

HOLOMORPHIC MAPS BETWEEN MODULI SPACES

JAVIER ARAMAYONA & JUAN SOUTO

ABSTRACT. We prove that every non-constant holomorphic map $\mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$ between moduli spaces of Riemann surfaces is a forgetful map, provided that $g \geq 6$ and $g' \leq 2g - 2$.

1

Let $\mathcal{M}_{g,p}$ be the moduli space of Riemann surfaces of genus g with p labelled marked points. Moduli space has a natural structure as a complex orbifold. In this paper we study holomorphic maps, in the category of orbifolds, between distinct moduli spaces. Examples of such maps are the so-called *forgetful maps* [7, 10]: given $(i_1, \dots, i_{p'})$ with $i_j \in \{1, \dots, p\}$ and $i_j \neq i_k$ for $j \neq k$, set

$$(1.1) \quad \mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g,p'}, \quad (X, x_1, \dots, x_p) \mapsto (X, x_{i_1}, \dots, x_{i_{p'}}).$$

We prove that under suitable genus bounds, there are no other non-constant holomorphic maps:

Theorem 1.1. *Suppose that $g \geq 6$ and $g' \leq 2g - 2$. Every non-constant holomorphic map $\mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$ is a forgetful map.*

As a direct consequence of Theorem 1.1 we obtain:

Corollary 1.2. *Suppose that $g \geq 6$ and $g' \leq 2g - 2$. If there is a non-constant holomorphic map $\mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$, then $g = g'$ and $p \geq p'$. \square*

Theorem 1.1 remains true under slightly more generous conditions - compare with the discussion at the end of section 6. On the other hand, some genus bounds are necessary for Theorem 1.1 to hold. For instance, it follows from [2] that for all $g \geq 2$ there are $g' > g$ and a holomorphic embedding $\mathcal{M}_{g,0} \hookrightarrow \mathcal{M}_{g',0}$.

Moduli space is not compact, but a natural compactification $\bar{\mathcal{M}}_{g,p}$, a projective variety, was constructed in [9] by Deligne and Mumford. Morphisms between Deligne-Mumford compactifications have been studied by several authors (see for example [8, 14]). Notice that in Theorem 1.1 we assume neither that the holomorphic maps in question extend to the Deligne-Mumford compactification, nor that they are surjective or have connected fibers.

Juan Souto has been partially supported by NSERC Discovery and Accelerator Supplement grants.

We sketch briefly the proof of Theorem 1.1. Since we are working in the category of orbifolds, every continuous map $f : \mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$ induces a homomorphism $f_* : \text{Map}_{g,p} \rightarrow \text{Map}_{g',p'}$ between the associated mapping class groups. In [3] we classified all homomorphisms between mapping class groups under the genus bounds in Theorem 1.1; it follows easily from this classification that f is either homotopically trivial or homotopic to a forgetful map h . The following result, the main technical observation of this note, yields immediately that, if f is holomorphic, then $f = h$ unless f is constant.

Proposition 1.3. *Let $M \rightarrow \mathcal{M}_{g,p}$ and $N \rightarrow \mathcal{M}_{g',p'}$ be finite covers and suppose that $f_1, f_2 : M \rightarrow N$ are homotopic holomorphic maps. If f_1 is not constant, then $f_1 = f_2$.*

To prove Proposition 1.3 we proceed as follows. Endowing the target N with the Weil-Petersson metric and the domain M with McMullen's Kähler hyperbolic metric [19], we derive from a result of Royden [21] that the maps f_1, f_2 have finite energy. Let $(f_t)_{t \in [1,2]}$ be the straight homotopy between f_1 and f_2 , and note that the energy function $t \mapsto E(f_t)$ is convex because the Weil-Petersson metric is negatively curved. Proposition 1.3 follows easily once we prove that $E(f_t)$ attains its minimum at $t \in \{1, 2\}$. To see that this is the case, we adapt an argument due to Eells-Sampson [12], who derived from a version of Wirtinger's inequality and Stokes' theorem that holomorphic maps between Kähler manifolds are harmonic, meaning that they minimize energy for every deformation with compact support. A priori, the straight homotopy (f_t) does not have compact support and so we have to control the boundary terms that appear when applying Stokes' theorem - this is what we do.

The content of this paper was originally included in our paper [3]. Following the suggestion of the journal and the referee, we decided to rewrite it and present Theorem 1.1 independently.

Acknowledgements. This paper was written during the program "Automorphisms of Free Groups: Algorithms, Geometry and Dynamics" at the CRM, Barcelona. We would like to thank the organizers of the program, as well as to express our gratitude to the CRM for its hospitality.

