ON THE GEOMETRY OF GRAPHS ASSOCIATED TO INFINITE-TYPE SURFACES

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Abstract. Consider a connected orientable surface S of infinite topological type, i.e. with infinitely-generated fundamental group. We describe the large-scale geometry of arbitrary connected subgraphs of $\mathcal{A}(S)$ and $\mathcal{C}(S)$, provided they are invariant under a sufficiently big subgroup of the mapping class group $\mathrm{Mod}(S)$. We obtain a number of consequences; in particular we recover the main results of J. Bavard [2] and Aramayona-Fossas-Parlier [1].

1. Introduction

There has been a recent surge of activity around mapping class groups of infinite-type surfaces, i.e. with infinitely-generated fundamental group. The motivation for studying these groups stems from several places, as we now briefly describe.

First, infinite-type surfaces appear as inverse limits of surfaces of finite type. In particular, infinite-type mapping class groups are useful in the study asymptotic and/or stable properties of their finite-type counterparts. This is the approach taken by Funar-Kapoudjian [7], where the authors identify the homology of an infinite-type mapping class group with the stable homology of the mapping class groups of its finite-type subsurfaces.

In a related direction, a number of well-known groups appear as subgroups of the mapping class group of infinite-type surfaces. For instance, Funar-Kapoudjian [8] realized Thompson's group T as a topologically-defined subgroup of the mapping class group of a certain infinitely-punctured sphere.

A third piece of motivation for studying mapping class groups of infinite-type surfaces comes from dynamics, as explained by Calegari in [3]. More concretely, let S be a closed surface, $P \subset S$ a finite subset, and consider the group $\mathrm{Homeo}(S, P)$ of those homeomorphisms of S that preserve P setwise. Let $G < \mathrm{Homeo}(S, P)$ be a subgroup that acts freely on $S - P$. Then G admits a natural homeomorphism to $\mathrm{Mod}(S - K, P)$, where K is either a finite set or a Cantor set. See [3] for more details.

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1.1. **Combinatorial models.** An extremely successful tool for understanding finite-type mapping class groups is the use of the various complexes built from arcs and/or curves on the surface. Notable examples of these are the *curve graph* $\mathcal{C}(S)$ and the *arc graph* $\mathcal{A}(S)$; see Section 3 for definitions. When S has finite type, a useful feature of these complexes is that, with respect to their standard path-metric, they are hyperbolic spaces of infinite diameter; see [14] and [12], respectively.

In sharp contrast, in the case of an infinite-type surface these complexes normally have finite diameter; see Section 3. In the case when S is a sphere minus the union of the north pole and a Cantor set, J. Bavard [2] proved that a certain subgraph of $\mathcal{A}(S)$ is hyperbolic and has infinite diameter, and used this to construct non-trivial quasi-morphisms from $\mathrm{Mod}(S)$. Subsequently, Aramayona-Fossas-Parlier [1] have produced similar graphs for arbitrary surfaces, subject to certain conditions on the set of punctures of S. However, the definition of these subgraphs is surprisingly subtle, and small variations in the definition may produce graphs that have finite diameter or are not hyperbolic.

Our first goal is to describe the geometry of arbitrary subgraphs of $\mathcal{A}(S)$, subject to some general conditions on them. First, we say that a connected subgraph $\mathcal{G}(S) \subset \mathcal{A}(S)$ is *sufficiently invariant* if it is invariant by $\mathrm{Mod}(S, P)$, for some (possibly empty) finite set P of punctures. In addition, we will assume that every such graph satisfies the *projection property.* This property is needed only for technical reasons, and thus we refer the reader to Section 4 for details. However, we stress that this restriction is easy to check, and often automatically satisfied, once one is given an explicit subgraph of $\mathcal{A}(S)$, see Remark 4.6 below.

Before we state our result, recall from [16] that a *witness* [1] of $\mathcal{G}(S)$ is an essential subsurface of S such that every vertex of $\mathcal{G}(S)$ intersects Y. We will prove:

Theorem 1.1. *Let S be a connected orientable surface of infinite type, and $\mathcal{G}(S)$ a connected, sufficiently invariant subgraph of $\mathcal{A}(S)$ with the projection property. Assume that every two witnesses of $\mathcal{G}(S)$ intersect.*

(1) If every witness of $\mathcal{G}(S)$ has infinitely many punctures, then $\mathcal{G}(S)$ has finite diameter.
(2) Otherwise, $\mathrm{diam}(\mathcal{G}(S)) = \infty$*. Moreover, $\mathcal{G}(S)$ is hyperbolic if and only if $\mathcal{G}(Y)$ is uniformly hyperbolic, for every finite-type witness Y.*

We stress that, once one is given an *explicit* subgraph $\mathcal{G}(S)$ of $\mathcal{A}(S)$, it is trivial to decide what the witnesses of $\mathcal{G}(S)$ are and, in particular, where $\mathcal{G}(S)$ falls in the description offered by Theorem 1.1.

[1] This definition is due to Schleimer [16], who referred to witnesses as *holes*. The word “witness” has been suggested to us by S. Schleimer.

Remark 1.2. As we will see in Section 4, the assumptions that $\mathcal{G}(S)$ is sufficiently invariant and has the projection property will not be used in the proof of part (1) of Theorem 1.1.

The main reason why in the theorem above we demand that every two witnesses intersect is the following manifestation of Schleimer's *Disjoint Witnesses Principle* [16, 14]:

Proposition 1.3. *Let S be a connected orientable surface of infinite type, and $\mathcal{G}(S)$ a connected, sufficiently invariant subgraph of $\mathcal{A}(S)$ with the projection property. If $\mathcal{G}(S)$ has two disjoint witnesses of finite type, then it contains a quasi-isometrically embedded copy of $\mathbb{Z}^2$. In particular, $\mathcal{G}(S)$ is not hyperbolic.*

For completeness, we will give a proof of Proposition 1.3 in Section 4.

