Hyperbolic structures on surfaces and geodesic currents

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Abstract

These are lecture notes for a course given by the authors during the program *Automorphisms of Free Groups: Geometry, Topology, and Dynamics*, held at the CRM (Barcelona) in 2012. The main objective of the notes is to describe Bonahon's construction of Thurston's compactification of Teichmüller space, in terms of geodesic currents on surfaces. In the final section, we present several extensions of the notion of geodesic current to various other more general settings.

1 Introduction

This chapter contains the lecture notes from the course "Hyperbolic structures on surfaces and geodesic currents", given by the authors during the summer school on *Automorphisms of Free Groups: Geometry, Topology, and Dynamics*, held at the CRM (Barcelona) in September 2012. The main objective of the notes is to give an account of Bonahon's description [5] of Thurston's compactification of Teichmüller space in terms of geodesic currents on surfaces. The plan of the chapter is as follows. Section 2 deals with hyperbolic structures on surfaces, explaining why a surface equipped with a complete hyperbolic structure is isometric to the quotient of $\mathbb{H}^2$ by a Fuchsian group. In Section 3 we will review some basic features of Teichmüller spaces and measured geodesic laminations, ending with some words about the "classical" construction of Thurston's compactification. In Section 4, we will introduce geodesic currents, and explain Bonahon's interpretation of the compactification of Teichmüller space. Finally, in Section 5 we will present some generalizations of the notion of geodesic currents to other settings, such as negatively curved metrics on surfaces, flat metrics on surfaces, and free groups.

The main references for the material presented here are [1,4,5,14,15,29].

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2 Hyperbolic structures on surfaces

In this section, we briefly discuss the notion of a *hyperbolic structure* on an oriented topological surface S, and explain why a surface equipped with a complete hyperbolic surface is isometric to the quotient of the hyperbolic plane $\mathbb{H}^2$ by a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, the group of orientation-preserving isometries of $\mathbb{H}^2$. We refer the reader to the sources [2, 3, 10, 11, 24, 33] for more detailed accounts of the topics treated here.

Definition 2.1 (Hyperbolic structure)**.** *Let S be an oriented topological surface without boundary. A* hyperbolic structure *on S consists of an open cover $(U_i)_{i\in I}$ of S, together with maps $\psi_i : U_i \to \mathbb{H}^2$, such that*

1. *ψ_i is an orientation-preserving homeomorphism onto its image, for all $i \in I$, and*

2. *Whenever $U_i \cap U_j \neq \emptyset$, the restriction of $\psi_j \circ \psi_i^{-1}$ to each connected component of $\psi_i(U_i \cap U_j)$ is an orientation-preserving isometry of $\mathbb{H}^2$, that is, an element of* $\mathrm{PSL}(2,\mathbb{R})$.

A surface equipped with a hyperbolic structure will be called a hyperbolic surface.

Each pair (U_i, ψ_i) is called a *chart.* Observe that a hyperbolic structure determines a Riemannian metric on S by declaring each map ψ_i in the definition above to be an isometry (in particular, this endows S with a smooth structure). In this way we can talk about the length of a (rectifiable) path on S; we say that a path on S is a *geodesic* if it locally minimizes distance between its points.

We next present two constructions of hyperbolic surfaces.

Example 1. Let Γ be a torsion free Fuchsian group, that is, a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ acting freely and properly discontinuously on $\mathbb{H}^2$. Then $S = \mathbb{H}^2/\Gamma$ is a surface and $p : \mathbb{H}^2 \to S$ is a covering map. Let $(U_i)_{i\in I}$ be a collection of evenly covered open sets of S whose union covers S, and let $\psi_i : U_i \to \mathbb{H}^2$ be a continuous 1–sided inverse to p (which exists since U_i is evenly covered). In this way, we see that the open sets (U_i), together with the maps ψ_i, give an example of a hyperbolic structure $S = \mathbb{H}^2/\Gamma$.

Example 2. Let P be a regular hyperbolic $4g$–gon with internal angle $\pi/2g$, where $g \geq 2$ is an integer. Let S be the closed orientable surface of genus g obtained by gluing the opposite sides of P by hyperbolic isometries; see Figure 1. The surface S admits a hyperbolic structure determined by charts defined on open subsets obtained from disks in the interior of P, or constructed by gluing half-disks or sectors of disks about points in the boundary of P; see again Figure 1. In fact, *Poincaré's Polygon Theorem* (see, for instance, [24]) tells us that this example is essentially equivalent to the one above; for instance, the Fuchsian group Γ of Example 1 is generated by the hyperbolic isometries used to identify the sides of the polygon.

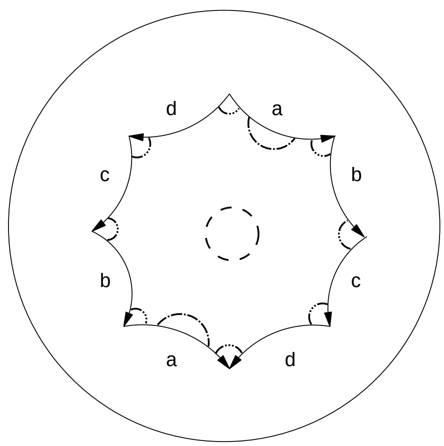

Figure 1: A regular hyperbolic octagon with internal angle $\pi/4$. Gluing opposite edges of the polygon renders a closed surface of genus 2, which is naturally equipped with a hyperbolic structure: the diagram shows charts around a point, depending on whether a lift of the point lies in the interior, on a side, or at a vertex of the polygon.

The following theorem states that Example 1 is the unique source of *complete* hyperbolic structures, that is, those hyperbolic structures that make the surface into a complete metric space; we refer the reader to [8] for a nice discussion of incomplete hyperbolic structures.

Theorem 2.2 (Cartan-Hadamard)**.** *Suppose S is equipped with a complete hyperbolic structure. Then S is isometrically diffeomorphic to $\mathbb{H}^2/\Gamma$, where Γ is a torsion free Fuchsian group.*

In particular, if S is a closed surface equipped with a hyperbolic structure, then S is (isometrically diffeomorphic to) the quotient of $\mathbb{H}^2$ by a Fuchsian group.

The idea of the proof of Theorem 2.2 is as follows; see [3] for a detailed proof. Suppose that S is equipped with a hyperbolic structure, with charts $(U_i, \psi_i)_{i\in I}$. Choose $x \in S$ and, up to relabeling, assume we have a chart defined on an open set U_1 containing x. Given a path $\alpha\colon [0,1] \to S$ with $\gamma(0) = x$, we cover it with domains of charts $U_1, \ldots, U_n$ such that only consecutive sets intersect and such that there is a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$ with $\alpha([t_{i-1}, t_i]) \subset U_i$ for each $i = 1, \ldots, n$; furthermore, up to refining the cover, we may assume that the intersection of any two consecutive open sets is connected. Using the fact that ψ_i and ψ_{i+1} differ by an element $g_i \in \mathrm{PSL}(2,\mathbb{R})$

$$\psi_i = g_i \circ \psi_{i+1}$$

we may construct a path $\alpha_H\colon [0,1] \to \mathbb{H}^2$ starting at $\psi_1(x)$ so that for each i, $\alpha_H|_{[t_{i-1},t_i]}$ agrees with $\psi_i \circ \gamma|_{[t_{i-1},t_i]}$, up to composing with an element of $\mathrm{PSL}(2,\mathbb{R})$; one can check that the terminal endpoint of α_H only depends on the homotopy class of α, rel endpoints. In this way we obtain a map

$$\mathrm{Dev} : \tilde{S} \to \mathbb{H}^2,$$

where $\tilde{S}$ denotes the universal cover of S, which associates to each $\alpha \in \tilde{S}$, the endpoint of α_H. By construction, the map Dev, called the *developing map*, is a locally isometric diffeomorphism onto its image, where $\tilde{S}$ has been equipped with the unique Riemannian metric that turns the natural covering map $\tilde{S} \to S$ into a Riemannian covering map.

In a similar fashion, to each $\alpha \in \pi_1(S, x)$ we associate a unique element of $\mathrm{PSL}(2,\mathbb{R})$, namely the composition $g_1 \circ \ldots \circ g_{n-1}$ of the maps associated to the consecutive pairs of the sets $U_1, \ldots, U_n$ chosen to cover α, where we assume $U_1 = U_n$. As above, the element of $\mathrm{PSL}(2,\mathbb{R})$ depends only on the homotopy class of α rel basepoint, and thus we obtain a map

$$\mathrm{Hol} : \pi_1(S, x) \to \mathrm{PSL}(2,\mathbb{R}),$$

called the *holonomy homomorphism.*

One can check that with the action of $\pi_1(S)$ on $\tilde{S}$, Dev is equivariant with respect to Hol:

$$\mathrm{Dev}(\alpha \cdot z) = \mathrm{Hol}(\alpha) \cdot \mathrm{Dev}(z).$$

The completeness assumption shows that Dev is a covering map. Since $\tilde{S}$ and $\mathbb{H}^2$ are simply connected, it is a homeomorphism and hence an isometric diffeomorphism. At this point, it follows from standard covering space theory that Hol is an injective homomorphism and the image is a Fuchsian group.

Remark 1. The construction of the maps Dev and Hol depend on the choice of basepoint and initial chart. However, the developing maps corresponding to two different choices differ by an isometry and the holonomy homomorphisms differ by conjugation in $\mathrm{PSL}(2, \mathbb{R})$. See [3] for details.

3 Teichmüller space

Throughout the remainder of these notes, S will be assumed to be a closed oriented surface of genus $g \geq 2$. This is for simplicity in this section, but in later sections this assumption is necessary not just for the proofs, but for the results themselves.

Theorem 2.2 yields that S admits a complete hyperbolic structure. In this section, we introduce the Teichmüller space of all hyperbolic structures on S, up to equivalence, and explain why it is homeomorphic to an open subset of $\mathbb{R}^{6g-6}$. We will end the section by briefly introducing Thurston's compactification of Teichmüller space in terms of *measured geodesic laminations* on S.

A convenient way to think of a hyperbolic structure on S is as a marked hyperbolic surface.

Definition 3.1. *A* marked hyperbolic surface *is a pair* (X, f) *where*

1. $X = \mathbb{H}^2/\Gamma$ *is a hyperbolic surface, and*
2. $f\colon S \to X$ *is an orientation-preserving homeomorphism.*

Given a marked hyperbolic surface (X, f), we can pull back the hyperbolic structure on X by f to one on S. Conversely, given a hyperbolic structure on S, the identity map $id\colon S \to S$ makes (S, id) into a marked hyperbolic surface. That said, the notion of marked hyperbolic surface is more convenient for many purposes. Now we give the definition of Teichmüller space, as a set.

Definition 3.2. *The* Teichmüller space *of S is the set*

$$\mathcal{T}(S) = \{(X, f)\}/\sim$$

of equivalence classes of marked hyperbolic surfaces, where two marked hyperbolic surfaces (X, f) and (Y, h) are deemed equivalent if $h \circ f^{-1}$ is homotopic to an isometry $X \to Y$.

In order to relax notation, we will denote a point $[(X, f)] \in \mathcal{T}(S)$ by (X, f), or simply by X with the marking homeomorphism implicit.

This definition of Teichmüller space gives a natural way to endow it with a topology. Given $[(X, f)] \in \mathcal{T}(S)$, we can choose an isomorphism $\pi_1(X) = \Gamma < \mathrm{PSL}(2, \mathbb{R})$, and then we have

$$f_*\colon \pi_1(S) \to \pi_1(X) = \Gamma.$$

Note that this is just a holonomy homomorphism for the associated hyperbolic structure on S obtained by pulling back via f. An equivalent marked surface $(Y, h) \sim (X, f)$ gives rise to a conjugate homomorphism $h_*\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$, and thus we obtain a map

$$\mathcal{T}(S) \to \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))/\text{conjugation}.$$

In fact this map is an injection. The set $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$ can be given the compact-open topology, where $\mathrm{PSL}(2, \mathbb{R})$ is given its natural topology as a quotient of the matrix group $\mathrm{SL}(2, \mathbb{R})$ and $\pi_1(S)$ is given the discrete topology. Equivalently, choosing $2g$ generators for $\pi_1(S)$, we obtain an injection

$$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R})) \to \mathrm{PSL}(2, \mathbb{R})^{2g}$$

where a homomorphism ρ is sent to the $2g$–tuple of ρ–images of the generators. The quotient $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))/\text{conjugation}$ is then given the quotient topology, and so $\mathcal{T}(S)$ is topologized as a subset of this space via the injection above.

The *mapping class group* $\mathrm{Mod}(S)$ of S is the group of isotopy classes of orientation-preserving homeomorphisms of S; in other words,

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S),$$

where $\mathrm{Homeo}_0(S)$ denotes the connected component of the identity map in the orientation-preserving homeomorphism group $\mathrm{Homeo}^+(S)$. The mapping class group acts naturally on $\mathcal{T}(S)$ by changing the marking; namely, given $\phi \in \mathrm{Mod}(S)$ and $X = [(X, f)]$, we define

$$\phi(X) = [(X, f \circ \hat{\phi}^{-1})],$$

where $\hat{\phi}$ denotes a representative of ϕ. This action is discrete and properly discontinuous (see [15], for instance); the quotient $\mathcal{T}(S)/\mathrm{Mod}(S)$ is the classical *moduli space* of S.

3.1 Length functions

As we shall see, $\mathcal{T}(S)$ is not compact. Thurston [33] constructed a $\mathrm{Mod}(S)$-invariant compactification of $\mathcal{T}(S)$ in terms of *measured laminations* on S, which may be regarded as limits of simple closed curves on the surface; compare with the paragraph preceding Theorem 3.6 below. Thurston's strategy to construct the compactification is to use *length functions* to embed $\mathcal{T}(S)$ into the function space $\mathbb{R}^{\mathcal{S}(S)}$, where $\mathcal{S}(S)$ denotes the set of all isotopy classes of simple closed curves on S, and then understand the closure of $\mathcal{T}(S)$ in the projectivized space $\mathbb{P}\mathbb{R}^{\mathcal{S}(S)}$; this is explained in detail in [1], for instance. Although this is not the approach that we will follow here, we give a brief account of this construction; along the way, we will obtain a more concrete, geometric description of Teichmüller space and its topology. We refer the reader to [1] for a complete discussion of the material discussed here.

Let (X, f) be a marked hyperbolic surface, where $X = \mathbb{H}^2/\Gamma$ and Γ is a Fuchsian group. Given an (isotopy class of) simple closed curve α on S, there is a unique simple closed geodesic on X homotopic to $f(\alpha)$. This is the projection of the axis of the element $f_*(\alpha) \in \Gamma$, where we have chosen a basepoint on α and an isomorphism $\pi_1(X) = \Gamma$. Equivalently, $f_*(\alpha)$ is a hyperbolic isometry fixing the (ideal) endpoints of a lift of $f(\alpha)$. We note that $f_*(\alpha)$ is only well-defined up to conjugacy in Γ depending on the choices involved, but the resulting geodesic in X is independent of these choices.

For every simple closed curve $\alpha \subset S$ and marked hyperbolic surface (X, f), we define

$$\ell_\alpha([(X, f)]) = \mathrm{length}(f(\alpha))$$

where $\mathrm{length}(f(\alpha))$ denotes the length, measured in the hyperbolic metric on X, of the unique simple closed geodesic homotopic to $f(\alpha)$. An equivalent marked hyperbolic surface $(Y, h) \sim (X, f)$ gives the same value, so for every simple closed curve $\alpha \subset S$ there is a well-defined function

$$\ell_\alpha \colon \mathcal{T}(S) \to \mathbb{R}.$$

To relax notation, we often write $\ell_\alpha(X)$ for $\ell_\alpha([(X, f)])$, with the marking implicit.