Throughout this paper we make use of standard facts about Teichmüller spaces and mapping class groups. We refer to [13, 15, 16, 20] for an extensive treatment of these subjects, and to [6] for a nice survey.

Let $S_{g,p}$ be the closed surface of genus g with p distinct labeled marked points. The mapping class group $\text{Map}_{g,p}$ is the group of isotopy classes of orientation preserving self-homeomorphisms of $S_{g,p}$ fixing each marked point. It acts discretely on Teichmüller space and this action preserves the standard complex structure on $\mathcal{T}_{g,p}$ (see [1]). Moduli space is the complex

orbifold

$$\mathcal{M}_{g,p} = \mathcal{T}_{g,p} / \text{Map}_{g,p}$$

We stress that we consider $\mathcal{M}_{g,p}$ always as an orbifold - in a language that we are not going to use, $\mathcal{M}_{g,p}$ is the moduli stack of algebraic curves of genus g with p points labelled. In particular, maps and covers are taken in the category of orbifolds. For instance, for every continuous map

$$f : \mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$$

there are a homomorphism

$$f_* : \text{Map}_{g,p} \rightarrow \text{Map}_{g',p'}$$

and a continuous f_* -equivariant map

$$\tilde{f} : \mathcal{T}_{g,p} \rightarrow \mathcal{T}_{g',p'}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}_{g,p} & \xrightarrow{\tilde{f}} & \mathcal{T}_{g',p'} \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,p} & \xrightarrow{f} & \mathcal{M}_{g',p'} \end{array}$$

The homomorphism f_* and the lift \tilde{f} are unique up to simultaneous conjugation by a mapping class.

Example 1. For the forgetful map $f : \mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g,p'}$ defined in (1.1), the lift $\tilde{f} : \mathcal{T}_{g,p} \rightarrow \mathcal{T}_{g,p'}$ is given by the same formula, and the homomorphism $f_* : \text{Map}_{g,p} \rightarrow \text{Map}_{g,p'}$ is the one given by forgetting the marked points $\{x_1, \dots, x_p\} \setminus \{x_{i_1}, \dots, x_{i_{p'}}\}$ [3, 13]. In fact, both f and \tilde{f} are holomorphic fiber bundles, and the long exact sequence of homotopy groups corresponding to the fiber bundle f yields the Birman exact sequence for f_* .

The above discussion amounts to saying that Teichmüller space is the (orbifold) universal cover of moduli space. In fact, more is true: Teichmüller space is a classifying space for proper actions $\underline{E}(\text{Map}_{g,p})$ of the mapping class group. In particular, the homotopy class of f is determined by f_* . We give a simple proof of this fact. Suppose that

$$f, h : \mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$$

have lifts $\tilde{f}, \tilde{h} : \mathcal{T}_{g,p} \rightarrow \mathcal{T}_{g',p'}$ that are equivariant under the same homomorphism $\rho : \text{Map}_{g,p} \rightarrow \text{Map}_{g',p'}$. For each $X \in \mathcal{T}_{g,p}$ let $\gamma_X : [0, 1] \rightarrow \mathcal{T}_{g',p'}$ be the unique Weil-Petersson (or Teichmüller) geodesic with $\gamma_X(0) = \tilde{f}(X)$ and $\gamma_X(1) = \tilde{h}(X)$. The uniqueness of γ_X implies that the map

$$(2.1) \quad F : [0, 1] \times \mathcal{T}_{g,p} \rightarrow \mathcal{T}_{g',p'}, \quad F(X, t) = \gamma_X(t)$$

is continuous and ρ -equivariant, and thus descends to a homotopy between f and h .

3

A key ingredient in the proof of Proposition 1.3 is the existence of a compact exhaustion of $\mathcal{M}(X)$ that behaves well with respect to the Teichmüller metric d_T ; see [15, 16] for a discussion of this metric.

Proposition 3.1. *There are $c > 0$ and a collection $\{K_n\}_{n \in \mathbb{N}}$ of compact subsets of $\mathcal{M}_{g,p}$ with the following properties:*

- (1) $\mathcal{M}_{g,p} = \bigcup_{n \in \mathbb{N}} K_n$, and $K_n \subset K_{n+1}$ for all n ,
- (2) $\text{vol}_T(\partial K_n) \leq ce^{-n}$ for all n , and
- (3) K_n is contained within d_T -distance $c + n$ of K_0 for all n .

In the statement of Proposition 3.1, $\text{vol}_T(\cdot)$ denotes the co-dimension one d_T -volume.