As an immediate consequence of Theorem 1.1 we recover the main results of [2] and [1]; see Section 4 for the precise statements. Another corollary of Theorem 1.1 is that there are no geometrically interesting, $\mathrm{Mod}(S)$-invariant subgraphs of $\mathcal{A}(S)$ for many surfaces with infinitely many punctures:

Corollary 1.4. *Let S be a connected orientable surface with punctures, such that the* $\mathrm{Mod}(S)$*-orbit of every puncture of S is infinite. If $\mathcal{G}(S)$ is a connected* $\mathrm{Mod}(S)$*-invariant subgraph of $\mathcal{A}(S)$, then $\mathcal{G}(S)$ has finite diameter.*

In sharp contrast, if S has finitely many punctures, then $\mathcal{A}(S)$ is a hyperbolic graph of infinite diameter by Theorem 1.1. Even if S has infinitely many punctures, the requirement that the orbit of every puncture be infinite is still necessary; compare with Remark 4.8 below.

As a special case of the corollary above, we obtain:

Corollary 1.5. *Let S closed orientable surface of genus $g \geq 0$ minus a Cantor set. If $\mathcal{G}(S)$ is a connected,* $\mathrm{Mod}(S)$*-invariant subgraph of $\mathcal{A}(S)$, then $\mathcal{G}(S)$ has finite diameter.*

In other words, there are no geometrically interesting $\mathrm{Mod}(S)$-invariant subgraphs of $\mathcal{A}(S)$ if S has infinitely many punctures and none of them are isolated (in the set of punctures). In particular, for such surfaces no subgraph of $\mathcal{A}(S)$ will be useful for understanding groups acting *freely* by homeomorphisms on S; compare with the discussion before subsection 1.1 above.

For this reason, we are going to study $\mathrm{Mod}(S)$-invariant subgraphs of the curve graph $\mathcal{C}(S)$ instead. As we will see below, the situation will depend heavily on whether the number of punctures (resp. the genus) of S is finite or infinite. Before we state our result, we denote by $\mathrm{NonSep}(S)$ the *nonseparating curve graph* of S, namely the subgraph of $\mathcal{C}(S)$ spanned by all nonseparating curves on S. Similarly, denote by $\mathrm{Outer}(S)$ the subgraph of $\mathcal{C}(S)$ spanned by all the *outer curves* on S, namely those which cut off a disk with punctures. See Section 3 for further definitions.

We start with the case when the genus of S is finite:

Theorem 1.6. *Let S be a connected orientable punctured surface of finite genus $g \geq 0$, such that the $\mathrm{Mod}(S)$-orbit of every puncture is infinite. Then, a $\mathrm{Mod}(S)$-invariant subgraph $\mathcal{G}(S) \subset \mathcal{C}(S)$ has infinite diameter if and only if $\mathcal{G}(S) \cap \mathrm{Outer}(S) = \emptyset$. Moreover, in this case:*

1. *If $\mathcal{G}(S) \cap \mathrm{NonSep}(S) = \emptyset$ then $\mathcal{G}(S)$ is not hyperbolic.*
2. *If $\mathcal{G}(S) \cap \mathrm{NonSep}(S) \neq \emptyset$ then $\mathcal{G}(S)$ is quasi-isometric to $\mathrm{NonSep}(S)$.*

Since every curve on a surface of genus 0 is separating, we have:

Corollary 1.7. *Let S be a sphere with punctures, such that the $\mathrm{Mod}(S)$-orbit of every puncture is infinite. Then any connected, $\mathrm{Mod}(S)$-invariant subgraph of $\mathcal{C}(S)$ has finite diameter.*

In particular, the corollary above applies when S is a sphere minus a Cantor set; compare with Corollary 1.5 above.

Remark 1.8. In sharp contrast to Corollary 1.7, if S is a punctured sphere with a finite number of *isolated* punctures, Durham-Fanoni-Vlamis [4] have recently identified a hyperbolic, infinite-diameter subgraph of $\mathcal{C}(S)$. More concretely, they consider a certain subgraph $\mathrm{Outer}(S)$, which they show is uniformly quasi-isometric to the relative arc graphs of [2] and [1].

In light of Theorem 1.6, a natural problem is to decide whether $\mathrm{NonSep}(S)$ is hyperbolic for S a surface of genus g and with infinitely many punctures. As we will see in Proposition 5.1 below, the answer is positive if and only if $\mathrm{NonSep}(S_{g,n})$ is hyperbolic *uniformly* in n; compare with part (3) of Theorem 1.1 above. We remark that $\mathrm{NonSep}(S_{g,n})$ is known to be hyperbolic by the work of Masur-Schleimer [14] and Hamensdädt [10].

We now deal with the case when the number of punctures of S is finite; however, we remind the reader than in this case $A(S)$ is itself a hyperbolic $\mathrm{Mod}(S)$-invariant graph of infinite diameter; see the comment after Corollary 1.5. We will prove:

Theorem 1.9. *Let S be a connected orientable surface of infinite genus and with finitely many punctures, and $\mathcal{G}(S)$ a $\mathrm{Mod}(S)$-invariant subgraph of $\mathcal{C}(S)$. If $\mathrm{diam}(\mathcal{G}(S)) = \infty$ then $\mathcal{G}(S) \subset \mathrm{Outer}(S)$.*

As $\mathrm{Outer}(S) = \emptyset$ if S has exactly one puncture, we deduce:

Corollary 1.10. *Suppose S is a connected orientable surface of infinite genus with exactly one puncture*[2]*. Then any $\mathrm{Mod}(S)$-invariant subgraph of $\mathcal{C}(S)$ has finite diameter.*

In light of Theorem 1.9, an interesting problem is to decide whether $\mathrm{Outer}(S)$ is hyperbolic, for a surface of infinite genus and a finite number n of punctures. Here, the answer will be positive if and only if $\mathrm{Outer}(S_{g,n})$

[2]This surface is sometimes referred to in the literature as the *Loch Ness Monster*.

is hyperbolic *uniformly in g*. See Proposition 5.3 below and the discussion around it for more details and known partial results about $\mathrm{Outer}(S)$ and some related graphs.

