COMBINATORIAL SUPERRIGIDITY FOR GRAPHS ASSOCIATED TO SURFACES

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Abstract. We introduce a number of conditions on graphs built from arcs and/or curves on a topological surface, which informally correspond to saying that the given graph behaves like the pants graph. We prove that, under these conditions, every injective and *alternating* simplicial map between two of these graphs is induced by a subsurface inclusion between the underlying surfaces.

We then explain why pants graphs satisfy these conditions, thus recovering the superrigidity result obtained in [1]. We also describe a more sophisticated version of the conditions that gives a similar result for flip graphs, as obtained in [2]. Finally, we explain why Hatcher-Thurston graphs and curve graphs do not satisfy the conditions, and explain a more general rigidity phenomenon for Hatcher-Thurston graphs, recently obtained by J. Hernández [7].

1. Introduction

A large number of problems in the study of Teichmüller spaces and mapping class groups may be understood in combinatorial terms, through the various graphs of arcs and/or curves that one can associate to a surface. Prominent examples of such graphs include the *curve graph*, the *arc graph*, the *pants graph*, etc.

A common feature of these graphs is that they are *simplicially rigid*: every automorphism of the graph is induced by an element of the mapping class group of the underlying surface. In fact, Brendle-Margalit [3] have recently proved that this is the case for any complex of curves that satisfies certain general conditions.

Expanding on these rigidity results, it was shown in [1] that pants graphs are *superrigid*; see Section 4 for a definition of these graphs. More concretely, if S has complexity at least 2, then every injective simplicial map $\mathcal{P}(S) \to \mathcal{P}(S')$ between pants graphs is induced by a subsurface inclusion $S \hookrightarrow S'$.

Using a similar array of ideas, although in a significantly more complicated setting, the analogous result holds for *flip graphs* of surfaces (see again Section 4): if S is "complicated enough", then every injective simplicial map between flip graphs is induced by a subsurface inclusion [2].

The purpose of this note is to "abstract out" the ideas behind the proof of the main result in [1], in terms of rather general conditions on the class of graphs associated to the surface, with the hope that these ideas may be applicable to other classes of graphs.

Rather informally, suppose that to each connected orientable surface S we have associated a simplicial graph $\mathcal{G}(S)$, built from arcs and/or curves on S, which satisfies certain axiomatic conditions that make it "pants-like": these are the conditions (G1) – (G7)

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described in the next section. In this setting, we will prove that a certain class of injective maps between two of these graphs are always induced by an inclusion between the underlying surfaces:

Theorem 1.1. *Suppose the class of graphs $\mathcal{G}(\cdot)$ satisfies the conditions (G1) – (G7) introduced in Section 2. Suppose S and S' are connected orientable surfaces for which there exists an injective alternating map $\mathcal{G}(S) \to \mathcal{G}(S')$. Then ϕ is induced by a subsurface inclusion $S \hookrightarrow S'$.*

Remark 1.2. In recent work, Erlandsson-Fanoni [4] have proved that every injective simplicial map between two "multicurve graphs" is induced by a subsurface inclusion, subject to certain conditions on the topology of the underlying surfaces and the number of curves defining a vertex of the graph. As the authors point out, it is not true that an *arbitrary* injection between multicurve graphs arises in this way. In particular, the graphs considered in [4] do not satisfy the axioms given in Section 2, more concretely axiom (G7); see [4] for details.

The plan of the paper is as follows. In Section 2 we will introduce all the necessary background and terminology, as well as the conditions (G1) – (G7) mentioned in Theorem 1.1. In Section 3 we will give a proof of Theorem 1.1. Finally, in Section 4, we will explore a number of known classes of graphs (e.g. curve graphs, pants graphs, etc), explaining in each case whether they (do not) satisfy the conditions (G1) – (G7) of Theorem 1.1.

2. Graphs

2.1. **Arcs and curves.** Throughout, S will denote a connected, orientable surface of finite topological type, and P a (possibly empty) set of marked points on S. By a *curve* on S we mean a free isotopy class of essential simple closed curves on S; here, a simple closed curve is said to be *essential* if it does not bound a disk with at most one marked point. By an *arc* we mean an isotopy class, relative endpoints, of simple arcs on S with both endpoints on P, and whose interior is disjoint from P.