The key tool of the proof of Proposition 3.1 is a result due to Royden [21] asserting that the Teichmüller metric d_T agrees with the Kobayashi metric of $\mathcal{T}_{g,p}$. Recall that if M is a complex manifold, the Kobayashi pseudometric [18] is the largest pseudometric on M such that every holomorphic map $\mathbb{D} \rightarrow M$ is 1-Lipschitz; here \mathbb{D} is the unit disk in \mathbb{C} endowed with the Poincaré metric.

Remark. In general, holomorphic maps between complex manifolds endowed with the Kobayashi pseudometric are 1-Lipschitz. Thus it follows from Royden's theorem that holomorphic maps between Teichmüller spaces equipped with the Teichmüller metric are 1-Lipschitz.

Proof. To simplify notation, we write $\mathcal{M} = \mathcal{M}_{g,p}$ and let $\bar{\mathcal{M}}$ be its Deligne-Mumford compactification. Let also \mathbb{D}^* and \mathbb{D} be, respectively, the punctured and unpunctured open unit disks in \mathbb{C} endowed with their complete hyperbolic metrics.

By a result of Wolpert [23], every point in $\bar{\mathcal{M}} \setminus \mathcal{M}$ has a neighborhood \bar{U} in $\bar{\mathcal{M}}$ whose intersection $U = \bar{U} \cap \mathcal{M}$ with moduli space is bi-holomorphic to

$$U \simeq ((\mathbb{D}^*)^k \times \mathbb{D}^{d-k})/G$$

where $k \geq 1$, $d = \dim_{\mathbb{C}} \mathcal{M} = 3g - 3 + p$, and G is a finite group. Let $d_{\mathcal{H}}$ be the distance on U induced by the product of hyperbolic metrics.

Let $D_0 \subset \mathbb{D}$ and $D_0^* \subset \mathbb{D}^*$ be, respectively, the disk and punctured disk of Euclidean radius $\frac{1}{2}$, and denote by $D_n^* \subset D_0^*$ the punctured disk such that the hyperbolic distance between ∂D_0^* and ∂D_n^* is equal to n . We set

$$W_n^k = (D_n^* \times D_0^* \times \cdots \times D_0^*) \cup \dots \cup (D_0^* \times \cdots \times D_0^* \times D_n^*) \subset (D_0^*)^k$$

and

$$U_n = (W_n^k \times D_0^{d-k})/G \subset U$$

By construction ∂U_n is at $d_{\mathcal{H}}$ -distance n from ∂U_0 and its co-dimension one volume decreases exponentially: $\text{vol}_{\mathcal{H}}(\partial U_n) \simeq e^{-n}$.

Since $\bar{\mathcal{M}} \setminus \mathcal{M}$ is compact, we can pick finitely many sets U^1, \dots, U^r such that $\mathcal{M} \setminus \cup_i U_0^i$ is compact. For $n \in \mathbb{N}$ we set

$$K_n = \mathcal{M} \setminus \cup_i U_n^i$$

By construction, K_n is compact, $\mathcal{M} = \cup_n K_n$, and $K_n \subset K_{n+1}$ for all n . In other words, $\{K_n\}_{n \in \mathbb{N}}$ satisfies (1).

To prove (2), note that Royden's theorem [21] implies that the inclusion

$$(U^i, d_{\mathcal{H}}) \hookrightarrow (\mathcal{M}, d_{\mathbb{T}})$$

is 1-Lipschitz for all i . In particular we have

$$\text{vol}_{\mathbb{T}}(\partial K_n) \leq \sum_{i=1}^r \text{vol}_{\mathbb{T}}(\partial U_n^i) \leq \sum_{i=1}^r \text{vol}_{\mathcal{H}}(\partial U_n^i)$$

As mentioned, each summand on the right side decreases exponentially in n and thus there is a constant c with

$$\text{vol}_{\mathbb{T}}(\partial K_n) \leq ce^{-n} \quad \text{for all } n,$$

as claimed in (2). It remains to prove that (3) is satisfied.

Since $K_n \subset K_0 \cup_i (U_0^i \setminus U_n^i)$ and ∂U_0^i is compact for each i , we can enlarge c so that

$$d_{\mathbb{T}}(X, K_0) \leq c + \max_i \max_{Y \in U_0^i \setminus U_n^i} d_{\mathbb{T}}(Y, \partial U_0^i)$$

for every $X \in K_n$. Applying once again Royden's theorem, we get

$$\max_{Y \in U_0^i \setminus U_n^i} d_{\mathbb{T}}(Y, \partial U_0^i) \leq \max_{Y \in U_0^i \setminus U_n^i} d_{\mathcal{H}}(Y, \partial U_0^i) = n,$$

from where we obtain

$$d_{\mathbb{T}}(X, K_0) \leq c + n \quad \text{for all } X \in K_n,$$

as we needed to prove. \square

4

In this section we discuss some geometric facts about holomorphic maps $M \rightarrow N$ between finite (orbifold) covers of moduli spaces:

$$M \rightarrow \mathcal{M}_{g,p}, \quad N \rightarrow \mathcal{M}_{g',p'}.$$

We will endow the domain M with McMullen's Kähler hyperbolic metric and the target N with the Weil-Petersson metric.