There is an explicit relation between $\ell_\alpha(X)$ and the trace $\mathrm{tr}(\mathrm{f}_*(\alpha))$ of $f_*(\alpha)$, viewed as an element of Γ (up to conjugacy); namely

$$\mathrm{tr}^2(\mathrm{f}_*(\alpha)) = 4\cosh^2\left(\frac{\ell_\alpha(\mathrm{X})}{2}\right) \qquad (1)$$

(see, for instance, [3]). As a consequence, length functions are continuous functions on Teichmüller space.

Consider the map

$$L : \mathcal{T}(S) \to \mathbb{R}_{\geq 0}^{\mathcal{S}(S)}$$

defined as $L(X) = (\ell_\alpha(X))_{\alpha \in \mathcal{S}(S)}$. Giving $\mathbb{R}_{\geq 0}^{\mathcal{S}(S)}$ the product topology, turns L into a continuous map. Observe that there is a natural action of $\mathbb{R}_+$ on $\mathbb{R}_{\geq 0}^{\mathcal{S}(S)} \setminus \{0\}$. Denoting the quotient space of this action by $\mathbb{P}\mathbb{R}_{\geq 0}^{\mathcal{S}(S)}$, we have the following result [1]:

Theorem 3.3. *The map*

$$L : \mathcal{T}(S) \to \mathbb{R}_{\geq 0}^{\mathcal{S}(S)}$$

is a proper embedding. Furthermore the composition of L *with the quotient map* $\mathbb{R}_{\geq 0}^{\mathcal{S}(S)} \setminus \{0\} \to \mathbb{P}\mathbb{R}_{\geq 0}^{\mathcal{S}(S)}$ *gives an embedding* $\mathcal{T}(S) \to \mathbb{P}\mathbb{R}_{\geq 0}^{\mathcal{S}(S)}$.

Our next goal is to give an idea of the proof of Theorem 3.3. Along the way we introduce *Fenchel-Nielsen coordinates*, which serve to give a concrete description of $\mathcal{T}(S)$ as an open subset of $\mathbb{R}^{6g-6}$.

3.2 Fenchel-Nielsen coordinates

It is a standard fact in plane hyperbolic geometry (see, for instance, Proposition 10.4 of [15]) that, for any $a, b, c > 0$, there is a right-angled hyperbolic hexagon with three non-consecutive sides of length a, b, c, respectively; moreover, such a hexagon is unique up to an isometry of $\mathbb{H}^2$. As a consequence, any triple $(a, b, c) \in \mathbb{R}_+^3$ determines a unique hyperbolic structure with geodesic boundary[1] on a sphere with three boundary components, or *pair of pants*, in such a way that the boundary components have prescribed lengths a, b, c.

A *pants decomposition of* S is a set of (isotopy classes of) simple closed curves $\alpha_1, \ldots, \alpha_{3g-3}$ such that every connected component of $S \setminus \cup \alpha_i$ is homeomorphic to the interior of a pair of pants. Now, fix a pants decomposition $\alpha_1, \ldots, \alpha_{3g-3}$ of S and let $X \in \mathcal{T}(S)$. If $\alpha_i, \alpha_j, \alpha_k$ bound a pair of pants in S, then by the discussion above, the hyperbolic structure on the subsurface of X bounded by the geodesic representatives of $f(\alpha_i), f(\alpha_j), f(\alpha_k)$ is uniquely determined by $\ell_{\alpha_i}(X), \ell_{\alpha_j}(X), \ell_{\alpha_k}(X)$. However, the hyperbolic structure on X is *not* uniquely determined by these numbers: we need another $3g - 3$ real numbers $t_1(X), \ldots, t_{3g-3}(X)$, called the *twist parameters*

[1] The definition of hyperbolic structure with geodesic boundary is analogous to Definition 2.1, with the modification that the elements of the cover are open subsets of a closed half-space of $\mathbb{H}^2$.

of X that measure "how twisted" on X the different pairs of pants are with respect to one another; see, for instance, Section 10.6 of [15] for details. The tuple

$$(\ell_{\alpha_1}(X), \ldots, \ell_{\alpha_{3g-3}}(X), t_1(X), \ldots, t_{3g-3}(X))$$

is called the *Fenchel-Nielsen coordinates* of the point $X \in \mathcal{T}(S)$. One then has:

Theorem 3.4. *The map*

$$\mathcal{F} : \mathcal{T}(S) \to \mathbb{R}_{+}^{3g-3} \times \mathbb{R}^{3g-3},$$

given by

$$\mathcal{F}(X) = \left(\ell_{\alpha_1}(X), \ldots, \ell_{\alpha_{3g-3}}(X), t_1(X), \ldots, t_{3g-3}(X)\right),$$

is a homeomorphism.

The proof of Theorem 3.4 follows quickly from the construction of the coordinates. Indeed, $\mathcal{F}$ is continuous since the first $3g-3$ are length functions, and the last $3g-3$ coordinates have simple expressions in terms of the associated homomorphism to $\mathrm{PSL}(2,\mathbb{R})$ showing that these are also continuous. Moreover, it is a bijection, as it admits an "obvious" inverse map: one first starts with a collection of pairs of pants whose boundary components have length prescribed by the first $3g-3$ parameters, and then glues them appropriately according to the twist parameters; again, see Section 10.6 of [15] for details.

Armed with Theorem 3.4, one can show the following stronger version of the first statement of Theorem 3.3:

Theorem 3.5. *There exist $9g-9$ simple closed curves $\gamma_1, \ldots, \gamma_{9g-9}$ on S such that the map*

$$\mathcal{N} : \mathcal{T}(S) \to \mathbb{R}_{\geq 0}^{9g-9},$$

defined as $\mathcal{N}(X) = (\ell_{\gamma_i}(X))_{i=1}^{9g-9}$, is a proper embedding.

The collection of curves of Theorem 3.5 may be chosen as follows. First, consider a pants decomposition $\alpha_1, \ldots, \alpha_{3g-3}$, together with $3g-3$ *transverse curves* $\beta_1, \ldots, \beta_{3g-3}$: these are curves with $i(\alpha_j, \beta_k) = 0$ if $j \neq k$ and $i(\alpha_j, \beta_j) = 1$ or 2, depending on the topological type of the subsurface filled by α_j and β_j. In addition consider, for each j, the curve β_j' obtained by performing a positive *Dehn twist* to β_j along α_j; see [15]. The curves $\{\alpha_j, \beta_j, \beta_j'\}_{j=1}^{3g-3}$ so constructed form the collection of $9g-9$ curves whose existence is claimed by Theorem 3.5. Again, $\mathcal{N}$ is continuous since it is given by length functions. The fact that it is injective follows from a result of Kerckhoff [25], which states that the length functions of β_j and β'_j are strictly convex functions of the j^{th} twist parameter t_j; see Section 10.6 of [15] for details. Finally, $\mathcal{N}$ is proper since the map $\mathcal{F}$ of Theorem 3.4 is a homeomorphism and the lengths of β_j, β'_j tend to infinity with t_j, whenever the length of α_j is bounded.

Proof of Theorem 3.3. As mentioned above, the first statement is a direct consequence of Theorem 3.5. To see the injectivity of the map $\mathcal{T}(S) \to \mathbb{PR}^{\mathcal{S}(S)}$, one proceeds as follows. Let $[(X, f)] \in \mathcal{T}(S)$, and choose simple closed curves $\alpha, \beta \subset S$ which transversely intersect once. Write $X = \mathbb{H}^2/\Gamma$, where Γ is a Fuchsian group, and denote by A and B the realization of $f_*(\alpha)$ and $f_*(\beta)$, respectively, as elements of $\Gamma < \mathrm{PSL}(2, \mathbb{R})$. Now, let γ and δ be, respectively, the positive and negative Dehn twists of α along β. Then γ and δ are realized by the matrices AB and AB^{-1}, respectively. From the trace relation

$$\mathrm{tr}(A) + \mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$$

in $\mathrm{PSL}(2, \mathbb{R})$, plus the relation (1) between trace and length, we obtain a relation between the lengths of $\alpha, \beta, \gamma, \delta$ which is not invariant by scaling for all choices of $\alpha, \beta, \gamma, \delta$. See [1] for details. □

3.3 Measured laminations and Thurston's compactification of $\mathcal{T}(S)$

We now recall some basic facts about measured geodesic laminations on surfaces, referring the reader to [7, 11, 31] for a detailed account.

We continue to assume S is a closed oriented surface of genus $g \geq 2$, which we endow with a fixed hyperbolic structure defining a metric σ. A complete, simple σ–geodesic on S is an injectively immersed geodesic with respect to σ isometric to $\mathbb{R}$ or a circle of some length. A *(geodesic) lamination* on S is a closed subset $\mathcal{L} \subset S$ which may be decomposed as a disjoint union of pairwise disjoint, complete, simple σ-geodesics on S. The decomposition into simple geodesics depends only on the subset, so the subset $\mathcal{L}$ determines the structure as a geodesic lamination. Each geodesic in the decomposition is called a *leaf* of the lamination.

A *transverse measure* λ on a lamination $\mathcal{L}$ is an assignment of a Radon measure $\lambda|_k$ to each arc k on S transverse to $\mathcal{L}$, in a way that:

1. If k' is a subarc of an arc k, then $\lambda|_{k'}$ is the restriction to k' of $\lambda|_k$;

2. If k, k' are arcs that are homotopic via a homotopy F_t, such that $F_1 \colon k \to k'$ is a homeomorphism and $F_t(k)$ is transverse to the leaves of $\mathcal{L}$ for all t, then $\lambda|_{k'} = (F_1)_*(\lambda|_k)$.

As a consequence of (2), it follows that for any arc k, the support of the measure is contained in the intersection $k \cap \mathcal{L}$. A *measured lamination* is a pair $(\mathcal{L}, \lambda)$ of a geodesic lamination together with a transverse measure. We often abuse notation and simply write λ instead of $(\mathcal{L}, \lambda)$.

We denote by $\mathcal{ML}(S)$ the set of measured geodesic laminations on S. Topologize $\mathcal{ML}(S)$ by declaring that a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of measured laminations converges to another measured lamination λ if

$$\int_k f d\lambda_n \to \int_k f d\lambda;$$

for every continuous function $f : k \to \mathbb{R}$ with compact support defined on a generic transverse geodesic arc $k \subset S$ (that is, a geodesic arc transverse to all simple complete geodesics, or equivalently, not contained in any simple complete geodesic).

Remark 2. Observe that the notation $\mathcal{ML}(S)$ does not make reference to the fixed hyperbolic metric σ on S that we used to define measured geodesic laminations. The reason for using such notation is that, as we shall see in Section 4 below, for any two choices of hyperbolic metric there is a canonical homeomorphism between the corresponding spaces of measured geodesic laminations.

A first example of a measured geodesic lamination on S is a simple closed geodesic $\alpha \subset S$, where the transverse measure assigns, to each arc k transverse to α, the Dirac measure $\mu_\alpha|_k$ that counts the intersection with α:

$$\mu_\alpha|_k(E) = |E \cap \alpha|,$$

for any Borel subset $E \subset k$. More generally, we can consider the *weighted* measure $t\mu_\alpha$, where t is a positive real number. Since every isotopy class of simple closed curves has a unique geodesic representative, we thus obtain an injective map

$$\mathcal{S}(S) \times \mathbb{R}_+ \to \mathcal{ML}(S),$$

given by $(\alpha, t) \mapsto t\mu_\alpha$. As it turns out, the image of the above map is dense in $\mathcal{ML}(S)$ (see, for instance, Theorem 3.1.3 of [31]). There is a slightly weaker result whose proof is less involved, which we explain next. If $\alpha_1, \ldots, \alpha_n$ are pairwise disjoint simple closed curves and $t_1, \ldots, t_n \in \mathbb{R}_+$, then $\sum_i t_i\mu_{\alpha_i}$ also defines a measured geodesic lamination called a *weighted multicurve*.

Theorem 3.6. *The subset of $\mathcal{ML}(S)$ consisting of weighted multicurves is dense in $\mathcal{ML}(S)$.*

This theorem follows from the construction of the so-called *Dehn-Thurston coordinates* for $\mathcal{ML}(S)$ (see, for instance, [31]), which may be regarded as an analog for laminations of Fenchel-Nielsen coordinates for Teichmüller space. Before describing these coordinates we need some definitions. Given $\alpha \in \mathcal{S}(S)$ and $\lambda \in \mathcal{ML}(S)$, the intersection number of α and λ is defined by

$$i(\alpha, \lambda) = \int_{\alpha^*} d\lambda$$

where the integral is over the geodesic representative α^* of α. This generalizes the notion of *geometric intersection number* $i(\alpha, \beta)$ between two simple closed curves α and β, which is defined as the minimal number of transverse intersection points between representatives of α and β. Indeed, with the notation above,

$$i(\alpha, \beta) = i(\alpha, \mu_\beta).$$

More generally, there is a continuous, symmetric, bilinear form

$$i : \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R},$$

called the *intersection number* form, which extends the usual geometric intersection number for pairs $(\alpha, \beta) \in \mathcal{S}(S) \times \mathcal{S}(S)$. In the next section we will give an explicit definition for (a generalization of) i in terms of geodesic currents.

Given a fixed pants decomposition $\alpha_1, \ldots, \alpha_{3g-3}$, the associated Dehn-Thurston coordinates for $\mathcal{ML}(S)$ is a homeomorphism

$$\mathcal{ML}(S) \to (\mathbb{R}_{\geq 0}^{3g-3} \times \mathbb{R}^{3g-3} \setminus \{0\})/\sim \tag{2}$$

where the first $3g-3$ coordinates of $\lambda \in \mathcal{ML}(S)$ are the intersection numbers $i(\alpha_i, \lambda)$ and the last $3g-3$ coordinates $\{tw_i(\lambda)\}_{i=1}^{3g-3}$ provide a measurement of the twisting of λ around the curves α_i. As with twisting in Fenchel-Nielsen coordinates, the twisting here can be expressed in terms of (intersection numbers with) transverse curves; see [31]. The equivalence relation $\sim$ in (2) is generated by

$$\begin{aligned}&(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{3g-3}, \tau_1, \ldots, \tau_i, \ldots, \tau_{3g-3})\\ &\quad\sim (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{3g-3}, \tau_1, \ldots, -\tau_i, \ldots, \tau_{3g-3})\end{aligned}$$

for all i. The point is that if the i^{th} coordinate of $\lambda \in \mathcal{ML}(S)$ is zero, that is $i(\alpha_i, \lambda) = 0$, then the twisting has no well-defined sign. In fact, in this case

λ can be written as $\lambda' + c_i \mu_{\alpha_i}$ for some (largest possible) $c_i > 0$, and the $(3g-3+i)^{th}$ coordinate of λ is $\pm c_i$. Finally, points in $\mathbb{R}^{3g-3}_{\geq 0} \times \mathbb{R}^{3g-3}$ with rational coordinates correspond to a subset of the weighted multicurves, and thus Theorem 3.6 follows.