2.2. **A class of graphs.** We now proceed to describe the class of graphs that we will consider. To each orientable surface S we will associate an abstract simplicial graph $\mathcal{G}(S)$ built from arcs and/or curves on S, and which satisfies the conditions (G1) – (G7) below. The first one describes what is the vertex set of $\mathcal{G}(S)$:

(G1) There exists a positive integer $d = d(\mathcal{G}(S))$ such that each vertex of $\mathcal{G}(S)$ is a set of d arcs and/or curves on S. In other words, every vertex v of $\mathcal{G}(S)$ has the form

$$\{a_1, \dots, a_d\},$$

where each a_i is either an arc or a curve on S.

The next condition informally asserts that edges of $\mathcal{G}(S)$ correspond to "flipping" an arc or a curve on S:

(G2) For every vertices $u, v \in \mathcal{G}(S)$ with $v \in \operatorname{lk}(u)$, the cardinality of $u \cap v$ is $d - 1$.

Also, we want to consider connected graphs only:

(G3) The graph $\mathcal{G}(S)$ is connected.

Roughly speaking, the next condition ensures that one can flip every element of every vertex of $\mathcal{G}(S)$:

(G4) For every vertex $v \in \mathcal{G}(S)$ and all $a \in v$, there exists $v' \in \mathrm{lk}(v)$ with

$$v \setminus (v \cap v') = \{a\}.$$

Here, $\mathrm{lk}(v)$ denotes the *link* of the vertex v, that is the set of vertices adjacent to it. Before we state the next condition, we need the following definition:

Definition 2.1 (Extendable set)**.** Let M be a non-empty finite set of arcs and/or curves on S. We say that M is *extendable* if there exists a vertex v of $\mathcal{G}(S)$ with $M \subsetneq v$. We say that M has *deficiency* $k \geq 1$ if there exists an extendable set M' on S such that M' has k elements and $M \cup M'$ is a vertex of $\mathcal{G}(S)$.

Remark 2.2. Observe that if an extendable set on S has deficiency k, then it has $d(\mathcal{G}(S)) - k$ elements. Observe also that if u, u', v are vertices with $u, u' \in \mathrm{lk}(v)$ then

$$u \cap u' = u \cap u' \cap v$$

is either empty or else is an extendable set, in which case it has deficiency 1 or 2.

The following condition asserts that the graphs $\mathcal{G}(S)$ behave well with respect to considering subsurfaces.

(G5) If S is an essential subsurface of S', then $\mathcal{G}(S) \subset \mathcal{G}(S')$. Moreover, if M is an extendable set on S then

$$\mathcal{G}_M(S) \cong \mathcal{G}(S - M),$$

where $\mathcal{G}_M(S)$ denotes the subgraph of $\mathcal{G}(S)$ spanned by those vertices of $\mathcal{G}(S)$ that contain M.

The next property provides a "base case" for the superrigidity of the graphs $\mathcal{G}(S)$:

(G6) Let S, S' be surfaces, with S connected and $d(\mathcal{G}(S)) = d(\mathcal{G}(S'))$. If there exists an injective simplicial map

$$\mathcal{G}(S) \to \mathcal{G}(S')$$

then S' is the disjoint union of connected surfaces $S'_1, \dots, S'_k$, in such way that:

(a) $\mathcal{G}(S_i) = \emptyset$ for $i = 2, \dots, k$, and

(b) The restricted (injective) map $\mathcal{G}(S) \to \mathcal{G}(S'_1)$ is induced by a homeomorphism $S \to S'_1$.

We now proceed to state our final condition, which guarantees the existence of certain special closed paths in $\mathcal{G}(S)$. Before doing so we need the following definition, which is a direct translation to our setting of the concept of "alternating tuple" for the pants graph from [1].

Definition 2.3 (Alternating circuit)**.** An alternating circuit τ in $\mathcal{G}(S)$ consists of $n \in \{4, 5\}$ vertices $v_1, \dots, v_n$, and paths $\gamma_1, \dots, \gamma_n$ in $\mathcal{G}(S)$ such that (counting indices mod n):

(1) γ_i is a path from v_i to v_{i+1}.
(2) $\gamma_i \cap \gamma_j = \emptyset$ if $|i - j| > 1$, and $\gamma_i \cap \gamma_{i+1} = \{v_{i+1}\}$.
(3) The set $\cap\{u \mid u \in \gamma_i\}$ is an extendable set of deficiency 1.
(4) If $u \in p_i - \{v_i + 1\}$ and $v \in p_{i+1} - \{v_i + 1\}$ then $u \cap v$ is an extendable set of deficiency 2.