The Weil-Petersson distance d_{WP} is induced by a negatively curved, although unfortunately incomplete, Riemannian metric. However, d_{WP} is geodesically convex, meaning that any two points in Teichmüller space $\mathcal{T}_{g,p}$ are connected by a unique d_{WP} -geodesic segment. Moreover, the identity map

$$(4.1) \quad \text{Id} : (\mathcal{T}_{g,p}, d_{\mathbb{T}}) \rightarrow (\mathcal{T}_{g,p}, d_{\text{WP}})$$

is Lipschitz [19, Proposition 2.4]. McMullen's Kähler hyperbolic metric d_{KH} on $\mathcal{T}_{g,p}$ is again induced by a Riemannian metric, and the identity map

$$(4.2) \quad \text{Id} : (\mathcal{T}_{g,p}, d_{\text{KH}}) \rightarrow (\mathcal{T}_{g,p}, d_{\text{T}})$$

is bi-Lipschitz [19, Theorem 1.1]. Hence, $(\mathcal{M}_{g,p}, d_{\text{KH}})$ is complete and has finite volume. See [15, 16] for background on the Weil-Petersson metric d_{WP} , and [19] for the definition and properties of d_{KH} .

As remarked earlier, Royden's theorem [21] implies that holomorphic maps between Teichmüller spaces endowed with the Teichmüller metric are 1-Lipschitz. In particular, (4.1) and (4.2) imply that there is $L > 0$ such that every holomorphic map $f : (M, d_{\text{KH}}) \rightarrow (N, d_{\text{WP}})$ is L -Lipschitz.

Suppose from now on that

$$f_1, f_2 : (M, d_{\text{KH}}) \rightarrow (N, d_{\text{WP}})$$

are homotopic holomorphic maps, and let

$$\hat{F} : [1, 2] \times M \rightarrow N$$

be a homotopy between them. Since the Weil-Petersson metric is negatively curved and geodesically convex, we can replace \hat{F} with the straight homotopy

$$(4.3) \quad F : [1, 2] \times M \rightarrow N, \quad F(t, X) = f_t(X)$$

determined by the fact that $t \mapsto f_t(X)$ is the d_{WP} -geodesic segment joining $f_1(X)$ and $f_2(X)$ in the homotopy class of $\hat{F}([1, 2] \times \{X\})$.

Note that f_t is L -Lipschitz for all t because f_1 and f_2 are. Indeed, for $v \in T_X M$ the vector field $t \mapsto d(f_t)_X v$ is a Jacobi field along $t \mapsto f_t(X)$. Since the Weil-Petersson metric is negatively curved, the length of Jacobi fields is a convex function, and hence attains its maximum at $t \in \{1, 2\}$.

A priori, the map F itself need not be Lipschitz: the norm of $dF_{(t,X)} \frac{\partial}{\partial t}$ is equal to the length of the geodesic arc $t \mapsto F(t, X)$, and there is no reason for this to be bounded, as M is not compact. However, fixing $X_0 \in M$ there is a constant k , independent of X , such that the segment $t \mapsto F(t, X)$ has length at most $k + 2Ld_{\text{KH}}(X, X_0)$ because f_1, f_2 are L -Lipschitz. It follows that there are constants A, B with

$$\|dF_{(t,X)}\|^2 \leq A \cdot d_{\text{KH}}(X_0, X)^2 + B$$

for all $(t, X) \in [1, 2] \times M$. Here, $\|dF_{(t,X)}\|$ is the operator norm of $dF_{(t,X)}$.

The convexity of Jacobi fields also implies the convexity of the *energy density*

$$t \mapsto E_X(f_t) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=1}^{\dim_{\mathbb{R}} M} \|d(f_t)_X v_i\|_{\text{WP}}^2$$

where $v_1, \dots, v_{\dim_{\mathbb{R}} M}$ is an arbitrary orthonormal basis of $T_X M$ with respect to the Kähler hyperbolic metric. This function is strictly convex if one of

$d(f_1)_X$ or $d(f_2)_X$ has rank at least 2. In particular, if the holomorphic maps f_1, f_2 are distinct and one of them is non-constant, then the *energy*

$$t \mapsto E(f_t) \stackrel{\text{def}}{=} \int_M E_X(f) \nu_{\text{KH}}$$

is strictly convex. Here ν_{KH} is the Riemannian volume form of (M, d_{KH}) , and $E(f_t) < \infty$ because f_t is L -Lipschitz and M has finite d_{KH} -volume.

