A REMARK ON HOMOMORPHISMS FROM RIGHT-ANGLED ARTIN GROUPS TO MAPPING CLASS GROUPS

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Abstract. We study rigidity properties of certain homomorphisms from right-angled Artin groups to mapping class groups. As an application we show that if $\Gamma \subset \text{Map}(S)$ is a subgroup that contains some power of every Dehn twist, then any injective homomorphism $\Gamma \to \text{Map}(S)$ is a restriction of an automorphism of $\text{Map}(S)$.

Résumé. Nous examinons la rigidité de certains homomorphismes entre groupes d’Artin rectangulaires et groupes modulaires. Nous démontrons que si $\Gamma \subset \text{Map}(S)$ est un sous-groupe qui contient quelque puissance de tout twist de Dehn, alors tout homomorphisme injectif $\Gamma \to \text{Map}(S)$ est la restriction d’un automorphisme de $\text{Map}(S)$.

Version française abrégée. Soit $S$ une surface connexe et orientable, de genre $g$ et avec $n$ pointes. Nous supposerons que $S$ est non-exceptionnelle, c’est-à-dire que $3g + n \geq 5$ et $(g, n) \neq (1, 2)$. Le groupe modulaire étendu $\text{Map}^*(S)$ est le groupe de difféomorphismes de $S$ à isotopie près. Le groupe modulaire $\text{Map}(S)$ est le sous-groupe d’indice 2 en $\text{Map}^*(S)$ dont les éléments sont représentés par les difféomorphismes préservant l’orientation de $S$. Finalement, le groupe modulaire $\text{Map}(S)$ est le sous-groupe des éléments de $\text{Map}(S)$ qui fixent chaque pointe de $S$.

Le groupe d’Artin rectangulaire $\mathbb{A}(X)$ associé à un complexe simplicial $X$ est le groupe engendré par l’ensemble $X^0$ des sommets de $X$, et tel que les éléments correspondant à deux sommets voisins commutent. On remarque que si $\Delta \subset X$ est un simplex, alors $\mathbb{A}(\Delta)$ est un sous-groupe de $\mathbb{A}(X)$ isomorphe à $\mathbb{Z}^{\dim(\Delta)+1}$.

Dans cet article nous nous intéressons aux homomorphismes faiblement injectifs $\rho : \mathbb{A}(X) \to \text{Map}(S)$, où $X$ est un sous-complexe rigide [1] du complexe des courbes $\mathcal{C}(S)$ de $S$ [14]. Ici, nous disons que $X \subset \mathcal{C}(S)$ est rigide si toute application injective et simpliciale $\omega : X \to \mathcal{C}(S)$ est la restriction d’un automorphisme de $\mathcal{C}(S)$. Un homomorphisme $\rho$ est faiblement injectif si pour tous les simplexes $\Delta, \Delta' \subset X$ et pour tous les éléments $\gamma, \gamma' \in \mathbb{A}(\Delta), \gamma' \in \mathbb{A}(\Delta')$, si $\rho(\gamma) = \rho(\gamma')$ alors $\gamma = \gamma'$. Dénominateur par $\delta_{\gamma}$ le twist de Dehn le long de $\gamma \in \mathcal{C}(S)$, on montre:

Théorème 1. Soit $S$ une surface connexe, orientable et non-exceptionnelle. Soit aussi $X \subset \mathcal{C}(S)$ un sous-complexe rigide avec $\dim(X) = \dim(C)$, et tel que chaque simplexe de $X$ est l’intersection des simplexes de dimension maximale de $X$ qui le contiennent. Pour tout homomorphisme faiblement injectif $\rho : \mathbb{A}(X) \to \text{Map}(S)$ il existe $f \in \text{Map}^*(S)$ et fonctions $a, b : X^0 \to \mathbb{Z} \setminus \{0\}$ telles que $\rho(\gamma^n(\gamma)) = f_\gamma\delta_{\gamma}^b(\gamma) f^{-1}$ pour tout $\gamma \in X^0$. De plus, $f$ est unique si $S$ n’est pas la surface fermée de genre 2.

