

## A characterization by optimization of the orthocenter of a triangle

JEAN-BAPTISTE HIRIART-URRUTY & PIERRE-JEAN LAURENT

“Nothing in the world takes place without optimization, and there is no doubt that all aspects of the world that have a rational basis can be explained by optimization methods” used to say L.EULER in 1744. This is confirmed by numerous applications of mathematics in physics, mechanics, economy, etc. In this note, we show that it is also the case for the classical centers of the triangle, more specifically for the orthocenter. To our best knowledge, the characterization of the orthocenter of a triangle by optimization, that we are going to present, is new.

Let us begin by revisiting the usual centers of a triangle and their characterizations by optimization. Let  $\mathcal{T} = ABC$  be a triangle (the points  $A, B, C$  are supposed non aligned, of course).

- The *centroid* or *isobarycenter* of  $\mathcal{T}$  is the point of concurrence of the medians of  $\mathcal{T}$ , which are the line-segments joining the vertices of  $\mathcal{T}$  to the mid-points of the respective opposite sides. It is the point which minimizes on  $\mathcal{T}$  the following objective function or criterion

$$P \mapsto (PA)^2 + (PB)^2 + (PC)^2. \quad (1)$$

- The *incenter* of  $\mathcal{T}$  is the point of concurrence of the internal bisectors of  $\mathcal{T}$ ; it is also the point equidistant from the sides of  $\mathcal{T}$ . It is the point which minimizes on  $\mathcal{T}$  the following function

$$P \mapsto \max(PA', PB', PC'), \quad (2)$$

where  $A', B', C'$  denote the projections of  $P$  on the sides of  $\mathcal{T}$ . The triangle  $A'B'C'$  is called the *pedal triangle* of  $P$ .

- The *circumcenter* is the point of concurrence of the perpendicular bisectors of the sides of  $\mathcal{T}$ ; it is also the point equidistant from the vertices of  $\mathcal{T}$ . Assuming that the triangle  $\mathcal{T}$  is acute-angled, the circumcenter lies in the interior of  $\mathcal{T}$ . It is the point which minimizes on  $\mathcal{T}$  the following function

$$P \mapsto \max(PA, PB, PC). \quad (3)$$

The above characterizations of the incenter and circumcenter in terms of optimization do not seem to be well-known. The difficulty there is that they involve a nonsmooth convex function (to be minimized), while the function to be minimized in (1) is convex and smooth. See [1] for more on that. We add two further interesting points to our list.

- The *LEMOINE point* of  $\mathcal{T}$  is the one which minimizes on  $\mathcal{T}$  the function

$$P \mapsto (PA')^2 + (PB')^2 + (PC')^2, \quad (4)$$

where  $A'B'C'$  is the pedal triangle of  $P$ .

- The FERMAT or FERMAT & TORRICELLI *point* of  $\mathcal{T}$  is the one which minimizes on  $\mathcal{T}$  the function

$$P \mapsto PA + PB + PC. \quad (5)$$

Again the functions appearing in (4) and (5) are convex, but the one in (5) is not differentiable at the points  $A, B$  or  $C$ . So, characterizing the FERMAT point is different if a vertex of the triangle is candidate for optimality or not. It is a classical result that this depends on the angles of  $\mathcal{T}$  : if the three angles of  $\mathcal{T}$  are less than  $120^\circ$ , then the FERMAT point lies in the interior of  $\mathcal{T}$  ; if one of the angles of  $\mathcal{T}$ , say at  $A$ , is greater or equal  $120^\circ$ , then  $A$  is the FERMAT point of  $\mathcal{T}$ .

Many centers of the triangle have been discovered and studied during the past centuries, and that continues; see for that the website *Encyclopedia of Triangle Centers (ETC)* [2]. However, quite a few of them enjoy a “variational” characterization, that is to say : minimizing some criteria built up from the elements of the triangle (like vertices or sides).

There is a classical triangle center missing in our list above, namely the *orthocenter*. Indeed, the orthocenter of  $\mathcal{T}$  is defined as the concurrence of the “altitudes” of  $\mathcal{T}$ , that is the perpendiculars dropped from the vertices to respective opposite sides. So, our question here is : *can the orthocenter be viewed as the minimizer on  $\mathcal{T}$  of some appropriate criterion ?* It happens that the answer is “yes”. In spite of a thorough searching in the literature, we did not find any mention of it. Here it is. In order to simplify a little bit our presentation, we assume that the triangle  $\mathcal{T}$  is acute-angled ; hence, the orthocenter lies in the interior of  $\mathcal{T}$  (like in all the previous examples of centers of triangles).