We summarize this discussion in the following lemma:

Lemma 4.1. *Let $M \rightarrow \mathcal{M}_{g,p}$ and $N \rightarrow \mathcal{M}_{g',p'}$ be finite covers and*

$$f_1, f_2 : (M, d_{\text{KH}}) \rightarrow (N, d_{\text{WP}})$$

homotopic holomorphic maps. Consider the straight homotopy

$$F : [1, 2] \times M \rightarrow N, \quad F(t, X) = f_t(X)$$

between them. Then:

- (1) *There is $L > 0$ such that $f_t : M \rightarrow N$ is L -Lipschitz and has finite energy $E(f_t) < \infty$ for all t .*
- (2) *For $X_0 \in M$, there are $A, B > 0$ with*

$$\|dF_{(t,X)}\|^2 \leq A \cdot d_{\text{KH}}(X_0, X)^2 + B$$

for all $(t, X) \in [1, 2] \times M$.

- (3) *The energy function $t \mapsto E(f_t)$ is convex. Moreover, it is strictly convex unless either $f_1 = f_2$ or both are constant. \square*

So far, we have only used Royden's theorem, the comparison between the different metrics, and the curvature properties of d_{WP} . We will also make key use of the fact that both the Weil-Petersson metric and McMullen's metric are Kähler. This means that the Kähler form, i.e. the 2-form $\omega = \langle \cdot, J \cdot \rangle$, is closed, where $\langle \cdot, \cdot \rangle$ is the relevant Riemannian metric and J is the endomorphism of the tangent bundle given by complex multiplication. See [5, 22] for facts on Kähler manifolds.

We need an observation due to Eells-Sampson [12]. Suppose that

$$f : \mathcal{T}_{g,p} \rightarrow \mathcal{T}_{g',p'}$$

is a smooth map, and write ω_{WP} and ω_{KH} for the Kähler forms of the Weil-Petersson metric and of McMullen's metric respectively. The Riemannian volume form induced by d_{KH} may be expressed as

$$\nu_{\text{KH}} = \frac{1}{d!} \omega_{\text{KH}}^d = \frac{1}{d!} \omega_{\text{KH}} \wedge \cdots \wedge \omega_{\text{KH}},$$

where $d = \dim_{\mathbb{C}} \mathcal{T}_{g,p}$. Pulling back the Kähler form ω_{WP} via f we can also consider the top-dimensional form $(f^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1}$ on $\mathcal{T}_{g,p}$. An infinitesimal computation due to Eells and Sampson [12] - valid for maps between arbitrary Kähler manifolds - proves that:

Proposition 4.2 (Eells-Sampson). *Let $f : (\mathcal{T}_{g,p}, d_{\text{KH}}) \rightarrow (\mathcal{T}_{g',p'}, d_{\text{WP}})$ be a smooth map. For every $X \in \mathcal{T}_{g,p}$ we have*

$$(4.4) \quad E_X(f)\nu_{\text{KH}} \geq \frac{1}{d!} (f^*\omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1}$$

where $d = \dim_{\mathbb{C}} \mathcal{T}_{g,p}$. Moreover, equality holds in (4.4) if and only if f is holomorphic at X .

Proposition 4.2 is basically an incarnation of the classical Wirtinger inequality. In other words, it is a purely linear algebra fact which follows from the observation that, whenever $\Lambda : \mathbb{C} \rightarrow \mathbb{C}^n$ is \mathbb{R} -linear, then

$$E_0(\Lambda)\omega_{\mathbb{C}} \geq \det(\Lambda)\omega_{\mathbb{C}} \geq \Lambda^*(\omega_{\mathbb{C}^n})$$

where $\omega_{\mathbb{C}} = dx \wedge dy$ and $\omega_{\mathbb{C}^n} = \sum_{i=1}^n dx_i \wedge dy_i$ are the standard Kähler forms.

5

We are now ready to prove Proposition 1.3:

Proposition 1.3. *Let $M \rightarrow \mathcal{M}_{g,p}$ and $N \rightarrow \mathcal{M}_{g',p'}$ be finite covers and suppose that $f_1, f_2 : M \rightarrow N$ are homotopic holomorphic maps. If f_1 is not constant, then $f_1 = f_2$.*

We recall from the introduction that the proof of Proposition 1.3 uses an argument due to Eells and Sampson [12] based on Proposition 4.2 and Stokes' theorem. Here we have to integrate over moduli space, but as we mentioned above, $\mathcal{M}_{g,p}$ is non-compact. We apply Stokes' theorem to the compact subsets K_n provided by Proposition 3.1 and show that the arising boundary terms tend to 0 when $n \rightarrow \infty$.

