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# A CHARACTERISATION OF PLANE QUASICONFORMAL MAPS USING TRIANGLES

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Abstract. We show that an injective continuous map between planar regions which distorts vertices of equilateral triangles by a small amount is quasiconformal.

Quasiconformal maps have become an important class of homeomorphisms, for they arise in many fields of mathematics, such as pde's, Teichmüller theory, hyperbolic geometry, complex dynamics etc. Their involvement may be explained from the numerous characterisations of quasiconformality involving different flavours, which generally amount to loosening characterisations of conformal maps.

Let $\Omega \subset \mathbb{C}$ be a domain in the plane, and let us first define $\delta_\Omega(z) = \mathrm{dist}(z, \mathbb{C} \setminus \Omega)$. Let $f : \Omega \to \mathbb{C}$ be an injective continuous map. For $z \in \Omega$ and $r \in (0, \delta_\Omega(z))$, one may consider

$$L_f(z,r) = \sup\{|f(z) - f(w)|,\ |z-w| = r\}\,,\ \text{and}$$
$$\ell_f(z,r) = \inf\{|f(z) - f(w)|,\ |z-w| = r\}\,.$$

Let us set $H_f(z,r) = L_f(z,r)/\ell_f(z,r)$ and

$$H_f(z) = \limsup_{r\to 0} H_f(z,r) \quad \in [1,\infty]\,.$$

The metric definition of F.W. Gehring asserts that $f$ is $K$-quasiconformal if $H_f$ is finite everywhere, and if $H_f \le K$ a.e. [3].

In his monograph [5], J.H. Hubbard proposes a new formulation of plane quasiconformal maps in terms of distortion of triangles (Definition 4.5.1 therein). If $T$ is a Euclidean triangle in $\mathbb{C}$ with vertices $V(T) = \{z_1, z_2, z_3\}$, i.e., the convex hull of $V(T)$, one defines the skew of $T$ as

$$\mathrm{skew}(T) = \mathrm{skew}(V(T)) = \inf\{L \in [1,\infty],\ |z_i - z_j| \le L|z_i - z_k|,\ \{i,j,k\} = \{1,2,3\}\}\,.$$

He proves that if $f : \Omega \to f(\Omega)$ is a homeomorphism for which there is an increasing homeomorphism $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathrm{skew}(f(T)) = \mathrm{skew}(f(V(T))) \le h(\mathrm{skew}(T))$$

for any $T \subset \Omega$, then $f$ is quasiconformal. He then mentions that it suffices to control the distortion of triangles with skew bounded above by $\sqrt{7/3}$ in order to establish the result, and he poses the question of whether it is enough to have control only on equilateral triangles (see the remark following Exercise 4.5.12 in [5]).

In this note, we prove

**Theorem.** — *There is a constant $\varepsilon_0 > 0$ such that, if $\varepsilon \in [0, \varepsilon_0)$ and if $f : \Omega \to f(\Omega)$ is a homeomorphism such that*

$$\mathrm{skew}(f(T)) \le 1 + \varepsilon$$

*for any equilateral triangle $T \subset \Omega$, then $f$ is $K$-quasiconformal for some $K = K(\varepsilon)$.*

This statement is essentially local. We were unable to prove or disprove the quasiconformality for arbitrary $\epsilon > 0$. At first glance, imposing a uniform upper bound on the distortion of equilateral

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triangles looks like a slight variation of standard metric characterisations of quasiconformal maps. However, it turns out to be a much weaker condition: for example, knowing the distance between the images of 0 and 1 only gives control on the images of the 6th roots of unity, saying nothing about the images of points closer to the origin. In fact, it is not at all clear that a map which distorts equilateral triangles by a uniform amount should even be continuous if this is not required a priori. This highlights some of the difficulties one would have to overcome in order to prove quasiconformality in general, which does not seem obvious even for $\varepsilon = 1$. The construction of potential counterexamples is also a delicate issue.

**Outline of the paper.** We first establish a local criterion of quasiconformality (Corollary 1.2). We then prove the theorem under an additional property, but for any $\varepsilon > 0$ (Proposition 2.1). We prove this property is satisfied if $\varepsilon$ is small enough (Proposition 2.2), completing the proof of our main result. In the appendix, we will provide a short proof of the following fact, which improves the bound of $\sqrt{7/3}$ given in [5]: Let $\mu > 0$ and $\lambda < \infty$. If $f : \Omega \to f(\Omega)$ is a homeomorphism such that

$$\text{skew}(f(T)) \leq \lambda$$

for any triangle $T \subset \Omega$ with $\text{skew}(T) \leq 1 + \mu$, then $f$ is $K$-quasiconformal for some $K = K(\lambda, \mu)$.