Finally, we deal with surfaces with infinite genus and infinitely many punctures:

Theorem 1.11. *Let S be a connected orientable punctured surface of infinite genus, such that the $\mathrm{Mod}(S)$-orbit of every puncture is infinite. If $\mathcal{G}(S)$ is a $\mathrm{Mod}(S)$-invariant subgraph of $\mathcal{C}(S)$, then $\mathrm{diam}(\mathcal{G}(S)) = 2$.*

Remark 1.12. Again, the assumption that the $\mathrm{Mod}(S)$-orbit of every puncture be infinite is most definitely necessary; see Remark 1.8 above.

The plan of the paper is as follows. Section 2 provides the necessary background on δ-hyperbolic spaces and quasi-isometries. In Section 3 we briefly introduce mapping class groups and various combinatorial complexes one can associate to a surface. In Section 4 we prove Theorem 1.1 and discuss some of its consequences. Finally, Section 5 contains the proofs of Theorems 1.6, 1.9, and 1.11, as well as some concrete applications and open questions.

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2. Metric spaces

We briefly recall some notions on large-scale geometry that will be used in the sequel. For a thorough discussion, see [9].

Definition 2.1 (Hyperbolic space)**.** Let X be a geodesic metric space. We say that X is *δ-hyperbolic* if there exists $\delta \geq 0$ such that every triangle $T \subset X$ is *δ-thin*: there exists a point $c \in X$ at distance at most δ from every side of T.

We will simply say that a geodesic metric space is *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Definition 2.2 (Quasi-isometry)**.** Let $(X, d_X), (Y, d_Y)$ be two geodesic metric spaces. We say that a map $f : (X, d_X) \to (Y, d_Y)$ is a *quasi-isometric embedding* if there exist $\lambda \geq 1$ and $C \geq 0$ such that

$$\frac{1}{\lambda} d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + C, \tag{1}$$

for all $x, x' \in X$. We say that f is a *quasi-isometry* if, in addition to (1), there exists $D \geq 0$ such that Y is contained in the D-neighbourhood of $f(X)$. More concretely, for all $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq D$.

We say that two spaces are *quasi-isometric* if there exists a quasi-isometry between them. The following is well-known:

Proposition 2.3. *Suppose that two geodesic metric spaces X, Y are quasi-isometric to each other. Then X is hyperbolic if and only if Y is hyperbolic.*

3. Arcs, curves, and witnesses

In this section we will introduce the necessary definitions about surfaces, arcs, and curves that appear in our results. Throughout, let S be a connected, orientable surface of infinite topological type. Let Π be a (possibly empty) set of marked points on S, which we feel free to regard as marked points, punctures, or topological ends of S. We recall that, up to homeomorphism, S is completely determined by its genus and its space of ends; see [15].

3.1. **Mapping class group.** The mapping class group $\mathrm{Mod}(S)$ is the group of self-homeomorphisms of S that preserve Π setwise, up to isotopy preserving Π setwise. Given a (possibly empty) finite subset P of Π, we define $\mathrm{Mod}(S, P)$ to be the subgroup of $\mathrm{Mod}(S)$ whose every element preserves P setwise. Observe that $\mathrm{Mod}(S, \emptyset) = \mathrm{Mod}(S)$.

3.2. **Arcs and curves.** By a *curve* on S we mean the isotopy class of a simple closed curve on S that does not bound a disk with at most one puncture. An *arc* on S is the isotopy class of a simple arc on S with both endpoints in Π.

The *arc and curve graph* $\mathcal{AC}(S)$ of S is the simplicial graph whose vertices are all arcs and curves on S, and where two vertices are adjacent in $\mathcal{AC}(S)$ if they have disjoint representatives on S. As is often the case, we turn $\mathcal{AC}(S)$ into a geodesic metric space by declaring the length of each edge to be 1.

Observe that $\mathrm{Mod}(S)$ acts on $\mathcal{AC}(S)$ by isometries. As mentioned in the introduction, we will be interested in subgraphs of $\mathcal{A}(S)$ that are invariant under big subgroups of $\mathrm{Mod}(S)$. More concretely, we have the following definition:

Definition 3.1 (Sufficient invariance)**.** We say that a subgraph $\mathcal{G}(S)$ of $\mathcal{AC}(S)$ is *sufficiently invariant* if there exists a (possibly empty) subset P of Π such that $\mathrm{Mod}(S, P)$ acts on $\mathcal{G}(S)$.

We will be interested in various standard $\mathrm{Mod}(S)$-invariant subgraphs of $\mathcal{AC}(S)$, whose definition we now recall.

The *arc graph* $\mathcal{A}(S)$ is the subgraph of $\mathcal{AC}(S)$ spanned by all vertices of $\mathcal{AC}(S)$ that correspond to arcs on S; note that $\mathcal{A}(S) = \emptyset$ if and only if $\Pi = \emptyset$. Observe that if S has infinitely many punctures then $\mathcal{A}(S)$ has finite diameter.

Similarly, the *curve graph* $\mathcal{C}(S)$ is the subgraph spanned by those vertices that correspond to curves on S. Note that $\mathcal{C}(S)$ has diameter 2 for every surface of infinite type.

A further subgraph is the *nonseparating curve graph* $\mathrm{NonSep}(S)$, whose vertices are curves on S whose complement is connected. This graph has diameter 2 if S has infinite genus.

Finally, the *outer curve graph* $\mathrm{Outer}(S)$ is the subgraph of $\mathcal{C}(S)$ spanned by those curves that bound a disk with punctures on S. Observe that $\mathrm{Outer}(S) = \emptyset$ if S is closed or has exactly one puncture, and that $\mathrm{Outer}(S)$ has finite diameter if S has infinitely many punctures.

