SIMPLICIAL EMBEDDINGS BETWEEN PANTS GRAPHS

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Abstract

We prove that, except in some low-complexity cases, every locally injective simplicial map between pants graphs is induced by a π_1-injective embedding between the corresponding surfaces.

1 Introduction and main results

To a surface Σ one may associate a number of naturally defined objects – its Teichmüller space, mapping class group, curve or pants graph, etc. An obvious problem is then to study embeddings between objects in the same category, where the term "embedding" is to be interpreted suitably in each case, for instance "isometric embedding" in the case of Teichmüller spaces, "injective homomorphism" in the case of mapping class groups, and "injective simplicial map" in the case of curve and pants graphs.

For pants graphs, this problem was first studied by D. Margalit [Mar], who showed that every automorphism of the pants graph is induced by a surface homeomorphism. More concretely, let Σ be a compact orientable surface whose every connected component has negative Euler characteristic. The pants graph $\mathcal{P}(\Sigma)$ of Σ is a simplicial graph whose vertices are pants decompositions of Σ and whose edges correspond to elementary moves on pants decompositions; see Section 2.2 for an expanded definition. Let $\mathrm{Mod}(\Sigma)$ be the mapping class group of Σ, that is, the group of homotopy classes of self-homeomorphisms of Σ. Note that $\mathrm{Mod}(\Sigma)$ acts on $\mathcal{P}(\Sigma)$ by simplicial automorphisms. Denote by $\mathrm{Aut}(\mathcal{P}(\Sigma))$ the group of all simplicial automorphisms of $\mathcal{P}(\Sigma)$. Let $\kappa(\Sigma)$ be the complexity of Σ, that is, the cardinality of a pants decomposition of Σ. The following is part of Theorem 1 of [Mar]:

Theorem 1 ([Mar])**.** *If Σ is a compact, connected, orientable surface with $\kappa(\Sigma) > 0$, then the natural homomorphism $\mathrm{Mod}(\Sigma) \to \mathrm{Aut}(\mathcal{P}(\Sigma))$ is surjective. Moreover, if $\kappa(\Sigma) > 3$ then it is an isomorphism.*

The main purpose of this article is to extend Margalit's result to injective simplicial maps between pants graphs. We note that examples of such maps

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are plentiful. Indeed, let Σ_1 be an essential subsurface of Σ_2 (see Section 2 for definitions) whose every connected component has positive complexity. Then one may construct an injective simplicial map $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ by first choosing a multicurve Q that extends any pants decomposition of Σ_1 to a pants decomposition of Σ_2 and then setting $\phi(v) = v \cup Q$.

Our main result asserts that, except in some low-complexity cases, this is the only way in which injective simplicial maps of pants graphs arise. Given a simplicial map $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ and a π_1-injective embedding $h : \Sigma_1 \to \Sigma_2$, we say that ϕ is *induced* by h if there exists a multicurve Q on Σ_2, disjoint from $h(\Sigma_1)$, such that $\phi(v) = h(v) \cup Q$ for all vertices v of $\mathcal{P}(\Sigma_1)$. In particular, Q has cardinality $\kappa(\Sigma_2) - \kappa(\Sigma_1)$. We will show:

Theorem A. *Let Σ_1 and Σ_2 be compact orientable surfaces of negative Euler characteristic, such that each connected component of Σ_1 has complexity at least 2. Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be an injective simplicial map. Then there exists a π_1-injective embedding $h : \Sigma_1 \to \Sigma_2$ that induces ϕ.*

We note that the hypothesis that all connected components of Σ_1 have complexity at least 2 is necessary, since the pants graph of the 1-holed torus and the 4-holed sphere are isomorphic (see [Min], for instance).

Remark. In the case of curve graphs, Teichmüller spaces and mapping class groups, there exist embeddings for which there are no π_1-injective embeddings of the corresponding surfaces. First, one may construct an injective simplicial map from the curve graph of a closed surface X to that of $X - p$, by considering a point p in the complement of the union of all simple closed geodesics on X. Next, any finite-degree cover $\tilde{Y} \to Y$ gives rise to an isometric embedding $T(Y) \to T(\tilde{Y})$ of Teichmüller spaces, so we may take Y to be a closed surface in order to produce the desired example. Finally, there exist injective homomorphisms of mapping class groups with no π_1-injective embeddings between the corresponding surfaces, see [ALS].

We also remark that, even though the results in this article are stated for injective simplicial maps, our arguments will only require the maps to be simplicial and *locally injective*, that is, injective on the *star* of every vertex of $\mathcal{P}(\Sigma_1)$ (the star of a vertex is defined as the union of all edges incident on it). In particular, we have:

Theorem B. *Let Σ_1 and Σ_2 be compact orientable surfaces of negative Euler characteristic, such that each connected component of Σ_1 has complexity at least 2. Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be a locally injective simplicial map. Then there exists a π_1-injective embedding $h : \Sigma_1 \to \Sigma_2$ that induces ϕ.*

In order to prove Theorem A, we will closely follow Margalit's strategy in [Mar] for proving Theorem 1. In Section 2 we will introduce the pants graph and its natural subgraphs. In Section 3 we will study some objects in the pants graph, namely *Farey graphs* and *alternating tuples*, which appear, or at least have their origin, in [Mar]. As we will see, the structure of these objects is preserved by injective simplicial maps.

The main ingredient of the proof of Theorem A will be Theorem C in Section 4. Roughly speaking, Theorem C states that if $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ is an injective simplicial map, then there exists a multicurve Q on Σ_2, satisfying certain conditions, such the image of $\mathcal{P}(\Sigma_1)$ under ϕ is the subgraph of $\mathcal{P}(\Sigma_2)$ spanned by those vertices of $\mathcal{P}(\Sigma_2)$ that contain Q.

Theorem C itself will have an interesting application to pants graph automorphisms. More concretely, in Corollary 11 of Section 5 we will see that pants graph automorphisms preserve the pants graph stratification (see Section 2 for definitions). This implies Theorem 1 if Σ is not the 2-holed torus (that is, the torus with two boundary components). The case of the 2-holed torus needs some extra care but it also follows from Theorem C by applying the same strategy of [Mar], Section 5.

Finally, in Section 6 we will prove Theorem A, which will follow easily from Theorem C and the classification of pants graphs up to isomorphism, stated as Lemma 12 in Section 6. We remark that, in most cases, it is possible to distinguish pants graphs up to isomorphism using Theorem 1 and the classification of mapping class groups.