Informally, exactly $d-1$ arcs/curves do not change along γ_i , and it is one of these arcs/curves that is "flipped" when passing from γ_i to γ_{i+1}. We remark that the existence of an alternating circuit in $\mathcal{G}(S)$ immediately implies that $d(\mathcal{G}(S)) \geq 2$. The fact that $n \leq 5$ in the definition above has the following observation as an immediate consequence; we state it as a separate lemma as it will be useful in the proof of Theorem 1.1:

Lemma 2.4. *Suppose $\tau \subset \mathcal{G}(S)$ is an alternating circuit. Then*

$$M = \cap\{u \mid u \in \tau\}$$

is either empty or else is an extendable set of deficiency 2.

Armed with the definition above, the last condition is:

(G7) Let u, v, w be vertices with $u, w \in \mathrm{lk}(v)$, and such that $u \cap v \cap w$ is an extendable set of deficiency 2. Then there exists an alternating circuit in $\mathcal{G}(S)$ containing u, v, w.

2.3. **Alternating maps.** Before closing this section, we define the notion of an *alternating map*, and observe that such maps send alternating circuits to alternating circuits.

Definition 2.5 (Alternating map)**.** We say that a map $\phi : \mathcal{G}(S) \to \mathcal{G}(S')$ is *alternating* if, for every u, v, w vertices of $\mathcal{G}(S)$ with $u, w \in \mathrm{lk}(v)$, we have:

$$u \cap v \cap w \text{ has deficiency } 2 \iff \phi(u) \cap \phi(v) \cap \phi(w) \text{ has deficiency } 2.$$

The following is an immediate consequence of the definitions of alternating map and alternating circuit:

Lemma 2.6. *Let $\phi : \mathcal{G}(S) \to \mathcal{G}(S')$ be an injective alternating map. For every alternating circuit $\tau \subset \mathcal{G}(S)$, the path $\phi(\tau) \subset \mathcal{G}(S')$ is an alternating circuit.*

Proof. Let $\tau \subset \mathcal{G}(S)$ be an alternating circuit. Making reference to the notation in the definition of alternating circuit above, let $v_1, \ldots, v_n$ be the vertices and $\gamma_1, \ldots, \gamma_n$ be the paths between them, with $n \in \{4, 5\}$. We consider the path $\phi(\tau) \subset \mathcal{G}(S')$ which, again with respect to the same notation, has vertices $\phi(v_1), \ldots, \phi(v_n)$ and paths $\phi(\gamma_1), \ldots, \phi(\gamma_n)$ between them.

We now verify that $\phi(\tau)$ satisfies conditions (1) – (4) in Definition 2.5. First, it is immediate that it satisfies (1), since ϕ is simplicial, and (2), as it is injective. For (3), let $w_1, \ldots, w_k$ be the vertices of γ_i, where $w_1 = v_i$ and $w_k = v_{i+1}$. Since ϕ is alternating, we have that $\phi(w_j) \cap \phi(w_{j+1}) \cap \phi(w_{j+2})$ has deficiency 1. Since the same is true for $\phi(w_{j+1}) \cap \phi(w_{j+2}) \cap \phi(w_{j+3})$, it follows that

$$\phi(w_j) \cap \phi(w_{j+1}) \cap \phi(w_{j+2}) \cap \phi(w_{j+3}).$$

Repeating this argument we obtain that $\phi(\tau)$ satisfies property (3). This, combined with the fact that ϕ is alternating, implies that it also satisfies (4), which finishes the proof. □

3. Proof of Theorem 1.1

In this section we will prove of Theorem 1.1. A large part of the argument needed is contained in the following lemma:

Lemma 3.1. *Suppose that the class of graphs $\mathcal{G}(\cdot)$ satisfies conditions (G1)–(G7) above, and that $d(\mathcal{G}(S)) \geq 2$. Let S, S' be surfaces for which there exists an alternating injective map*

$$\phi : \mathcal{G}(S) \to \mathcal{G}(S').$$

Then $d(\mathcal{G}(S)) \leq d(\mathcal{G}(S'))$. Moreover, if the inequality is strict then there exists an extendable set M on S', with $d(\mathcal{G}(S')) - d(\mathcal{G}(S))$ elements, such that $M \subset \phi(v)$ for every vertex v of $\mathcal{G}(S)$.