Classic references on quasiconformal maps include the monographs [1, 6, 7].

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## 1. A criterion of quasiconformality

Let $Aut(\mathbb{C})$ denote the group of conformal automorphisms of the plane, i.e., affine maps $z \mapsto az+b$, where $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. Note the derivative of an affine map is constant. Given $z \in \mathbb{C}$ and $r > 0$, we denote by $D(z, r)$ the open Euclidean disc centred at $z$ of radius $r$.

**Proposition 1.1.** *Let $\Omega \subset \mathbb{C}$ be a domain, and $\mathcal{F}$ a family of injective continuous maps $f : \Omega \to \mathbb{C}$ which satisfies the following property. For any sequence $(f_n)$ of $\mathcal{F}$, and for any $\alpha_n \in Aut(\mathbb{C})$ such that $\alpha_n(0) \in \Omega$ and $|\alpha_n'(0)| \leq (1/3)\delta_\Omega(\alpha_n(0))$, there are a subsequence $(n_k)_k$ and a sequence $(\beta_k)_k$ of elements of $Aut(\mathbb{C})$ such that the sequence $(\beta_k \circ f_{n_k} \circ \alpha_{n_k})_k$ converges uniformly on the disk $D(0, 2)$ to an injective continuous map $g : D(0, 2) \to \mathbb{C}$.*

*Then there is some $K = K(\mathcal{F})$ such that each element of $\mathcal{F}$ is $K$-quasiconformal.*

This proposition is similar to criteria given for global homeomorphisms in § 2.2–2.5 in [2] and [4, Th. 18 and Cor. 8].

Proof. Suppose, for contradiction, that the result were not true. Then one can find a sequence of maps $(f_n)_n$ and a sequence of points $(z_n)_n$ in $\Omega$ such that $H_{f_n}(z_n) \geq 2n$. Let us consider $r_n \in (0, \delta_\Omega(z_n)/3)$ such that $H_{f_n}(z_n, r_n) \geq n$.

Let us consider $\alpha_n(z) = z_n + r_n z$. It follows that $r_n = |\alpha_n'(0)| \leq (1/3)\delta_\Omega(\alpha_n(0))$. Therefore, extracting a subsequence if necessary, one may assume that there is a sequence $(\beta_n)_n$ of affine maps such that the maps $g_n = \beta_n \circ f_n \circ \alpha_n$ form a sequence which converges uniformly on $D(0, 2)$ to an injective continuous map $g : D(0, 2) \to \mathbb{C}$. It follows from the uniform convergence and the injectivity of $g$ that

$$\lim H_{g_n}(0, 1) = H_g(0, 1) < \infty\,.$$

But $H_{g_n}(0, 1) = H_{f_n}(z_n, r_n) \geq n$, a contradiction. ■

We derive a corollary as follows. Let $\mathcal{P}$ be a property which can be satisfied by a continuous complex-valued function defined on a planar region. We say that $\mathcal{P}$ is a *conformal property* if the following propositions hold:

(1) If $f$ satisfies $\mathcal{P}$, then either $f$ is locally injective or constant.
(2) The property $\mathcal{P}$ is closed under uniform convergence.
(3) The property $\mathcal{P}$ is preserved under pre- and post-composition by affine maps.

We will denote by $\mathcal{P}(\Omega)$ the set of continuous complex-valued functions defined on $\Omega$ which satisfy the property $\mathcal{P}$. We also define its *Schwarz class* $\mathcal{S}_{\mathcal{P}}$ as the subset of injective functions of $\mathcal{P}(D(0,3))$ normalised by $f(0)=0$ and $f(1)=1$.

**Corollary 1.2.** *Let $\mathcal{P}$ be a conformal property. Suppose that $\mathcal{S}_{\mathcal{P}}$ is equicontinuous at $0$. Then $\mathcal{S}_{\mathcal{P}}$ is compact with respect to the topology of uniform convergence of compact subsets of $D(0,3)$ and there is some finite constant $K$ such that any non-constant map $f$ on any domain $\Omega$ which satisfies $\mathcal{P}$ is locally $K$-quasiconformal.*

Given $\mathcal{P}$, $\Omega$, $w, w' \in \Omega$ such that $|w-w'| \leq \delta_{\Omega}(w)/3$, we define using (3) the operator $T_{w,w'} : \mathcal{P}(\Omega) \to \mathcal{P}(D(0,3))$ by

$$T_{w,w'}f(z) = \frac{f(w+(w'-w)z)-f(w)}{f(w')-f(w)}\,.$$

Proof. We will first establish the corollary for $\mathcal{S}_{\mathcal{P}}$. Thus we take $\Omega = D(0,3)$. We define, for $z \in D(0,3)$,

$$M_z = \sup\{|f(z)|,\ f \in \mathcal{S}_{\mathcal{P}}\} \in [0,\infty]\,.$$

We will prove that $M_z$ is finite for all $z$. Let $\mathcal{B}$ be the set of points $z \in D(0,3)$ such that $M_z$ is finite, and $\mathcal{B}'$ those points of $\mathcal{B}$ such that there is another point $w' \in \mathcal{B}$ such that $|w-w'| \leq \delta_{D(0,3)}(w)/3$.

