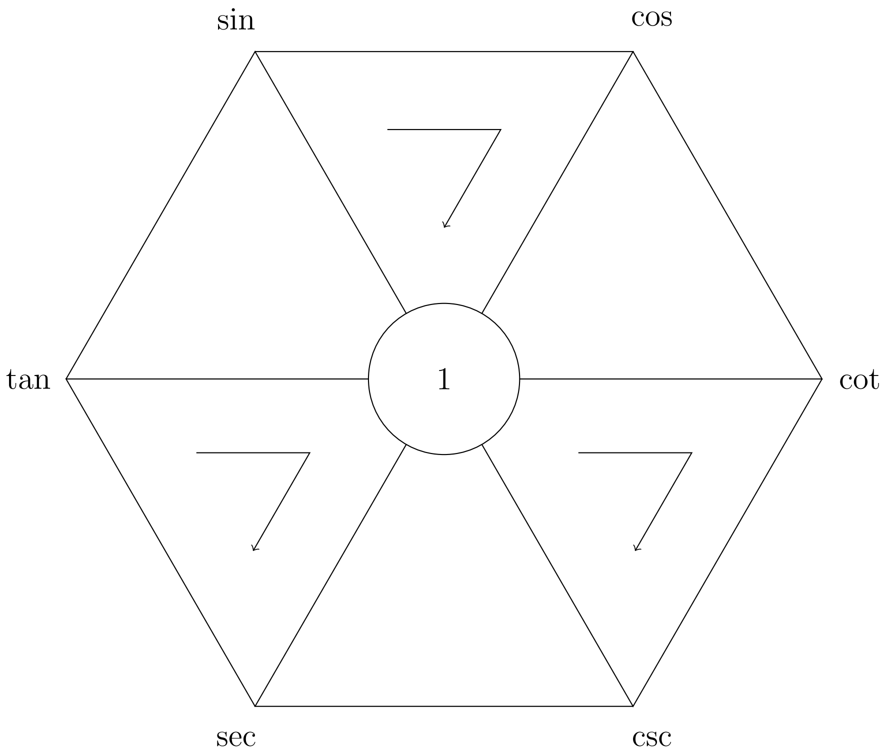

Figure 1

– *Diametrically opposed functions*
  Each function placed at a vertex is the inverse of the diametrically opposed one : for example, *sin* and *csc* are inverse one of the other. Said otherwise : the product of diametrically opposed functions equal the central value 1. There are three relations like that.
– *Neighboring functions*
  Each function placed at a vertex is the product of its two neighboring functions (one from its right, one from its left) : for example, *cos* is the product of *sin* and *cot*, *cosec* is the product of *cot* and *sec*, etc. There are six relations of this sort.
– *Following functions*
  Each function placed at a vertex is the quotient of the two functions coming after it (move to the right or to the left) : for example, *tan* is the quotient of *sin* by *cos (*or of *sec* by *csc)*, *csc* is the quotient of *sec* by *tan*, *cos* is the quotient of *sin* by *tan*, etc. There are twelve relations of this type.
– *Adding up the squares of functions*
  In the hexagon of the six trigonometric functions, there are three equilateral triangles placed north, south-east and south-west. The functions placed at the two outer vertices and (the constant function) 1 placed at the center vertex of these three triangles obey the next rule : by following the arrow (see Figure 1), the sum of squares of the

two first vertices equal the square of the third vertex :

$$\sin^2 + \cos^2 = 1;\ 1 + \cot^2 = \csc^2;\ \tan^2 + 1 = \sec^2 . \tag{2}$$

The six trigonometric functions, plus the constant function 1, are involved in the three relations of (2).

3 *The hexagon of the six hyperbolic functions*

The so-called hyperbolic functions are analogs of the trigonometric functions. The four first hyperbolic functions are : the hyperbolic sine function (*sinh* in short), the hyperbolic cosine function (*cosh*), the hyperbolic tangent (*tanh*) and the hyperbolic cotangent (*coth*). Like in the previous sections, one can add two further ones : the hyperbolic secant (*sech* in short) and the hyperbolic cosecant (*csch*) :

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}; \ \operatorname{csch}(x) = \frac{1}{\sinh(x)}. \tag{3}$$

– Usually, one learns definitions and properties of trigonometric functions (in high-schools for example) before those of hyperbolic functions (first years of universities). There is a general rule allowing to discover any identity on hyperbolic functions from the corresponding one on trigonometric functions. We have tested and practiced it many times in our teachings, it works perfectly. Here it is : consider any formula or identity on trigonometric functions; write the same one with hyperbolic functions instead of trigonometric functions (so, just adding the letter h) ; change the sign if, and only if, the sine function appears twice as a product (for example in $\sin^2$ or $\sin\tan$). Let us illustrate this rule with the next examples :

$$\begin{aligned}
\cos^2 + \sin^2 &= 1 \text{ gives } \cosh^2 - \sinh^2 = 1 \\
&\quad \text{(change of sign for the square of the } \textit{sine}\text{) ;} \\
\sin(2x) &= 2\sin(x)\cos(x) \text{ gives } \sinh(2x) = 2\sinh(x)\cosh(x) \\
&\quad \text{(no change of sign since the } \textit{sine} \text{ appears once) ;} \\
\tan(2x) &= \frac{2\tan(x)}{1-\tan^2(x)} \text{ gives } \tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)} \\
&\quad \text{(change of sign for the square of the } \textit{tangent}\text{,} \\
&\quad \text{containing the square of the } \textit{sine} \text{ in a hidden form) ;} \\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \text{ gives} \\
\cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \\
&\quad \text{(change of sign due to the product of two } \textit{sine}\text{) ;} \\
\sin(x) + \sin(y) &= 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) \text{ gives} \\
\sinh(x) + \sinh(y) &= 2\sinh(\frac{x+y}{2})\cosh(\frac{x-y}{2}) \\
&\quad \text{(no change of sign since the } \textit{sine} \text{ appears once) ;} \\
&\quad \text{etc.}
\end{aligned}$$

There is a reason behind that : passing from the trigonometric sine function to the hyperbolic one, at least for complex variables, involves a multiplication by the complex number i, whose square is -1 (in fact, $\sinh z = i\sin(iz)$; but we do not dwell on it).