**Theorem.** *The orthocenter of  $\mathcal{T}$  minimizes on  $\mathcal{T}$  the function*

$$P \mapsto f(P) = PA + PB + PC + PA' + PB' + PC', \quad (6)$$

*that is the sum of the distances from  $P$  to the vertices of the triangle and to those of the pedal triangle.*

*It is the only point inside  $\mathcal{T}$  enjoying such a variational property.*

Once we know the result, the proof is easy... as it frequently happens in mathematics. We provide here two proofs, one with the help of optimization techniques, another one quite straightforward from geometrical considerations on the triangle.

*Proof 1.* The function  $f$  defined in (6) is the sum of six distance functions : three distances to the vertices of  $\mathcal{T}$ , three distances to the sides of  $\mathcal{T}$ . Each of these functions is differentiable at  $P$  with a gradient vector at  $P$  which is unitary : an unitary vector pointed at the vertex of  $\mathcal{T}$  (marked in black in Figure 1) or an unitary vector pointed at the side of  $\mathcal{T}$  (marked in red in Figure 1). Such unitary vectors can be assembled two by two, a red one with a black one : since they are opposite at the orthocenter, they “kill” each other. As a consequence, the gradient vector of  $f$  at the orthocenter is null. Moreover, the function  $f$  is clearly convex, it is even strictly convex because the three points  $A, B, C$  are no aligned [3]. Whence the announced result is proved.

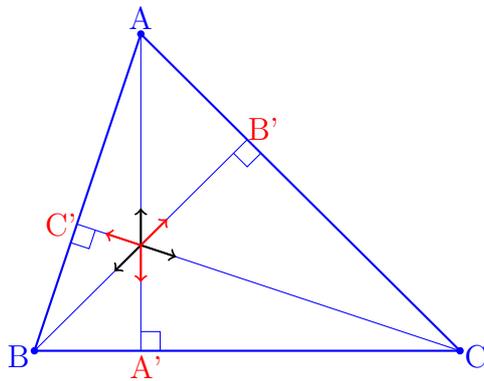


FIGURE 1 –

*Proof 2.* Let  $P$  be a point inside the triangle  $\mathcal{T}$ . See Figure 2. Clearly, the function

$$P \mapsto f_1(P) = PA + PA'$$

is minimized when the points  $A, P$  and  $A'$  are aligned, that is to say, when  $AA'$  is an altitude of  $\mathcal{T}$ . All the minimizers of this function are the points lying on the line-segment  $AA'$ ; the orthocenter of  $\mathcal{T}$  is one of them. Similarly, we consider the functions

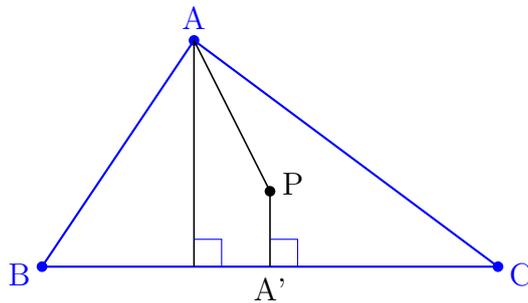


FIGURE 2 –

$$P \mapsto f_2(P) = PB + PB',$$

$$P \mapsto f_3(P) = PC + PC'.$$

Again, the orthocenter minimizes  $f_2$  and  $f_3$ . Finally, the orthocenter minimizes  $f_1 + f_2 + f_3$ , as announced.

**Remark 1.** Note the particular property of the orthocenter in a string of three “nested” optimization problems : it minimizes  $f_1$  (but it is not the only minimizer),  $f_1 + f_2$ , and finally  $f_1 + f_2 + f_3$ .

**Remark 2.** In view of what has been presented above, it is tempting to look at the (convex and differentiable) function

$$P \longmapsto g(P) = (PA)^2 + (PB)^2 + (PC)^2 + (PA')^2 + (PB')^2 + (PC')^2. \quad (7)$$

Indeed, we have done that : there is an unique point inside  $\mathcal{T}$  which minimizes the  $g$  function ; we have proposed it to the ETC in [2], it is indexed as X(5544).

### References

- [1] J.-B.HIRIART-URRUTY, *Quand la géométrie du triangle et l'optimisation convexe font bon ménage*. Submitted (2013).
- [2] C.KIMBERLING, *Encyclopedia of Triangle Centers (ETC)*,  
<http://faculty.evansville.edu/ck6/encyclopedia/>
- [3] J.-B.HIRIART-URRUTY et C.LEMARÉCHAL, *Fundamentals of Convex Analysis*. Grundlehren Text Editions. Springer (2001).

### The authors :

JEAN-BAPTISTE HIRIART-URRUTY  
Institut de mathématiques  
Université PAUL SABATIER  
118, route de Narbonne  
31062 TOULOUSE Cedex 9, France

[www.math.univ-toulouse.fr/~jbhu/](http://www.math.univ-toulouse.fr/~jbhu/)  
PIERRE-JEAN LAURENT  
Laboratoire Jean Kuntzman (LJK)  
Campus de Saint Martin d'Hères  
Université JOSEPH FOURIER  
38041 GRENOBLE Cedex 9, France