By assumption, for any $\varepsilon > 0$, there is some $\eta_\varepsilon \in (0,1)$ such that $M_z \leq \varepsilon$ whenever $z \in D(0,\eta_\varepsilon)$. We let $\eta = \eta_1$.

It follows that if $w \in \mathcal{B}'$ and $w' \in \mathcal{B}$ are as above, then $|T_{w,w'}f(z)| \leq \varepsilon$ when $|z| \leq \eta_\varepsilon$ for any $f \in \mathcal{S}_{\mathcal{P}}$. Therefore, if $z \in D(w, |w-w'|\eta_\varepsilon)$, then, for any $f \in \mathcal{S}_{\mathcal{P}}$,

$$|f(z)-f(w)| \leq |f(w)-f(w')|\varepsilon \leq (M_w + M_{w'})\varepsilon\,.$$

Thus $\mathcal{S}_{\mathcal{P}}$ is equicontinuous at every point of $\mathcal{B}'$ and $\mathcal{B}'$ is an open set (it contains $D(w,|w-w'|\eta)$).

Let us now prove that $\mathcal{B}'$ is closed in $D(0,3)$. Let $(w_n)_n$ be a sequence in $\mathcal{B}'$ which converges to $w \in D(0,3)$. If $n$ is large enough, then $\delta_{D(0,3)}(w_n) \geq \delta_{D(0,3)}(w)/2$ and $w \in D(w_n, \eta\delta_{D(0,3)}(w_n)/3)$. Hence $w \in \mathcal{B}$. Taking $n$ even larger implies that $w_n \in D(w, \delta_{D(0,3)}(w)/3)$ so that $w \in \mathcal{B}'$ (we may take such a $w_n$ as second point).

Finally, we note that $D(0,\eta) \subset \mathcal{B}'$ so that $\mathcal{B}' = D(0,3)$ since it is a non-empty open and closed set. It follows that $\mathcal{S}_{\mathcal{P}}$ is equicontinuous at every point, so uniformly equicontinuous on every compact subset of $D(0,3)$.

Ascoli's theorem implies that $\mathcal{S}_{\mathcal{P}}$ is a normal family. Moreover, from (1) and (2), any limit satisfies $\mathcal{P}$ and is not constant. Since it is a limit of injective functions, the limit is also injective, so it belongs to $\mathcal{S}_{\mathcal{P}}$, and $\mathcal{S}_{\mathcal{P}}$ is compact.

Let us now consider sequences $(f_n)_n$ of $\mathcal{S}_{\mathcal{P}}$ and $(\alpha_n)_n$ of $Aut(\mathbb{C})$ such that $\alpha_n(0) \in D(0,3)$ and $|\alpha_n'(0)| \leq \delta_{D(0,3)}(\alpha_n(0))/3$. By compactness of $\mathcal{S}_{\mathcal{P}}$, the sequence $(T_{0,1}(f_n \circ \alpha_n))$ has a convergent subsequence with an injective limit. It follows that $\mathcal{S}_{\mathcal{P}}$ satisfies the assumptions of Proposition 1.1. Therefore there is some $K$ such that every element of $\mathcal{S}_{\mathcal{P}}$ is $K$-quasiconformal.

Let us fix a domain $\Omega$, and let $f \in \mathcal{P}(\Omega)$. If $D$ is a disc on which $f$ is injective, then one can find $\alpha \in Aut(\mathbb{C})$ such that $\alpha(D(0,3)) = D$. It follows that $T_{0,1}(f \circ \alpha) \in \mathcal{S}_{\mathcal{P}}$ and $f|_D$ is $K$-quasiconformal. Thus, maps in $\mathcal{P}(\Omega)$ are locally $K$-quasiconformal. ■

## 2. Skewed maps

**Definition 1** ($\varepsilon$-skewed and $(\varepsilon,\rho)$-skewed map)**.** *Let $\varepsilon>0$ and $\rho\in[1/2,1)$. We say that a map $f:\Omega\to\mathbb{C}$ is* $\varepsilon$-skewed *if $f$ is continuous and* $\mathrm{skew}(f(T))\le 1+\varepsilon$ *for every equilateral triangle $T\subset\Omega$. We say that $f$ is $(\varepsilon,\rho)$-skewed if furthermore, for any $z,w\in\Omega$ with $|w-z|\le(1/3)\min\{\delta_\Omega(z),\delta_\Omega(w)\}$,*

$$\left|f\left(\frac{z+w}{2}\right)-f(z)\right|\le\rho|f(z)-f(w)|\,.$$

The proof of the theorem will follow from the next three propositions.