Proof. Fix an arbitrary vertex v of $\mathcal{G}(S)$, which we will use to identify the extendable set M of the statement. By (G1), we may write

$$v = \{a_1, \ldots, a_d\},$$

where $d = d(\mathcal{G}(S))$. From (G2) and (G4), we know that there exist $v_1, \ldots, v_d \in \mathrm{lk}(v)$ such that

$$v \setminus (v \cap v_i) = \{a_i\}$$

for all $i = 1, \ldots, d$. By construction, if $i \neq j$ then $v_i \cap v_j = v \cap v_i \cap v_j$ has deficiency 2, and therefore $\phi(v) \cap \phi(v_i) \cap \phi(v_j)$ also has deficiency 2 as ϕ is alternating. It follows that the intersection of k distinct $\phi(v_i)$'s is either empty or else is an extendable set of deficiency k. Since every vertex of $\mathcal{G}(S')$ has $d' = d(\mathcal{G}(S'))$ elements, we deduce that

$$d \leq d',$$

so the first part of the theorem holds. Suppose from now on that $d < d'$, which in turn implies that

$$M := \phi(v_1) \cap \ldots \cap \phi(v_d)$$

is is an extendable set of $d' - d$ elements.

Once we have identified a candidate extendable set M on S, we now claim if u is any vertex of $\mathcal{G}(S)$, then $M \subset \phi(w)$ for every vertex $w \in \mathrm{lk}(u)$. We first prove the claim in the special case when $u = v$. Let $w \in \mathrm{lk}(v)$, and $v_1, \ldots, v_d$ be the vertices identified above. Then there exists exactly one $i = 1, \ldots, d$ such that

$$v \cap v_i = v \cap w = \{a_i\}.$$

Since ϕ is alternating, then $\phi(v) \cap \phi(v_i) \cap \phi(w)$ has deficiency 1, and as such

$$M \subset \phi(v) \cap \phi(v_i) = \phi(v) \cap \phi(v_i) \cap \phi(w).$$

In particular, $M \subset \phi(w)$, as desired.

Consider now the general case $u \neq v$. As $\mathcal{G}(S)$ is connected, in the light of (G3), it suffices to establish the claim in the case when $u \in \mathrm{lk}(v)$. Let $w \in \mathrm{lk}(u)$; we want to show that $M \subset \phi(w)$. To this end, observe first that $u \cap v \cap w$ is an extendable set of deficiency 1 or 2. Suppose first that $u \cap v \cap w$ has deficiency 1. In this case, $\phi(u) \cap \phi(v) \cap \phi(w)$ has deficiency 1 as well, since ϕ is alternating, and therefore $M \subset \phi(w)$, as

$$M \subset \phi(u) \cap \phi(v) = \phi(u) \cap \phi(v) \cap \phi(w).$$

Therefore, it remains to consider the case when $u \cap v \cap w$ has deficiency 2, which in turn implies that $\phi(u) \cap \phi(v) \cap \phi(w)$ has deficiency 2 also, again since ϕ is alternating. By condition (G7) above, there exists an alternating circuit $\tau \subset \mathcal{G}(S)$ containing u, v, w. Using Lemma 2.6, the image path $\phi(\tau)$ is also an alternating circuit. Let z be the unique vertex of τ that is distinct from u and spans an edge with v, noting that $M \subset \phi(z)$ by the discussion in the paragraph above; in particular,

$$M \subset \phi(z) \cap \phi(v) \cap \phi(u).$$

As $\phi(z) \cap \phi(v) \cap \phi(u)$ has deficiency 2, Lemma 2.4 tells us that

$$\phi(z) \cap \phi(v) \cap \phi(u) = \phi(z) \cap \phi(v) \cap \phi(u) \cap \phi(w),$$

and thus $M \subset \phi(w)$, as desired.