The Dehn-Thurston coordinates (2) are homogeneous of degree 1, and so the space of *projective measured laminations*

$$\mathbb{P}\mathcal{ML}(S) = (\mathcal{ML}(S) \setminus \{0\})/\mathbb{R}_+$$

is homeomorphic to $(\mathbb{R}^{3g-3}_{\geq 0} \times \mathbb{R}^{3g-3}/\sim \setminus \{0\})/\mathbb{R}_+$. It is not difficult to see that this space is homeomorphic to a $(6g-7)$–dimensional sphere $\mathbb{S}^{6g-7}$, and thus

$$\mathbb{P}\mathcal{ML}(S) \cong \mathbb{S}^{6g-7}.$$

We are finally in a position to say some words about Thurston's compactification. First, using intersection numbers, one proves an analog for $\mathcal{ML}(S)$ of Theorem 3.5; as it turns out, one may use the same $9g-9$ curves as in that theorem. That is, the intersection numbers determine an injective map

$$\mathcal{ML}(S) \to \mathbb{R}^{\mathcal{S}(S)}_{\geq 0}. \tag{3}$$

This map is homogeneous of degree 1 and hence remains injective after positively projectivizing both domain and range:

$$\mathbb{P}\mathcal{ML}(S) \to \mathbb{P}\mathbb{R}^{\mathcal{S}(S)}_{\geq 0}.$$

By Theorem 3.3, $\mathcal{T}(S)$ embeds into $\mathbb{P}\mathbb{R}^{\mathcal{S}(S)}_{\geq 0}$. The images of $\mathcal{T}(S)$ and $\mathbb{P}\mathcal{ML}(S)$ in $\mathbb{P}\mathbb{R}^{\mathcal{S}(S)}_{\geq 0}$ are readily seen to be disjoint. With much more work (see [1]), one proves that the closure of $\mathcal{T}(S)$ in $\mathbb{P}\mathbb{R}^{\mathcal{S}(S)}$ is precisely the image of $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$:

Theorem 3.7. *$\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$ is a $\mathrm{Mod}(S)$-invariant compactification of $\mathcal{T}(S)$. Moreover, $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$ is homeomorphic to the closed unit ball $\overline{\mathbb{B}}^{6g-6} = \mathbb{B}^{6g-6} \cup \mathbb{S}^{6g-7}$.*

Here the action of $\mathrm{Mod}(S)$ on $\mathcal{ML}(S)$, and consequently $\mathbb{P}\mathcal{ML}(S)$, is the natural extension of the action on simple closed curves.

4 Geodesic currents

Throughout this section S will denote a closed oriented surface of genus $g \geq 2$, endowed with a hyperbolic metric σ. By Cartan-Hadamard's Theorem

2.2, the universal cover $\tilde{S}$ is isometrically diffeomorphic to $\mathbb{H}^2$, and thus we have a homeomorphism

$$\tilde{S} \cup \mathbb{S}^1_\infty \to \mathbb{H}^2 \cup \mathbb{S}^1,$$

where $\mathbb{S}^1_\infty$ and $\mathbb{S}^1$ denote, respectively, the ideal boundaries of $\tilde{S}$ and $\mathbb{H}^2$.

Let $G(\tilde{S})$ be the set of unoriented, bi-infinite geodesics on $\tilde{S}$. Each such geodesic is determined uniquely by its endpoints, which are necessarily distinct, and therefore

$$G(\tilde{S}) \cong (\mathbb{S}^1_\infty \times \mathbb{S}^1_\infty \setminus \Delta)/\mathbb{Z}_2,$$

where Δ denotes the diagonal in $\mathbb{S}^1_\infty \times \mathbb{S}^1_\infty$, and $\mathbb{Z}_2$ is the order-two group action that swaps the coordinates. Given $\{a, b\} \in (\mathbb{S}^1_\infty \times \mathbb{S}^1_\infty \setminus \Delta)/\mathbb{Z}_2$, let $\overline{ab}$ denote the unoriented biinfinite geodesic in $\tilde{S}$ with these endpoints. Observe that the action of $\pi_1(S)$ extends to an action on $\tilde{S} \cup \mathbb{S}^1_\infty$, and thus there is a natural action of $\pi_1(S)$ on $G(\tilde{S})$.

Remark 3. The notations $G(\tilde{S})$ or $\mathbb{S}^1_\infty$ are may seem rather ambiguous, as they do not make any reference to the hyperbolic metric on S that we have fixed to define these objects. However, if S_1 and S_2 denote the result of equipping S with two hyperbolic metrics σ_1 and σ_2, then the universal covers $\tilde{S}_1$ and $\tilde{S}_2$ are $\pi_1(S)$-equivariantly quasi-isometric to each other, by the Svarč-Milnor Lemma (see, for instance, [9]). Such a quasi-isometry extends to a $\pi_1(S)$-equivariant homeomorphism between their ideal boundaries, and thus there is a $\pi_1(S)$–equivariant homeomorphism $G(\tilde{S}_1)$ and $G(\tilde{S}_2)$.

Following Bonahon [5] we will define geodesic currents as certain $\pi_1(S)$-invariant measures on the space $G(\tilde{S})$. In order to motivate their definition, we present an alternative viewpoint on measured geodesic laminations to the one given in the previous section.

4.1 Measured laminations as measures on $G(\tilde{S})$

Given a measured geodesic lamination $(\mathcal{L}, \lambda) \in \mathcal{ML}(S)$, consider the preimage $\tilde{\mathcal{L}} = p^{-1}(\mathcal{L})$. As $\mathcal{L}$ is a disjoint union of simple complete geodesics, $\tilde{\mathcal{L}}$ is also a disjoint union of bi-infinite geodesics, and is invariant by $\pi_1(S)$. As such, we can view $\tilde{\mathcal{L}}$ as a $\pi_1(S)$-invariant closed subset of $G(\tilde{S})$, and we do so whenever it is convenient.

Next we explain how λ determines a $\pi_1(S)$-invariant Radon measure on $G(\tilde{S})$ with support equal to $\tilde{\mathcal{L}} \subset G(\tilde{S})$. For this, note that a small arc $\tilde{k}$

transverse to $\tilde{\mathcal{L}}$ descends to an arc k transverse to $\mathcal{L}$. We define the λ–measure of the set of geodesics intersecting $\tilde{k}$ to be $\lambda|_k(k)$, and this extends to a measure on all of $G(\tilde{S})$ supported on $\tilde{\mathcal{L}}$.

Conversely, suppose we are given a $\pi_1(S)$-invariant Radon measure λ on $G(\tilde{S})$ with support $\tilde{\mathcal{L}} = p^{-1}(\mathcal{L})$, for some geodesic lamination $\mathcal{L}$ in S. Then λ determines a transverse measure on $\mathcal{L}$, in the sense of Section 3.3, as follows. Given a small arc $k \subset S$ transverse to $\mathcal{L}$ we need to describe a measure $\lambda|_k$ on k. For this, let $\tilde{k}$ be a lift of k to $\tilde{S}$. If $E \subset k$ is any Borel subset, let $\tilde{E} \subset \tilde{k}$ be its lift to $\tilde{k}$, and define $\lambda|_k(E)$ to be the λ–measure of the set of geodesics in $G(\tilde{S})$ intersecting $\tilde{k}$ in $\tilde{E}$. It is straightforward to check that this defines a transverse measure as in Section 3.3. This is the basis for the following; see [4, 5].

Theorem 4.1. *The above construction defines an injective map*

$$\mathcal{ML}(S) \to \{\pi_1(S)\text{-}\textit{invariant Radon measures on } G(\tilde{S})\}$$

assigning to each $(\mathcal{L}, \lambda) \in \mathcal{ML}(S)$*, a measure on* $G(\tilde{S})$*, also denoted* λ*, with* $\mathrm{supp}(\lambda) = \tilde{\mathcal{L}}$*.*

4.2 Currents

Following Bonahon [5], we define a *geodesic current* on S as a $\pi_1(S)$-invariant Radon measure on $G(\tilde{S})$. We denote by $\mathrm{Curr}(S)$ the set of geodesic currents on S; although this technically depends on an initial choice of hyperbolic metric on S, the canonical $\pi_1(S)$–equivariant homeomorphism mentioned in Remark 3 between corresponding spaces of geodesics $G(\tilde{S})$ determines a canonical homeomorphism between the corresponding spaces of currents, and so we continue to ignore this dependence in the notation. We endow $\mathrm{Curr}(S)$ with the weak* topology; that is, a sequence $(\mu_n)_{n\in\mathbb{N}}$ of geodesic currents converges to $\mu \in \mathrm{Curr}(S)$ if and only if

$$\int_{G(\tilde{S})} f d\mu_n \to \int_{G(\tilde{S})} f d\mu,$$

for every $f \in C_c(G(\tilde{S}), \mathbb{R})$, the space of continuous $\mathbb{R}$-valued functions on $G(\tilde{S})$ with compact support.

As a simple closed geodesic on S determines a measured geodesic lamination, an arbitrary primitive closed geodesic $\gamma \subset S$ (not necessarily simple) determines a geodesic current on S; here primitive means not a nontrivial power of another closed geodesic. Indeed, the preimage $\tilde{\gamma} = p^{-1}(\gamma)$ determines a $\pi_1(S)$–invariant closed *discrete* subset of $G(\tilde{S})$ of the same name, and we can define a measure μ_γ on $G(\tilde{S})$ with support $\tilde{\gamma}$ that simply counts points of intersection with $\tilde{\gamma}$

$$\mu_\gamma(E) = |E \cap \tilde{\gamma}|$$

where $E \subset G(\tilde{S})$ is an arbitrary Borel set. Since $\tilde{\gamma}$ is invariant by $\pi_1(S)$, μ_γ is also $\pi_1(S)$–invariant. In the special case that γ is simple, this agrees with previous construction of a measured lamination associated to γ followed by the identification of the latter with a $\pi_1(S)$–invariant measure on $G(\tilde{S})$. For notational purposes, in the sequel we will blur the difference between a closed geodesic γ and the geodesic current μ_γ it defines, denoting both as γ when it is convenient to do so.

Analogous to the fact that weighted simple closed curves are dense in $\mathcal{ML}(S)$ (see Theorem 3.6 and the paragraph preceding it), Bonahon [5] proved that positive real multiples of (geodesic currents defined by) primitive closed geodesics form a dense subset of $\mathrm{Curr}(S)$.

Theorem 4.2. *The subset of* $\mathrm{Curr}(S)$ *consisting of positive real multiples of primitive closed geodesics on* S *is dense in* $\mathrm{Curr}(S)$.

Using the embedding $\mathcal{ML}(S) \to \mathrm{Curr}(S)$ from Theorem 4.1, we will identify $\mathcal{ML}(S)$ and its image under this embedding, denoting the image of $(\mathcal{L}, \lambda)$ as $\lambda \in \mathrm{Curr}(S)$. Bonahon [5] proved that the geometric intersection number between closed geodesics has a unique continuous extension to a symmetric bilinear form on $\mathrm{Curr}(S)$, which simultaneously extends the geometric intersection number between simple closed curves and the bilinear intersection form

$$\mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}$$

eluded to in Section 3.3. Moreover he showed that measured geodesic laminations are characterized as those geodesic currents that have zero intersection number with themselves. Before we proceed to define the intersection number form on $\mathrm{Curr}(S)$ and study some of its features, it is useful to have an alternative perspective on the space of geodesic currents.

4.3 Alternative definition of geodesic currents

As we will see, geodesic currents may also be defined as certain $\pi_1(S)$-invariant transverse measures on the *projective tangent bundle* $\mathbb{P}\mathrm{T}(\tilde{S})$ of the universal cover $\tilde{S}$. We need some preliminaries before we are able to present this alternative description.

Recall that the unit tangent bundle of a Riemannian manifold M is

$$T^1(M) = \{(x, v) \mid x \in M, v \in T_x(M), ||v|| = 1\},$$

where $T_x(M)$ denotes the tangent space to M at x, and $|| \cdot ||$ is the norm (on $T_x(M)$) determined by the Riemannian metric.

For the universal cover $\tilde{S}$ of our surface S (equipped with its hyperbolic metric), the unit tangent bundle $T^1(\tilde{S})$ may be identified with $\Theta_3^+(\mathbb{S}^1_\infty)$, the space of positively (i.e. counterclockwise) oriented distinct triples on the circle at infinity $\mathbb{S}^1_\infty$. We describe this identification concretely as follows. Let $(x, v) \in T^1(\tilde{S})$, and let δ be the unique bi-infinite geodesic on $\tilde{S}$ passing through x with direction v. Let $a, b \in \mathbb{S}^1_\infty$ be the (necessarily distinct) endpoints of δ, labelled so that that δ goes from a to b. Let δ' be the unique bi-infinite geodesic on $\tilde{S}$ that passes through x and is orthogonal to δ, and let c be the endpoint of δ' such that a, b, c appear in this (cyclic) order when traveling counterclockwise along $\mathbb{S}^1_\infty$; see Figure 2. Then the rule

$$(x, v) \mapsto (a, b, c)$$

provides the desired homeomorphism $T^1(\tilde{S}) \to \Theta_3^+(\mathbb{S}^1_\infty)$.

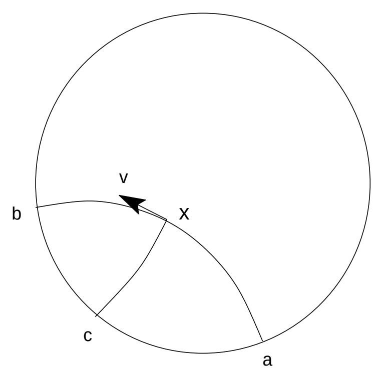

Figure 2: The rule $(x, v) \mapsto (a, b, c)$ defines a homeomorphism between $T^1(\tilde{S})$ and $\Theta_3^+(\mathbb{S}^1_\infty)$.

The *projective tangent bundle* $\mathbb{P}\mathrm{T}(\tilde{S})$ of $\tilde{S}$ is obtained from $T^1(\tilde{S})$ by forgetting the sign of tangent vectors. More precisely, we have

$$\mathbb{P}\mathrm{T}(\tilde{S}) = \{(x, [v]) \mid (x, v) \in T^1(\tilde{S}), [v] = \{\pm v\}\}.$$

Observe that $\pi_1(S)$ acts on $T^1(\tilde{S})$ and $\mathbb{P}\mathrm{T}(\tilde{S})$; the quotient spaces are $T^1(S)$ and $\mathbb{P}\mathrm{T}(S)$, the unit tangent bundle and projective tangent bundle of S, respectively. We remark that these are both compact spaces since S is compact; we will make use of this fact when we describe the properties of the intersection number form on $\mathrm{Curr}(S)$.

The metric determines a geodesic flow on $T^1(\tilde{S})$, whose trajectories are the images of lifts $t \mapsto (\gamma(t), \gamma'(t))$ to $T^1(\tilde{S})$ of (unit speed parameterizations of) geodesics $t \mapsto \gamma(t)$ in $\tilde{S}$. This defines a foliation of $T^1(\tilde{S})$ by flow lines. With respect to the homeomorphism $T^1(\tilde{S}) \to \Theta_3^+(\mathbb{S}^1_\infty)$, the leaves are precisely the fibers of the map onto the last coordinate. This foliation descends to a 1-dimensional foliation $\mathcal{F}$ of $\mathbb{P}\mathrm{T}(\tilde{S})$, called the *geodesic foliation* of $\mathbb{P}\mathrm{T}(\tilde{S})$.