**Proposition 2.1.** *Let $\varepsilon>0$ and $\rho\in[1/2,1)$. The family of $(\varepsilon,\rho)$-skewed maps on $D(0,3)$ which fix $0$ and $1$ is equicontinuous at the origin.*

**Proposition 2.2.** *There are $\varepsilon_0>0$ and a function $\rho:[0,\varepsilon_0)\to\left[\frac12,1\right)$ such that if $0<\varepsilon<\varepsilon_0$ then any injective $\varepsilon$-skewed map is $(\varepsilon,\rho(\varepsilon))$-skewed.*

**Proposition 2.3.** *Both skewedness properties are conformal properties.*

Using the three propositions above, we are able to prove our main theorem.

Proof of the theorem. First, both skewedness properties are conformal properties by Proposition 2.3. Therefore, it follows from Corollary 1.2 that it is enough to prove the equicontinuity at the origin. By Proposition 2.2, an $\varepsilon$-skewed map is $(\varepsilon,\rho)$-skewed for some $\rho\in[1/2,1)$, so Proposition 2.1 implies the equicontinuity at the origin, and hence, by Corollary 1.2, the quasiconformality of injective $\varepsilon$-skewed maps. ■

The proof of the propositions will follow after we establish several lemmas. Let us begin by giving a definition which will be convenient for our purposes.

**Definition 2** ($n$-connectivity)**.** *Let $z_1,z_2,w_1,w_2\in\mathbb{C}$ such that $z_1\ne z_2$, $w_1\ne w_2$ and $|z_1-z_2|=|w_1-w_2|$. We say that the segments $[z_1,z_2]$ and $[w_1,w_2]$ are* $n$-connected *if there exist $n$ equilateral triangles $T_1,\dots,T_n$ of sidelength $|z_1-z_2|$ such that $[z_1,z_2]\subset T_1$, $[w_1,w_2]\subset T_n$, and $T_j$ and $T_{j+1}$ have exactly one edge in common, for all $j=1,\dots,n-1$.*

We state the following observation as a separate lemma, since we will make use of it later.

**Lemma 2.4.** *Let $\varepsilon>0$ and let $f:\mathbb{C}\to\mathbb{C}$ be an $\varepsilon$-skewed map. If two segments $[z_1,z_2]$ and $[w_1,w_2]$ are $n$-connected, for some $n\in\mathbb{N}$, then*

$$\frac{1}{(1+\varepsilon)^n}|f(z_1)-f(z_2)|\le|f(w_1)-f(w_2)|\le(1+\varepsilon)^n|f(z_1)-f(z_2)|\,.$$

We start with Proposition 2.3.

Proof of Proposition 2.3. The fact that a non-constant $\varepsilon$-skewed or $(\varepsilon,\rho)$-skewed map is locally injective is the only non-trivial property to establish.

We let $\Omega$ be a domain, and we consider two distinct points $z,w\in\Omega$. Let $\Lambda_{z,w}$ be the set of points $z'\in\Omega$ for which there is $w'\in\Omega$ such that the segments $[z,w]$ and $[z',w']$ are connected, in the sense of Definition 2, by a chain of triangles whose vertices all lie in $\Omega$.

First, if $f$ is $\varepsilon$-skewed and $f(z)=f(w)$ for some distinct points, then $f$ is constant on $\Lambda_{z,w}$. If $f$ is not locally injective at a point $\zeta\in\Omega$, then one can find a sequence $(z_n,w_n)_n$ of distinct points which converge to $\zeta$ such that $f(z_n)=f(w_n)$. Thus, $f$ is constant on each $\Lambda_n=\Lambda_{z_n,w_n}$. As $n$ tends to infinity, $\Lambda_n$ tends in the Hausdorff topology to $\overline{\Omega}$; furthermore, since $(z_n)$ tends to $\zeta$, it follows from the continuity of $f$ that $f$ turns out to be constant equal to $f(\zeta)$. Thus, a non-constant $\varepsilon$-skewed or $(\varepsilon,\rho)$-skewed map is locally injective.