As the reader may suspect at this point, these observations constitute the main source of inspiration behind the statements of Theorems 1.6, 1.9, and 1.11

3.3. **Witnesses.** Let S be a connected orientable surface of infinite type, and $\mathcal{G}(S)$ a connected subgraph of $\mathcal{AC}(S)$. As mentioned in the introduction, we will use the following notion, originally due to Schleimer [16]:

Definition 3.2 (Witness)**.** A *witness* of $\mathcal{G}(S)$ is an essential subsurface $Y \subset S$ such that every vertex of $\mathcal{G}(S)$ intersects Y essentially.

Remark 3.3. Observe that if Y is a witness of $\mathcal{G}(S)$ and Z is a subsurface of S such that $Y \subset Z$, then Z is also a witness.

Example 3.4. For the sake of concreteness, let S be a connected orientable surface of finite genus g, possibly with infinitely many punctures.

1. If $\mathcal{G}(S) = \mathcal{A}(S)$, then $Y \subset S$ is a witness if and only if Y contains every puncture of S.
2. If $\mathcal{G}(S) = \mathcal{C}(S)$, then $Y \subset S$ is a witness if and only if $Y = S$.
3. If $\mathcal{G}(S) = \mathrm{NonSep}(S)$, then $Y \subset S$ is a witness if and only if Y has genus g.

On the other hand, if S has a finite number n of punctures, then Y is a witness of $\mathrm{Outer}(S)$ if and only if Y contains at least $n-1$ punctures.

4. Subgraphs of the arc complex

In this section we give a proof of Theorem 1.1. As hinted to in the introduction, the main tool is the following variant of Masur-Minsky's *subsurface projections* [13]:

Subsurface projections. Let Y be a witness of $\mathcal{G}(S)$. There is a natural projection

$$\pi_Y : \mathcal{G}(S) \to \mathcal{A}(Y)$$

defined by setting $\pi_Y(v)$ to be any connected component of $v \cap Y$. In particular, $\pi_Y(v) = v$ for every $v \subset Y$; in other words, the restriction of π_Y to $\mathcal{G}(Y)$ is the identity. Observe that the definition of π_Y involves a choice, but any two such choices are disjoint and therefore at distance 1 in $\mathcal{A}(Y)$. The same argument gives:

Lemma 4.1. *Let S be a surface and Y an essential subsurface. If u, v are disjoint arcs which intersect Y essentially, then $\pi_Y(u)$ and $\pi_Y(v)$ are disjoint.*

For technical reasons, which will become apparent in the proof of Lemma 4.3 below, we will be interested in subgraphs of $\mathcal{A}(S)$ for which the subsurface projections defined above satisfy the following property:

Definition 4.2 (Projection property)**.** We say that a subgraph $\mathcal{G}(S) \subset \mathcal{A}(S)$ has the *projection property* if, for every finite-type witness Y of $\mathcal{G}(S)$, the graphs $\pi_Y(\mathcal{G}(S))$ and $\mathcal{G}(Y)$ are uniformly quasi-isometric, via a quasi-isometry that is the identity on $\mathcal{G}(Y)$.

As mentioned in the introduction, we remark that deciding whether a given *explicit* subgraph of $\mathcal{A}(S)$ has the projection property is normally easy to check; see Remark 4.6 below.

The following lemma, which is a small variation of Corollary 4.2 in [1], is the main ingredient in the proof of Theorem 1.1. We note that this is the sole instance in which we will make use of the assumption that $\mathcal{G}(S)$ has the projection property.

Lemma 4.3. *Let S be a surface of infinite type, and $\mathcal{G}(S) \subset \mathcal{A}(S)$ a connected subgraph with the projection property. Then, for every finite-type witness Y of $\mathcal{G}(S)$, the subgraph $\mathcal{G}(Y)$ is uniformly quasi-isometrically embedded in $\mathcal{G}(S)$.*

Proof. Let u, v be arbitrary vertices of $\mathcal{G}(Y)$. First, observe that since $\mathcal{G}(Y) \subset \mathcal{G}(S)$, we have

$$d_{\mathcal{G}(S)}(u, v) \leq d_{\mathcal{G}(Y)}(u, v).$$

To show a reverse coarse inequality, we proceed as follows. Consider a geodesic $\gamma \subset \mathcal{G}(S)$ between u and v. The projected path $\pi_Y(\gamma)$ is a path in $\pi_Y(\mathcal{G}(S))$ between $u = \pi_Y(u)$ and $v = \pi_Y(v)$, and

$$\text{length}_{\pi_Y(\mathcal{G}(S))}(\pi_Y(\gamma)) \leq \text{length}_{\mathcal{G}(S)}(\gamma),$$

by Lemma 4.1. In particular,

$$d_{\pi_Y(\mathcal{G}(S))}(u, v) \leq d_{\mathcal{G}(S)}(u, v).$$

Since $\mathcal{G}(S)$ has the projection property, there exist constants $L \geq 1$ and $C \geq 0$ (which depend only on S) such that

$$d_{\mathcal{G}(Y)}(u, v) \leq L \cdot d_{\pi_Y(\mathcal{G}(S))}(u, v) + C,$$

and thus the result follows by combining the above two inequalities. $\square$

Remark 4.4. Observe that if Y has finite type and $\text{Mod}(Y)$ is infinite, then any of the (finitely many) $\text{Mod}(Y)$-invariant subgraphs of $\mathcal{A}(Y)$ has infinite diameter; a proof of this is essentially contained Corollary 2.25 of [16], for instance. The proof of the first claim part (2) of Theorem 1.1 boils down to this fact.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let S be a connected, orientable surface of infinite type, and denote by Π the set of marked points of S. Let $\mathcal{G}(S)$ be a connected subgraph of $\mathcal{A}(S)$ with the projection property, and invariant under $\mathrm{Mod}(S, P)$ for some $P \subset \Pi$ finite (possibly empty). Further, suppose that every two witnesses of $\mathcal{G}(S)$ intersect.