We conclude the introduction by pointing out that a number of authors have studied embeddings in the context of mapping class groups and other complexes associated to surfaces. References include [ALS], [BehMa], [BelMa], [Irm1], [Irm2], [Irm3], [IrmKo], [IrmMc], [Iva], [IvaMc], [Ko], [Luo], [PaRo], [Sch], [Sha].

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2 Definitions and basic results

2.1 Surfaces and curves

Let Σ be a compact orientable surface whose every connected component has negative Euler characteristic. If g and b are, respectively, the genus and number of boundary components of Σ, we will refer to the number $\kappa(\Sigma) = 3g - 3 + b$ as the *complexity* of Σ.

A connected component of Σ is said to be *nontrivial* if it has positive complexity. We will say that two subsurfaces of Σ are *disjoint* if they can be homotoped into disjoint subsurfaces. A subsurface $X \subset \Sigma$ is said to be *essential* if it has no connected components homeomorphic to an annulus with core curve homotopic to a component of $\partial\Sigma$, and every component of ∂X is either a homotopically nontrivial simple closed curve on Σ, or is homotopic to a component of $\partial\Sigma$. Throughout this article we will only consider essential subsurfaces whose every connected component has negative Euler characteristic.

A simple closed curve on Σ is said to be *peripheral* if it is homotopic to a component of $\partial\Sigma$. By a *curve* on Σ we will mean a homotopy class of nontrivial and nonperipheral simple closed curves on Σ. The *intersection number* between two curves α and β is defined as

$$i(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}.$$

If $i(\alpha, \beta) = 0$, we say that α and β are *disjoint*. A *multicurve* is a collection of pairwise distinct and pairwise disjoint curves on Σ. Given a multicurve Q on Σ, the *deficiency* of Q is defined to be $\kappa(\Sigma) - |Q|$. A *pants decomposition* is a multicurve of cardinality $\kappa(\Sigma)$ (and so maximal with respect to inclusion). Note, if Q is a pants decomposition then $\Sigma - Q$ is a disjoint union of 3-holed spheres, or *pairs of pants*.

If $X \subset \Sigma$ is a (not necessarily proper) subsurface, we say that a collection $\mathcal{A}$ of curves on X *fills* X if, for every curve γ on X, there exists $\alpha \in \mathcal{A}$ with $i(\alpha, \gamma) > 0$. In particular, if $\kappa(X) = 1$ then any pair of distinct curves fill X.

2.2 The pants graph

We say that two pants decompositions are related by an *elementary move* if they have a deficiency 1 multicurve in common, and the remaining two curves either fill a 4-holed sphere and intersect exactly twice, or they fill a 1-holed torus and intersect exactly once. See Figure 1.

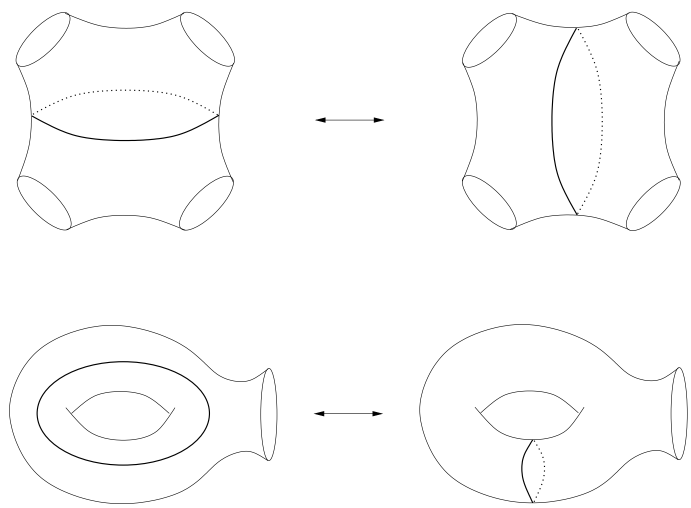

Figure 1: The two types of elementary move.

The *pants graph* $\mathcal{P}(\Sigma)$ of Σ is the simplicial graph whose vertex set is the set of all pants decompositions of Σ and where two vertices are connected by an edge if the corresponding pants decompositions are related by an elementary move. Abusing notation, we will not distinguish between vertices of $\mathcal{P}(\Sigma)$ and the corresponding multicurves on Σ.

A *path* in $\mathcal{P}(\Sigma)$ is a sequence $v_1, \dots, v_n$ of adjacent vertices of $\mathcal{P}(\Sigma)$. A *circuit* is a path $v_1, \dots, v_n$ such that $v_1 = v_n$ and $v_i \neq v_j$ for all other i, j.

The pants graph was introduced by Hatcher–Thurston in [HatTh], who proved it is connected (see the remark on the last page of [HatTh]). A detailed proof was then given by Hatcher-Lochak-Schneps in [HLS], where they proved that attaching 2-cells to finitely many types of circuits in $\mathcal{P}(\Sigma)$ produces a simply-connected 2-complex, known as the *pants complex*. The graph $\mathcal{P}(\Sigma)$ becomes a geodesic metric space by declaring each edge to have length 1, and Brock [Br] recently showed that $\mathcal{P}(\Sigma)$ is quasi-isometric to the Weil-Petersson metric on the Teichmüller space of Σ.

2.3 Natural subgraphs

As mentioned in the introduction, if $Y \subset \Sigma$ is an essential subsurface with no trivial components, then the inclusion map induces an injective simplicial map $\mathcal{P}(Y) \to \mathcal{P}(\Sigma)$, and so we can regard $\mathcal{P}(Y)$ as a connected subgraph of

$\mathcal{P}(\Sigma)$. If $Y_1, Y_2 \subset \Sigma$ are disjoint essential subsurfaces of positive complexity, then $\mathcal{P}(Y_1) \times \mathcal{P}(Y_2)$, which we define as the 1-skeleton of the product of $\mathcal{P}(Y_1)$ and $\mathcal{P}(Y_2)$, is a connected subgraph of $\mathcal{P}(\Sigma)$. Moreover, if Σ is not connected, then $\mathcal{P}(\Sigma)$ is the 1-skeleton of the product of the pants graphs of its nontrivial components.

Given a multicurve Q on Σ, let $\mathcal{P}_Q$ be the subgraph of $\mathcal{P}(\Sigma)$ spanned by those vertices of $\mathcal{P}(\Sigma)$ that contain Q. It will be convenient to consider the empty set as a multicurve, in which case we set $\mathcal{P}_\emptyset$ to be equal to $\mathcal{P}(\Sigma)$. Note that $\mathcal{P}_Q$ is connected for all multicurves Q; indeed, if Q is strictly contained in a pants decomposition then $\mathcal{P}_Q$ is naturally isomorphic to $\mathcal{P}(\Sigma - Q)$, and if Q is itself a pants decomposition then $\mathcal{P}_Q$ is equal to Q.