Actually, if $f$ is $(\varepsilon, \rho)$-skewed in some disc $D(0,R)$ and is not constant, then $f$ is injective on $D(0,R/2)$. ■

**Lemma 2.5.** *Let $\varepsilon > 0$ and $\rho \in [1/2, 1)$. For every $(\varepsilon, \rho)$-skewed map $f : D(0,3) \to \mathbb{C}$,*

$$\operatorname{diam} f([z,w]) \leq \frac{|f(z) - f(w)|}{1-\rho}$$

*for all $z, w \in D(0,3)$ with $|w - z| \leq (1/3)\min\{\delta_{D(0,3)}(z), \delta_{D(0,3)}(w)\}$.*

Proof. It is enough to show the result for $z = 0$ and $w = 1$. Let $f : D(0,3) \to \mathbb{C}$ be an $(\varepsilon, \rho)$-skewed map. Consider, for $x \in [0,1]$, the binary expansion of $x$. Let $x_n$ be the truncation of the expansion of $x$ after the $n-th$ digit. Then, for all $n \in \mathbb{N}$,

$$|f(x) - f(0)| \ \leq |f(x) - f(x_n)| + \sum_{0 \leq k \leq n-1} |f(x_k) - f(x_{k+1})|\,.$$

Observe that $|f(x) - f(x_n)| \to 0$ as $n \to \infty$ and that $|f(x_k) - f(x_{k+1})| \leq \rho^k |f(0) - f(1)|$. Therefore

$$|f(x) - f(0)| \leq \frac{|f(0) - f(1)|}{1-\rho}\,.$$

■

**Lemma 2.6.** *Let $\varepsilon > 0$ and $\rho \in [1/2, 1)$. For every $(\varepsilon, \rho)$-skewed map $f : D(0,3) \to \mathbb{C}$, if $T$ is an equilateral triangle with vertices $z_1, z_2, z_3 \in D(0,3)$ such that $|z_i - z_j| \leq (1/3)\min\{\delta_{D(0,3)}(z_i), \delta_{D(0,3)}(z_j)\}$ whenever $i, j \in \{1,2,3\}$ with $i \neq j$, then*

$$\operatorname{diam} f(T) \leq \frac{2(1+\varepsilon)}{1-\rho}|f(z_1) - f(z_2)|\,.$$

Proof. Let $f : D(0,3) \to \mathbb{C}$ be an $(\varepsilon, \rho)$-skewed map.

We may assume that $f$ is not constant. Since $f$ is a continuous open mapping, the diameter of $f(T)$ is given by the diameter of $f(\partial T)$.

Let $z, w \in \partial T$. Perhaps after relabeling the vertices of $T$ we may assume that $z \in [z_1, z_2]$ and $w \in [z_1, z_j]$, for some $j = 2, 3$. Then, by Lemma 2.5,

$$\begin{aligned}|f(z) - f(w)| &\leq |f(z_1) - f(z)| + |f(z_1) - f(w)| \leq \frac{1}{1-\rho}(|f(z_1) - f(z_2)| + |f(z_1) - f(z_j)|) \\ &\leq \frac{2}{1-\rho} \max_{k \in \{2,3\}} |f(z_1) - f(z_k)| \leq \frac{2(1+\varepsilon)}{1-\rho}|f(z_1) - f(z_2)|\,.\end{aligned}$$

■

Proof of Proposition 2.1. Let $f : D(0,3) \to \mathbb{C}$ be an $(\varepsilon, \rho)$-skewed map, normalised so that $f(0) = 0$ and $f(1) = 1$. Given $n \in \mathbb{N}$, let $H_n$ be the regular hexagon centred at 0 and with vertex $\frac{1}{2^n}$. Divide $H_n$ into six equilateral triangles, each of sidelength $\frac{1}{2^n}$, so that the vertices of each such triangle are 0 and two consecutive vertices of $H_n$. Let $\mathcal{T}_n$ be the set of these six triangles so obtained. We note that each edge of a triangle in $\mathcal{T}_n$ is at most 3-connected to $[0, 1/2^n]$.

Then, by Lemma 2.6,

$$\begin{aligned}
\mathrm{diam} f(H_n) &\leq 2 \sup_{T\in\mathcal{T}_n} \mathrm{diam} f(T) \\
&\leq \frac{4(1+\varepsilon)^4}{1-\rho}|f(0)-f(1/2^n)| \\
&\leq \frac{4(1+\varepsilon)^4}{1-\rho}\rho^n\,.
\end{aligned}$$

This proves the equicontinuity at the origin. ■

We now turn to the proof of Proposition 2.2. Let us start with the following lemma.