Nous remarquons qu’il y a des constructions d’ensembles rigides finis de $\mathcal{C}(S)$ [1] qui satisfont les hypothèses du Théorème 1. Par exemple, soit $X_n$ le complexe simplicial dont les simplexes de dimension $k$ correspondent aux ensembles de $k+1$ diagonales disjointes du polygone avec $n$ sommets. Pour $n \geq 5$, $X_n$ est un sous-complexe rigide du complexe de courbes de la sphère $S_{0,n}$ avec $n$ pointes [1]. Donc, si $\rho_0 : \mathbb{A}(X_n) \to P \text{Map}(S_{0,n})$ est l’homomorphisme faiblement injectif donné par $\rho_0(\gamma) = \delta_{\gamma}$ pour tout $\gamma$, alors tout homomorphisme injectif $\rho : \mathbb{A}(X_n) \to P \text{Map}(S_{0,n})$ a la forme $\rho(\cdot) = f((\rho_{X_n} \circ \tau)(\cdot)) f^{-1}$, où $f \in \text{Map}^*(S_{0,n})$ et $\tau : \mathbb{A}(X_n) \to \mathbb{A}(X_n)$ est un monomorphisme.

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Par ailleurs, le complexe des courbes lui même est rigide [22]. En lui appliquant le Théorème 1, on montre:

**Corollaire 2.** Soit $S$ une surface connexe, orientable et non exceptionnelle, autre que la surface fermée de genre 2. Soit aussi $\Gamma \subset \text{Map}(S)$ un sous-groupe tel que pour toute $\gamma \in \mathcal{C}(S)$ il y a $n(\gamma) \in \mathbb{N}$ avec $\delta_n^{\gamma} \in \Gamma$. Pour tout homomorphisme injectif $\sigma : \Gamma \to \text{Map}(S)$ il existe un unique élément $f \in \text{Map}^*(S)$ tel que $\sigma(g) = fgf^{-1}$ pour tout $g \in \Gamma$.

Rappelons que si $S$ a genre au moins 3, les noyaux des homomorphismes du groupe modulaire dans des groupes de Lie compacts - par exemple les représentations quantiques [21] - satisfont la condition du Corollaire 2 [3].

On observe aussi que le Corollaire 2 implique que le groupe $\Gamma$ est cohéopien et que son commensurateur abstrait est isomorphe à $\text{Map}^*(S)$. Ces résultats sont déjà connus pour le group de Torelli [11] et pour quelques autres sous-groupes normaux de $\text{Map}(S)$ d’indice infini [8, 9]. Toutefois, la rigidité qu’on trouve dans le Corollaire 2 est bien plus forte que ce qu’on connait dans ces cas là: ici on ne suppose pas que $\sigma(\Gamma) \subset \Gamma$, ni que $\Gamma$ et $\sigma(\Gamma)$ soient commensurables.

**1. Introduction**

Let $S$ be a connected, orientable surface of genus $g$ with $n$ punctures. Throughout, we will assume that $S$ is **non-exceptional**, that is, $3g + n \geq 5$ and $(g, n) \neq (1, 2)$. Denote by $\text{Map}^*(S)$ the **extended mapping class group**, that is the group of isotopy classes of self-diffeomorphisms of $S$. The **mapping class group** $\text{Map}(S) \subset \text{Map}^*(S)$ the index 2 subgroup consisting of isotopy classes of those diffeomorphisms that preserve the orientation of $S$; finally, the **pure mapping class group** $P \text{Map}(S) \subset \text{Map}(S)$ is the subgroup of those mapping classes fixing each puncture of $S$.

Given a simplicial complex $X$, the **right-angled Artin group** $A(X)$ associated to $X$ is the group generated by the set $X^{(0)}$ of vertices of $X$, subject to the relation that $\gamma_i, \gamma_j \in X^{(0)}$ commute if and only if they are adjacent in $X$. Note that every simplex $\Delta$ of $X$ determines an abelian subgroup $A(\Delta)$ of $A(X)$, isomorphic to $\mathbb{Z}^{\dim(\Delta) + 1}$.

In this note we are interested in homomorphisms from right-angled Artin groups to mapping class groups. We remark that there are numerous examples of such homomorphisms: for instance, if $X \neq \emptyset$ then $A(X)$ surjects onto $\mathbb{Z}$, and hence we obtain infinitely many homomorphisms $A(X) \to \text{Map}(S)$. In addition, so long as $X$ has at least two non-adjacent vertices, $A(X)$ surjects onto the non-abelian free group of rank 2 and thus we obtain still more homomorphisms $A(X) \to \text{Map}(S)$.

Observe, however, that the homomorphisms just described fail to be injective. On the other hand, Koberda [18] and Clay-Leininger-Mangahas [10] showed that every finitely generated right-angled Artin group embeds as a subgroup of some mapping class group.