If Q_1 and Q_2 are multicurves on Σ, then $\mathcal{P}_{Q_1} \cap \mathcal{P}_{Q_2} \neq \emptyset$ if and only if $Q_1 \cup Q_2$ is a multicurve, in which case $\mathcal{P}_{Q_1} \cap \mathcal{P}_{Q_2} = \mathcal{P}_{Q_1 \cup Q_2}$. Furthermore, $\mathcal{P}_{Q_1} \subset \mathcal{P}_{Q_2}$ if and only if $Q_2 \subset Q_1$. This endows the pants graph with a *stratified structure*, analogous to the stratification of the Weil-Petersson completion (see [Wol]), with strata all the subgraphs of the form $\mathcal{P}_Q$, for some multicurve Q. Then $\mathcal{P}(\Sigma)$ is the union of all strata, and two strata intersect over a stratum if at all.

3 Farey graphs and alternating tuples.

3.1 Farey graphs

The *standard Farey graph* $\mathcal{F}$ is the simplicial graph whose vertices are all rational numbers including $\frac{1}{0} = \infty$, and where two vertices p/q and r/s, in lowest terms, are adjacent if $|ps - rq| = 1$. It is usually represented as an ideal triangulation of the Poincaré disc model of the hyperbolic plane.

If $\kappa(\Sigma) > 0$, the pants graph $\mathcal{P}(\Sigma)$ contains many subgraphs isomorphic to $\mathcal{F}$. To see this, consider a deficiency 1 multicurve Q on Σ. Then $\Sigma - Q$ has a unique nontrivial component X, which has complexity 1, and so it is either a 1-holed torus or a 4-holed sphere. In either case, $\mathcal{P}(X)$ is easily seen to be isomorphic to $\mathcal{F}$ (see, for instance, [Min], Section 3), and thus and $\mathcal{P}_Q \cong \mathcal{P}(\Sigma - Q) \cong \mathcal{P}(X)$.

By a *Farey graph* in $\mathcal{P}(\Sigma)$ we will mean a subgraph of $\mathcal{P}(\Sigma)$ isomorphic to $\mathcal{F}$. The next lemma implies that all Farey graphs in the pants graph arise in the manner just described. We remark that this result is essentially Lemma 2 of [Mar].

Lemma 2. *Let $\phi : \mathcal{F} \to \mathcal{P}(\Sigma)$ be a simplicial embedding. Then there exists a deficiency 1 multicurve Q such that $\phi(\mathcal{F}) = \mathcal{P}_Q$.*

Proof. Let Δ be the subgraph of $\mathcal{F}$ spanned by three pairwise adjacent vertices of $\mathcal{F}$; we will refer to Δ as a "triangle" in $\mathcal{F}$. Since any three pairwise adjacent vertices of $\mathcal{P}(\Sigma)$ have exactly $\kappa(\Sigma)-1$ curves in common, and since ϕ is a simplicial embedding, there exists a deficiency 1 multicurve Q on Σ such that $\phi(\Delta) \subset \mathcal{P}_Q$. Now, for any two vertices x, y of $\mathcal{F}$ there is a sequence $\Delta_0, \ldots, \Delta_n$ of triangles in $\mathcal{F}$ such that x is a vertex of Δ_0, y is a vertex of Δ_n, and Δ_i and Δ_{i+1} have exactly two vertices in common, for all $i = 0, \ldots, n-1$. Hence $\phi(\mathcal{F}) \subset \mathcal{P}_Q$. Thus the result follows, since $\mathcal{P}_Q$ is isomorphic to $\mathcal{F}$ and every injective simplicial map $\mathcal{F} \to \mathcal{F}$ is an isomorphism. $\square$

With the notation of Lemma 2, we will say that the Farey graph $F = \phi(\mathcal{F})$ is *determined* by Q. The following observation asserts that if a Farey graph F intersects a stratum in the pants graph, then either F is contained in the stratum or else is "transversal" to it.

Lemma 3. *Let F be a Farey graph in $\mathcal{P}(\Sigma)$ and let T be a multicurve. Suppose that $F \cap \mathcal{P}_T$ has at least 2 vertices. Then $F \subseteq \mathcal{P}_T$.*

Proof. Lemma 2 implies that $F = \mathcal{P}_Q$, for some deficiency 1 multicurve Q. Suppose there exist two distinct vertices u, v in $\mathcal{P}_Q \cap \mathcal{P}_T$. Write $u = Q \cup \alpha$, $v = Q \cup \beta$, noting $\alpha \neq \beta$. Since $u, v \in \mathcal{P}_T$ then $T \subseteq Q$, and therefore $\mathcal{P}_Q \subseteq \mathcal{P}_T$. $\square$

Let e be an edge of $\mathcal{P}(\Sigma)$, and let u and v be its endpoints. By Lemma 3, e is contained in a unique Farey graph, determined by the deficiency 1 multicurve $u \cap v$. Given a vertex u of $\mathcal{P}(\Sigma)$, observe that there are exactly $\kappa(\Sigma)$ distinct Farey graphs containing u, determined by the $\kappa(\Sigma)$ distinct deficiency 1 multicurves contained in u. We state this observation as a separate lemma, as we will make extensive use of it later.

Lemma 4. *Given any vertex u of $\mathcal{P}(\Sigma)$, there are exactly $\kappa(\Sigma)$ distinct Farey graphs containing u.* $\square$

As mentioned in Section 1, the *star* $\mathrm{St}(u)$ of a vertex u of $\mathcal{P}(\Sigma)$ is the union of all edges of $\mathcal{P}(\Sigma)$ incident on u. By the discussion preceding Lemma 4, each edge of $\mathrm{St}(u)$ is contained in exactly one of $\kappa(\Sigma)$ Farey graphs. The following remark offers a characterisation of when two edges of $\mathrm{St}(u)$ are contained in the same Farey graph, and makes apparent that such a property is preserved by locally injective simplicial maps. The proof is immediate.

Lemma 5. *Let u be a vertex of $\mathcal{P}(\Sigma)$. Two edges $e, e' \in \mathrm{St}(u)$ are contained in the same Farey graph if and only if there exists a sequence of edges $e = e_0, e_1, \ldots, e_n = e'$ in $\mathrm{St}(u)$ such that e_i and e_{i+1} are edges of the same circuit of length 3 in $\mathcal{P}(\Sigma)$, for all $i = 0, \ldots, n-1$.* □

3.2 Alternating tuples in the pants graph

We now introduce the notion of *alternating tuple* in the pants graph, a slight generalisation of what Margalit refers to as "alternating circuit" in [Mar].