**Lemma 2.7.** *There is some $\varepsilon_1 > 0$ which satisfies the following property. For all $0 < \varepsilon < \varepsilon_1$ there exists $c_\varepsilon > 0$ such that for every $\varepsilon$-skewed map $f : \Omega \to \mathbb{C}$,*

$$\left| f\left(z + (w-z)e^{i\pi/3}\right) - f\left(z + (w-z)e^{-i\pi/3}\right)\right| \geq \sqrt{3}(1-c_\varepsilon)|f(w)-f(z)|,$$

*for all $z \in \Omega$ and $w \in D(z, \delta_\Omega(z))$ such that $f(z) \neq f(w)$. Moreover, $c_\varepsilon$ can be chosen so that $c_\varepsilon \to 0$ as $\varepsilon \to 0$.*

Proof. Let $\varepsilon > 0$ be given. Without loss of generality, we may assume that $\overline{D(0,1)} \subset \Omega$, $z = 0$, $w = 1$, $f(0) = 0$ and $f(1) = 1$. In this case, $z + (w-z)e^{i\pi/3} = e^{i\pi/3}$ and $z + (w-z)e^{-i\pi/3} = e^{-i\pi/3}$. Let us denote the complex number $e^{i\pi/3}$ by $\alpha$.

Since the triangles with vertices $(0, 1, \alpha)$ and $(0, 1, \overline{\alpha})$ are both equilateral, we have that $\mathrm{skew}(0, 1, f(\alpha)) \leq \lambda$ and $\mathrm{skew}(0, 1, f(\overline{\alpha})) \leq \lambda$. Thus, there exists $r_\varepsilon > 0$ such that

$$f(\alpha), f(\overline{\alpha}) \in D(\alpha, r_\varepsilon) \cup D(\overline{\alpha}, r_\varepsilon)$$

and $r_\varepsilon \to 0$ as $\varepsilon \to 0$. To conclude the proof, we only need to prove that each image is in a different disc.

Note that if $\varepsilon > 0$ is small enough then $\varepsilon$-skewed maps transform the vertices of equilateral triangles into three non-aligned points. Therefore, if $\{z_1(t), z_2(t), z_3(t)\}$ is a continuous motion of equilateral triangles in $\Omega$, where $t$ ranges over some interval $I$, then the vectors $[(f(z_2(t)) - f(z_1(t))), (f(z_3(t)) - f(z_1(t)))]$, $t \in I$, form a continuous family of bases of $\mathbb{C}$ over $\mathbb{R}$ with a fixed orientation. Applying this remark to $z_1(t) = 0$, $z_2(t) = e^{-it\pi/3}$ and $z_3(t) = e^{i(1-t)\pi/3}$ for $t \in [0, 1]$, we see that both bases $[1, f(\alpha)]$ and $[f(\overline{\alpha}), 1]$ have the same orientation, so the points $f(\alpha)$ and $f(\overline{\alpha})$ belong to two opposite discs. ■

Let us now state the following elementary lemma, omitting its proof.

**Lemma 2.8.** *Let $\lambda > 1$. The region $E_\lambda = \{z \in \mathbb{C} \mid \frac{1}{\lambda}|z| \leq |z-1| \leq \lambda|z|\}$ is the complement of the union of two disjoint open discs in $\mathbb{C}$, with centers $\frac{\lambda^2}{\lambda^2-1}$ and $\frac{1}{1-\lambda^2}$, respectively, and with radii both equal to $\frac{\lambda}{\lambda^2-1}$.*

If $w, w' \in \mathbb{C}$, we define $E_\lambda(w, w') = \phi(E_\lambda)$ where $\phi(z) = (w - w')z + w'$.

**Lemma 2.9.** *There is some $\varepsilon_2 > 0$ which satisfies the following property. For all $0 < \varepsilon < \varepsilon_2$ there exists $s_\varepsilon > 0$ such that for every $\varepsilon$-skewed map $f : \Omega \to \mathbb{C}$,*

$$f\left(\frac{z+w}{2}\right) \in D\left(\frac{f(z)+f(w)}{2}, s_\varepsilon|f(z)-f(w)|\right),$$

*for all $z, w \in \Omega$ such that $|w - z| \leq (1/3)\min\{\delta_\Omega(z), \delta_\Omega(w)\}$ and $f(z) \neq f(w)$. Moreover, $s_\varepsilon$ can be chosen so that $s_\varepsilon \to 0$ as $\varepsilon \to 0$.*

PROOF. It suffices to show the result for $z = 0$ and $w = 1$. Let $\varepsilon > 0$ and let $f : \Omega \to \mathbb{C}$ be an $\varepsilon$-skewed map, normalised so that $f(0) = 0$ and $f(1) = 1$. Set $\lambda = 1 + \varepsilon$ and $\alpha = e^{\frac{i\pi}{3}}$.