We first prove part (1); in fact, we will show that the diameter of $\mathcal{G}(S)$ is at most 4. Let u, v be two arbitrary distinct vertices of $\mathcal{G}(S)$. Since $\mathcal{G}(S)$ is connected, there exists $w \in \mathcal{G}(S)$ that intersects both u and v a finite number of times. We now claim that there is a vertex $z \in \mathcal{G}(S)$ that is disjoint from v and w. Indeed, consider the surface $F(v, w)$ filled by v and w, which has finite type since v and w intersect finitely many times. Since every witness of $\mathcal{G}(S)$ has infinitely many punctures, by assumption, we deduce that $F(v, w)$ is not a witness, and therefore there exists a vertex $z \in \mathcal{G}(S)$ that does not intersect $F(v, w)$. Using the same reasoning, there exists a vertex $z' \in \mathcal{G}(S)$ that is disjoint from u and w. Thus,

$$u \to z' \to w \to z \to v$$

is a path of length at most 4 in $\mathcal{G}(S)$ between u and v, as desired.

We now proceed to prove part (2), arguing along similar lines to [1]. By assumption, there exists a witness Y of $\mathcal{G}(S)$ with finitely many punctures. After modifying Y if necessary, we may assume that Y has finite type and $\mathrm{Mod}(Y)$ is infinite. In particular $\mathcal{G}(Y)$ has infinite (intrinsic) diameter; see Remark 4.4 above. Since $\mathcal{G}(Y)$ is quasi-isometrically embedded in $\mathcal{G}(S)$, by Lemma 4.3, it follows that

$$\mathrm{diam}(\mathcal{G}(S)) = \infty,$$

and thus the first part of the claim follows.

It remains to prove the second part of the claim. To this end, suppose first that there exists $\delta = \delta(S)$ such that $\mathcal{G}(Y)$ is δ-hyperbolic, for every finite-type witness Y. We will prove that $\mathcal{G}(S)$ is δ-hyperbolic. Indeed, consider a geodesic triangle $T \subset \mathcal{G}(S)$. Since T has finitely many vertices, there exists a finite-type witness $Z = Z(T)$ of $\mathcal{G}(S)$ such that every vertex of T is contained in Z; we remark that by enlargening the vertex set of T if necessary, we may assume that Z is connected. We may thus regard T as a geodesic triangle in $\mathcal{G}(Z)$. Since $\mathcal{G}(Z)$ is δ-hyperbolic, by assumption, T has a δ-center $c \in \mathcal{G}(Z)$. Now, since $\mathcal{G}(Z) \subset \mathcal{G}(S)$, the $\mathcal{G}(S)$-distance from c to each of the three sides of T is at most δ; in other words, c also serves as a δ-center for T when the latter is viewed as a geodesic triangle in $\mathcal{G}(S)$. Since T is arbitrary and δ is uniform, we obtain that $\mathcal{G}(S)$ is δ-hyperbolic, as claimed.

Finally, using a very similar argument to the one just given, we also deduce that the hyperbolicity of $\mathcal{G}(S)$ implies that of $\mathcal{G}(Y)$, for every finite-type witness Y of $\mathcal{G}(S)$. This finishes the proof of Theorem 1.1. $\square$

Remark 4.5. Observe that in the proof of part (1) of Theorem 1.1 we have not used that $\mathcal{G}(S)$ is sufficiently invariant or that it has the projection property, only that it is connected and every witness has infinitely many punctures. Thus this part of Theorem 1.1 holds in more generality.

As mentioned in the introduction, Proposition 1.3 asserts that the assumption that every two witnesses of $\mathcal{G}(S)$ intersect is necessary. While this proposition is merely a manifestation of Schleimer's *Disjoint Witness Principle* [16], we include a proof for completeness:

Proof of Proposition 1.3. We first prove part (1). Assume that $\mathcal{G}(S)$ has two disjoint witnesses $Y, Z \subset S$, each of finite type. After enlarging Y and/or Z if necessary, we may assume in addition that both $\mathrm{Mod}(Y)$ and $\mathrm{Mod}(Z)$ are infinite; note that these groups act on $\mathcal{G}(Y)$ and $\mathcal{G}(Z)$, respectively, which in turn gives that $\mathcal{G}(Y)$ and $\mathcal{G}(Z)$ have infinite (intrinsic) diameter; see again Remark 4.4 above.

Since Y and Z are witnesses, the projection maps π_Y and π_Z are well-defined. Therefore there is a projection map

$$\pi : \mathcal{G}(S) \to \mathcal{A}(Y) \times \mathcal{A}(Z)$$

which is simply the map $\pi_Y \times \pi_Z$. Using this projection and the same arguments as in the proof of Lemma 4.3, the fact that $\mathcal{G}(S)$ has the projection property implies that $\mathcal{G}(S)$ contains a quasi-isometrically embedded copy of $\mathcal{G}(Y) \times \mathcal{G}(Z)$. By choosing a bi-infinite quasi-geodesic in $\mathcal{G}(Y)$ and in $\mathcal{G}(Z)$, we obtain $\mathcal{G}(S)$ contains a quasi-isometrically embedded copy of $\mathbb{Z}^2$, as claimed. □

4.1. **Consequences.** We now discuss a number of applications of Theorem 1.1 to more concrete situations. To start with, we have the following corollaries, mentioned in the introduction:

Corollary 1.4. Let S be a connected orientable surface with punctures, such that the $\mathrm{Mod}(S)$-orbit of every puncture of S is infinite. If $\mathcal{G}(S)$ is a connected $\mathrm{Mod}(S)$-invariant subgraph of $\mathcal{A}(S)$, then $\mathcal{G}(S)$ has finite diameter.