Proof. Suppose that f_1 is not constant and $f_1 \neq f_2$, and let

$$F : [1, 2] \times M \rightarrow N, \quad F(t, X) = f_t(X)$$

be the straight homotopy (4.3) between them. From Lemma 4.1 we know that the function $t \mapsto E(f_t)$ is strictly convex; we may hence assume that

$$(5.1) \quad E(f_t) < E(f_1)$$

for all $t \in (1, 2)$. We are going to contradict this assertion.

Let $\{K_n\}$ be the compact exhaustion of $\mathcal{M}_{g,p}$ provided by Proposition 3.1. Abusing terminology, we denote also by K_n the preimage of K_n under the covering map $M \rightarrow \mathcal{M}_{g,p}$. By (4.2), the Teichmüller metric is bi-Lipschitz to McMullen's Kähler hyperbolic metric and thus we get from Proposition 3.1 that there are constants c and L such that:

- (1) $M = \bigcup_{n \in \mathbb{N}} K_n$, and $K_n \subset K_{n+1}$ for all n ,
- (2) $\text{vol}_{\text{KH}}(\partial K_n) \leq ce^{-n}$ for all n , and
- (3) K_n is contained within d_{KH} -distance $c + L \cdot n$ of K_0 for all n .

Write ω_{KH} and ω_{WP} for the Kähler forms of McMullen's metric and the Weil-Petersson metric respectively, and set $d = \dim_{\mathbb{C}} M = 3g - 3 + p$. Since Kähler forms are closed, we deduce from Stokes theorem that

$$\begin{aligned}
0 &= \int_{[1,t] \times K_n} d \left((F^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} \right) \\
&= \int_{\partial([1,t] \times K_n)} (F^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} \\
&= \int_{\{t\} \times K_n} (F^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} - \int_{\{1\} \times K_n} (F^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} \\
&\quad + \int_{[1,t] \times \partial K_n} (F^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} \\
&= \int_{K_n} (f_t^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} - \int_{K_n} (f_1^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} \\
&\quad + \int_{[1,t] \times \partial K_n} (F^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1}
\end{aligned}$$

Below we will prove:

$$(5.2) \quad \lim_{n \rightarrow \infty} \int_{[1,t] \times \partial K_n} (F^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} = 0.$$

Assuming (5.2), we obtain from the computation above that

$$\lim_{n \rightarrow \infty} \left(\int_{K_n} (f_t^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} - \int_{K_n} (f_1^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} \right) = 0$$

Taking into account that f_t and f_1 are Lipschitz and that (M, d_{KH}) has finite volume, we deduce that

$$\int_M (f_t^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} = \int_M (f_1^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1}.$$

From Proposition 4.2 we get

$$\begin{aligned}
E(f_t) &\geq \frac{1}{d!} \int_M (f_t^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} \\
&= \frac{1}{d!} \int_M (f_1^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} = E(f_1)
\end{aligned}$$

where the last equality holds because f_1 is holomorphic. This contradicts (5.1). It remains to prove (5.2).

Fix $(t, X) \in [0, 1] \times \partial K_n$ and let v_1, \dots, v_{2d} be an orthonormal basis of $T_{(t,X)}([0, 1] \times \partial K_n)$. We have

$$\begin{aligned}
&\left| (F^* \omega_{\text{WP}})(v_1, v_2) \cdot \omega_{\text{KH}}(v_3, v_4) \cdot \dots \cdot \omega_{\text{KH}}(v_{2d-1}, v_{2d}) \right| \\
&= \left| \langle dF_{(t,X)} v_1, JdF_{(t,X)} v_2 \rangle_{\text{WP}} \cdot \langle v_3, Jv_4 \rangle_{\text{KH}} \cdot \dots \cdot \langle v_{2d-1}, Jv_{2d} \rangle_{\text{KH}} \right| \\
&\leq \|dF_{(t,X)}\|^2
\end{aligned}$$

where $\|dF_{(t,X)}\|$ is the operator norm of $dF_{(t,X)}$. Fixing $X_0 \in M$ we get from Lemma 4.1 that there are $A, B > 0$ with

$$\|dF_{(t,X)}\|^2 \leq A \cdot d_{\text{KH}}(X_0, X)^2 + B$$

for all $(t, X) \in [1, 2] \times M$. We deduce that

$$\begin{aligned} \left| \int_{[0,t] \times \partial K_n} (F^* \omega_{\text{WP}}) \wedge \omega_{\text{KH}}^{d-1} \right| &\leq (2d)! \int_{[0,t] \times \partial K_n} \|dF_{(t,X)}\|^2 \nu_{[0,t] \times \partial K_n} \\ &\leq (2d)! \left(A \cdot \max_{X \in \partial K_n} d_{\text{KH}}(X, X_0)^2 + B \right) \text{vol}_{\text{KH}}(\partial K_n) \end{aligned}$$