We now (re-)define a geodesic current as a $\pi_1(S)$-invariant Radon transverse measure on $(\mathbb{P}\mathrm{T}(\tilde{S}), \mathcal{F})$ that is transverse to $\mathcal{F}$. More precisely, a geodesic current is a $\pi_1(S)$-invariant assignment of a measure to each submanifold $V \subset \mathbb{P}\mathrm{T}(\tilde{S})$ of codimension 1 that is transverse to $\mathcal{F}$; furthermore, we require that such assignment be invariant under homotopy transverse to $\mathcal{F}$. The latter condition means that given two codimension-1 submanifolds $V, V' \subset \mathbb{P}\mathrm{T}(\tilde{S})$ transverse to $\mathcal{F}$ and a homeomorphism $h : V \to V'$ homotopic to the inclusion of V into $\mathbb{P}\mathrm{T}(\tilde{S})$ by a homotopy h_t preserving intersections with each leaf of $\mathcal{F}$ (i.e. for each $x \in V$, $h_t(x)$ and $h_{t'}(x)$ are contained in the same leaf, for all t, t'), then the measure assigned to V' coincides with the push forward by h of the measure assigned to V (compare with the definition of transverse measure on a geodesic lamination from Section 3.3).

We now explain how one goes back and forth between this notion of geodesic current and the one given in Section 4.2. First, we remark that there is a homeomorphism

$$P : \mathbb{P}\mathrm{T}(\tilde{S}) \to \Pi(\tilde{S}) \subset G(\tilde{S}) \times G(\tilde{S}),$$

where $\Pi(\tilde{S})$ is the set of all ordered pairs of unoriented bi-infinite geodesics in $\tilde{S}$ that are orthogonal to each other. In other words,

$$\Pi(\tilde{S}) = \left\{ (\{a,b\}, \{c,d\}) \mid a,b,c,d \in \mathbb{S}^1_\infty \text{ pairwise distinct and } \overline{ab} \perp \overline{cd} \right\};$$

where $\overline{ab}$ is the unoriented geodesic with endpoints a and b, and $\perp$ denotes orthogonality. Indeed, P may be obtained by setting

$$P\left((x, [v])\right) = (\{a,b\}, \{c,d\}),$$

where a, b are the endpoints of the unique unoriented bi-infinite geodesic δ in $\tilde{S}$ through x and with direction v, and c, d are the endpoints of the unique unoriented bi-infinite geodesic through x and orthogonal to δ; see Figure 3. Observe that, in the identification of $\mathbb{P}\mathrm{T}(\tilde{S})$ with $\Pi(\tilde{S})$, any leaf of the

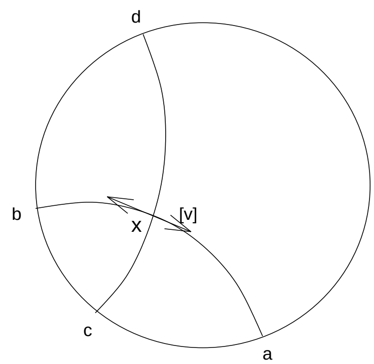

Figure 3: The rule $(x, [v]) \mapsto (\{a, b\}, \{c, d\})$ defines an identification between $\mathbb{P}\mathrm{T}(\tilde{S})$ and the subset of $G(\tilde{S}) \times G(\tilde{S})$ consisting of those pairs of unoriented geodesics that are orthogonal to each other.

geodesic foliation $\mathcal{F}$ consists precisely of a set of points with image in $\Pi(\tilde{S})$ having the same first coordinate. Said differently, the map

$$\mathbb{P}\mathrm{T}(\tilde{S}) \to G(\tilde{S})$$

given by composing P with the projection onto the first factor is a submersion, and the fibers are precisely the leaves of $\mathcal{F}$. It follows that there is a bijection between the set of $\pi_1(S)$-invariant, transverse Radon measures on $(\mathbb{P}\mathrm{T}(\tilde{S}), \mathcal{F})$ that are transverse to $\mathcal{F}$, and the set of $\pi_1(S)$-invariant measures on $G(\tilde{S})$. In other words, we see that the two definitions of geodesic current that we have given are in fact equivalent.

This provides us with yet another formulation that can be understood without going to the universal cover. Namely, we can consider the unit tangent bundle $T^1(S)$ of S and the geodesic flow on it. This descends to a geodesic foliation of the projective tangent bundle $\mathbb{P}\mathrm{T}(S)$, which we also denote $\mathcal{F}$. The covering map $\tilde{S} \to S$ induces a covering map $\mathbb{P}\mathrm{T}(\tilde{S}) \to \mathbb{P}\mathrm{T}(S)$, and the geodesic foliation descends to the geodesic foliation. A $\pi_1(S)$–invariant transverse measure to $\mathcal{F}$ on $\mathbb{P}\mathrm{T}(\tilde{S})$ descends to a transverse measure to $\mathcal{F}$ on $\mathbb{P}\mathrm{T}(S)$. In this way, we can also think of a geodesic current as a transverse measure to $\mathcal{F}$ on $\mathbb{P}\mathrm{T}(S)$.

4.4 The flow-box topology on $\mathrm{Curr}(S)$

A useful way to understand the topology on $\mathbb{P}\mathrm{T}(\tilde{S})$ is through the notion of a *flow box* on $\mathbb{P}\mathrm{T}(\tilde{S})$, as described by Bonahon in [4]. We now briefly explain how this works.

An *H-shape* on S consists of three arcs (τ_L, γ, τ_R) on S subject to the following conditions:

- γ is a geodesic arc on S transverse to τ_L and τ_R with one endpoint on τ_L and the other on τ_R;
- Each geodesic arc on S that connects τ_L and τ_R and is homotopic to γ rel $\tau_L \cup \tau_R$ intersects each τ_L and τ_R transversely.

The *flow box* $B = B_H \subset \mathbb{P}\mathrm{T}(S)$ defined by an H-shape (τ_L, γ, τ_R) consists of the lifts to $\mathbb{P}\mathrm{T}(S)$ of all the geodesic segments that connect τ_L and τ_R and are homotopic to γ rel $\tau_L \cup \tau_R$. By a flow box in $\mathbb{P}\mathrm{T}(\tilde{S})$ we mean a lift of a flow box in $\mathbb{P}\mathrm{T}(S)$; therefore, a flow box in $\mathbb{P}\mathrm{T}(\tilde{S})$ is defined by a set of geodesic segments in $\tilde{S}$ with endpoints on a pair of (small, close-by) arcs. See Figure 4. In order to keep notation under control, we will use the same notation for a flow box in $\mathbb{P}\mathrm{T}(S)$ or $\mathbb{P}\mathrm{T}(\tilde{S})$.

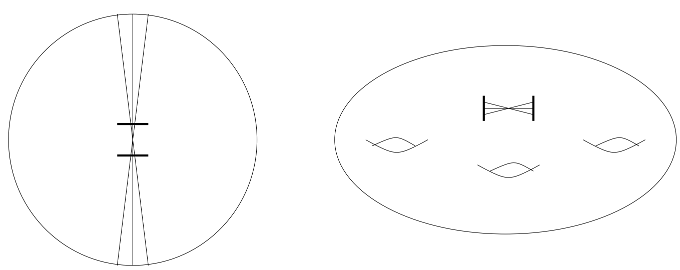

Figure 4: A flow box in $\mathbb{P}\mathrm{T}(\tilde{S})$ (left) consists of lifts to $\mathbb{P}\mathrm{T}(\tilde{S})$ of segments of biinfinite geodesics with endpoints on the two arcs, while a flow box in $\mathbb{P}\mathrm{T}(S)$ (right) consists of lifts to $\mathbb{P}\mathrm{T}(S)$ of geodesic segments having endpoints on the arcs, *and* in the correct relative homotopy class.

Observe that a flow box B in $\mathbb{P}\mathrm{T}(\tilde{S})$ (or in $\mathbb{P}\mathrm{T}(S)$, for the same reason) is homeomorphic to $Q \times [0,1]$, where $Q = [0,1] \times [0,1]$. Informally, each point of Q specifies a pair of points on the small, close-by pair of arcs defining B, respectively, thus determining a unique geodesic segment connecting the two arcs; finally, the third coordinate specifies the position along such geodesic segment.

If $B \cong Q \times [0,1]$ is a flow box in $\mathbb{PT}(\tilde{S})$, then Q may be lifted to a codimension-1 submanifold of $\mathbb{PT}(\tilde{S})$ transversal to the geodesic foliation $\mathcal{F}$, simply by specifying a third coordinate $t_0 \in [0,1]$ on B. In the light of this we define, for a flow box $B \subset \mathbb{PT}(\tilde{S})$ and a geodesic current $\mu \in \mathrm{Curr}(S)$ (thought of as a transverse measure on $\mathbb{PT}(\tilde{S})$), the *μ-measure* of B as

$$\mu(B) = \mu(Q).$$

The fact that $\mu(B)$ does not depend on the chosen lift of Q follows from the definition of geodesic current, as such lift is unique up to homotopy transverse to the geodesic foliation. Observe that, in the particular case where $\mu = \mu_\alpha$ with α a primitive closed geodesic on S, the number $\mu_\alpha(B)$ is the number of subarcs of α that connect the pair of transversal arcs defining the flow box B and are homotopic to γ rel τ_L, τ_R.

Using this, one can give a local description of the topology on $\mathrm{Curr}(S)$ in terms of measures of flow boxes. Here, one has to impose a standard condition that the boundaries of flow boxes have measure zero with respect to the given current; more concretely, given a geodesic current $\mu \in \mathrm{Curr}(S)$, we say that a flow box $B \cong Q \times [0,1] \subset \mathbb{PT}(\tilde{S})$ is *μ-admissible* if $\mu(\partial Q \times \{t_0\}) = 0$, where we are applying the measure μ defined on the transversal $Q \times \{t_0\}$ as described above.

Lemma 4.3. *Let $\mu \in \mathrm{Curr}(S)$. The collection of sets of the form*

$$U(\mu; \mathcal{B}, \epsilon) = \{\nu \in \mathrm{Curr}(S) \mid \forall B \in \mathcal{B}, |\mu(B) - \nu(B)| < \epsilon\},$$

where ϵ ranges over all positive real numbers and $\mathcal{B}$ ranges over all finite collections of μ-admissible flow boxes, forms a basis of neighborhoods for μ.

4.5 Intersection number between geodesic currents

Let $\mathcal{D}G(\tilde{S})$ be the subset of $G(\tilde{S}) \times G(\tilde{S})$ consisting of those pairs of geodesics in $\tilde{S}$ that intersect transversely in $\tilde{S}$. In other words, $\mathcal{D}G(\tilde{S})$ consists of those pairs $(\{a,b\},\{c,d\})$, where $a,b,c,d \in \mathbb{S}^1_\infty$ are distinct and the pairs $\{a,b\}$ and $\{c,d\}$ *link at infinity*, meaning that a and b lie in different connected components of $\mathbb{S}^1_\infty \setminus \{c,d\}$. Given a pair $\mu, \nu \in \mathrm{Curr}(S)$, we can restrict the product measure $\mu \times \nu$ to $\mathcal{D}G(\tilde{S})$, which we also denote $\mu \times \nu$.

Observe that $\mathcal{D}G(\tilde{S})$ is $\pi_1(S)$–equivariantly homeomorphic to a subset of the Whitney sum $\mathbb{PT}(\tilde{S}) \oplus \mathbb{PT}(\tilde{S})$:

$$\mathcal{D}G(\tilde{S}) \cong \{(x,[u],[v]) \in \mathbb{PT}(\tilde{S}) \oplus \mathbb{PT}(\tilde{S}) \mid [u] \neq [v]\}.$$

Here the homeomorphism sends a pair $(\{a,b\},\{c,d\})$ to the triple $(x,[u],[v])$ where x is the point of intersection of the geodesics $\overline{ab}$ and $\overline{cd}$ and $[u],[v]$ are the respective tangent directions of $\overline{ab}$ and $\overline{cd}$. The advantage of thinking about $\mathcal{D}G(\tilde{S})$ in this way is that the action of $\pi_1(S)$ on the Whitney sum (and hence on $\mathcal{D}G(\tilde{S})$) is properly discontinuous and free, and hence the quotient

$$\mathcal{D}G(\tilde{S}) \to \mathcal{D}G(S) = \mathcal{D}G(\tilde{S})/\pi_1(S).$$

is a covering map. In fact, $\mathcal{D}G(S)$ is nothing but $\mathbb{P}\mathrm{T}(S) \oplus \mathbb{P}\mathrm{T}(S)$.

Given two geodesic currents $\mu, \nu \in \mathrm{Curr}(S)$, the measure $\mu \times \nu$ on $\mathcal{D}G(\tilde{S})$ descends to a measure on on $\mathcal{D}G(S)$ that we also denote $\mu \times \nu$. This is defined by *locally* pushing forward $\mu \times \nu$ on open subsets of $\mathcal{D}G(\tilde{S})$ on which the quotient is a homeomorphism. Now one defines the *intersection number* between μ and ν, denoted by $i(\mu,\nu)$, as the $(\mu\times\nu)$-mass of the space $\mathcal{D}G(S)$, that is

$$i(\mu,\nu) = \int_{\mathcal{D}G(S)} d\mu \times d\nu.$$

To get some feeling for this, let us consider the case that α and β are primitive closed geodesics on S and μ_α, μ_β are the corresponding geodesic currents. Note that on $\mathcal{D}G(\tilde{S})$, $\mu_\alpha \times \mu_\beta$ is the Dirac measure that counts, for any Borel subset $E \subset \mathcal{D}G(\tilde{S})$, the number of points $(\{a,b\},\{c,d\}) \in E$ where $\{a,b\}$ and $\{c,d\}$ are endpoints of geodesics in the preimages of α and β, respectively. This descends to a measure on $\mathcal{D}G(S) \subset \mathbb{P}\mathrm{T}(S) \oplus \mathbb{P}\mathrm{T}(S)$ which is the Dirac measure on the *finite* set of points $(x,[u],[v])$ where x is a point of intersection of α and β and $[u],[v]$ are lines tangent to these two geodesics, respectively, at the point x. Thus, $i(\mu_\alpha,\mu_\beta)$ is precisely the geometric intersection number $i(\alpha,\beta)$.

It is not obvious from the definition that the intersection number between any two currents is finite. We prove this in the following lemma from [4]:

Lemma 4.4. *For all $\mu,\nu \in \mathrm{Curr}(S)$, $i(\mu,\nu)$ is finite.*

Proof. Let $p_k : \mathbb{P}\mathrm{T}(S) \oplus \mathbb{P}\mathrm{T}(S) \to \mathbb{P}\mathrm{T}(S)$ be the projection to the k-th factor, for $k = 1,2$. Given two flow boxes B, B' in $\mathbb{P}\mathrm{T}(S)$, let $B \oplus B' = p_1^{-1}(B) \cap p_2^{-1}(B')$, which is the set of all points $(x,[u],[v])$ where $(x,[u]) \in B$ and $(x,[v]) \in B'$.

Since $\mathbb{P}\mathrm{T}(S)$ is compact, it may be covered by finitely many flow boxes $B_1, \ldots, B_n$. Therefore,

$$\mathcal{D}G(S) \subset \bigcup_{i,j=1}^{n} B_i \oplus B_j.$$

Hence,

$$\begin{aligned} i(\mu,\nu) &= \int_{\mathcal{D}G(S)} \mathrm{d}\mu \times \mathrm{d}\nu \leq \sum_{\mathrm{i,j=1}}^{\mathrm{n}} \int_{(\mathrm{B_i}\oplus\mathrm{B_j})\cap\mathcal{D}\mathrm{G(S)}} \mathrm{d}\mu \times \mathrm{d}\nu \\ &\leq \sum_{i,j=1}^{n} \mu(B_i)\nu(B_j), \end{aligned}$$

which is finite, as desired. □

Bonahon proved the following theorem in [4], which extends the aforementioned result of Thurston on the intersection number form for measured geodesic laminations:

Theorem 4.5. *The function* $i : \mathrm{Curr}(S) \times \mathrm{Curr}(S) \to \mathbb{R}$ *is continuous, symmetric, and bilinear.*

The proof that the intersection number form i is continuous requires a fair amount of work due to the fact that $\mathcal{D}G(S)$, which is the complement of the diagonal in $\mathbb{P}\mathrm{T}(S) \oplus \mathbb{P}\mathrm{T}(S)$, is not compact; see [4] for details.