First, the segments $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ are 3-connected, as is shown by considering the three triangles with vertices $\frac{1}{2}(0, 1, \alpha)$, $\frac{1}{2}(1, \alpha, 1 + \alpha)$ and $\frac{1}{2}(1, 1 + \alpha, 2)$, respectively. Therefore, $f\left(\frac{1}{2}\right) \in E_{\lambda^3}(0, 1)$, by Lemma 2.4. By a similar argument, the segments $\left[\alpha, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, \overline{\alpha}\right]$ are 3-connected and thus $f\left(\frac{1}{2}\right) \in E_{\lambda^3}(f(\alpha), f(\overline{\alpha}))$.

Now observe that the region $E_\lambda$ is a neighbourhood of $\{\Re z = 1/2\}$, and as $\varepsilon$ tends to 0, $E_\lambda$ tends to this line on any compact set in the Hausdorff topology. Therefore, for $\varepsilon$ small enough, the region $E_{\lambda^3}(0, 1) \cap E_{\lambda^3}(\alpha, \overline{\alpha})$ is the union of two components: one compact, which contains $1/2$ and the diameter of which tends to zero as $\varepsilon$ tends to zero, and the second is unbounded, but its diameter in the spherical metric tends to 0 as well with $\varepsilon$.

Thus, there exist $s'_\varepsilon, R'_\varepsilon > 0$ such that

$$E_{\lambda^3}(0, 1) \cap E_{\lambda^3}(\alpha, \overline{\alpha}) \subset D\left(\frac{1}{2}, s'_\varepsilon\right) \cup \{z \in \mathbb{C}, |z| \geq R'_\varepsilon\}$$

where $s'_\varepsilon \to 0$ and $R'_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Lemma 2.7 and its proof imply that we may as well assume, replacing $f$ by $\overline{f}$, if necessary, that $f(\alpha)$ (resp. $f(\overline{\alpha})$) lies in $D(\alpha, r_\varepsilon)$ (resp. in $D(\overline{\alpha}, r_\varepsilon)$) where $r_\varepsilon$ tends to 0 with $\varepsilon$. It follows that there exist $s_\varepsilon > 0$ and $R_\varepsilon$, depending only on $s'_\varepsilon$, $R'_\varepsilon$ and $r_\varepsilon$, and therefore depending only on $\varepsilon$, such that

$$E_{\lambda^3}(0, 1) \cap E_{\lambda^3}(f(\alpha), f(\overline{\alpha})) \subset D\left(\frac{1}{2}, s_\varepsilon\right) \cup \{z \in \mathbb{C}, |z| \geq R_\varepsilon\}$$

where $s_\varepsilon \to 0$ and $R_\varepsilon \to \infty$ as $\varepsilon \to 0$.

To conclude, we need to prove that $f(1/2)$ belongs to the bounded component of $E_{\lambda^3}(0, 1) \cap E_{\lambda^3}(f(\alpha), f(\overline{\alpha}))$. Let $\beta = (1/2)e^{i\pi/3}$ and $\gamma = (1/2)(1 + e^{i\pi/3})$. Note that the triangles $\{0, 1/2, \beta\}$ and $\{\gamma, 1/2, \beta\}$ are both equilateral and have one edge in common. Recalling that $f(0) = 0$ and $f(1) = 1$ and applying Lemma 2.7 for $z = \beta$ and $w = 1/2$, one gets that that

$$|f(\gamma)| \geq \sqrt{3}(1 - c_\varepsilon)|f(\beta) - f(1/2)|$$

and thus

$$|f(\gamma)| \geq \frac{\sqrt{3}(1 - c_\varepsilon)}{\lambda^2}|1 - f(1/2)|$$

since the segments $[\beta, 1/2]$ and $[1/2, 1]$ are 2-connected. Also,

$$|f(\gamma)| \leq 1 + |f(\gamma) - 1| \leq 1 + \lambda|1 - f(1/2)|$$

Combining these two inequalities we get that

$$|1 - f(1/2)| \leq B_\varepsilon = \frac{\lambda^2}{\sqrt{3}(1 - c_\varepsilon) - \lambda^3}$$

which remains bounded as $\varepsilon \to 0$ since this quantity tends to $(\sqrt{3} - 1)^{-1}$ as $\varepsilon$ tends to 0.