Definition 6 (Alternating tuple)**.** *Let $n > 3$ and let $(v_1, \ldots, v_n)$ be a cyclically ordered n-tuple of distinct vertices of $\mathcal{P}(\Sigma)$. We say that $(v_1, \ldots, v_n)$ is* alternating *if v_i and v_{i+1} belong to the same Farey graph F_i, and $F_i \neq F_{i+1}$ (counting subindices modulo n).*

Observe that, in the definition, one has $F_i \cap F_{i+1} = \{v_{i+1}\}$. Intuitively, v_{i-1} and v_i can be joined by a path whose every vertex contains the same deficiency 1 multicurve, so all the elementary moves along this path happen in the same complexity 1 subsurface, and that subsurface changes after v_{i+1}.

The following lemma follows immediately from Lemmas 2 and 5, plus the definition of an alternating tuple.

Lemma 7. *Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be an injective simplicial map and $n > 3$. Then ϕ maps alternating n-tuples to alternating n-tuples.* □

An alternating 5-tuple will be called a *pentagon* if v_i and v_{i+1} are adjacent for all i. The following lemma describes the structure of alternating 4- and 5-tuples in the pants graph, and will be crucial in the proof of our main results. We remark that this result is implicit in [Mar] (see Lemmas 4, 7 and 8 therein).

Lemma 8 (Structure of alternating 4- and 5-tuples)**.** *Let $(v_1, \ldots, v_n)$ be an alternating n-tuple, where $n \in \{4, 5\}$. Then there exists a deficiency 2 multicurve T such that $v_i \in \mathcal{P}_T$ for all i. Moreover, if $n = 4$ then $\Sigma - T$ has exactly 2 nontrivial components, each of complexity 1; if $n = 5$ then $\Sigma - T$ has exactly 1 nontrivial component, which has complexity 2.*

Proof. For the first part, note there is nothing to show if $\kappa(\Sigma) = 2$, for in that case we let $T = \emptyset$, so that $\mathcal{P}_T = \mathcal{P}(\Sigma)$. So assume $\kappa(\Sigma) \geq 3$. Since v_1, v_2, v_3 do not belong to the same Farey graph, then $T = v_1 \cap v_2 \cap v_3$ is a deficiency 2 multicurve. Now $T \subset v_4$ as well; otherwise v_1 and v_4 would differ by 3 curves and thus one could not connect v_1 and v_4 by a path entirely

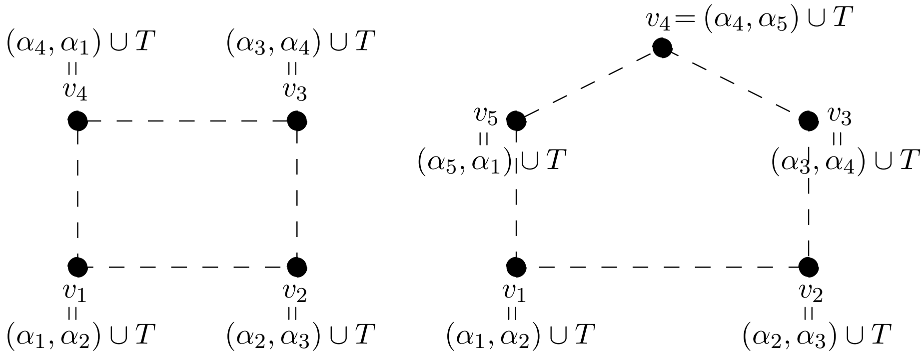

Figure 2: Alternating 4- and 5-tuples in the pants graph. Here T is a deficiency 2 multicurve, and the dashed line between v_i and v_{i+1} represents a path between them, entirely contained in a Farey graph F_i.

contained in at most 2 Farey graphs. Similarly, $T \subset v_5$ in the case of an alternating 5-tuple, and so the first part of the result follows.

Note that, in particular, one can write $v_i = (\alpha_i, \alpha_{i+1}) \cup T$ for all i, as in Figure 2. Since T has deficiency 2, then $\Sigma - T$ either has one nontrivial component of complexity 2, or two nontrivial components of complexity 1. Let X (resp. Y) be the complexity 1 subsurface filled by α_1 and α_3 (resp. α_2 and α_4), noting $X \neq Y$ and $X, Y \subset \Sigma - T$.

If $n = 4$ then $i(\alpha_j, \alpha_2) = i(\alpha_j, \alpha_4) = 0$ for $j \in \{1, 3\}$ (see Figure 2). Thus X and Y are disjoint and thus the result follows.

Now assume $n = 5$. We claim that $X \cup Y$ is connected. If not, then α_5 is contained in either X or Y. But α_5 is distinct and disjoint from both α_1 and α_4 (see Figure 2), and so α_5 cannot be contained in either X or Y, which is a contradiction. □

Let F_1, F_2 be distinct Farey graphs of $\mathcal{P}(\Sigma)$ intersecting at a vertex u, where F_i is determined by the deficiency 1 multicurve Q_i, for $i = 1, 2$. If the nontrivial components of $\Sigma - Q_1$ and $\Sigma - Q_2$ are disjoint, we say that F_1 and F_2 *commute*. Now, Lemma 8 implies that if $(v_1, \ldots, v_4)$ is an alternating 4-tuple, then F_i and F_{i+1} commute for all i, where F_i is the Farey graph containing v_i and v_{i+1}. The following converse is an immediate consequence of the proof of Lemma 8:

Corollary 9. *Let F_1, F_2 be Farey graphs that commute. For any $u_i \in F_i - \{u\}$, the vertices u, u_1, u_2 are elements of an alternating 4-tuple.*

Proof. Let $u_i \in F_i - \{u\}$ for $i = 1, 2$. Since $F_1 \neq F_2$ then $u = (\alpha_1, \alpha_2) \cup T$,

$u_1 = (\alpha_1', \alpha_2) \cup T$ and $u_2 = (\alpha_1, \alpha_2') \cup T$, for some deficiency 2 multicurve T. Now F_1 and F_2 commute, and so $i(\alpha_1', \alpha_2') = 0$. Therefore, (u_2, u, u_1, w) is an alternating 4-tuple, where $w = (\alpha_1', \alpha_2') \cup T$. □

In particular, there exists an alternating 4-tuple in $\mathcal{P}(\Sigma)$ if and only if $\kappa(\Sigma) \geq 3$. The next technical result will be very important in the next section:

Lemma 10 (Extending adjacent vertices to alternating tuples)**.** *Let Σ be a surface of complexity at least 2. Let u and v be adjacent vertices of $\mathcal{P}(\Sigma)$ and let G be a Farey graph containing v but not u. Then there exists $n \in \{4, 5\}$ and a vertex $w \in G - \{v\}$, such that u, v, w are elements of an alternating n-tuple.*

Proof. Write $k = \kappa(\Sigma)$, $u = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $v = (\alpha_1', \alpha_2, \ldots, \alpha_k)$. Let F be the Farey graph that contains u and v, thus determined by $u \cap v = v - \alpha_1'$. Since $u \notin G$ then, up to relabeling the curves of v, G is determined by $v - \alpha_2$. Let T denote the deficiency 2 multicurve $v - (\alpha_1', \alpha_2)$. There are two cases to consider:

Case 1. *F and G commute.* In this case the result follows from Corollary 9 by considering any $w \in G$, with $w \neq v$.