Below, we will prove a rigidity result for a certain class of homomorphisms $A(X) \to \text{Map}(S)$, called **weakly injective**, in the case when $X$ is a **rigid** subset of the curve complex $\mathcal{C}(S)$. We need a couple of definitions before stating our main result:

**Definition** (Weak injectivity). Let $X$ be a simplicial complex, and $G$ a group. A homomorphism $\rho : A(X) \to G$ is **weakly injective** if the following holds: for all simplices $\Delta, \Delta' \subset X$, and for all $\gamma \in A(\Delta), \gamma' \in A(\Delta')$, if $\rho(\gamma) = \rho(\gamma')$ then $\gamma = \gamma'$.

Recall that the curve complex $\mathcal{C}(S)$ is the simplicial complex whose $k$-simplices correspond to sets of $k+1$ distinct free isotopy classes of essential simple closed curves on $S$ with pairwise disjoint representatives [14]. Denote by $\delta_\gamma$ the right Dehn twist along the simple closed curve $\gamma \in \mathcal{C}(S)$. If $X \subset \mathcal{C}(S)$ is an arbitrary subcomplex, then the homomorphism

$$\rho_0 : A(X) \to \text{Map}(S), \quad \rho_0(\gamma) = \delta_\gamma \text{ for every vertex } \gamma \in X^{(0)}$$
is weakly injective; see [12], in particular Section 3.3, for basic facts about Dehn twists. Note, however, that the map $\rho_0$ is not injective in general; compare with [13]. As mentioned earlier, we will be interested in subcomplexes of $C(S)$ that are rigid:

**Definition (Rigid subcomplex).** A simplicial subcomplex $X$ of $C(S)$ is rigid if for every injective simplicial map $\omega : X \to C(S)$ there is an automorphism $\phi \in \text{Aut}(C(S))$ of $C(S)$ with $\omega(\gamma) = \phi(\gamma)$ for all $\gamma \in X^{(0)}$.

Since $S$ is assumed to be non-exceptional, the combination of results of Ivanov [17], Korkmaz [19], and Luo [20], implies that every automorphism of $C(S)$ is induced by an element of $\text{Map}^*(S)$. Furthermore, if $S$ is not the closed surface of genus 2, then the said element is unique – see [20].

We are finally ready to state our main result:

**Theorem 1.** Let $S$ be a connected, orientable and non-exceptional surface. Suppose that $X$ is a rigid subcomplex of $C(S)$, with $\dim(X) = \dim(C(S))$, and such that every simplex of $X$ is equal to the intersection of all maximal dimensional simplices of $X$ that contain it. For every weakly injective homomorphism $\rho : A(X) \to \text{Map}(S)$ there are $f \in \text{Map}^*(S)$ and functions $a,b : X^{(0)} \to \mathbb{Z} \setminus \{0\}$ with $\rho(a(\gamma)) = f_0 \delta^{b(\gamma)} f^{-1}$, for every $\gamma \in X^{(0)}$. Moreover, $f$ is unique unless $S$ is a closed surface of genus 2.

The equality $\rho(a(\gamma)) = f_0 \delta^{b(\gamma)} f^{-1}$ asserts that $\rho(\gamma)$ is a root of a power of the Dehn twist along $f(\gamma)$. In the absence of roots – for instance if $\rho$ takes values is the pure mapping class group $P\text{Map}(S_{0,n})$ of the $n$-punctured sphere – we deduce that $\rho(\gamma)$ is in fact a power of a Dehn twist; with the notation of Theorem 1 this means that $a(\gamma) = 1$.

Concrete examples of finite rigid subsets of $C(S_{0,n})$ were given in [1]. Indeed, the simplicial complex $X_n$ whose $k$-simplices correspond to sets of $k+1$ pairwise disjoint diagonals of the polygon with $n$ vertices is a rigid subcomplex of $C(S_{0,n})$ for $n \geq 5$. The complex $X_n$ is the dual polytope to the associahedron, and hence every simplex is equal to the intersection of all maximal dimensional simplices containing it. We thus deduce from Theorem 1 that, for $n \geq 5$, every weakly injective homomorphism $\rho : A(X_n) \to P\text{Map}(S_{0,n})$ is of the form $\rho(\cdot) = f ( (\rho_0 \circ \tau)(\cdot)) f^{-1}$, where $f \in \text{Map}^*(S_{0,n})$, $\rho_0$ is as in (1), and $\tau : A(X_n) \to A(X_n)$ is the injective homomorphism determined by $\tau(\gamma) = \gamma^{b(\gamma)}$ for $\gamma \in X_n^{(0)}$.