□

Remark 3.2. Observe that, in fact, in the above lemma we have not made use of conditions (G5) and (G6), and hence the result holds in slightly more generality.

At this point, we are in a position to prove Theorem 1.1:

Proof of Theorem 1.1. Let S and S' be connected orientable surfaces for which there exists an injective alternating map

$$\phi : \mathcal{G}(S) \to \mathcal{G}(S').$$

Write $d = d(\mathcal{G}(S))$ and $d' = d(\mathcal{G}(S'))$. By Lemma 3.1, we know that

$$d \leq d'.$$

If $d = d'$, then (G6) implies

$$S' = S'_1 \sqcup \ldots \sqcup S'_k,$$

in such way that $\mathcal{G}(S_i) = \emptyset$ for $i = 2, \ldots, k$, and the restricted (injective) map $\mathcal{G}(S) \to \mathcal{G}(S'_1)$ is induced by a homeomorphism $S \to S'_1$. As a consequence, the map ϕ is induced by a subsurface embedding $S \hookrightarrow S'$, as we wanted to prove.

On the other hand, if $d < d'$ then Lemma 3.1 again implies that there exists an extendable set $M \subset S'$, with $d' - d$ elements, such that

$$\phi(\mathcal{G}(S)) \subset \mathcal{G}_M(S') \cong \mathcal{G}(S' - M),$$

where the above isomorphism is guaranteed by condition (G5). Now, $d(\mathcal{G}(S' - M)) = d$, and we conclude as above with $S' - M$ instead of S'. □

4. Examples and non-examples

In this section we will discuss certain well-known classes of graphs of arcs or curves on S and, in each case, we will explain why they (do not) satisfy the conditions (G1) – (G7) described above. To the best of our knowledge the only example of a class of graphs that satisfy such conditions is that of *pants graphs* of surfaces; see below. Thus we ask:

Problem 4.1. *Is there a natural class of graphs associated to a surface, other than pants graphs, that satisfies conditions (G1) – (G7) above?*

Next, we will examine the interesting case of the *flip graph*, as it does not satisfy conditions (G1) – (G7) but still forms a superrigid class of graphs, as shown in [2]. Although the ideas of the proof of this result are similar in spirit to those discussed here, the situation is significantly more involved, especially to due to the presence of vertices with non-isomorphic links.

Next, we discuss the case of *Hatcher-Thurston graphs* and explain why they do not satisfy conditions (G1) – (G7). We will comment on a more general classification, due to

Jesús Hernández [7], of the possible simplicial injections between two Hatcher-Thurston graphs in the case where the domain surface is closed.

Finally, we discuss the case of curve graphs, and explore some possible versions of superrigidity for them.

4.1. **Pants graphs.** Let S be a connected orientable surface. The *pants graph* $\mathcal{P}(S)$ is the simplicial graph whose vertices are pants decompositions on S, up to isotopy, and where two such decompositions are adjacent in $\mathcal{P}(S)$ if and only if they are related by an *elementary move*. Recall that the latter means that the two decompositions have all but one curves in common, and the remaining two curves either intersect once and fill a one-holed torus, or intersect twice and fill a four-holed sphere. See Figure 1.

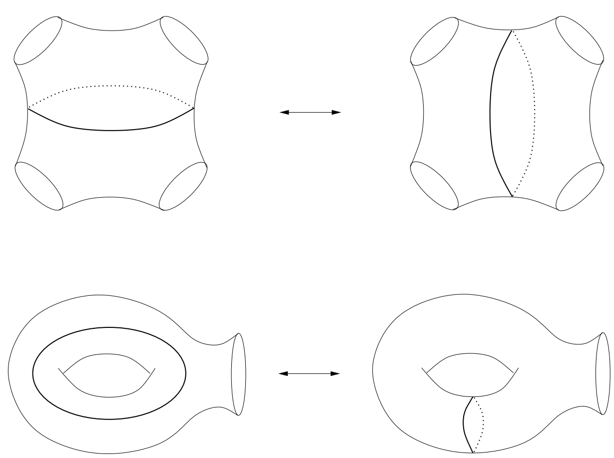

FIGURE 1. The two types of elementary move.