Case 2. *F and G do not commute.* Then $\Sigma - T$ has exactly one nontrivial component S, of complexity 2, and (α_1, α_2) and (α_1', α_2) are adjacent vertices of $\mathcal{P}(S)$. There are two possibilities for S, namely S is a 5-holed sphere or S is a 2-holed torus.

If S is a 5-holed sphere then, up to the action of $\mathrm{Mod}(S)$, the curves $\alpha_1, \alpha_2, \alpha_1'$ are, respectively, the curves α, β, γ on the left of Figure 3. Consider the curves δ and η, also from the left of Figure 3, and set $v_i = w_i \cup T$, where w_i is defined as in Figure 3. Then $(v_1, \ldots, v_5)$ is an alternating 5-tuple (in this case, a pentagon) in $\mathcal{P}(\Sigma)$, and observe that $v_1 = u$, $v_2 = v$ and $v_3 \in G$. Thus we can take $w = v_3$ and so the result follows.

The case of S a 2-holed torus is dealt with along the exact same lines, using the curves on the 2-holed torus of Figure 3; in this case, the 5-tuple we obtain is not a pentagon since $i(\delta, \beta) = 4$ (in fact, there are no pentagons in the pants graph of the 2-holed torus; see the proof of Lemma 8 in [Mar]) □

4 Statement and proof of Theorem C

We now proceed to state and prove Theorem C, from which Theorem A will easily follow.

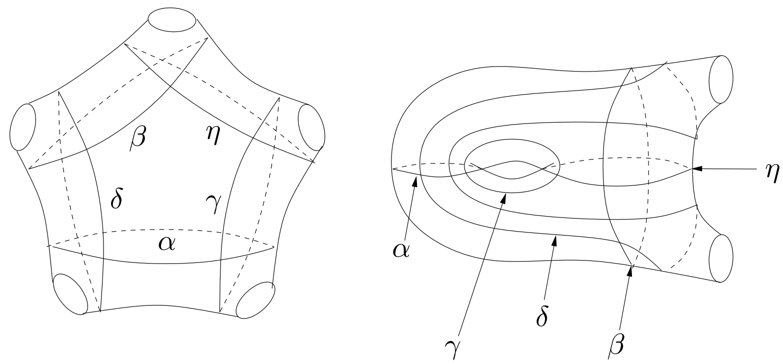

Figure 3: Curves giving rise to an alternating 5-tuple $(w_1, \ldots, w_5)$ in a 5-holed sphere (left) and a 2-holed torus (right), where $w_1 = (\alpha, \beta)$, $w_2 = (\beta, \gamma)$, $w_3 = (\gamma, \delta)$, $w_4 = (\delta, \eta)$ and $w_5 = (\eta, \alpha)$.

Theorem C. *Let Σ_1 and Σ_2 be compact orientable surfaces such that every connected component of Σ_1 has positive complexity. Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be an injective simplicial map. Then the following hold:*

(1) $\kappa(\Sigma_1) \leq \kappa(\Sigma_2)$,

(2) There exists a multicurve Q on Σ_2, of cardinality $\kappa(\Sigma_2) - \kappa(\Sigma_1)$, such that $\phi(\mathcal{P}(\Sigma_1)) = \mathcal{P}_Q$. In particular, $\mathcal{P}(\Sigma_1) \cong \mathcal{P}(\Sigma_2 - Q)$;

(3) Σ_1 and $\Sigma_2 - Q$ have the same number of nontrivial components. Moreover, if $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_r$ are, respectively, the nontrivial components of Σ_1 and $\Sigma_2 - Q$ then, up to reordering the indices, ϕ induces an isomorphism $\phi_i : \mathcal{P}(X_i) \to \mathcal{P}(Y_i)$. In particular, $\kappa(X_i) = \kappa(Y_i)$.

Proof. Observe that if $\kappa(\Sigma_1) = 1$ then the result follows from Lemma 2. Therefore, from now on we will assume that $\kappa(\Sigma_1) \geq 2$. Let $\kappa_i = \kappa(\Sigma_i)$, for $i = 1, 2$.

Part (1) is immediate, since Lemmas 2 and 5 imply that ϕ maps distinct Farey graphs containing a vertex u (and there are κ_1 of these, by Lemma 4) to distinct Farey graphs containing $\phi(u)$ (and there are κ_2 of these, again by Lemma 4).

We will now prove part (2). For clarity, its proof will be broken down into 3 separate claims.

Claim I. *Let u be a vertex of $\mathcal{P}(\Sigma_1)$. Then there exists a multicurve $Q(u)$ on Σ_2, of cardinality $\kappa_2 - \kappa_1$, such that $\phi(\mathrm{St}(u)) \subset \mathcal{P}_{Q(u)}$.*

Proof. Let $e \in \mathrm{St}(u)$. By Lemmas 3 and 4, there are κ_1 distinct Farey graphs $F_1, \ldots, F_{\kappa_1}$ containing u, and e belongs to exactly one of them. By Lemma 2, $\phi(F_i) = \mathcal{P}_{Q_i}$, for some deficiency 1 multicurve $Q_i \subset \phi(u)$. Observe that $Q_i \neq Q_j$ for all $i \neq j$, by Lemma 5. Consider the multicurve $Q(u) = Q_1 \cap \cdots \cap Q_{\kappa_1}$, which has cardinality $\kappa_2 - \kappa_1$. Since $Q(u) \subset Q_i$ then $\phi(F_i) = \mathcal{P}_{Q_i} \subset \mathcal{P}_{Q(u)}$. In particular, $\phi(e) \subset \mathcal{P}_{Q(u)}$. $\diamond$

Claim II. *If $v \in \mathrm{St}(u)$ then $\phi(\mathrm{St}(v)) \subset \mathcal{P}_{Q(u)}$.*