As mentioned above, it is possible to characterize the subset of measured geodesic laminations as the "light-cone" in $\mathrm{Curr}(S)$ with respect to the intersection number form. Specifically, one has:

Proposition 4.6. *Let* $\mu \in \mathrm{Curr}(S)$. *Then* $i(\mu,\mu) = 0$ *if and only if* $\mu \in \mathcal{ML}(S)$.

Proof. Suppose that $\mu \in \mathrm{Curr}(S)$ satisfies $i(\mu,\mu) = 0$. By the definition of intersection number, we deduce that the support of μ is a $\pi_1(S)$-invariant closed set $\tilde{\mathcal{L}}$ of pairwise non-intersecting geodesics in $\tilde{S}$. Therefore, $p(\tilde{\mathcal{L}}) = \mathcal{L} \subset S$ is a closed subset which is a union of pairwise disjoint, complete, simple geodesics (a transverse intersection between two geodesics would give rise to intersecting geodesics in $\tilde{\mathcal{L}}$). Therefore, $\mathcal{L}$ is a geodesic lamination and μ defines a transverse measure on it.

For the other direction, note that given $(\mathcal{L},\lambda) \in \mathcal{ML}(S)$, the support $\tilde{\mathcal{L}} \subset G(\tilde{S})$ consists of pairwise non-intersecting geodesics, and therefore $i(\lambda,\lambda) = 0$. □

4.6 Projective currents

We say that a geodesic current $\mu \in \mathrm{Curr}(S)$ is *filling* if, for all $g \in G(\tilde{S})$, there exists $h \in G(\tilde{S})$ in the support of μ such that g and h intersect transversely in $\tilde{S}$.

Remark 4. The reader should be reminded of the homonymous condition for a collection of closed geodesics on S. Indeed, a first example of a filling current is given by $\alpha_1 + \ldots + \alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are closed geodesics on S such that $S - (\alpha_1 \cup \ldots \cup \alpha_n)$ is a collection of topological disks, and in the sum $\alpha_1 + \ldots + \alpha_n$, we view each term as a geodesic current.

A useful fact about filling currents is the following from [5]:

Theorem 4.7. *Let $\mu \in \mathrm{Curr}(S)$ be a filling current and $R > 0$. Then the set*

$$C_R = C_R(\mu) = \{\nu \in \mathrm{Curr}(S) \mid i(\mu, \nu) \leq R\}$$

is compact.

Proof. Let $\Omega = C_c(G(\tilde{S}), \mathbb{R}_{\geq 0})$ be the set of continuous, nonnegative real valued functions from $G(\tilde{S})$ with compact support. We consider the embedding $\mathrm{Curr}(S) \to \mathbb{R}^{\Omega}$ given by

$$\nu \mapsto \left(\int_{G(\tilde{S})} f d\nu \right)_{f \in \Omega},$$

for all $\nu \in \mathrm{Curr}(S)$, where the target is given the product topology (that this is an embedding follows from the definition of the weak* topology, and the fact that integrating functions separates points). Since

$$\int_{G(\tilde{S})} f d\nu \leq \max(f) \nu(\mathrm{supp}(f))$$

and f is continuous and has compact support, it suffices to show that the set

$$\{\nu(\mathrm{supp(f)}) \mid \nu \in \mathrm{C_R}\}$$

is bounded in $\mathbb{R}$ for all $f \in C_c(G(\tilde{S}), \mathbb{R})$, by the Tychonoff Theorem. In particular, it is enough to show the following:

Claim. *For all $K \subset G(\tilde{S})$ compact, the set $\{\nu(K) \mid \nu \in C_R\}$ is bounded in $\mathbb{R}$.*

To prove the claim, it suffices (by compactness of K) to prove that every geodesic $\delta \in G(\tilde{S})$ has a neighborhood U_δ such that $\{\nu(U_\delta) \mid \nu \in C_R\}$ is bounded. Let $\delta \in G(\tilde{S})$. Since μ is a filling current, we may choose $\epsilon \in G(\tilde{S})$ in the support of μ such that δ and ϵ intersect transversely in $\tilde{S}$. Let $U_\delta, U_\epsilon \subset \mathcal{D}G(\tilde{S})$ be neighborhoods of δ and ϵ, respectively, such that every element of U_δ intersects every element of U_ϵ; in particular, $U_\delta \times U_\epsilon \subset DG(\tilde{S})$.

Moreover, by reducing U_δ and U_ϵ if necessary, we may assume that $U_\delta \times U_\epsilon$ projects homeomorphically to $DG(S)$. Thus,

$$i(\mu, \nu) \geq (\nu \times \mu)(U_\delta \times U_\epsilon) = \nu(U_\delta)\mu(U_\epsilon).$$

Since ϵ is in the support of μ, then $\mu(U_\epsilon) \neq 0$, and hence

$$\nu(U_\delta) \leq \frac{i(\mu, \nu)}{\mu(U_\epsilon)} \leq \frac{R}{\mu(U_\epsilon)}.$$

Thus $\{\nu(U_\delta) \mid \nu \in C_R\}$ is a bounded subset of $\mathbb{R}$, as claimed. This finishes the proof of the theorem. □

Let $\mathbb{P}\mathrm{Curr}(S) := (\mathrm{Curr}(S) \setminus \{0\})/\mathbb{R}_+$ be the space of *projective geodesic currents*. We have:

Corollary 4.8. *The space* $\mathbb{P}\mathrm{Curr}(S)$*, endowed with the quotient topology, is compact.*

Proof. Let $\mu \in \mathrm{Curr}(S)$ be a filling geodesic current, and consider the set

$$S_1(\mu) = \{\nu \in \mathrm{Curr}(S) \mid i(\mu, \nu) = 1\},$$

which is a closed subset of $C_1(\mu)$ by Theorem 4.5, and hence compact by Theorem 4.7. Now, the restriction of the projectivization $\mathrm{Curr}(S) \setminus \{0\} \to \mathbb{P}\mathrm{Curr}(S)$ to the subset $S_1(\mu)$ is surjective (in fact, the restriction is a homeomorphism onto $\mathbb{P}\mathrm{Curr}(S)$), and hence the image $\mathbb{P}\mathrm{Curr}(S)$ is compact, and we are done. □

4.7 Determining currents from intersection numbers

In the sequel, we will make use of the following result of Otal [29], which gives an extension of the embeddings of Theorem 3.3 and Fact (3). Denote by $\mathcal{C}(S)$ the set of all closed curves on S. We have:

Theorem 4.9. *Every* $\mu \in \mathrm{Curr}(S)$ *is determined by* $\{i(\alpha, \mu)\}_{\alpha \in \mathcal{C}(S)}$*. In fact, the map* $i_* : \mathrm{Curr}(S) \to \mathbb{R}^{\mathcal{C}(S)}$*, given by*

$$i_*(\mu) = \{i(\alpha, \mu)\}_{\alpha \in \mathcal{C}(S)}$$

is a proper embedding.

Before we give the proof, we need a lemma. Given a primitive closed geodesic $\gamma \subset S$, let $\tilde{\gamma} \subset p^{-1}(\gamma) \subset \tilde{S}$ denote a bi-infinite geodesic in the preimage of γ and let $\gamma_* \in \pi_1(S)$ be an element that generates the stabilizer of $\tilde{\gamma}$ in $\pi_1(S)$. Let $[\gamma) \subset \tilde{\gamma}$ denote a half-open arc of length ℓ_γ, which is a fundamental domain for the action of $\langle \gamma_* \rangle$ on $\tilde{\gamma}$. Let $E_{[\gamma)} \subset G(\tilde{S})$ denote the set of geodesics that transversely intersect $[\gamma)$ nontrivially.

Lemma 4.10. *Suppose γ is a primitive closed geodesic, and let $E_{[\gamma)} \subset G(\tilde{S})$ be as above. For any $\mu \in \mathrm{Curr}(S)$, we have*

$$i(\mu_\gamma, \mu) = \mu(E_{[\gamma)}).$$

Proof. Let $\mu \in \mathrm{Curr}(S)$. The support of $\mu_\gamma \times \mu$ in $\mathcal{D}G(\tilde{S})$ is the set of pairs of geodesics $(\tilde{\delta}_1, \tilde{\delta}_2)$ where $\tilde{\delta}_1, \tilde{\delta}_2$ are geodesics in the supports of μ_γ, μ respectively. Since the support of μ_γ is exactly the full preimage $p^{-1}(\gamma)$, it follows that

$$\{\tilde{\gamma}\} \times (E_{[\gamma)} \cap \mathrm{supp}(\mu)) \subset \mathcal{D}G(\tilde{S})$$

projects bijectively onto the support of $\mu_\gamma \times \mu$. Therefore,

$$i(\mu_\gamma, \mu) = \int_{\mathcal{D}G(S)} d\mu_\gamma \times d\mu = (\mu_\gamma \times \mu)(\{\tilde{\gamma}\} \times (E_{[\gamma)} \cap \mathrm{supp}(\mu))) = \mu(E_{[\gamma)}).$$

□

Sketch of proof of Theorem 4.9. Given a geodesic arc, ray, or line $\tau \subset \tilde{S}$, let $E_\tau \subset G(\tilde{S})$ denote the set of geodesics that nontrivially transversely intersect τ.

Let $\delta \subset T^1(S)$ denote a complete geodesic which is dense in both forward and backward time in $T^1(S)$. It follows that the set of all lifts $\tilde{\delta}$ of δ is dense in $G(\tilde{S})$. Considering two disjoint lifts $\tilde{\delta}_1 = \overline{ad}, \tilde{\delta}_2 = \overline{bc}$, the set of bi-infinite geodesics that lie strictly between $\tilde{\delta}_1, \tilde{\delta}_2$ is the set $(a,b) \times (c,d) \subset G(\tilde{S})$ consisting of all the geodesics with one endpoint in the arc $(a,b) \subset \mathbb{S}^1_\infty$ and the other in the second arc $(c,d) \subset \mathbb{S}^1_\infty$; see Figure 5.

Density of the set of lifts $\tilde{\delta}$ of δ in $G(\tilde{S})$ implies that μ is determined by the set $\{\mu((a,b) \times (c,d))\}$ over all $(a,b) \times (c,d)$ where $\tilde{\delta}_1 = \overline{ad}, \tilde{\delta}_2 = \overline{bc}$ are disjoint lifts of δ. The proof of injectivity of i_* is then a consequence of the following claim, by taking $\epsilon \to 0$.

Claim. Given disjoint lifts $\tilde{\delta}_1 = \overline{ad}, \tilde{\delta}_2 = \overline{bc}$ of δ and $\epsilon > 0$, there exists $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{C}(S)$ such that

$$\left| \mu((a,b) \times (c,d)) - \frac{i(\alpha_1, \mu) + i(\alpha_2, \mu) - i(\alpha_3, \mu) - i(\alpha_4, \mu)}{2} \right| < \epsilon.$$

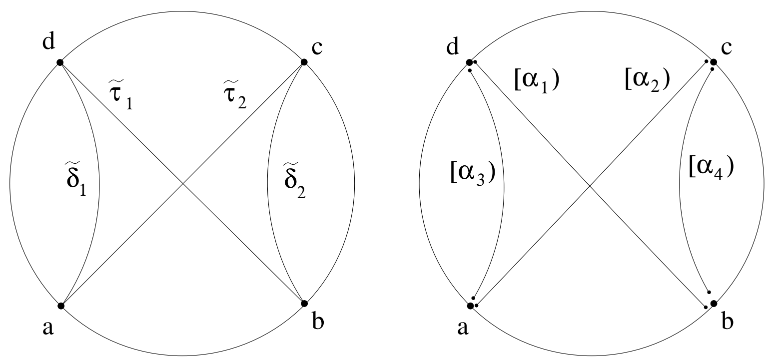

Figure 5: On the left: disjoint lifts $\tilde{\delta}_1, \tilde{\delta}_2$ of δ defining a subset of $G(\tilde{S})$ of the form $(a, b) \times (c, d) \subset G(\tilde{S})$ and the corresponding diagonal arcs $\tilde{\tau}_1, \tilde{\tau}_2$. On the right: fundamental domains for $(\alpha_1)_*, \ldots, (\alpha_4)_* \in \pi_1(S)$ approximating the four geodesics

Proof of Claim. We sketch the idea; see [29] for the details.

Consider the geodesics $\tilde{\tau}_1 = \overline{ad}, \tilde{\tau}_2 = \overline{bc}$ as shown in Figure 5. Note that the set of geodesics $(a, b) \times (c, d)$ consists of precisely those geodesics that transversely intersect *both* $\tilde{\tau}_1$ and $\tilde{\tau}_2$, and *neither* $\tilde{\delta}_1$ nor $\tilde{\delta}_2$. So, one is tempted to write

$$\mu((a, b) \times (c, d)) = \frac{\mu(E_{\tilde{\tau}_1}) + \mu(E_{\tilde{\tau}_2}) - \mu(E_{\tilde{\delta}_1}) - \mu(E_{\tilde{\delta}_2})}{2}$$

However, the measures on the right are all infinite. For any $\epsilon > 0$, we approximate the left-hand side to within $\epsilon > 0$ by a formula similar to the right-hand side, where we replace each of the bi-infinite geodesics by appropriately chosen long, but finite length, subarcs $[\tilde{\tau}_i) \subset \tilde{\tau}_i$ and $[\tilde{\delta}_i) \subset \tilde{\delta}_i$, for $i = 1, 2$. That is

$$\left| \mu((a, b) \times (c, d)) - \frac{\mu(E_{[\tilde{\tau}_1)}) + \mu(E_{[\tilde{\tau}_2)}) - \mu(E_{[\tilde{\delta}_1)}) - \mu(E_{[\tilde{\delta}_2)})}{2} \right| < \epsilon.$$

Density of δ in forward and backward time tells us that these arcs can be chosen so that the tangent vectors to these arcs (oriented from the "bottom" to the "top" in the figure) at the endpoints are as close as we like to four vectors projecting to the same vector in $T^1(S)$. The arcs $[\tilde{\tau}_1), [\tilde{\tau}_2), [\tilde{\delta}_1), [\tilde{\delta}_2)$ are therefore as close as we like to arcs $[\alpha_1), [\alpha_2), [\alpha_3), [\alpha_4)$, respectively, which are fundamental domains for primitive closed geodesics $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, respectively. With enough care, we can guarantee that the μ–measures of these also satisfy an inequality analogous to the previous one:

$$\left|\mu((a,b)\times(c,d)) - \frac{\mu(E_{[\alpha_1]}) + \mu(E_{[\alpha_2]}) - \mu(E_{[\alpha_3]}) - \mu(E_{[\alpha_4]})}{2}\right| < \epsilon.$$

Applying Lemma 4.10 proves the claim. $\square$

Thus, i_* is injective. The continuity of i_* follows from continuity of i. To prove that i_* is proper, let $K \subset \mathbb{R}_{\geq 0}^{\mathcal{C}(S)}$ be a compact set. Then for any curve α, there exists $R > 0$ so that $i_*^{-1}(K)$ is a closed subset of

$$C_R(\alpha) = \{\mu \in \mathrm{Curr}(S) \mid i(\alpha,\mu) \leq R\}.$$

By Theorem 4.7, $C_R(\alpha)$ is compact if α is a filling curve. Choosing α to be a filling curve, it therefore follows that $i_*^{-1}(K)$ is compact, and hence i_* is proper. $\square$

4.8 Liouville currents

For more on the results of this section, see [5]. Recall that

$$G(\mathbb{H}^2) = (\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta)/\mathbb{Z}_2,$$

where Δ denotes the diagonal, and $\mathbb{Z}_2$ is the order-two group swapping the two coordinates. Given disjoint intervals $[a,b]$ and $[c,d]$ in $\mathbb{S}^1$, we define the *Liouville measure* of $[a,b]\times[c,d] \subset G(\mathbb{H}^2)$ as the modulus of the logarithm of the cross ratio of a, b, c, d:

$$L([a,b]\times[c,d]) = \left|\log\left|\frac{(a-c)(b-d)}{(a-d)(b-c)}\right|\right|. \tag{4}$$

We refer to L as the *Liouville current* for $\mathbb{H}^2$. In the upper half-plane model of $\mathbb{H}^2$, we can take (local) coordinates $(x,y) \in \mathbb{R}\times\mathbb{R}\setminus\Delta \subset \widehat{\mathbb{R}}\times\widehat{\mathbb{R}}\setminus\Delta$. The current L is absolutely continuous with respect to Lebesgue measure $dx\,dy$ and a simple calculation verifies that

$$L = \frac{dx\,dy}{(x-y)^2}.$$

In the disk model, with $(e^{i\alpha}, e^{i\beta}) \in \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$, one has

$$L = \frac{d\alpha\,d\beta}{|e^{i\alpha} - e^{i\beta}|^2}.$$

Another useful formula which can be derived from either of the expressions above is the following. Given a (unit speed) geodesic segment $\delta : (-\epsilon, \epsilon) \to \mathbb{H}^2$, we can parameterize the set of geodesics E_δ transversely intersecting δ nontrivially by the point of intersection $\delta(t)$ (or just t), and the angle $\theta \in (0, \pi)$ made with the tangent vector $\delta'(t)$. Then

$$L|_{E_\delta} = \frac{1}{2}\sin(\theta)d\theta dt.$$

The following is then immediate.