We stress that in Theorem 1 we are not assuming that the subcomplex $X$ be finite. In particular, applying the theorem to $X = C(S)$, which is itself rigid by the work of Shackleton [22], we prove:

**Corollary 2.** Let $S$ be a connected, orientable and non-exceptional surface, other than the closed surface of genus 2. Let $\Gamma \subset \text{Map}(S)$ be a subgroup such that for every $\gamma \in C(S)$ there is $n(\gamma) \in \mathbb{N}$ with $\delta^{n(\gamma)} \in \Gamma$. For every injective homomorphism $\sigma : \Gamma \to \text{Map}(S)$ there is a unique $f \in \text{Map}^*(S)$ such that $\sigma(g) = f gf^{-1}$, for all $g \in \Gamma$.

Since any finite index subgroup $\Gamma \subset \text{Map}(S)$ automatically satisfies the hypothesis above, Corollary 2 implies the results in [6, 7, 15, 16, 22] about injections of finite index subgroups of mapping class groups.

In addition, there are numerous subgroups $\Gamma \subset \text{Map}(S)$ of infinite index that satisfy the condition of Corollary 2, for example the kernel of any representation of $\text{Map}(S)$ to a compact Lie group, provided that $S$ has genus at least 3 – see Corollary 2.6 of [3]. This applies for instance to the so-called quantum representations [21] of $\text{Map}(S)$, many of which have infinite image. Note also that Corollary 2 implies that the subgroup $\Gamma$ is co-Hopfian, and that its abstract commensurator is isomorphic to $\text{Map}^*(S)$. Such results were already known for the Torelli group [11], as well as for other infinite index normal subgroups of $\text{Map}(S)$ [8, 9]. However, we remark that the rigidity statement in Corollary 2 is more powerful than any of the existing ones, since we do not assume that $\sigma(\Gamma) \subset \Gamma$, or that $\Gamma$ and $\sigma(\Gamma)$ are commensurable.
2. Abelian subgroups of the mapping class group.

We recall a few standard facts about abelian subgroups of the mapping class group. See [12] for basic facts on the mapping class group and [5] for details on its abelian subgroups.

Let $S$ be a connected orientable surface, and $A$ an abelian subgroup of $\text{Map}(S)$. By the rank of $A$ we understand the dimension of $A \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space. A reducing system for $A$ is an $A$-invariant multicurve $\lambda \subset S$. If there is no reducing system for $A$, then $A$ contains a pseudo-Anosov and hence $\text{rank}(A) = 1$. Thus, every abelian subgroup $A$ of $\text{Map}(S)$ with $\text{rank}(A) \geq 2$ is reducible. Given any reducing system $\lambda$ for $A$ we have the exact sequence:

$$1 \to A \cap T_\lambda \to A \to \text{Map}(S \setminus \lambda),$$

where $T_\lambda$ is the group generated by the Dehn twists along the components of $\lambda$ (or half-twists in the case when the given component bounds a twice-punctured disk or a once-punctured torus). We say that $\lambda$ is a complete reducing system for $A$ if, for every component $W$ of $S \setminus \lambda$, either:

(a) there are $d > 0$ and $f$ in the image of the third homomorphism in (2), such that $f^d|_W = f^d$ is pseudo-Anosov, or

(b) there is $d > 0$ with $f^d|_W = \text{Id}$ for every $f$ in the image of the third homomorphism in (2).

There is a unique complete reducing system $\lambda(A)$ for $A$, the canonical reducing system [5], contained in every other complete reducing system for $A$. The active surface $S(A)$ of $A$ is the union of those components of $S \setminus \lambda(A)$ for which (a) above is satisfied. Noting that no component of $S(A)$ is homeomorphic to a three-times punctured sphere, the pigeonhole principle and (2) together imply:

**Fact 1.** Suppose that $S$ has genus $g$ and $n$ punctures. Every abelian subgroup $A$ of $\text{Map}(S)$ satisfies $\text{rank}(A) \leq 3g + n - 3 = \text{dim}(\mathcal{C}(S)) + 1$. Moreover, in the equality case one has:

1. $3g + n - 3 = l(A) + s(A)$, where $l(A)$ and $s(A)$ are, respectively, the number of components of $\lambda(A)$ and $S(A)$;