Proof. Let $e \in \mathrm{St}(u)$ be the edge with endpoints u and v. Let $e' \in \mathrm{St}(v)$ and let G be the unique (by Lemma 3) Farey graph containing e', see Figure 4. If $e \subset G$ then $\phi(e') \in \mathcal{P}_{Q(u)}$, by Lemma 3 and Claim I. Thus suppose e is not contained in G, so $u \notin G$. By Lemma 10, there exists a number $n \in \{4, 5\}$ and a vertex $w \in G$, with $w \neq v$, such that u, v, w are elements of an alternating n-tuple in $\mathcal{P}(\Sigma_1)$, which we denote by τ. By Lemma 7, $\phi(u), \phi(v), \phi(w)$ are also elements of an alternating n-tuple in $\mathcal{P}(\Sigma_2)$, namely $\phi(\tau)$. Therefore there is a deficiency 2 multicurve T on Σ_2 such that $\phi(\tau) \subset \mathcal{P}_T$, by Lemma 8.

We now claim that $Q(u) \subseteq T$. To see this, let z be the unique element of $\tau - \{v\}$ contained in the same Farey graph as u, noting z, u, v are not contained in the same Farey graph of $\mathcal{P}(\Sigma_1)$ by the definition of an alternating tuple. Therefore $\phi(z), \phi(u), \phi(v)$ are not contained in the same Farey graph of $\mathcal{P}(\Sigma_2)$ and so $\phi(z) \cap \phi(u) \cap \phi(v) = T$, since $\phi(\tau) \subset \mathcal{P}_T$ and T has deficiency 2. Finally, $Q(u) \subseteq \phi(z) \cap \phi(u) \cap \phi(v)$ since ϕ maps every Farey graph containing u into $\mathcal{P}_{Q(u)}$ and u, v (resp. u, z) are contained in a common Farey graph. Thus $Q(u) \subseteq T$, as desired.

Since $Q(u) \subseteq T$ then $\phi(\tau) \subset \mathcal{P}_T \subseteq \mathcal{P}_{Q(u)}$. In particular, $\phi(w)$ is contained in $\mathcal{P}_{Q(u)}$ and thus in $\phi(G) \cap \mathcal{P}_{Q(u)}$. Since $\phi(v) \in \phi(G) \cap \mathcal{P}_{Q(u)}$ as well, we conclude that $\phi(G) \subset \mathcal{P}_{Q(u)}$ by Lemma 3. In particular, $\phi(e') \subset \mathcal{P}_{Q(u)}$ and thus Claim II follows. $\diamond$

As a consequence, and since $\mathcal{P}(\Sigma_1)$ is connected, it follows that $\phi(\mathcal{P}(\Sigma_1)) \subseteq \mathcal{P}_Q$, where $Q = Q(u)$ for some, and hence any, vertex u of $\mathcal{P}(\Sigma_1)$. Therefore we can view ϕ as a an injective simplicial map $\mathcal{P}(\Sigma_1) \to \mathcal{P}_Q$. We will now prove that this map is also surjective, thus completing the proof of part (2).

Claim III. The map $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}_Q$ is surjective.

Proof. Let e be an edge of $\mathcal{P}_Q$; we want to show that $e \in \mathrm{Im}(\phi)$. Since $\phi(\mathcal{P}(\Sigma_1))$ and $\mathcal{P}_Q$ are connected, and since $\phi(\mathcal{P}(\Sigma_1)) \subseteq \mathcal{P}_Q$, we can assume

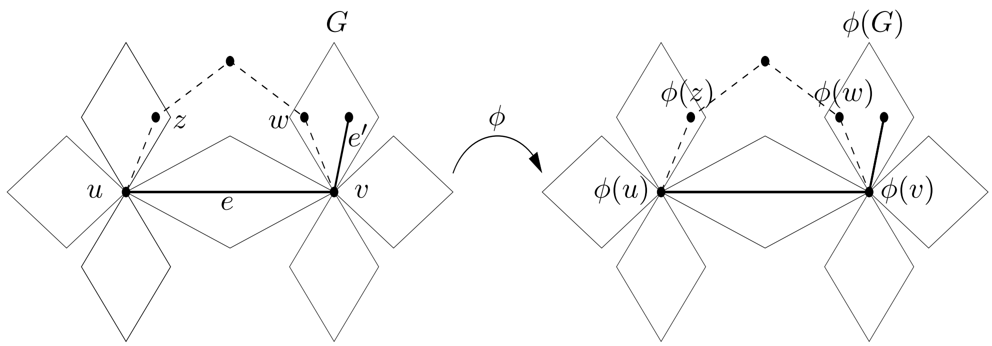

Figure 4: The situation of Claim II. Each rhombus represents a Farey graph in the pants graph containing the vertex on which it is incident. The dashed line on the left figure corresponds to the alternating tuple τ (in this case, a 5-tuple) containing u, v and w.

$e \in \mathrm{St}(\phi(u))$ for some vertex u of $\mathcal{P}(\Sigma_1)$. Note that e is contained in a unique Farey graph H and that $H \subset \mathcal{P}_Q$, by Lemma 3.

Since $\mathcal{P}_Q \cong \mathcal{P}(\Sigma_2 - Q)$ and $\kappa_1 = \kappa(\Sigma_2 - Q)$, there are exactly κ_1 distinct Farey graphs in $\mathcal{P}(\Sigma_2)$ which are contained in $\mathcal{P}_Q$ and contain $\phi(u)$, by Lemma 4. Again by Lemma 4, there are exactly κ_1 distinct Farey graphs in $\mathcal{P}(\Sigma_1)$ containing u. Since ϕ maps distinct Farey graphs containing u to distinct Farey graphs containing $\phi(u)$, we get that $H = \phi(F)$ for some Farey graph F in $\mathcal{P}(\Sigma_1)$ containing u. In particular, $e \in \mathrm{Im}(\phi)$, as desired. $\diamond$

Finally, we will prove part (3). Let $X_1, \ldots, X_r$ be the nontrivial components of Σ_1. Observe that every pants decomposition of Σ_1 has the form $(v_1, \ldots, v_r)$, where v_i is a pants decomposition of X_i, and so $\mathcal{P}(\Sigma_1) = \Pi_{i=1}^r \mathcal{P}(X_i)$, which recall is defined as the 1-skeleton of the product of the pants graphs of the X_i. Fix a pants decomposition $v = (v_1, \ldots, v_r)$ of Σ_1. Then ϕ induces an injective simplicial map

$$\phi_i : \mathcal{P}(X_i) \to \mathcal{P}_Q \cong \mathcal{P}(\Sigma_2 - Q),$$

by setting $\phi_i(w) = \phi(v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_r)$ for all vertices w of X_i. Applying part (2) of Theorem C to ϕ_i, we deduce that there exists an essential subsurface Y_i of $\Sigma_2 - Q$ such that $\phi_i(\mathcal{P}(X_i)) = \mathcal{P}(Y_i)$. In particular, $\kappa(Y_i) = \kappa(X_i)$, by part (1). Moreover, by discarding those connected components of Y_i homeomorphic to a 3-holed sphere, we can assume that Y_i has no trivial components.