Proposition 4.11. $L(E_\delta) = \ell_\delta$.

Proof. Calculate

$$\int_{E_\delta} dL = \int_{-\epsilon}^{\epsilon}\int_0^{\pi} \frac{1}{2}\sin(\theta)d\theta dt = \int_{-\epsilon}^{\epsilon} dt = 2\epsilon = \ell_\delta.$$

□

Remark 5. Even without the specific expression for L used in the proof, it is easy to see that $L(E_\delta)$ is a fixed constant multiple of ℓ_δ. For this, note that since L is invariant under the action of the entire isometry group of $\mathbb{H}^2$, $\delta \mapsto L(E_\delta)$ defines a measure on any geodesic, and this measure is invariant under any isometry. Therefore, it is a constant multiple of the Lebesgue (length) measure. Because $\mathrm{PSL}(2,\mathbb{R})$ acts transitively on the set of geodesics, the constant is independent of the geodesic.

4.9 Teichmüller space

Given any marked hyperbolic surface $f : S \to X$, we can lift f to a diffeomorphism $\tilde{f} : \tilde{S} \to \mathbb{H}^2$, which restricts to a homeomorphism $\mathbb{S}^1_\infty \to \mathbb{S}^1$ between the corresponding ideal boundaries. This homeomorphism, in turn, determines a homeomorphism between the corresponding spaces of geodesics, which we denote

$$\Phi_X \colon G(\tilde{S}) \to G(\mathbb{H}^2).$$

Let $L_X = (\Phi_X^{-1})_*(L)$ denote the push-forward of the Liouville measure L by the homeomorphism Φ_X^{-1}. The following theorem, due to Bonahon [5], explains how Teichmüller space may be embedded in the space of geodesic currents:

Theorem 4.12. *The map $(X, f) \mapsto L_X$ defines a proper embedding of $\mathcal{T}(S)$ into* $\mathrm{Curr}(S)$ *and, for all $\alpha \in \mathcal{C}(S)$,*

$$i(\alpha, L_X) = \ell_\alpha(X).$$

Proof. Write $X = \mathbb{H}^2/\Gamma$. For any $\alpha \in \mathcal{C}(S)$, let $f_*(\alpha)$ denote the X–geodesic representative of the homotopy class of the same name. Let $[f_*(\alpha)) \subset \mathbb{H}^2$ be a geodesic arc that is a fundamental domain in some axis for a covering transformation corresponding to (a representative of) $f_*(\alpha)$ in $\pi_1(X) = \Gamma$. By Proposition 4.11, the L–measure of $E_{[f_*(\alpha))}$ is precisely the length of $[f_*(\alpha))$ which is also equal to $\ell_\alpha(X)$, the X–length of α.

The map Φ_X pushes currents on S forward to currents on X preserving intersection number. Viewing α as both a current and a homotopy class of curve, and appealing to Lemma 4.10, we have

$$\begin{aligned} i(\alpha, L_X) &= i((\Phi_X)_*(\alpha), L) = i(f_*(\alpha), L) \\ &= L(E_{[f_*(\alpha)]}) = \ell_{[f_*(\alpha)]} \\ &= \ell_\alpha(X). \end{aligned} \tag{5}$$

Now suppose that we have two marked hyperbolic surfaces $f\colon S \to X$ and $h\colon S \to Y$. From Theorem 3.3 we know that these represent the same point in Teichmüller space $[f\colon S \to X] = [h\colon S \to Y]$ if and only if $\ell_\alpha(X) = \ell_\alpha(Y)$ for all $\alpha \in \mathcal{S}(S) \subset \mathcal{C}(S)$. From (5) and Theorem 4.9, it follows that $[f\colon S \to X] = [h\colon S \to Y]$ if and only if $L_X = L_Y$. Thus, the rule $[f\colon S \to X] \mapsto X$ yields a well-defined map

$$\mathcal{T}(S) \to \mathrm{Curr}(S).$$

Appealing Theorem 3.3, Theorem 4.9, and continuity of length functions, one can prove that this is in fact a proper embedding. $\square$

As a consequence of Theorem 4.5 and Theorem 4.12, we have the following result of Thurston.

Corollary 4.13. *The length function*

$$\ell\colon \mathcal{T}(S) \times \mathcal{S}(S) \to \mathbb{R}$$

extends to a continuous function

$$\ell\colon \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathbb{R}$$

which is homogeneous in the second factor.

In fact, the length function $\ell\colon \mathcal{T}(S) \times \mathcal{C}(S) \to \mathbb{R}$ extends continuously and homogeneously to all geodesic currents $\ell\colon \mathcal{T}(S) \times \mathrm{Curr}(S) \to \mathbb{R}$.

4.10 Thurston's compactification

In [5], Bonahon gives an alternative construction of the Thurston compactification of $\mathcal{T}(S)$ from Theorem 3.7 which we now explain. For this, we will also need the following:

Proposition 4.14. *For any marked hyperbolic surface $f : S \to X$,*

$$i(L_X, L_X) = \pi^2 |\chi(S)|,$$

where $\chi(S)$ denotes the Euler characteristic of S.

Proof. As in the proof of Theorem 4.12, intersection numbers with L_X can be calculated by pushing-forward by Φ_X:

$$i(L_X, L_X) = i((\Phi_X)_*(L_X), (\Phi_X)_*(L_X)) = i(L, L).$$

The latter number is computed as the integral

$$i(L, L) = \int_{\mathcal{D}G(X)} dL \times dL.$$

On the other hand, $L \times L$ is invariant under the action of the whole group $\mathrm{PSL}(2, \mathbb{R})$ on $\mathcal{D}G(\mathbb{H}^2)$. We can use the projection $\mathcal{D}G(\mathbb{H}^2) \to \mathbb{H}^2$ to push $L \times L$ forward to a Radon measure on $\mathbb{H}^2$. As $L \times L$ is invariant by $\mathrm{PSL}(2, \mathbb{R})$, so is the push-forward, and therefore this measure is a constant multiple of the hyperbolic area. It follows that the area of X is a fixed constant multiple (independent of X) of the number $i(L, L)$. By the Gauss-Bonnet Theorem, this is a constant multiple of $|\chi(S)|$. A calculation shows that the constant is in fact π^2, but as we will carry out a more general computation at the end of Section 5.2, we do not do this calculation here. $\square$

Corollary 4.15. *The map $\mathcal{T}(S) \to \mathrm{Curr}(S)$ remains injective after projectivizing. That is, $\mathcal{T}(S) \to \mathbb{P}\mathrm{Curr}(S)$, is injective.*

Proof. Suppose, for contradiction, that $L_X = tL_Y$ for two points $[f\colon S \to X]$ and $[h\colon S \to Y]$, and some $t > 0$. Then, by Proposition 4.14,

$$\pi^2 |\chi(S)| = i(L_X, L_X) = t^2 i(L_Y, L_Y) = t^2 \pi^2 |\chi(S)|,$$

so $t = 1$, hence $L_X = L_Y$ and $[f : S \to X] = [h : S \to Y]$. $\square$

Another consequence is:

Corollary 4.16. The images of $\mathcal{T}(S)$ and $\mathcal{ML}(S)$ in $\mathrm{Curr}(S)$ are disjoint.

Proof. For every $[f\colon S \to X] \in \mathcal{T}(S)$, $i(L_X, L_X) = \pi^2|\chi(S)| > 0$, by Proposition 4.14. On the other hand, Proposition 4.6 gives $i(\lambda, \lambda) = 0$ for all $\lambda \in \mathcal{ML}(S)$. □

When convenient in what follows, we will identify $\mathcal{T}(S)$ and $\mathcal{ML}(S)$ as (disjoint) subsets of $\mathrm{Curr}(S)$, and $\mathcal{T}(S)$ and $\mathbb{P}\mathcal{ML}(S)$ as disjoint subsets of $\mathbb{P}\mathrm{Curr}(S)$. The following result is also due to Bonahon [5]:

Theorem 4.17. *The closure of* $\mathcal{T}(S)$ *in* $\mathbb{P}\mathrm{Curr}(S)$ *is precisely* $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$.

Proof. Let $\{[f_n\colon S \to X_n]\}_{n=1}^{\infty}$ be a divergent sequence in $\mathcal{T}(S)$. Because $\mathbb{P}\mathrm{Curr}(S)$ is compact, by Theorem 4.7, we may pass to a subsequence so that $\{L_{X_n}\}$ converges in $\mathbb{P}\mathrm{Curr}(S)$. Let $\{t_n\} \subset \mathbb{R}_+$ be such that

$$t_n L_{X_n} \to \mu \in \mathrm{Curr}(S).$$

By Theorem 3.3, divergence in $\mathcal{T}(S)$ means that there is some curve α in $\mathcal{C}(S)$ so that $\ell_\alpha(X_n) \to \infty$, and therefore by continuity of i

$$t_n \ell_\alpha(X_n) = t_n i(\alpha, L_{X_n}) = i(\alpha, t_n L_{X_n}) \to i(\alpha, \mu) < \infty.$$

It follows that $t_n \to 0$. Combining this fact with Proposition 4.14, we have

$$t_n^2 \pi^2 |\chi(S)| = i(t_n L_{X_n}, t_n L_{X_n}) \to i(\mu, \mu).$$

The left-hand side tends to 0 (since t_n does), and hence $i(\mu, \mu) = 0$. That is, $\mu \in \mathcal{ML}(S)$. Therefore the closure $\overline{\mathcal{T}}(S)$ of $\mathcal{T}(S)$ in $\mathbb{P}\mathrm{Curr}(S)$ is contained in $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$.

Since $\mathcal{S}(S)$ is dense in $\mathbb{P}\mathcal{ML}(S)$ (see the comment before Theorem 3.6), to get the equality $\overline{\mathcal{T}}(S) = \mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$ it suffices to show that for any $\alpha \in \mathcal{S}(S)$, one can construct a sequence $\{X_n\} \subset \mathcal{T}(S)$ so that $[L_{X_n}] \to \alpha$. This follows easily by appropriately "pinching the curve α" in a sequence of points in $\mathcal{T}(S)$, for example using Fenchel-Nielsen coordinates. Alternatively, one can apply Theorem 3.6 and appropriately pinch the components of an arbitrary weighted multicurve. □

It is worth mentioning that Thurston's result actually proves a little more. Namely, that the pair $(\overline{\mathcal{T}}(S), \mathbb{P}\mathcal{ML}(S))$ is homeomorphic to a closed ball and its sphere boundary $(\overline{\mathbb{B}}^{6g-6}, \mathbb{S}^{6g-7})$; see [1]. This does not follow from Theorem 4.17 and does indeed require more work.

5 Geodesic currents in other settings

In [29], Otal generalizes Bonahon's embedding $\mathcal{T}(S) \to \mathrm{Curr}(S)$ to an injective map $\mathcal{T}_{<0}(S) \to \mathrm{Curr}(S)$. Here $\mathcal{T}_{<0}(S)$ is the space of isotopy classes of negatively curved Riemannian metrics on S. Otal was primarily interested in the following corollary:

Theorem 5.1. *A negatively curved metric m on S is determined by the set $\{\ell_\alpha(m)\}_{\alpha\in\mathcal{C}(S)}$. That is, the length function*

$$\mathcal{T}_{<0}(S) \to \mathbb{R}^{\mathcal{C}(S)}$$

is injective.

This is sometimes known as *marked length spectral rigidity* for negatively curved metrics. This has been generalized to a much larger class of metrics on surfaces by Fathi [16], Croke–Fathi [12], Croke-Fathi-Feldman [13] Hersonsky–Paulin [18] as well as in many other settings.

In this section, we meander through the various constructions of geodesic currents. We will describe the Liouville current used by Otal, whose construction begins with the Liouville measure. We also mention briefly another construction of this due to Hersonsky-Paulin [18], valid for certain singular negatively metrics. Then we will turn to a seemingly different construction from [14] for certain families of Euclidean cone metrics on surfaces, only to see it as yet another incarnation of the same idea (see also Frazier's thesis [17] for further discussion of constructions of geodesic currents associated to singular metrics, in greater generality). Finally, we will give some references for one of the generalizations most relevant to the summer school, namely geodesic currents on free groups.

5.1 Liouville measures for Riemannian metrics.

For more on the topics discussed here, see [30, Chapter 1]. Given any Riemannian metric σ on S, let ϕ_t denote the geodesic flow on the unit tangent bundle $T^1(S)$ generated by the geodesic vector field $\mathcal{G}$ on $T^1(S)$. There is a canonical ϕ_t–invariant measure on $T^1(S)$, called the Liouville measure, which we denote ν_σ. We give a few different descriptions of this. Let $\pi : T^1(S) \to S$ denote the natural projection.

Write σ_0 for the metric on $T^1(S)$ induced by σ, which is defined as follows. Recall that the Levi-Civita connection of σ defines a horizontal distribution $\mathcal{H} \subset T(T^1(S))$: At any point $u \in T^1(S)$, the 2–plane $\mathcal{H}_u \subset$

$T_u(T^1(S))$ is spanned by the derivatives of parallel vector fields through u over paths through $\pi(u)$. That is, it is spanned by the derivatives of vector fields ξ over paths $\gamma\colon (-\epsilon, \epsilon) \to S$ with $\gamma(0) = \pi(u)$, $\xi_0 = u$, and covariant derivative zero $D\xi/dt|_t = 0$. The derivative π_* restricted to $\mathcal{H}_u$ at the point u is an isomorphism onto $T_{\pi(u)}(S)$. For example, $\mathcal{G}_u \in \mathcal{H}_u$ is the unique vector of $\mathcal{H}_u$ with $\pi_*(\mathcal{G}_u) = u$. We require σ_0 restricted to $\mathcal{H}_u$ to be such that π_* is an isometry from $\mathcal{H}_u$ to $T_{\pi(u)}(S)$. The kernel $\mathcal{V}_u = \ker(\pi_*|_{T_u(T^1(S))})$ is the tangent space of the fiber of π through u which is the circle $T^1_{\pi(u)}(S) \subset T_{\pi(u)}(S)$, and is thus naturally identified with a subspace

$$\mathcal{V}_u \subset T_{\pi(u)}(S).$$

We define the restriction of σ_0 to $\mathcal{V}_u$ so that this inclusion is an isometry. Finally, since $\mathcal{V}_u \oplus \mathcal{H}_u$, it makes sense to declare $\mathcal{V}_u \perp \mathcal{H}_u$, and we do so, thus determining σ_0.