Proof. Let $\mathcal{G}(S)$ be such a connected $\mathrm{Mod}(S)$-invariant subgraph of $\mathcal{A}(S)$. Since $\mathrm{Mod}(S)$ acts on $\mathcal{G}(S)$ and the $\mathrm{Mod}(S)$-orbit of every puncture is infinite, we get that any witness of $\mathcal{G}(S)$ must contain an infinite number of punctures. Thus part (1) of Theorem 1.1 applies (compare with Remark 4.5), and the result follows. □

Next, we show how Theorem 1.1 implies the main results of [2] and Aramayona-Fossas-Parlier [1]. Before doing so, we need some definitions from [1]. Throughout, we will assume that the set Π of marked points of S is not empty. We say that a marked point $p \in \Pi$ is *isolated* if it is isolated in Π, where the latter is equipped with the subspace topology (here we are viewing Π as a set of marked points on S). Let $P \subset \Pi$ be a non-empty finite subset of marked points on S. Define $\mathcal{A}(S, P)$ as the subgraph of $\mathcal{A}(S)$ spanned by those arcs that have at least one endpoint in P. Note that $\mathrm{Mod}(S, P)$ acts on $\mathcal{A}(S, P)$, and hence $\mathcal{A}(S, P)$ is sufficiently invariant.

Remark 4.6. The graphs $\mathcal{A}(S, P)$ have the projection property: if Y is an essential subsurface of S then $\pi_Y(\mathcal{A}(S, P))$ is uniformly quasi-isometric to $\mathcal{A}(Y, P \cap Y)$, which is $\mathcal{G}(Y)$ for $\mathcal{G}(S) = \mathcal{A}(S, P)$. The proof that both graphs are quasi-isometric boils down to the fact that, for $v \in \mathcal{A}(S, P)$, there is at least one component of $v \cap Y$ that has an endpoint in P, which we can use to define a subsurface projection map with nice properties.

Observe that $Y \subset S$ is a witness of $\mathcal{A}(S, P)$ if and only if it contains every element of P. First, we have:

Corollary 4.7 ([1], Theorem 1.1)**.** *If $P \subset \Pi$ is a finite set of isolated punctures, then $\mathcal{A}(S, P)$ is hyperbolic and has infinite diameter.*

Proof. Since P is finite and every puncture is isolated, there exists a witness containing only finitely many punctures (any finite-type surface containing P will do). Now, part (2) of Theorem 1.1 applies with $\mathcal{G}(S) = \mathcal{A}(S, P)$, and thus $\mathcal{A}(S, P)$ has infinite diameter. Moreover, if Y is a finite-type witness of $\mathcal{A}(S, P)$ then $\mathcal{G}(Y) = \mathcal{A}(Y, P \cap Y)$, which is 7-hyperbolic by [11]. □

We stress that, in the particular case when S is a sphere minus the union of the north pole and a Cantor set, the above result is originally due to Bavard [2].

Remark 4.8. Observe that $\mathrm{Mod}(S)$ preserves the set of isolated punctures of S. Therefore, if the set P of isolated punctures of S is finite, then $\mathcal{A}(S, P)$ is a connected, $\mathrm{Mod}(S)$-invariant which is hyperbolic and has infinite diameter. Therefore we see that, in Corollary 1.4, the requirement about the $\mathrm{Mod}(S)$-orbits of punctures of S being infinite is necessary.

On the other hand, if P contains punctures that are not isolated, then the situation is drastically different. More concretely, we recover the following observation due to Bavard (stated as Proposition 3.5 of [1])

Corollary 4.9. *Suppose $P \subset \Pi$ contains a puncture that is not isolated. Then $\mathcal{A}(S, P)$ has finite diameter.*

Proof. In this setting, every witness of Y contains infinitely many punctures. Now part (1) of Theorem 1.1 applies. □

Finally, one could define, for disjoint finite subsets P, Q of isolated punctures, let $\mathcal{A}(S, P, Q)$ be the subgraph of $\mathcal{A}(S)$ spanned by those arcs that have one endpoint in P and the other in Q. In this situation we have the following result, also due to Bavard (unpublished):

Corollary 4.10. *The graph $\mathcal{A}(S, P, Q)$ is not hyperbolic.*

Proof. Observe that Y is a witness of $\mathcal{A}(S, P, Q)$ if and only if it contains P or Q. In particular, there are two disjoint witnesses of finite type, and Proposition 1.3 applies. □

5. Subgraphs of the curve graph

In this section we deal with connected subgraphs of the curve graph, proving Theorems 1.6, 1.9, and 1.11. As mentioned in the introduction, we restrict our attention to the case when $\mathrm{Mod}(S)$ acts on the relevant subgraph.

We first prove Theorem 1.6. The arguments we will use are similar in spirit to those used in the previous section, but adapted to this particular setting.

Proof of Theorem 1.6. Let S be a surface as in the statement, and $\mathcal{G}(S)$ a connected, $\mathrm{Mod}(S)$-invariant subgraph of $\mathcal{C}(S)$.

Suppose first that $\mathcal{G}(S) \cap \mathrm{Outer}(S) \neq \emptyset$. We want to conclude that $\mathrm{diam}(\mathcal{G}(S)) = 2$. To this end, let α and β be arbitrary vertices of $\mathcal{G}(S)$. If α and β are disjoint, there is nothing to prove, so assume that $i(\alpha, \beta) \neq 0$. Let $F(\alpha, \beta)$ be the subsurface of S filled by α and β, which has finite topological type since α and β are compact. Therefore, there exists a connected component Y of $S - F(\alpha, \beta)$ that has infinitely many punctures. Let $\gamma \in \mathcal{G}(S) \cap \mathrm{Outer}(S)$. Since $\mathrm{Mod}(S)$ acts on $\mathcal{G}(S)$ and the $\mathrm{Mod}(S)$-orbit of every puncture is infinite, there exists $h \in \mathrm{Mod}(S)$ such that $h(\gamma) \subset Y$. In particular, $h(\gamma)$ is disjoint from both α and β and hence $d_{\mathcal{G}(S)}(\alpha, \beta) = 2$.