This last quantity tends to 0 as $n \rightarrow \infty$ by (2) and (3) above. Having proved (5.2), we are done with the proof of Proposition 1.3. \square

6. PROOF OF THEOREM 1.1

In this section we deduce Theorem 1.1 from Proposition 1.3 and a rigidity theorem for homomorphisms between mapping class groups proved in [3]. We remind the reader of some terminology from the said paper.

Let S and S' be compact surfaces, possibly with boundary, and P and P' finite sets of marked points in the interior of S and S' respectively. By an *embedding* $\iota : (S, P) \rightarrow (S', P')$ we understand a continuous injective map $\iota_{\text{top}} : S \rightarrow S'$ with the property that $\iota_{\text{top}}^{-1}(P') \subset P$. Every embedding $\iota : (S, P) \rightarrow (S', P')$ induces a (continuous) homomorphism $\text{Homeo}(S, P) \rightarrow \text{Homeo}(S', P')$ between the corresponding groups of self-homeomorphisms fixing pointwise the boundary and the set of marked points. In particular, ι induces a homomorphism

$$\iota_{\#} : \text{Map}(S, P) \rightarrow \text{Map}(S', P')$$

between the associated mapping class groups. The main result proved in [3] asserts that, subject to suitable genus bounds, every non-trivial homomorphism between mapping class groups is in fact induced by an embedding.

Theorem 6.1 (Aramayona-Souto). *Suppose that S and S' are compact surfaces, possibly with boundary, and that P and P' are finite sets of marked points in the interior of S and S' respectively. If S has genus $g \geq 6$ and S' has genus $g' \leq 2g - 2$, then every nontrivial homomorphism*

$$\text{Map}(S, P) \rightarrow \text{Map}(S', P')$$

is induced by an embedding $(S, P) \rightarrow (S', P')$.

Recall that $\mathcal{T}_{g,p}$ is a classifying space for proper actions $\underline{E}(\text{Map}_{g,p})$. In particular, as we discussed at the end of section 2, the homotopy type of a map $\mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$ is determined by the associated homomorphism between the corresponding mapping class groups. Armed with Theorem 6.1 we prove:

Proposition 6.2. *If $g \geq 6$ and $g' \leq 2g - 2$, then every map $f : \mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$ is either homotopically trivial or homotopic to a forgetful map.*

Proof. Let $f_* : \text{Map}_{g,p} \rightarrow \text{Map}_{g',p'}$ be the homomorphism associated to f and let $\tilde{f} : \mathcal{T}_{g,p} \rightarrow \mathcal{T}_{g',p'}$ be an f_* -equivariant lift of f . If f_* is trivial, then f is homotopically trivial and we have nothing to prove. Suppose from now on that this is not the case.

Let (S, P) and (S', P') be, respectively, closed surfaces of genus g and g' , with p and p' marked points. Identifying $\text{Map}(S, P) = \text{Map}_{g,p}$ and $\text{Map}(S', P') = \text{Map}_{g',p'}$, we obtain from Theorem 6.1 that the homomorphism f_* is induced by an embedding

$$\iota : (S, P) \rightarrow (S', P')$$

Since S and S' are closed, the underlying injective map $\iota_{top} : S \rightarrow S'$ is a homeomorphism and $\iota_{top}(P) \supset P'$. In other words, the embedding ι is obtained by forgetting marked points.

In the same way that we have identified mapping class groups, we also identify Teichmüller spaces $\mathcal{T}_{g,p} = \mathcal{T}(S, P)$ and $\mathcal{T}_{g',p'} = \mathcal{T}(S', P')$. The embedding ι induces an f_* -equivariant map

$$\tilde{h} : \mathcal{T}_{g,p} \rightarrow \mathcal{T}_{g',p'}$$

obtained again by forgetting marked points. By construction, \tilde{h} descends to a forgetful map

$$h : \mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$$

Since both \tilde{f} and \tilde{h} are f_* -equivariant, (2.1) yields a homotopy between f and h . \square

We are now ready to prove Theorem 1.1:

Theorem 1.1. *Suppose that $g \geq 6$ and $g' \leq 2g - 2$. Every non-constant holomorphic map $\mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$ is a forgetful map.*