Claim. Y_i is connected.

Proof. Suppose, for contradiction, that Y_i had $N \geq 2$ connected components $Z_1, \cdots, Z_N$. In particular, $\phi_i(\mathcal{P}(X_i)) = \Pi_{j=1}^N \mathcal{P}(Z_j)$, and $0 < \kappa(Z_j) < \kappa(Y_i) = \kappa(X_i)$, for all j.

We introduce some terminology. Given $k = 1, \ldots, N$, let

$$p_k : \Pi_{j=1}^N \mathcal{P}(Z_j) \to \mathcal{P}(Z_k)$$

be the natural projection onto the k-th factor. We say a subgraph G of $\Pi_{j=1}^N \mathcal{P}(Z_j)$ is *contained in one of the factors* of $\Pi_{j=1}^N \mathcal{P}(Z_j)$ if there exists $k = 1, \ldots, N$ such that the restriction of p_k to G is an isomorphism and, for all $m \neq k$, the restriction of p_m to G is constant.

Let e be an edge of $\mathcal{P}(X_i)$. Since the Z_j are pairwise disjoint, then $\phi(e)$ is contained in one of the factors of $\Pi_{j=1}^N \mathcal{P}(Z_j)$. Using this, plus Lemma 3, one deduces that if F is a Farey graph in $\mathcal{P}(X_i)$, then $\phi(F)$ is also contained in one of the factors of $\Pi_{j=1}^N \mathcal{P}(Z_j)$. Moreover, if F and F' do not commute, then $\phi_i(F)$ and $\phi_i(F')$ are contained in the same factor of $\Pi_{j=1}^N \mathcal{P}(Z_j)$, by Corollary 9.

Let u be a vertex of $\mathcal{P}(X_i)$. We now define the *adjacency graph* Γ of u, introduced independently by Behrstock-Margalit [BehMa] and Shackleton [Sha]. The vertices of Γ are exactly those curves in u, and two distinct curves are adjacent in Γ if they are boundary components of the same pair of pants determined by u. Observe that Γ is connected since X_i is.

Now a Farey graph containing u is is determined by a deficiency 1 multicurve contained in u or, equivalently, by a curve in u. Moreover, two curves in u are adjacent if and only if the Farey graphs they determine do not commute. Let $\mathcal{G}$ be the graph whose vertices are those Farey graphs containing u and whose edges correspond to distinct non-commuting Farey graphs. Note $\mathcal{G}$ is isomorphic to Γ and so it is connected.

By Lemma 4 and since $\mathcal{G}$ is connected, there exist $\kappa(X_i)$ Farey graphs in $\mathcal{P}(X_i)$, all containing u, whose images under ϕ_i are all contained in the same factor of $\Pi_{j=1}^N \mathcal{P}(Z_j)$. This contradicts Lemma 4, since $\kappa(Z_j) < \kappa(X_i)$ for all j, and thus the claim follows. $\diamond$

The discussion above implies that there are r connected subsurfaces $Y_1, \ldots, Y_r$ of $\Sigma_2 - Q$ such that, up to reordering, ϕ induces an isomorphism $\phi_i : \mathcal{P}(X_i) \to \mathcal{P}(Y_i)$ for $i = 1, \ldots, r$. In particular, $\kappa(X_i) = \kappa(Y_i)$. Now,

$$\Sigma_1 = X_1 \sqcup \ldots \sqcup X_r$$

and

$$\Sigma_2 - Q \supseteq Y_1 \cup \ldots \cup Y_r,$$

and therefore the Y_i are pairwise disjoint, since the X_i are pairwise disjoint and $\kappa(\Sigma_1) = \kappa(\Sigma_2 - Q)$. For the same reason, the Y_i are the only nontrivial connected components of $\Sigma_2 - Q$. This finishes the proof of Part (3) of Theorem C. □

5 A proof of Theorem 1

An immediate consequence of Theorem C is that every automorphism of the pants graph preserves the stratified structure of the pants graph. More concretely, let $\phi : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ be an injective simplicial map; by Theorem C, ϕ is in fact an isomorphism. Let α be a curve on Σ, and observe that $\mathcal{P}(\Sigma - \alpha) \cong \mathcal{P}_\alpha \subset \mathcal{P}(\Sigma)$. Then ϕ induces an injective simplicial map, which we also denote by ϕ, from $\mathcal{P}_\alpha$ to $\mathcal{P}(\Sigma)$. Applying Theorem C to $\Sigma_1 = \Sigma - \alpha$ and $\Sigma_2 = \Sigma$, we readily obtain the following:

Corollary 11. *Let $\phi : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ be an automorphism. Then, for every curve α, there exists a unique curve β such that $\phi(\mathcal{P}_\alpha) = \mathcal{P}_\beta$. Moreover, $\Sigma - \alpha$ and $\Sigma - \beta$ have the same number of nontrivial components.*

Corollary 11 provides a somewhat different proof of Theorem 1 from the one given in [Mar]. Let us briefly comment on this. In [Mar], Margalit introduced the notion of a *marked* Farey graph in the pants graph. A marked Farey graph is a pair (F, v), where F is a Farey graph in the pants graph and v is a vertex of F. A marked Farey graph singles out exactly one curve on Σ, although there are infinitely many marked Farey graphs in $\mathcal{P}(\Sigma)$ that single out a given curve. Margalit associates, to the pants graph automorphism ϕ, a curve graph automorphism ψ by defining $\psi(\alpha)$ to be the curve β singled out by $\phi(F)$, where F is a marked Farey graph that singles out α. One of the main steps in [Mar] is to show that this construction gives rise to a well-defined map between the pants graph automorphism group and the curve graph automorphism group, which Margalit then shows is an isomorphism. If Σ is not a 2-holed torus, Theorem 1 then follows from results of Ivanov [Iva], Korkmaz [Ko] and Luo [Luo] on the automorphism group of the curve graph. The case of the 2-holed torus requires separate treatment in [Mar], and boils down to showing that the curve graph automorphism induced by a pants graph automorphism maps nonseparating curves to nonseparating curves.