Therefore, for $\varepsilon$ small enough, one has that $B_\varepsilon < R_\varepsilon$ and thus $f\left(\frac{1}{2}\right) \in D\left(\frac{1}{2}, s_\varepsilon\right)$, as desired. ∎

PROOF OF PROPOSITION 2.2. Let $\varepsilon_0$ be such that $s_{\varepsilon_0} < \frac{1}{4}$, where $s_{\varepsilon_0}$ is the constant given by Lemma 2.9 for $\varepsilon_0$. Let $\varepsilon < \varepsilon_0$ and let $f$ be an injective $\varepsilon$-skewed map defined on a domain $\Omega$. Pick $z, w \in \Omega$ such that $|w - z| \leq (1/3)\min\{\delta_\Omega(z), \delta_\Omega(w)\}$.

Using the transformation $T_{z,w}f$, we may as well assume that $z = 0$, $w = 1$ and that $f$ is normalised so that $f(0) = 0$ and $f(1) = 1$. Note that $f\left(\frac{1}{2}\right) \in D\left(\frac{1}{2}, s_\varepsilon\right)$, by Lemma 2.9. The circle with centre $\frac{1}{2}$ and radius $s_\varepsilon$ intersects the segment $\left[\frac{1}{2}, 1\right]$ at a point $u_\varepsilon$. Let $\rho(\varepsilon) = u_\varepsilon$, noting that $\frac{1}{2} < \rho(\varepsilon) < 1$. Then $\left|f(0) - f\left(\frac{1}{2}\right)\right| \leq |f(0) - u_\varepsilon| = |u_\varepsilon| = \rho(\varepsilon)|f(0) - f(1)|$. The proposition follows. ∎

## Appendix A. Control of nearly equilateral triangles

In this appendix, we prove the following.

**Proposition A.1.** *Let $\mu > 0$ and $\lambda < \infty$. If $f : \Omega \to f(\Omega)$ is a homeomorphism such that*

$$\mathrm{skew}(f(T)) \leq \lambda$$

*for any triangle $T \subset \Omega$ with* $\mathrm{skew}(T) \leq 1 + \mu$*, then $f$ is $K$-quasiconformal for some $K = K(\lambda, \mu)$.*

Proof. Fix $z_0 \in \Omega$ and $r \in (0, \delta_\Omega(z_0))$. We claim that $H_f(z_0, r) \leq K$ for some constant $K$ which depends only on $\lambda$ and $\mu$. This will establish the proposition.

Without loss of generality, we may assume that $z_0 = 0$ and $r = 1$. There is some $\eta > 0$ of the form $\eta = \pi/(3n)$, $n \geq 1$, such that $\mathrm{skew}(0, 1, e^{it}) \leq 1 + \mu$ for all for $t \in [\pi/3 - \eta, \pi/3 + \eta]$.

Let us subdivide $[0, \pi/3]$ into $n$ intervals $I_j = [(j-1)\eta, j\eta]$ of length $\eta$, and similarly $[\pi/3, 2\pi/3]$ by $n$ intervals $J_j = [\pi/3 + (j-1)\eta, \pi/3 + j\eta]$ for $j \in \{1, \ldots, n\}$. Let us write, for $j \in \{1, \ldots, n\}$ and $k \in \{0, 1\}$,

$$z_j^k = e^{i(k\pi/3 + (j-1)\eta)} .$$

It follows that the skew of triangles with vertices 0, $z_j^0$ and $e^{it}$ for $t \in J_j$ and of triangles with vertices 0, $z_{j+1}^1$ and $e^{it}$ for $t \in I_j$ is bounded by $1 + \mu$. Therefore,

$$\lambda^{-2n}|f(0) - f(1)| \leq |f(0) - f(e^{it})| \leq \lambda^{2n}|f(0) - f(1)|$$

for $t \in [0, 2\pi/3]$.

Finally, any segment of the form $[0, e^{is}]$ is at most 2-connected to a segment of the form $[0, e^{it}]$ with $e^{it}$ in some $I_j$ or $J_j$, and thus the claim follows. ■

## References

[1] L.V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, 1966.

[2] A. Beurling & L.V. Ahlfors, The boundary correspondence under quasiconformal mappings, *Acta Math.* **96** (1956), 125–142.

[3] F.W. Gehring, The definitions and exceptional sets for quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. A I* **No. 281** (1960), 28 pp.

[4] F.W. Gehring, Rings and quasiconformal mappings in space, *Trans. Amer. Math. Soc.* **103** (1962), 353–393.

[5] J.H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1. Teichmüller theory*, with contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra. With forewords by William Thurston and Clifford Earle. Matrix Editions, Ithaca, NY, 2006.

[6] O. Lehto & K.I. Virtanen, *Quasiconformal mappings in the plane*, Second edition. Translated from the German by K. W. Lucas. Die Grundlehren der mathematischen Wissenschaften, Band 126. Springer-Verlag, New York-Heidelberg, 1973.

[7] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin-New York, 1971.

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