The Riemannian metric σ_0 has a natural volume form (so that the volume of a positively oriented orthonormal frame is 1) and the Liouville measure is defined by the 3–form $\nu = \nu_\sigma \in \Omega^3(T^1(S))$ which is *half* of this volume form. In terms of local trivializations of the bundle $T^1(S) \to S$ (with structure group $SO(2)$), ν is 1/2 times the product of the natural area measure (from σ) on the base times the angle measure on the fiber. Consequently, we have

Proposition 5.2. *For any metric σ, we have*

$$\int_{T^1(S)} d\nu_\sigma = \pi Area(S, \sigma).$$

A second description is as follows. There is a canonical 1–form Θ on $T^1(S)$ defined by

$$\Theta_u(v) = \sigma(u, \pi_*(v))$$

for every $u \in T^1(S)$ and $v \in T_u(T^1(S))$. Alternatively

$$\Theta_u(v) = \sigma_0(\mathcal{G}_u, v). \tag{6}$$

This 1–form is actually a *contact form* which means that $\Theta \wedge d\Theta$ is a (nowhere zero) volume form. The next proposition relates this volume form to ν.

Proposition 5.3. *We have $\nu = -\frac{1}{2}\Theta \wedge d\Theta$, and this volume form is ϕ_t–invariant.*

Before we give the proof, it is useful to define two more vector fields on $T^1(S)$, in addition to $\mathcal{G}$. First, let $\rho : T^1(S) \to T^1(S)$ denote the rotation of each fiber of $\pi\colon T^1(S) \to S$ by angle $\pi/2$. Let $\mathcal{G}^\perp$ be the vector field on $T^1(S)$ whose value $\mathcal{G}^\perp_u$ at any $u \in T^1(S)$ is the tangent vector of the parallel vector field ξ over the geodesic $\gamma_{\rho(u)}\colon (-\epsilon, \epsilon) \to S$, with $\xi(0) = u$ and $\gamma'_{\rho(u)}(0) = \rho(u)$. That is, ξ is the parallel transport of u along the geodesic $\gamma_{\rho(u)}$ whose vector tangent at $\gamma_{\rho(u)}(0) = \pi(u)$ is $\rho(u)$, and $\mathcal{G}^\perp_u = \xi'(0)$. Note that $\rho(\xi(t)) = \gamma'_{\rho(u)}(t)$ for every t, and hence $t \mapsto \xi_t$ is actually an arc of a flow line for $\mathcal{G}^\perp$. See Figure 6.

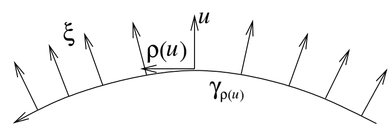

Figure 6: The geodesic $\gamma_{\rho(u)}$ and vector field ξ over this. The derivative ξ'_t is $\mathcal{G}^\perp_{\xi_t}$.

Therefore, $\mathcal{G}$ and $\mathcal{G}^\perp$ span $\mathcal{H}$ at every point, and are mutually orthonormal. The third vector field $\mathcal{R}$ is the unit vector field tangent to the fibers of π that generates a unit speed rotation R_s of each fiber (note that $\rho = R_{\pi/2}$). Then by definition of σ_0, $\mathcal{G}, \mathcal{G}^\perp, \mathcal{R}$ form an orthonormal basis at every point of $T^1(S)$. The orientation induces an orientation on $T^1(S)$, and this is in fact a positively oriented basis.

Next we calculate certain Lie derivatives with respect to $\mathcal{G}$.

Proposition 5.4. *We have $\mathcal{L}_\mathcal{G}(\Theta) = i_\mathcal{G} d\Theta = 0$. Consequently $\mathcal{L}_\mathcal{G} d\Theta = 0$.*

Proof. Applying (6) we can calculate the contractions of each of $\mathcal{G}, \mathcal{G}^\perp, \mathcal{R}$ into Θ as

$$\iota_{\mathcal{G}^\perp}\Theta = \iota_\mathcal{R}\Theta = 0 \text{ and } \iota_\mathcal{G}\Theta = 1. \tag{7}$$

Combining this with the Cartan formula for the Lie derivative gives

$$\mathcal{L}_\mathcal{G}\Theta = \iota_\mathcal{G} d\Theta + d\,\iota_\mathcal{G}\Theta = \iota_\mathcal{G} d\Theta.$$

Therefore, to prove the first part of the proposition, it suffices to show that the Lie derivative is zero.

For any vector field $\mathcal{W}$ on $T^1(S)$, consider the identity

$$\mathcal{L}_\mathcal{G}\Theta(\mathcal{W}) = \mathcal{G}(\Theta(\mathcal{W})) - \Theta(\mathcal{L}_\mathcal{G}(\mathcal{W})).$$

We will apply this identity for $\mathcal{W}$ equal to each of $\mathcal{G}, \mathcal{G}^\perp, \mathcal{R}$. For each of these three vector fields, (7) implies that this equation reduces to

$$\mathcal{L}_{\mathcal{G}}\Theta(\mathcal{W}) = -\Theta(\mathcal{L}_{\mathcal{G}}(\mathcal{W})).$$

Since $\mathcal{G}, \mathcal{G}^\perp, \mathcal{R}$ is a basis at every point, to prove $\mathcal{L}_{\mathcal{G}}\Theta = 0$, it therefore suffices to prove

$$\Theta(\mathcal{L}_{\mathcal{G}}(\mathcal{W})) = 0,$$

for each $\mathcal{W} \in \{\mathcal{G}, \mathcal{G}^\perp, \mathcal{R}\}$. Furthermore, since $\mathcal{L}_{\mathcal{G}}(\mathcal{G}) = 0$, we need only consider the cases $\mathcal{W} = \mathcal{G}^\perp$ and $\mathcal{W} = \mathcal{R}$. We explain the first case, and leave the second as an exercise.

To calculate $\mathcal{L}_{\mathcal{G}}(\mathcal{G}^\perp)$, we let ξ be an arc of a flow line for $\mathcal{G}^\perp$ as described above, which is a vector field over the geodesic $\gamma_{\rho(u)}$ (see Figure 6). Applying the geodesic flow ϕ_s generated by $\mathcal{G}$ to ξ, and projecting down to S we see that for fixed small s, the map $t \mapsto \pi(\phi_s(\xi(t)))$ is an equidistant arc from the geodesic $\gamma_{\rho(u)}$, and in particular is orthogonal to each of the geodesics $s \mapsto \pi(\phi_s(\xi(t)))$. Therefore $\pi_*(\frac{d}{dt}\phi_s(\xi(t)))$ is a multiple of $\pi_*(\mathcal{G}^\perp_{\phi_s(\xi(t))})$ for all small s, t. Applying the derivative of ϕ_{-s}, we therefore have

$$\pi_*((\phi_{-s})_*(\mathcal{G}^\perp_{\phi_s(u)})) \perp u$$

and so the Lie derivative satisfies

$$\begin{aligned} \pi_*(\mathcal{L}_{\mathcal{G}}(\mathcal{G}^\perp)_u) &= \pi_*\left(\frac{d}{ds}\bigg|_{s=0} ((\phi_{-s})_*(\mathcal{G}^\perp_{\phi_s(u)}))\right) \\ &= \frac{d}{ds}\bigg|_{s=0} (\pi_*((\phi_{-s})_*(\mathcal{G}^\perp_{\phi_s(u)}))), \end{aligned}$$

and this is orthogonal to u. From the definition of Θ we thus have

$$\Theta_u(\mathcal{L}_{\mathcal{G}}(\mathcal{G}^\perp)) = \sigma(\pi_*(\mathcal{L}_{\mathcal{G}}(\mathcal{G}^\perp)_u), u) = 0$$

as required. The case of $\mathcal{R}$ follows from a similar kind of geometric calculation, and carrying that out completes the proof of the first part of the proposition. The second part follows from the fact that $d\mathcal{L}_{\mathcal{G}} = \mathcal{L}_{\mathcal{G}}d$. □

Proof of Proposition 5.3. From Proposition 5.4 we know that $\mathcal{L}_{\mathcal{G}}(\Theta \wedge d\Theta) = 0$, and therefore $\Theta \wedge d\Theta$ is invariant by ϕ_t.

From (7) it follows that

$$(\Theta \wedge d\Theta)(\mathcal{G}, \mathcal{G}^\perp, \mathcal{R}) = (i_{\mathcal{G}}(\Theta \wedge d\Theta))(\mathcal{G}^\perp, \mathcal{R}) = d\Theta(\mathcal{G}^\perp, \mathcal{R}).$$

Therefore, to prove $\nu = -\frac{1}{2}\Theta \wedge d\Theta$, we need only prove that

$$d\Theta(\mathcal{G}^\perp, \mathcal{R}) = -1 \tag{8}$$

(recall that ν is defined as *half* of the volume form on $T^1(S)$). For this, we apply standard identities of differential forms to deduce

$$\begin{aligned} d\Theta(\mathcal{G}^\perp, \mathcal{R}) &= \mathcal{G}^\perp(\Theta(\mathcal{R})) - \mathcal{R}(\Theta(\mathcal{G}^\perp)) - \Theta(\mathcal{L}_{\mathcal{G}^\perp}(\mathcal{R})) \\ &= 0 - \Theta(\mathcal{L}_{\mathcal{G}^\perp}(\mathcal{R})) \\ &= \Theta(\mathcal{L}_{\mathcal{R}}(\mathcal{G}^\perp)). \end{aligned}$$

To prove that this is -1, we argue as in the previous proof. The flow for $\mathcal{R}$ is R_s, the rotation of the fibers through an angle s. From the description of $\mathcal{G}^\perp$ and its flow lines given above, one can check that for any $u \in T^1(S)$, we have

$$\pi_*(\mathcal{G}^\perp_{R_s(u)}) = \rho(R_s(u)) = R_{s+\pi/2}(u).$$

The map R_s descends to the identity on S by the map π (it just rotates every fiber), and hence we see that

$$\pi_*((R_{-s})_*(\mathcal{G}^\perp_{R_s(u)})) = R_{s+\pi/2}(u).$$

Therefore, calculating Lie derivatives and pushing forward to S, we have

$$\pi_*(\mathcal{L}_{\mathcal{R}}\mathcal{G}^\perp_u) = \pi_*\left(\frac{d}{ds}\bigg|_{s=0}(R_{-s})_*(\mathcal{G}^\perp_{R_s(u)})\right) = \frac{d}{ds}\bigg|_{s=0} R_{s+\pi/2}(u) = -u.$$

Thus $\Theta_u(\mathcal{L}_{\mathcal{R}}\mathcal{G}^\perp) = \sigma(u, -u) = -1$ and hence

$$(\Theta \wedge d\Theta)(\mathcal{G}, \mathcal{G}^\perp, \mathcal{R}) = -1.$$

□

5.2 From Liouville measure to Liouville current

Now suppose σ is a negatively curved Riemannian metric. To define the *Liouville current*, we note that since ν_σ is ϕ_t–invariant, we can contract with $\mathcal{G}$ and produce an invariant transverse measure L_σ for the geodesic foliation on $T^1(S)$ (which is also invariant by the antipodal map on the fibers, i.e. the map $\rho^2 = R_\pi$). Just as with hyperbolic metrics, the geodesic foliations of $T^1(S)$ and $\mathbb{P}T(S)$ coming from σ can be identified with that of a fixed reference hyperbolic metric, so L_σ is a geodesic current on S. Propositions 5.3 and 5.4, together with the discussion above, provide a nice description of this on $T^1(S)$:

Corollary 5.5. *For any $\sigma \in \mathcal{T}_{<0}(S)$, we have $L_\sigma = \frac{1}{2}|d\Theta|$.*

The calculations done so far easily lend themselves to a description in terms of lengths. For any σ–geodesic segment $\delta : (-\epsilon, \epsilon) \to \tilde{S}$ we parameterize the σ–geodesics E_δ through δ with $t \in (-\epsilon, \epsilon)$ and $\theta \in (0, \pi)$ as was done in the case of the hyperbolic plane in Section 4.8. We can also naturally view $E_\delta \subset T^1(\tilde{S})$, and as such, restricting $d\Theta$ to E_δ (which is transverse to the geodesic flow), we have

Proposition 5.6. *For any $\sigma \in \mathcal{T}_{<0}(S)$ and E_δ as above $d\Theta|_{E_\delta} = -\sin(\theta) d\theta dt$.*

Proof. Using the parameterization of E_δ by (θ, t), the tangent space to E_δ in $T^1(S)$ is spanned by $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial t}$. These are easily expressible in terms of our preferred basis at every point:

$$\left.\frac{\partial}{\partial\theta}\right|_{(\theta,t)} = \mathcal{R}_{(\theta,t)} \text{ and } \left.\frac{\partial}{\partial t}\right|_{(\theta,t)} = \cos(\theta)\mathcal{G}_{(\theta,t)} + \sin(\theta)\mathcal{G}^{\perp}_{(\theta,t)}.$$

Combining this with Proposition 5.4 and Equation (8), we have

$$\begin{aligned} d\Theta_{(\theta,t)}\left(\left.\frac{\partial}{\partial\theta}\right|_{(\theta,t)}, \left.\frac{\partial}{\partial t}\right|_{(\theta,t)}\right) &= d\Theta_{(\theta,t)}\left(\mathcal{R}_{(\theta,t)}, \cos(\theta)\mathcal{G}_{(\theta,t)} + \sin(\theta)\mathcal{G}^{\perp}_{(\theta,t)}\right) \\ &= \sin(\theta) d\Theta_{(\theta,t)}\left(\mathcal{R}_{(\theta,t)}, \mathcal{G}^{\perp}_{(\theta,t)}\right) \\ &= -\sin(\theta). \end{aligned}$$

This completes the proof. $\square$

From Corollary 5.5 and Proposition 5.6 we obtain the following familiar description of L_σ restricted to E_δ (c.f. Section 4.8).

Corollary 5.7. *For any $\sigma \in \mathcal{T}_{<0}(S)$, and any σ–geodesic $\delta \subset \widetilde{S}$ we have*

$$L_\sigma|_{E_\delta} = \frac{1}{2}\sin(\theta) d\theta dt.$$

Consequently, just as in the hyperbolic case this implies:

Corollary 5.8. *For any $\sigma \in \mathcal{T}_{<0}(S)$ and $\alpha \in \mathcal{C}(S)$ we have $i(\alpha, L_\sigma) = \ell_\alpha(\sigma)$.*

Another consequence is that, for any current $\mu \in \mathrm{Curr}(S)$, $i(\mu, L_\sigma)$ is obtained by integrating over all of $\mathbb{P}T(S)$ the measure which is the product of the Lebesgue (length) measure on the leaves of the geodesic foliation times the transverse measure μ (or equivalently, 1/2 times the total measure of this on $T^1(S)$). When $\mu = L_\sigma$, this product measure is precisely ν. The following is then immediate from Proposition 5.2:

Proposition 5.9. *For any $\sigma \in \mathcal{T}_{<0}(S)$, we have*

$$i(L_\sigma, L_\sigma) = \frac{\pi}{2} Area(S, \sigma).$$

If σ has constant curvature -1 (so $[id\colon S \to (S,\sigma)] \in \mathcal{T}(S)$), then $Area(S,\sigma) = 2\pi|\chi(S)|$ by the Gauss-Bonnet Theorem. This proposition thus proves Proposition 4.14 (with the correct constant).