Proof. Suppose that $f : \mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$ is holomorphic and not constant. Proposition 1.3 implies that f is not homotopically trivial and, in particular, f is homotopic to a forgetful map $h : \mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g',p'}$ by Proposition 6.2. Since both f and h are holomorphic and non-constant we get that $f = h$ from Proposition 1.3, as we needed to prove. \square

There is a number of rigidity results for homomorphisms between mapping class groups; see for example [4] for a survey of results in this direction. Combining any such theorem with Proposition 1.3 one obtains a rigidity result for holomorphic maps between the corresponding moduli spaces. For instance, the version of Theorem 6.1 proved in [3] covers a few more cases than the ones stated here. From this more general version, it follows that Theorem 1.1 also holds for maps $\mathcal{M}_{g,p} \rightarrow \mathcal{M}_{2g-1,p'}$ with $p' \geq 1$, and for maps $\mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g,p'}$ as long as $g \geq 4$. In particular, we have:

Corollary 6.3. *Suppose that $g \geq 4$. Every non-constant holomorphic map $\mathcal{M}_{g,p} \rightarrow \mathcal{M}_{g,p}$ is induced by a permutation of marked points, and is hence a biholomorphism. \square*

Note that the isomorphism, for $g \geq 2$, between the group of biholomorphisms of $\mathcal{M}_{g,p}$ and the symmetric group Σ_p follows also from Royden's characterization of the biholomorphism group of Teichmüller space [21, 11].

REFERENCES

- [1] L. Ahlfors, *The complex analytic structure of the space of closed Riemann surfaces*, in *Analytic Functions*, Princeton Univ. Press (1960).
- [2] J. Aramayona, C. Leininger and J. Souto, *Injections of mapping class groups*, *Geom. Topol.* 13 (2009), no. 5.
- [3] J. Aramayona and J. Souto, *Homomorphisms between mapping class groups*, to appear in *Geom. Topol.*
- [4] J. Aramayona and J. Souto, *Rigidity phenomena in the mapping class group*, preprint.
- [5] W. Ballmann, *Lectures on Kähler manifolds*, European Mathematical Society, (2006).
- [6] L. Bers, *Finite dimensional Teichmüller spaces and generalizations*, *Bull. AMS* 5 (1981).
- [7] L. Bers, *Fiber spaces over Teichmüller spaces*, *Acta Math.* 130 (1973).
- [8] A. Bruno and M. Mella, *The automorphism group of $\overline{M}_{0,n}$* , [arXiv:1006.0987](https://arxiv.org/abs/1006.0987).
- [9] P. Deligne and D. Mumford, *The irreducibility of the space of curves of a given genus*, *Inst. Hautes Études Sci. Publ. Math.* 36 (1969).
- [10] C. Earle and I. Kra, *On holomorphic mappings between Teichmüller spaces*, in *Contributions to Analysis*, Academic Press, New York, 1974.
- [11] C. Earle and I. Kra, *On isometries between Teichmüller spaces*, *Duke Math. J.* 41 (1974).
- [12] J. Eells, J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, *Amer. J. Math.* 86 (1964).
- [13] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, 49. Princeton University Press, 2012.
- [14] A. Gibney, S. Keel and I. Morrison, *Towards the ample cone of $\overline{M}_{g,p}$* , *J. Amer. Math. Soc.* 15 (2002).
- [15] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics*. Matrix Editions, 2006.
- [16] Y. Iwayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*. Springer-Verlag, Tokyo, 1992.
- [17] N. V. Ivanov, *Automorphisms of complexes of curves and of Teichmüller spaces*. *Internat. Math. Res. Notices* 14 (1997).
- [18] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, M. Dekker, 1970.
- [19] C. T. McMullen, *The moduli space of Riemann surfaces is Kähler hyperbolic*, *Ann. of Math. (2)* 151 (2000).
- [20] S. Nag, *The complex analytic theory of Teichmüller spaces*, John Wiley & Sons, Inc., 1988.
- [21] H. L. Royden, *Automorphisms and isometries of Teichmüller space*, in *Advances in the Theory of Riemann Surfaces*, *Ann. of Math. Studies*, No. 66. Princeton Univ. Press, 1971.
- [22] A. Weil, *Introduction à l'étude des variétés kählériennes*, Hermann, 1958.
- [23] S. Wolpert, *Riemann surfaces, moduli and hyperbolic geometry*, in *Lectures on Riemann Surfaces*, World Scientific, 1989.

School of Mathematics, Statistics and Applied Mathematics,
National University of Ireland, Galway.
`javier.aramayona@nuigalway.ie`

Department of Mathematics,
University of British Columbia, Vancouver.
`jsouto@math.ubc.ca`