Using Corollary 11, one may define a curve graph automorphism ψ from the pants graph automorphism ϕ, by setting $\psi(\alpha) = \beta$, where β is the curve such that $\phi(\mathcal{P}_\alpha) = \mathcal{P}_\beta$ in Corollary 11. One quickly checks that this produces an isomorphism between the pants graph automorphism group and the curve graph automorphism group, and thus Theorem 1 follows if the surface is not the 2-holed torus. The case of the 2-holed torus is also deduced from Corollary 11 by applying the same argument as in Section 5 of [Mar]. We remark that this approach to pants graph automorphisms is similar in spirit to those of Masur-Wolf [MasWo] and Brock-Margalit [BrMa] for showing that Weil-Petersson isometries are induced by surface self-homeomorphisms. Indeed, one of the key steps there is to prove that Weil-Petersson isometries preserve the stratification of the Weil-Petersson completion.

6 Proof of Theorem A

We are finally ready to give a proof of Theorem A. We will need the following lemma:

Lemma 12 (Classification of pants graphs up to isomorphism)**.** *Let Σ, Σ' be two compact connected orientable surfaces of complexity at least 2. Then $\mathcal{P}(\Sigma)$ and $\mathcal{P}(\Sigma')$ are isomorphic if and only if Σ and Σ' are homeomorphic.*

Proof. First, by part (1) of Theorem C, if $\mathcal{P}(\Sigma)$ and $\mathcal{P}(\Sigma')$ are isomorphic then $\kappa(\Sigma) = \kappa(\Sigma')$. We consider the following three cases:

(i) Suppose $\kappa(\Sigma) > 3$. By Theorem 1, $\mathrm{Aut}(\mathcal{P}(\Sigma)) \cong \mathrm{Mod}(\Sigma)$. Thus if $\mathcal{P}(\Sigma) \cong \mathcal{P}(\Sigma')$ then $\mathrm{Mod}(\Sigma) \cong \mathrm{Mod}(\Sigma')$. Since $\kappa(\Sigma) > 3$, Theorem 4 of [IvaMc] implies that Σ and Σ' are homeomorphic. (As pointed out by the referee, in the case of genus 3 or more, this can also be deduced by combining earlier results of Birman-Lubotzky-McCarthy [BLM] on the abelian rank of mapping class groups; of Harer [Har] on the virtual cohomological dimension of mapping class groups; and of Powell [Po] on mapping class groups of closed surfaces being perfect.)

(ii) Suppose that $\kappa(\Sigma) = 2$. Up to renaming the surfaces, Σ is a 5-holed sphere and Σ' is a 2-holed torus. The curves on the 5-holed sphere on the left of Figure 3 yield the existence of a pentagon in $\mathcal{P}(\Sigma)$, while there are no pentagons in the pants graph of the 2-holed torus (see the proof of Lemma 8 in [Mar]).

(iii) Finally, we consider the case $\kappa(\Sigma) = 3$. Let us denote by $S_{g,b}$ the surface of genus g with b boundary components. We have that

$$\Sigma, \Sigma' \in \{S_{0,6}, (S_{1,3}, S_{2,0}\}$$

Suppose, for contradiction, that there exists an isomorphism ϕ between $\mathcal{P}(S_{0,6})$ and $\mathcal{P}(S_{2,0})$. Choose a curve α on $S_{0,6}$ such that $S_{0,6}-\alpha = S_{0,3} \sqcup S_{0,5}$, noting that $\mathcal{P}_\alpha \cong \mathcal{P}(S_{0,5})$. By Theorem C, there exists a curve β on $S_{2,0}$ such that $\phi(\mathcal{P}_\alpha) = \mathcal{P}_\beta$. Moreover, $S_{2,0} - \beta$ has to be connected, and thus $S_{2,0} - \beta$ is homeomorphic to $S_{1,2}$. In particular, $\mathcal{P}(S_{2,0} - \beta) \cong \mathcal{P}(S_{1,2})$. Thus we get an isomorphism between $\mathcal{P}(S_{0,5})$ and $\mathcal{P}(S_{1,2})$, which contradicts (ii).

Now, suppose there is an isomorphism ϕ between $\mathcal{P}(S_{1,3})$ and $\mathcal{P}(S_{0,6})$. We choose a curve α on $S_{1,3}$ such that $S_{1,3} - \alpha = S_{0,3} \sqcup S_{1,2}$. Arguing as above, we get an isomorphism between $\mathcal{P}(S_{1,2})$ and $\mathcal{P}(S_{0,5})$, which does not exist, by (ii).

Finally, suppose that there exists an isomorphism between $\mathcal{P}(S_{1,3})$ and $\mathcal{P}(S_{2,0})$. Choose a non-separating curve γ on $S_{1,3}$, so that $S_{1,3} - \gamma = S_{0,5}$. By the same argument as above, we get an isomorphism between $\mathcal{P}(S_{0,5})$ and $\mathcal{P}(S_{1,2})$, contradicting (ii). $\square$.

As mentioned before, the pants graph of the two complexity 1 surfaces (that is, the 1-holed torus and the 4-holed sphere) are isomorphic. We remark that the isomorphism classification of pants graphs is different from that of curve complexes (see Lemma 2.1 in [Luo]).

Proof of Theorem A. Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be an injective simplicial map. By Theorem C, there exists a multicurve Q on Σ_2, of deficiency $\kappa(\Sigma_1)$, such that $\phi(\mathcal{P}(\Sigma_1)) = \mathcal{P}_Q \cong \mathcal{P}(\Sigma_2 - Q)$. Discarding the trivial components of $\Sigma_2 - Q$ we obtain an essential subsurface $Y \subset \Sigma_2 - Q$, with no trivial components, and such that $\mathcal{P}(\Sigma_1) \cong \mathcal{P}(Y)$. We can thus view ϕ as an isomorphism $\mathcal{P}(\Sigma_1) \to \mathcal{P}(Y)$. Let us first assume that Σ_1 is connected. In that case Y is connected as well, by part (3) of Theorem C. Since $\kappa(\Sigma_1) \geq 2$, Lemma 12 implies there exists a homeomorphism $g : \Sigma_1 \to Y$, which induces an isomorphism $\psi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(Y)$ by $\psi(v) = g(v)$. By Theorem 1, there exists $f \in \mathrm{Mod}(Y)$ such that $\phi = f \circ \psi$. Thus $f \circ g$ induces ϕ.

If Σ_1 is not connected, the result follows by applying the above argument to each connected component of Σ_1. $\square$

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