2. Every component of $S(A)$ is homeomorphic to either a once-punctured torus or a four-times punctured sphere;

3. The group $A$ does not permute the components of $\lambda(A)$ (resp. $S(A)$).

Suppose now that $f \in \text{Map}(S)$ has infinite order and is contained in some abelian subgroup $A < \text{Map}(S)$ of maximal rank. In particular, $f$ is not pseudo-Anosov and thus $\lambda((f)) \neq \emptyset$. Moreover, observe that $\lambda((f)) \subset \lambda(A)$, and that $S((f))$ is a union of connected components of $S(A)$. Moreover, both $\lambda((f))$ and $S((f))$ are preserved by the centralizer $Z_{\text{Map}(S)}(f)$ of $f$. In fact, the subgroup of $Z_{\text{Map}(S)}(f)$ that preserves each component of $\lambda(f)$ and each component of $S \setminus \lambda(f)$ has finite index in $Z_{\text{Map}(S)}(f)$ and contains $\langle f, T_{\lambda(f)} \rangle$ in its center. Notice that $\text{rank}((f, T_{\lambda(f)})) \geq 2$ unless $S((f)) = \emptyset$ and $\lambda(f)$ has a single component. Altogether we have:

**Fact 2.** Let $S$ be a connected, orientable and non-exceptional surface. Suppose that $f \in \text{Map}(S)$ has infinite order and is contained in an abelian group of maximal rank. Then, either $f$ is a root of a Dehn twist, or the centralizer of $f$ in $\text{Map}(S)$ has a finite index subgroup $G$ whose center $Z(G)$ satisfies $\text{rank}(Z(G)) \geq 2$.

We can now prove:

**Lemma 3.** Let $S$ be a connected, orientable and non-exceptional surface. If $\{A_i\}_{i \in I}$ is a collection of maximal rank abelian subgroups of $\text{Map}(S)$ such that $\text{rank}(\bigcap A_i) = 1$, then $\bigcap A_i$ is a cyclic group generated by a root of a power of a Dehn twist.

**Proof.** First, it follows from Fact 1 and the work of Birman-Hilden [4] (see Theorem 2.8 of [2] for an explicit statement) that the centralizer of a non-trivial finite order element of $\text{Map}(S)$ does not contain abelian groups of maximal rank. Therefore $\bigcap A_i$ is torsion free, and hence cyclic.
Let $G$ be a finite index subgroup of the centralizer of $\cap A_i$. Maximality of the rank of $A_i$ implies that $Z(G) \cap A_i$ has finite index in $Z(G)$ for all $i$. In particular, also $Z(G) \cap (\cap A_i)$ has finite index in $Z(G)$; this proves that $Z(G)$ has rank 1. By Fact 2, $\cap A_i$ is generated by a root of a power of a Dehn twist, as claimed.

**A remark on roots.** We remark that if $f$ is a half-twist along a curve that bounds a twice-punctured disk or a once-punctured torus in $S$, then $f$ is indeed contained in a maximal rank abelian subgroup of $\text{Map}(S)$. In fact, it is not difficult, albeit not so interesting and slightly cumbersome, to prove that these are the only roots which can appear in Lemma 3 as long as $(g, n) \neq (2, 0)$. It follows that Theorem 1 can be marginally improved to assert that $a(\gamma) \in \{1, 2\}$.

3. **Proofs**

Before proving the results announced in the introduction, we need a preparatory observation:

**Lemma 4.** Let $S$ be connected, orientable and non-exceptional surface. Suppose that $X$ is a simplicial complex with $\dim(X) = \dim(C(S))$, and whose every simplex is equal to the intersection of the maximal dimensional simplices of $X$ that contain it. Then every weakly injective homomorphism $\rho: A(X) \to \text{Map}(S)$ maps each standard generator of $A(X)$ to a root of a power of a Dehn twist along a single curve.

**Proof.** Let $\gamma \in X^{(0)}$ be a vertex, and consider the collection $\{\Delta_i\}_{i \in I}$ of maximal dimensional simplices of $X$ that contain $\gamma$. Our assumption implies that the cyclic group $\langle \gamma \rangle$ is equal to $\cap_i A(\Delta_i)$. Since $\rho$ is weakly injective, $\rho(\langle \gamma \rangle)$ is also an infinite cyclic subgroup of $\text{Map}(S)$, which moreover satisfies

$$\rho(\langle \gamma \rangle) = \cap_i \rho(A(\Delta_i)) \subset \text{Map}(S).$$