The rest of Otal's Spectral Rigidity Theorem involves showing that the map from $\mathcal{T}_{<0}(S)$ to $\mathrm{Curr}(S)$ given by $\sigma \mapsto L_\sigma$ is injective. This requires more work than in the case $T(S)$, and we leave the reader to read Otal's paper [29] for the details.

5.3 Möbius currents

Here we very briefly describe the construction of Hersonsky and Paulin [18], which associates a geodesic current to a locally CAT(-1) metric σ on S; see [9] for a discussion of CAT(-1) geometry. As a Riemannian metric of negative curvature is in particular CAT(-1), this is applicable in that setting as well. In fact, this is another construction of the Liouville current from the previous subsection in that setting. We refer the reader to [18] for details and further references.

To begin we fix a locally CAT(-1) metric σ on S. Pulling this back to the universal covering $\tilde{S}$, we obtain a CAT(-1) metric which we also denote σ, and which we use to define a boundary at infinity $\mathbb{S}^1_\infty$ of $\tilde{S}$. As with hyperbolic and negatively curved Riemannian metrics, the compactification of $\tilde{S}$ obtained from σ can be identified with the one coming from any fixed hyperbolic metric on S.

The *cross ratio* associated to σ is a function on distinct quadruples of points:

$$[a,b,c,d]_\sigma = \frac{1}{2} \lim_{i\to\infty} \sigma(a_i,c_i) - \sigma(c_i,b_i) + \sigma(b_i,d_i) - \sigma(d_i,a_i). \qquad (9)$$

where $\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}$ are sequences in $\tilde{S}$ converging to a, b, c, d respectively. One can show that the limit exists, it is finite, and it is independent of the sequences. From this it easily follows that it is invariant under the action of the isometry group of σ (and in particular, the action of $\pi_1(S)$).

For $\tilde{S} = \mathbb{H}^2$ and σ the hyperbolic metric, this is the logarithm of the usual cross ratio on $\mathbb{S}^1_{\mathbb{H}}$,

$$[a,b,c,d]_\sigma = \log\left|\frac{(a-d)(b-c)}{(a-c)(b-d)}\right|, \qquad (10)$$

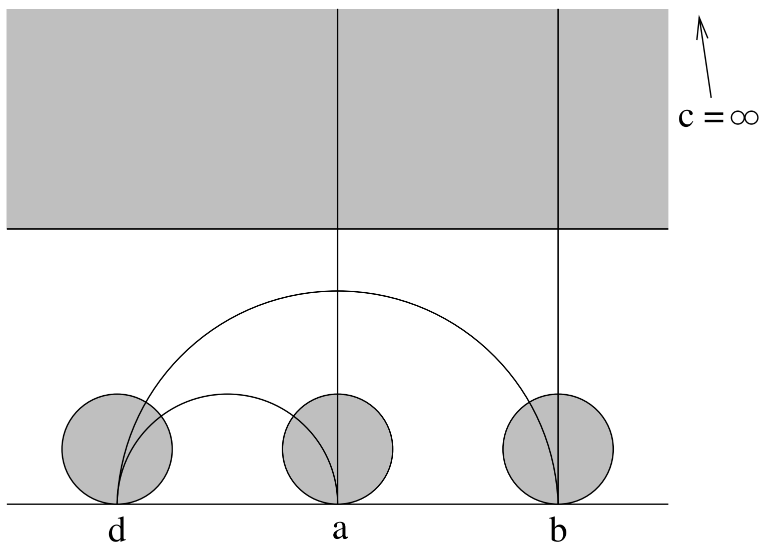

Figure 7: Horocycles centered at $a, b, c = \infty, d$. Expanding (or contracting) horocycles until the pairs (H_c, H_b), (H_b, H_d), and (H_d, H_a) are pairwise tangent allows one to calculate the right-hand side of (11) simply as $\frac{1}{2}\sigma(H_a, H_c)$. Applying an isometry, we can assume $a = 1, b > 1, d = 0$ keeping $c = \infty$ so that $\frac{(a-c)(b-d)}{(a-d)(b-c)} = b$ and hence the right hand side of (10) is $\log(b)$. An elementary calculation shows that this is precisely $\frac{1}{2}\sigma(H_a, H_c) = [a, b, c, d]_\sigma$.

where one views $\mathbb{S}^1_{\mathbb{H}} \subset \widehat{\mathbb{C}}$ either from the upper-half plane model or the Poincaré disk model of $\mathbb{H}^2$.

To prove this, note that instead of calculating the limit on the right-hand side of (9) for four sequences limiting to the points a, b, c, d, respectively, one can look at horocycles H_a, H_b, H_c, H_d centered at the points a, b, c, d respectively, and calculate the corresponding sum and differences of distances between the horocycles:

$$[a, b, c, d]_\sigma = \frac{1}{2}(\sigma(H_a, H_c) - \sigma(H_c, H_b) + \sigma(H_b, H_d) - \sigma(H_d, H_a)). \qquad (11)$$

From this and a calculation, (10) follows. See Figure 5.3.

Therefore, in light of (4), we see that for σ hyperbolic

$$L_\sigma([a, b] \times [c, d]) = [a, b, c, d]_\sigma.$$

Since the cross ratio was defined for any locally CAT(-1) metric, we can use this to define a Liouville current associated to σ, which Hersonsky-Paulin call the *Möbius current* associated to σ. The proof that this does indeed define a legitimate geodesic current is carried out in [18]. Furthermore, they prove:

Proposition 5.10. *For any locally CAT(-1) metric σ on S and any closed curve $\alpha \in \mathcal{C}(S)$*

$$i(L_\sigma, \alpha) = \ell_\alpha(\sigma)$$

where as usual, the right-hand side is the σ–length of the σ–geodesic representative of α.

This is the first step in the Hersonsky–Paulin proof that a certain class of CAT(-1)–metrics on S are spectrally rigid, generalizing Otal's Theorem 5.1. See [18] for the precise statement and more details.

5.4 Quadratic differentials

A unit norm holomorphic quadratic differential is a complex analytic object one can associate to a surface S equipped with a complex structure. In addition, it also determines a Euclidean cone metric on S for which the holonomy on $S - \mathrm{sing}(\sigma)$ (where $\mathrm{sing}(\sigma)$ are the cone points) lies in $\{\pm I\}$ and for which the cone angles are integral multiples of π, greater than 2π. This metric is obtained by first finding local coordinate charts in which the quadratic differential is simply dz^2, and then pulling back the Euclidean metric from such a chart. Conversely, every Euclidean cone metric with holonomy in $\{\pm I\}$ is obtained in this way from a complex structure and a quadratic differential which is unique up to scaling by complex number on the unit circle. In fact, the norm of the quadratic differential is just the area of the surface with respect to the given metric, and one can determine the quadratic differential uniquely from a choice of parallel line field on S, which is required to be vertical in the coordinate charts above; see e.g. [14, 28, 32] for details.

We let $\mathcal{Q}(S)$ denote the set of all unit norm (i.e. unit area) quadratic differentials on S up to isotopy (more precisely, points in $\mathcal{Q}(S)$ are determined by a complex structure and a unit norm holomorphic quadratic differential, up to isotopy) and $\mathrm{Flat}(S)$ denote the set of all isotopy classes of Euclidean cone metrics coming from quadratic differentials in $\mathcal{Q}(S)$. As an abuse of notation, we let $q \in \mathcal{Q}(S)$ also denote the metric $q \in \mathrm{Flat}(S)$ associated to q when the distinction is clear.

Given $q \in \mathcal{Q}(S)$ and angle $\theta \in [0, \pi)$, we can consider the foliation of $S \setminus \mathrm{sing}(\sigma)$ by geodesics in the direction of θ, measured with respect to the parallel line field determining q from the metric, which we take to have angle 0. The orthogonal distance between the leaves of this foliation defines a transverse measure which we denote $\nu_q(\theta)$. We can put the points of $\mathrm{sing}(\sigma)$ back in to produce a singular measured foliation of the same name. Thurston proved that any such foliation is associated to a measured geodesic lamination $\lambda_q(\theta)$ with the property that

$$i(\nu_q(\theta), \alpha) = i(\lambda_q(\theta), \alpha)$$

for every $\alpha \in \mathcal{C}(S)$; see also [27] and [26]. Here $i(\nu_q(\theta), \alpha)$ is the infimum of total variations, with respect to the measure $\nu_q(\theta)$, of representatives of α .

Therefore, given a quadratic differential q, we obtain a map

$$\lambda_q : [0, \pi) \to \mathcal{ML}(S) \subset \mathrm{Curr}(S).$$

The space $\mathrm{Curr}(S)$ is a positive cone (convex combinations of any set of elements are well-defined), and hence we can integrate this map to define

$$L_q = \frac{1}{2} \int_0^{\pi} \lambda_q(\theta) d\theta.$$

If we want to compute the intersection $i(L_q, \mu)$, this is given by

$$i(L_q, \mu) = \frac{1}{2} \int_0^{\pi} i(\lambda_q(\theta), \mu).$$

For example, suppose $\alpha \in \mathcal{C}(S)$ and let α_σ denote the σ–geodesic representative. Such a geodesic is either a closed Euclidean geodesic in $S \backslash \mathrm{sing}(\sigma)$, or else is a concatenation of Euclidean geodesic segments between points of $\mathrm{sing}(\sigma)$. It turns out that the infimum of the variations of α with $\nu_q(\theta)$ is realized by α_σ:

$$i(\lambda_q(\theta), \alpha) = i(\nu_q(\theta), \alpha) = \int_{\alpha_\sigma} d\nu_q(\theta).$$

Now write $\alpha_\sigma = \alpha_\sigma^1 \alpha_\sigma^2 \cdots \alpha_\sigma^k$, for the Euclidean segments in the geodesic representative (with $k = 1$ if α_σ is contained in the complement $\mathrm{sing}(\sigma)$). Suppose α_σ^i makes an angle θ^i with the preferred line field. Then if we parameterize $\alpha_\sigma^i : [0, T_i] \to S$ with respect to σ–arc length we have

$$\int_{\alpha_\sigma^i} d\nu_q(\theta) = \int_0^{T_i} |\sin(\theta - \theta_i)| dt = T_i |\sin(\theta - \theta_i)|.$$

Therefore

$$i(L_q,\alpha) = \frac{1}{2}\int_0^\pi i(\nu_q(\theta),\alpha)d\theta = \frac{1}{2}\sum_{i=1}^k \int_0^\pi T_i|\sin(\theta-\theta_i)|d\theta = \sum_{i=1}^k T_i = \ell_\alpha(\sigma).$$

This proves part of the following theorem of [14].

Theorem 5.11. *The map $q \mapsto L_q$ descends to an embedding* $\mathrm{Flat}(S) \to \mathrm{Curr}(S)$ *with the property that*

$$\ell_\alpha(\sigma(q)) = i(L_q,\alpha)$$

for every $\alpha \in \mathcal{C}(S)$. Moreover, $i(L_q, L_q) = \pi/2$.

The fact that the map to $\mathrm{Curr}(S)$ is an embedding is a consequence of the stronger statement that the length function

$$\mathrm{Flat}(S) \to \mathbb{R}^{\mathcal{S}(S)}$$

is injective. This should be compared with the case of $\mathcal{T}_{<0}(S)$ for which the analogous statement is false—one really needs to consider $\mathcal{C}(S)$ instead of $\mathcal{S}(S)$. On the other hand, comparing with $\mathcal{T}(S)$ where one really only needs a finite set of curves, for $\mathrm{Flat}(S)$ no finite set of curves suffices. In fact, it is shown in [14] that a subset $\Omega \subset \mathcal{S}(S)$ will have the property that the length function $\mathrm{Flat}(S) \to \mathbb{R}^\Omega$ is injective if and only if $\overline{\Omega} = \mathbb{P}\mathcal{ML}(S)$.

One can also use this theorem to provide an analogue of the Thurston compactification of Teichmüller space for the space $\mathrm{Flat}(S)$. In this case, a subset of the boundary turns out to be $\mathbb{P}\mathcal{ML}(S)$ again, but now there is a more general geometric object one can find in the boundary called a *mixed structure*. This is a sort of hybrid of measured lamination and Euclidean cone metric on disjoint subsurfaces; see [14].

We end this subsection by relating L_q to the previous descriptions of Liouville currents. Pull q back to the universal covering $\tilde{S} \to S$ and continue to denote it q. Fix any geodesic arc in the complement of the singularities $\delta : (-\epsilon,\epsilon) \to \tilde{S} \setminus \mathrm{sing}(\tilde{\sigma})$, parameterized with unit speed. Then we may parameterize the set E_δ of *nonsingular* biinfinite geodesics through δ by a full measure subset of $(0,\pi)\times(-\epsilon,\epsilon)$ just as in the case of hyperbolic and negatively curved metrics (where the subset was all of $(0,\pi)\times(-\epsilon,\epsilon)$). The above calculations easily yield an expression for L_q, familiar from both the hyperbolic and negatively curved setting:

$$L_q|_{E_\delta} = \frac{1}{2}\sin(\theta)d\theta dt.$$

5.5 Free groups

The definition of geodesic current easily generalized to an arbitrary Gromov hyperbolic group G; see [6]. In place of the action of $\pi_1(S)$ on $\mathbb{H}^2$ and the induced action on $\mathbb{S}^1$, we can consider the action of G on its Cayley graph $\Gamma(G)$ (with respect to some finite generating set) and the induced action on the Gromov boundary $\partial G = \partial\Gamma(G)$. G acts on the space of equivalence classes of geodesics $\mathcal{G}(\Gamma(G))$ in the Cayley graph $\Gamma(G)$, which is identified with $\partial G \times \partial G \setminus \Delta / \sim$ where $(x, y) \sim (y, x)$. A geodesic current on G is then a G–invariant measure on $\mathcal{G}(G)$, and the space of such is denoted $\mathrm{Curr}(G)$.

The case of $G = F_n$, a free group, has been extensively studied (a MathSciNet search for "free group" and "geodesic current" provides plenty of references, but we list a few in the next paragraph relevant to the current discussion). Many of the results we have described have analogues in that setting, and we briefly mention a few.

It was shown in [19] that a conjugacy class in F_n naturally defines a geodesic current, and that positive real multiples of such currents are dense in $\mathrm{Curr}(F_n)$. One might hope for a natural bilinear, symmetric intersection form as in the case of $\mathrm{Curr}(S)$. However, in [20], it is shown that this is not possible. On the other hand, in [22], it is shown that there is an "intersection form" defined on $\overline{\mathrm{cv}}_n \times \mathrm{Curr}(F_n)$, where $\overline{\mathrm{cv}}_n$ is the closure of the unprojectivized Culler–Vogtmann outer space in the space of length functions. In a different direction, there is a natural analogue of the Liouville current for free groups which provides an $\mathrm{Out}(F_n)$–equivariant embedding of Culler–Vogtman's outer space CV_n into $\mathbb{P}\mathrm{Curr}(F_n)$; this was defined and studied in [23]. This in turn provides a compactification of CV_n, but as is shown in [21], this is a different compactification than the length function compactification which is the natural analogue of the Thurston compactification.

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