Hence from now on, we assume that $\mathcal{G}(S) \cap \mathrm{Outer}(S) = \emptyset$. Suppose first that, in addition, $\mathcal{G}(S) \cap \mathrm{NonSep}(S) = \emptyset$, and so every element of $\mathcal{G}(S)$ is a curve that separates S into two surfaces of positive genus. As remarked by Schleimer (see Exercise 2.42 of [16]), $\mathcal{G}(S)$ has two disjoint witnesses, and thus will fail to be hyperbolic. To construct these witnesses, consider a multicurve M consisting of $g + 1$ non-separating curves on S such that $S - M$ is the disjoint union of two spheres W_1, W_2 with punctures which, by construction, are witnesses for $\mathcal{G}(S)$. Let P_i be the finite subset of punctures of W_i coming from the elements of M. Using subsurface projections as in the previous section gives a quasi-isometric embedding

$$\mathcal{A}(W_1, P_1) \times \mathcal{A}(W_2, P_2) \to \mathcal{G}(S),$$

thus obtaining a quasi-isometrically embedded copy of $\mathbb{Z}^2$ inside $\mathcal{G}(S)$. In particular, $\mathcal{G}(S)$ is not hyperbolic and has infinite diameter.

Suppose now that $\mathcal{G}(S) \cap \mathrm{NonSep}(S) \neq \emptyset$, which in particular implies that $\mathrm{NonSep}(S) \subset \mathcal{G}(S)$, since $\mathrm{Mod}(S)$ acts on $\mathcal{G}(S)$. Note that, in addition, S must have positive genus. We first claim:

Claim. The inclusion map $\mathrm{NonSep}(S) \hookrightarrow \mathcal{G}(S)$ is a quasi-isometry.

Proof of Claim. We begin by showing that the inclusion map is a quasi-isometric embedding. In fact, more is true: we will prove that, given $\alpha, \beta \in \mathrm{NonSep}(S)$ and a geodesic σ in $\mathcal{G}(S)$ between them, we can modify σ to a geodesic σ' in $\mathrm{NonSep}(S)$ of the same length. (We remark that this argument is contained in the proof that the nonseparating curve complex is connected; see Theorem 4.4 of [5].) Let $\gamma \in \sigma$ be a separating curve. Then $S - \gamma = Y \cup Z$, where Y and Z both have positive genus since $\gamma \notin \mathrm{Outer}(S)$. Let γ_L and γ_R be the vertices of σ preceding (resp. following) γ. The assumption that σ is geodesic implies that either $\gamma_L, \gamma_R \subset Y$ or $\gamma_L, \gamma_R \subset Z$; suppose for the sake of concreteness that we are in the former case. Since Z has positive genus, it contains a nonseparating curve γ' which, by construction, is disjoint from γ_L and γ_R. Replacing γ by γ' on σ produces a geodesic in $\mathcal{G}(S)$ with a strictly smaller number of separating curves.

At this point, we know that the inclusion map $\mathrm{NonSep}(S) \hookrightarrow \mathcal{G}(S)$ is a quasi-isometric embedding. To see that it is a quasi-isometry, observe that every element of $\mathcal{G}(S)$ is at distance at most 1 from an element of $\mathrm{NonSep}(S)$. This finishes the proof of the claim. □

In order to finish the proof of the theorem, it remains to show:

Claim. $\mathrm{diam}(\mathrm{NonSep}(S)) = \infty$.

Proof of Claim. In a similar fashion to what we did in the previous section, we are going to prove that, for every finite-type witness Y, the subgraph $\mathrm{NonSep}(Y)$ is quasi-isometrically embedded in $\mathrm{NonSep}(S)$; once this has been done, the claim will follow since $\mathrm{NonSep}(Y)$ has infinite diameter, a fact that follows from Corollary 2.25 of [16], for instance.

In this direction, let Y be a finite-type witness of $\mathrm{NonSep}(S)$; in other words, Y is a finite-type subsurface of S of the same genus as S; see Example 3.4 above. Let $\mathcal{A}(Y, \partial Y)$ be the subgraph of $\mathcal{A}(Y)$ spanned by those vertices that have both endpoints on ∂Y. Similarly, let $\mathcal{A}\,\mathrm{NonSep}(Y)$ be the subgraph of $\mathcal{AC}(Y)$ spanned by the vertices of $\mathrm{NonSep}(Y) \cup \mathcal{A}(Y, \partial Y)$. The inclusion map

$$\mathrm{NonSep}(Y) \hookrightarrow \mathcal{A}\,\mathrm{NonSep}(Y)$$

is a quasi-isometry, where the constants do not depend on Y; to see this, one may use the standard argument to show that the embedding of $C(Y)$ into $\mathcal{AC}(Y)$ is a uniform quasi-isometry (see for instance Exercise 3.15 of [16]).

Now, as in the previous section there is a subsurface projection

$$\pi_Y : \mathrm{NonSep}(S) \to \mathcal{A}\,\mathrm{NonSep}(Y)$$

that associates, to an element of $\mathrm{NonSep}(S)$, its intersection with Y. Using an analogous reasoning to that of Lemma 4.3, we obtain that $\mathcal{A}\,\mathrm{NonSep}(Y)$, and therefore, $\mathrm{NonSep}(Y)$, is uniformly quasi-isometrically embedded in $\mathrm{NonSep}(S)$, as desired. This finishes the proof of the claim, and thus that of Theorem 1.6. □

□

The graph $\mathrm{NonSep}(S)$ has an intriguing geometric structure. Indeed, using a small variation of the proof of Theorem 1.1, we obtain:

Proposition 5.1. *Let S be a connected surface of finite genus g and with infinitely many punctures. Then* $\mathrm{NonSep}(S)$ *is hyperbolic if and only if* $\mathrm{NonSep}(S_{g,n})$ *is hyperbolic uniformly in n.*

As remarked by Example 3.4 above, a subsurface Y of S is a witness for $\mathrm{NonSep}(S)$ if and only if Y has genus g. Thus the finite-type witnesses of $\mathrm{NonSep}(S)$ are precisely the subsurfaces of the form $S_{g,n}$; compare with part (3) of Theorem 1.1.