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We now explain why $\mathcal{P}(S)$ satisfies the conditions (G1) – (G7) above, provided S has complexity at least 2. In a nutshell, and rather informally, the reason boils down to the fact that $\mathcal{P}(S)$ is "built from" Farey graphs, which are simplicially rigid; moreover, there is a bijective correspondence between Farey graphs in $\mathcal{P}(S)$ and multicurves of cardinality one less than the complexity of S.

First, $\mathcal{P}(S)$ evidently satisfies (G1) and (G2), with

$$d = d(\mathcal{P}(S)) = 3g - 3 + p,$$

where g and p are, respectively, the genus and number of punctures of S. The fact that $\mathcal{P}(S)$ is connected is due originally to Hatcher-Thurston [5]; see also [8] for a recent and elementary proof. The fact that $\mathcal{P}(S)$ satisfies (G4) is also obvious: every curve in a pants decomposition may be the subject of an elementary move. Condition (G5) is easy as well: if S is an essential subsurface of S', then a pants decomposition P of S extends (in a non-unique way) to a pants decomposition P' of S' by first choosing a pants decomposition M of $S' - S$ and then setting $P' = P \cup M$.

As mentioned above, all the difficulty is reduced to proving that $\mathcal{P}(S)$ satisfies (G6) and (G7). We first treat condition (G6). Since $d(\mathcal{P}(S))$ is precisely the complexity of S, in this particular case condition (G6) asserts:

(G6) *Let S, S' be orientable surfaces of the same complexity ≥ 2, with S connected, and suppose there is an injective simplicial map $\mathcal{P}(S) \to \mathcal{P}(S)$. Then*

$$S' = S'_1 \sqcup \ldots \sqcup S'_k$$

in such way that:

(1) $\mathcal{P}(S_i) = \emptyset$ *for* $i = 2, \ldots, k$, *and*

(2) *The restricted (injective) map $\mathcal{P}(S) \to \mathcal{P}(S'_1)$ is induced by a homeomorphism $S \to S'_1$.*

Let S, S' be surfaces as above, and $\phi : \mathcal{P}(S) \to \mathcal{P}(S')$ an injective simplicial map. The fact that

$$S' = S'_1 \sqcup \ldots \sqcup S'_k,$$

with (up to reordering the indices) $\mathcal{P}(S_i) = \emptyset$ for $i = 2, \ldots, k$ is contained in the proof of Theorem 3(c) of [1], whose argument essentially boils down to the combination of Lemma 3.1 above and the fact that Farey graphs are simplicially rigid.

Thus we get that, abusing notation, ϕ gives an injective simplicial map

$$\phi : \mathcal{P}(S) \to \mathcal{P}(S'_1).$$

Now, S'_1 has the same complexity as S, by Theorem 3(b) of [1], whose proof is essentially contained that of Lemma 3.1. Therefore $d(\mathcal{P}(S)) = d(\mathcal{P}(S'_1))$. Again by the rigidity of Farey graphs (for details, see Claim III of the proof of Theorem 3 of [1]), the map $\phi : \mathcal{P}(S) \to \mathcal{P}(S'_1)$ is also surjective and therefore an isomorphism. Since $d(\mathcal{P}(S)) \geq 2$, the classification of pants graphs up to isomorphism (which appears as Lemma 12 in [1]) implies that there is a homeomorphism $S \to S'_1$ (and thus a subsurface inclusion $S \hookrightarrow S'$) which induces ϕ, as claimed.

Finally, the fact that $\mathcal{P}(S)$ has (G7) is Lemma 10 of [1], although the terminology is different; namely, in [1],"alternating circuits" are called "alternating n-tuples (with $n = 4, 5$)". See the case of the Hatcher-Thurston graph below for the proof of the analogous statement, which follows a similar, although simpler, argument.

In the light of the discussion above, we obtain from Theorem 1.1 that, so long as S has complexity 2, every alternating injective simplicial map $\mathcal{P}(S) \to \mathcal{P}(S)$ is induced by a subsurface inclusion $S \hookrightarrow S'$. Moreover, it turns out that every injective map between pants graphs is always alternating (see Lemma 7 of [1], whose proof rests again upon the simplicial rigidity of Farey graphs) and therefore we have the main result in [1]:

Theorem 4.2 ([1]). *Let S, S' be connected orientable surfaces, such that S has complexity at least 2. If there exists an injective simplicial map $\phi : \mathcal{P}(S) \to \mathcal{P}(S')$ then there exists a subsurface inclusion $S \hookrightarrow S'$ that induces ϕ.*

4.2. **Flip graphs.** Let S be a compact, connected and orientable surface, of genus $g \geq 0$ with $b \geq 0$ boundary components. Moreover, assume that S has $p + q > 0$ marked points, with $p \geq 0$ in the interior of S and the other $q \geq 0$ in ∂S, subject to the condition that every component of ∂S must contain at least one marked point.