Now, $\text{rank}(\rho(A(\Delta_i))) = \text{rank}(A(\Delta_i)) = \dim(X) + 1 = \dim(C(S)) + 1$. We can hence apply Lemma 3 to $\{\rho(\Delta_i)\}_{i \in I}$, thus deducing that $\rho(\langle \gamma \rangle)$ is generated by a root of a power of a Dehn twist, as we needed to prove. □

We are now ready to prove Theorem 1:

**Proof of Theorem 1.** By Lemma 4, $\rho(\gamma)$ is a root of a power of a Dehn twist along a single curve, for every $\gamma \in X^{(0)}$. In other words, there are $\rho_*(\gamma) \in C(S)$ and $a(\gamma), b(\gamma) \in \mathbb{Z} \setminus \{0\}$ with

$$\rho(\gamma^{a(\gamma)}) = \delta^{b(\gamma)}_{\rho_*(\gamma)}.$$  

Since the elements of $A(X)$ corresponding to adjacent vertices $\gamma, \eta \in X^{(0)}$ commute, $\rho_*(\gamma)$ and $\rho_*(\eta)$ do not intersect. Moreover, if $\gamma, \eta$ are arbitrary distinct vertices of $X$, then $\rho_*(\gamma) \neq \rho_*(\eta)$ because $\rho$ is weakly injective. Therefore, we deduce that the map $\rho_*: X \to C(S)$ is an injective simplicial map. Since $X \subset C(S)$ is assumed to be rigid, there is $\phi \in \text{Aut}(C(S))$ with $\rho_*(\gamma) = \phi(\gamma)$ for all $\gamma \in X$. As $S$ is not exceptional, the aforementioned results of Ivanov [17], Korkmaz [19] and Luo [20] together imply that there is $f \in \text{Map}^*(S)$ with $\phi(\gamma) = f(\gamma)$ for all $\gamma \in C(S)$; moreover, $f$ is unique unless $S$ is a closed surface of genus 2. Therefore, we obtain

$$\rho(\gamma^{a(\gamma)}) = \delta^{b(\gamma)}_{\rho_*(\gamma)} = \delta^{b(\gamma)}_{f(\gamma)} = f \delta^{b(\gamma)}_{f^{-1}}$$  

for all $\gamma \in X^{(0)}$, as desired. □

Finally, we prove Corollary 2:
Proof of Corollary 2. Let $\sigma : \Gamma \to \text{Map}(S)$ be an injective homomorphism, and $\rho : \mathcal{A}(\mathcal{C}(S)) \to \text{Map}(S)$ the homomorphism $\rho(\gamma) = \delta^a_{\gamma}(\gamma)$, noting that its image is contained in $\Gamma$. Hence, we can also consider the homomorphism

$$\rho' = \sigma \circ \rho : \mathcal{A}(\mathcal{C}(S)) \to \text{Map}(S).$$

As $S$ is assumed to be non-exceptional, $\mathcal{C}(S)$ is rigid [22] and thus Theorem 1 implies that there are $f \in \text{Map}^*(S)$ and functions $a, b : \mathcal{C}(S) \to \mathbb{Z} \setminus \{0\}$ with $\rho'(\gamma^a(\gamma)) = f\delta^b_{\gamma}(\gamma)f^{-1}$ for every $\gamma \in \mathcal{C}(S)$. Moreover, $f$ is unique since $S$ is not the closed surface of genus 2. Conjugating $\sigma$ by $f^{-1}$, we may in fact assume that

$$\rho'(\gamma^a(\gamma)) = \delta^b_{\gamma}(\gamma)$$

for every vertex $\gamma \in \mathcal{C}(S)$. After this normalization, $\sigma$ maps roots of powers of Dehn twists along a curve to roots of powers of Dehn twists along the same curve. We claim that $\sigma(h) = h$ for every $h \in \Gamma$. Indeed, note that for every $h \in \Gamma$ and $\gamma \in \mathcal{C}(S)$ there are $a, b, c \in \mathbb{Z}$ such that

$$\delta^a_{h(\gamma)} = \sigma(h_b(\gamma)) = \sigma(h_\gamma^b h^{-1}) = \sigma(h)\sigma(\delta^b_{\gamma})\sigma(h)^{-1} = \sigma(h)\delta^c_{\gamma}\sigma(h)^{-1} = \delta^c_{\sigma(h)(\gamma)}$$

This proves, in particular, that $h(\gamma) = \sigma(h)(\gamma)$. Since $\gamma \in \mathcal{C}(S)$ was arbitrary and $S$ is not the closed surface of genus 2, it follows that $\sigma(h) = h$, as we needed to prove. 

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