Proof of Proposition 5.1. Let T be a geodesic triangle in $\mathrm{NonSep}(S)$. Since T has finitely many vertices and curves are compact, there exists a finite-type subsurface Y of S that contains every element of T. Thus we can view T as a geodesic triangle in $\mathrm{NonSep}(Y)$. If $\mathrm{NonSep}(S_{g,n})$ is hyperbolic uniformly in n, there is $\delta = \delta(g)$ such that T has a δ-center $\alpha \in \mathrm{NonSep}(Y)$ (with respect to the distance function in $\mathrm{NonSep}(Y)$). In particular, α is at distance at most δ from the sides of T, where distance is measured in $\mathrm{NonSep}(Y)$, and hence is a δ-centre for T in $\mathrm{NonSep}(S)$. Thus, $\mathrm{NonSep}(S)$ is δ-hyperbolic.

The other direction is analogous. □

As mentioned in the introduction, it is known that $\mathrm{NonSep}(S_{g,n})$ is hyperbolic [10, 14], but in principle the hyperbolicity constant may well depend on n. Thus we ask:

Question 5.2. *For fixed g, is* $\mathrm{NonSep}(S_{g,n})$ *hyperbolic uniformly in n? More generally, is it hyperbolic uniformly in both g and n?*

We now proceed to prove Theorem 1.9:

Proof of Theorem 1.9. Let S be a surface with infinite genus and finitely many punctures. Let $\alpha, \beta \in \mathcal{G}(S)$ and observe that the surface $F(\alpha, \beta)$ they fill has finite topological type. Since $\mathcal{G}(S)$ is $\mathrm{Mod}(S)$-invariant, if $\mathcal{G}(S)$ has a vertex γ that is not an outer curve, we can find an element $h \in \mathrm{Mod}(S)$ such that $h(\gamma) \subset S - F(\alpha, \beta)$. Therefore, if

$$\mathcal{G}(S) \cap (\mathcal{C}(S) - \mathrm{Outer}(S)) \neq \emptyset$$

then $\mathrm{diam}(\mathcal{G}(S)) \leq 2$, as desired. □

As mentioned in the introduction, a natural problem after Theorem 1.9 is to decide whether $\mathrm{Outer}(S)$ is hyperbolic. Using an obvious modification of the argument behind Proposition 5.1, we have:

Proposition 5.3. *Let S be an infinite-genus surface with a finite number n of punctures. Then* $\mathrm{Outer}(S)$ *is hyperbolic if and only if* $\mathrm{Outer}(S_{g,n})$ *is hyperbolic uniformly in g.*

Thus we ask:

Question 5.4. *For a fixed n, is* $\mathrm{Outer}(S_{g,n})$ *hyperbolic uniformly in g? Is it hyperbolic uniformly both g and n?*

Contrary to the situation with the nonseparating curve graph, it is not even known whether $\mathrm{Outer}(S_{g.n})$ is hyperbolic if $g \geq 1$. However, it is plausible that this is the case, in light of Masur-Schleimer's conjectural piece [14] that the only obstruction to hyperbolicity is having a pair of disjoint holes; compare with the comment after Example 3.4 above.

On the other hand, we remark that $\mathrm{Outer}(S)$ generally contains natural subgraphs that are not hyperbolic, as we now explain. Given a surface with a finite number n of punctures, denote by $\mathrm{Outer}(S,k)$ the subgraph of $\mathrm{Outer}(S)$ spanned by those vertices that separate S into two components, where one contains k punctures and the other $n-k$. We have the following observation, which again follows from Scheimer's *Disjoint Witnesses Principle* [16]:

Proposition 5.5. *Let S be a connected orientable surface of positive genus, and with a finite number $n \geq 4$ of punctures. Then* $\mathrm{Outer}(S, n-1)$ *is not hyperbolic.*

Proof. Under the assumptions of the theorem, S contains two disjoint essential subsurfaces S_1 and S_2 such that S_i contains at least $[n/2]$ punctures of S. Indeed, as in the proof of Theorem 1.6, it suffices to consider a multicurve M on S consisting of $g+1$ nonseparating curves and such that the result of cutting S open along the elements of M is the disjoint union of two punctured spheres.

Now, S_i is a witness of $\mathrm{Outer}(S, n-1)$ and, again using subsurface projections, we get that $\mathrm{Outer}(S, n-1)$ contains a quasi-isometrically embedded copy of $\mathcal{A}(S_1) \times \mathcal{A}(S_2)$. Finally, we may assume that $\mathcal{A}(S_i)$ has infinite diameter, using for instance the construction of S_i in the paragraph above. Hence the result follows. □

We stress there are other values of k apart from $n-1$ that will do, provided one chooses n large enough; we urge the interested reader to find more such values.

Finally, we prove Theorem 1.11:

Proof. Let S be a connected orientable punctured surface of infinite genus, such that the $\mathrm{Mod}(S)$-orbit of every puncture is infinite. Consider a $\mathrm{Mod}(S)$-invariant subgraph $\mathcal{G}(S)$ of $\mathcal{C}(S)$. Let α and β be arbitrary vertices of $\mathcal{G}(S)$, noting again that the subsurface $F(\alpha, \beta)$ filled by them has finite type. As such, there exist two (possibly equal) connected components Y, Z of $S - F(\alpha, \beta)$ such that Y has infinitely many punctures, while Z has infinite genus.

Let $\gamma \in \mathcal{G}(S)$ be an arbitrary curve. If γ is an outer curve, then we can use the argument of the proof of Theorem 1.9 to find an element $h \in \mathrm{Mod}(S)$ such that $h(\gamma) \subset Y$, thus giving a path of length 2 in $\mathcal{G}(S)$ between α and β. Otherwise, γ is either non-separating or else it separates S into two subsurfaces of positive genus. In either case, by the reasoning in the proof of Theorem 1.6, we can find an element $h' \in \mathrm{Mod}(S)$ such that $h'(\gamma) \subset Z$. In particular, α and β are at distance 2 in $\mathcal{G}(S)$, and we are done. $\square$

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