A *triangulation* on S is a set of arcs on S that is maximal with respect to inclusion. Observe that a triangulation contains exactly $d(S) = 6g + 3b + 3p + q - 6$ arcs. The *flip graph* $\mathcal{F}(S)$ is the simplicial graph whose vertices are triangulations of S, and where two triangulations are adjacent if and only if they share exactly $d(S) - 1$ arcs. Note this implies that the remaining two arcs intersect exactly once; we say that the two triangulations differ by a “flip”. Observe that $\mathcal{F}(S)$ is locally finite, as every vertex has valence at most (but not always equal to) $d(S)$.

Flip graphs satisfy some, but not all, of the conditions (G1) – (G7). The major obstacle in this direction is the presence of triangulations with “unflippable” arcs. More concretely, there are triangulations v which contain an arc a with the property that there does not exist any vertex v' with

$$v - (v \cap v') = \{a\}$$

(see Figure 2). In other words, condition $(G4)$ does not hold for $\mathcal{F}(S)$.

It is immediate, however, that $\mathcal{F}(S)$ satisfies (G1) and (G2) with $d = 6g + 3b + 3p + q - 6$. Similarly, one sees that $\mathcal{F}(S)$ has properties (G3) (i.e. it is connected) and (G5). With considerable effort, and as long as S is not *exceptional*, it is possible to bypass the failure of condition (G3) and prove that $\mathcal{F}(S)$ satisfies a weakening of condition (G7); see Propositions 2.2 and 2.3 of [2]. Here, we say that the surface S is *exceptional* if it is an essential subsurface of (and possibly equal to) a torus with at most two marked points, or a sphere with at most four marked points. This property turns out to be enough to show, after a significant amount of work, that $\mathcal{F}(S)$ also has property (G6), see Theorem 1.4 of [2].

At this point one may apply a similar strategy to the one described in the previous section to prove that alternating injective maps between flip graphs are always induced by subsurface inclusions. However, as was the case with pants graph, alternating injective maps between flip graphs are automatically alternating, and thus one has the following result, proved in [2]

Theorem 4.3 ([2]). *Suppose S is non-exceptional. Then every injective simplicial map $\phi : \mathcal{F}(S) \to \mathcal{F}(S')$ is induced by a subsurface inclusion $S \to S'$.*

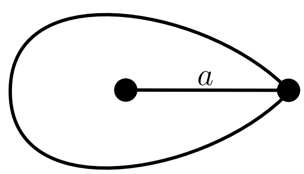

FIGURE 2. An unflippable arc of a triangulation

4.3. **Hatcher-Thurston graphs.** Let S be a connected orientable surface of genus $g \geq 2$, possibly with punctures. A *cut system* on S is a set M of g pairwise disjoint curves such that the result of cutting S open along the elements of M is homeomorphic to a sphere with punctures.

The *Hatcher-Thurston graph* (or *cut graph*) $\mathcal{K}(S)$ is the simplicial graph whose vertices are cut systems on S, and where two cut systems are adjacent in $\mathcal{K}(S)$ if and only if they have $g - 1$ curves in common, and the remaining two curves intersect exactly once. The graph $\mathcal{K}(S)$ (in fact, a certain 2-complex obtained from it) was used by Wajnryb [10] to study presentations of mapping class groups.

By construction, $\mathcal{K}(S)$ satisfies (G1), (G2) with $d(\mathcal{K}(S)) = g$. Wajnryb proved [10] that $\mathcal{K}(S)$ is connected and thus satisfies (G3). The fact that it also satisfies (G4) is also obvious. To verify (G5) we argue as in the case of pants graphs. Suppose we are given

$S \subset S'$, and choose a cut system M of $S' - S$; then every cut system v of S extends to the cut system $v \cup M$ of S'.