As one can imagine, there is an hexagon of hyperbolic functions like the one for trigonometric ones. Here it is (Figure 2).

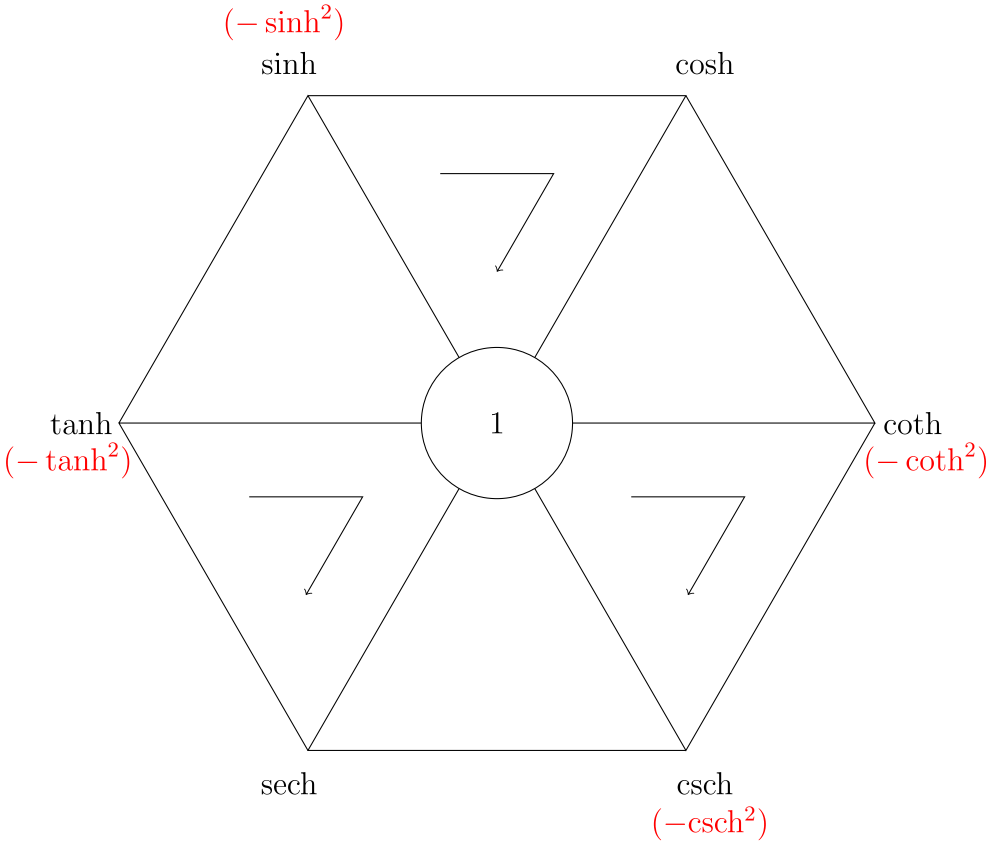

Figure 2

The relationships are very similar to those derived from Figure 1, with an exception relying on the "rule of thumb" explained just above.

– *Diametrically opposed functions*
  Each function placed at a vertex is the inverse of the diametrically opposed one : for example, *sinh* and *csch* are inverse one of the other. There are three relations like that.
– *Neighboring functions*
  Each function placed at a vertex is the product of its two neighboring functions (one from its right, one from its left) : for example, *cosh* is the product of *sinh* and *coth*, *csch* is the product of *coth* and *sech*, etc. There are six relations of this sort.
– *Following functions*
  Each function placed at a vertex is the quotient of the two functions coming after it (move to the right or to the left) : for example, *tanh* is the quotient of *sinh* by *cosh* *(*or of *sech* by *csch)*, *csch* is the quotient of *sech* by *tanh*, *cosh* is the quotient of *sinh* by *tanh*, etc. There are twelve relations of this type.
– *Adding up the signed squares of functions*

In the hexagon of the six hyperbolic functions, there are three equilateral triangles placed north, south-east and south-west. The functions placed at the two outer vertices and 1 placed at the center vertex of these three triangles obey the next rule : by following the arrow (see Figure 2), the sum of squares of the two first vertices (but with a minus sign if sinh appears twice, marked in red in the diagram) equal the square of the third vertex :

$$-\sinh^2 + \cosh^2 = 1;\ 1 - \coth^2 = -\operatorname{csch}^2;\ \ -\tanh^2 + 1 = \operatorname{sech}^2. \tag{4}$$

4 Conclusion

Although trigonometric functions are well-studied mathematical objects for long time, teaching them to various groups of students and in different countries allow us to exchange pedagogic experiences and tricks. This note was an example, like the one presented in [1] by the first author.

1. J.-B.Hiriart-Urruty, Les formules de trigonométrie sans pleurs... (Trigonometric formulas without tears...), *Bulletin de l'APMEP (Bulletin of the French Mathematical Teachers).* To appear in 2015.

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