To see that (G7) holds for $\mathcal{K}(S)$, let u, v, w be vertices of $\mathcal{K}(S)$ with $u, w \in \mathrm{lk}(v)$, such that $u \cap v \cap w$ is an extendable set of deficiency 2. As such, we may write:

$$u = \{a', b, c_3, \dots, c_g\},$$

$$v = \{a, b, c_3, \dots, c_g\},$$

and

$$w = \{a, b', c_3, \dots, c_g\},$$

with $i(a, a') = 1$ and $i(b, b') = 1$. If $i(a', b') = 0$, then the closed path

$$u \to v \to w \to z \to u$$

is an alternating circuit (with $n = 4$), where $z = \{a', b', c_3, \dots, c_g\}$. So suppose that $i(a', b') \neq 0$. Choose a nonseparating curve b'' such that

$$i(b'', a) = i(b'', a') = i(b'', c_j) = 0$$

for all j, and

$$i(b'', b) = 1.$$

Consider the vertex $w' = \{a, b'', c_3, \dots, c_g\}$. Now, w and w' may be connected by a path ρ in $\mathcal{K}(S)$ which misses w and whose every vertex contains the curves $a', c_3, \dots, c_g$; see, for instance, Lemma 3 of [6]. Considering the vertex

$$z = \{a', b'', c_3, \dots, c_g\},$$

we see that the closed path

$$u \to v \to w \xrightarrow{\rho} w' \to z \to u$$

is an alternating circuit (with $n = 4$ again).

In spite of the surge of optimism after this discussion, we remark the statement of Theorem 1.1 does not hold for Hatcher-Thurston graphs, as there exist injective maps between Hatcher-Thurston graphs that are not induced by subsurface inclusions. To construct examples of these, start with a closed surface S of genus $g \geq 2$ and endow S with a hyperbolic metric. We realize every simple closed curve on S by the unique geodesic in its isotopy class. The union of all such geodesics has measure zero, and thus we can choose a point p in the complement, thus obtaining an injective simplicial map

$$\mathcal{K}(S) \to \mathcal{K}(S - \{p\});$$

observe that there are no continuous injective maps $S \to S - \{p\}$. We may now repeat this process a finite number of times, and "attach" pairs of punctures on the new surface to obtain, for any $g \geq 2$, an injective simplicial map

$$\mathcal{K}(S_{g,0}) \to \mathcal{K}(S_{g',n}),$$

where $S_{h,k}$ denotes the surface of genus h and with k punctures. In ongoing work, Jesús Hernández [7] has proved that, in fact, every injective alternating map between Hatcher-Thurston graph arises in this way, as long as the domain surface is closed and has genus at least 3.

4.4. **Curve graphs.** The curve graph $\mathcal{C}(S)$ is the simplicial graph whose vertices are (isotopy classes of essential simple closed) curves on S, and where two such curves are adjacent in $\mathcal{C}(S)$ if they can be realized disjointly on S.

We see that $\mathcal{C}(S)$ satisfies (G1) and (G2) with $d = d(\mathcal{C}(S)) = 1$. A pleasant exercise shows that $\mathcal{C}(S)$ is connected as long as S has complexity at least 2.

Since there are no extendable sets with respect to $\mathcal{C}(S)$, we immediately get that $\mathcal{C}(S)$ satisfies (G4), (G5) and (G7).

However, curve graphs do not satisfy (G6), as there are simplicial injections between curve graphs that do not arise from homeomorphisms, or even inclusions, between the underlying surfaces. Indeed, using the same argument as in the case of Hatcher-Thurston graphs, we see that there are injective simplicial maps

$$\mathcal{C}(S) \to \mathcal{C}(S - \{p\}),$$

which cannot be induced by a subsurface inclusion if S is closed, for instance. Motivated by the result of J. Hernández on Hatcher-Thurston graphs mentioned above, we ask:

Problem 4.4. *Let S be a closed surface of genus $g \geq 2$. Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S - \{p\})$ be an injective simplicial map, and $\pi : \mathcal{C}(S - \{p\}) \to \mathcal{C}(S)$ the natural "puncture-forgetting" map. Is it true that $\pi \circ \phi$ is always an isomorphism?*

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