Descent for differential Galois theory of difference equations
Confluence and $q$-dependency

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Abstract

The present paper essentially contains two results that generalize and improve some of the constructions of [HS08]. First of all, in the case of one derivation, we descend the differential Galois theory for difference equations constructed in [HS08] from a differentially closed to an algebraically closed field. In the second part of the paper, we show that the theory can be applied to deformations of $q$-series, to study the differential dependency with respect to $x\frac{d}{dx}$ and $q\frac{d}{dq}$. We show that the differential Galois group of the Jacobi Theta function can be considered as the galoisian counterpart of the heat equation.

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Introduction

The present paper essentially contains two results that generalize and improve some of the constructions of [HS08]. First of all, in the case of one derivation, we descend the constructions in [HS08] from a differentially closed to an algebraically closed field. The latter being much smaller, this is a good help in the applications. In the second part of the paper, we show that the theory can be applied to deformations of $q$-series, which appears in many setting, like quantum invariants, modular forms, etc. etc., to study the differential dependency with respect to $x\frac{d}{dx}$ and $q\frac{d}{dq}$.

In [HS08], the authors construct a specific Galois theory to study the differential relations among solutions of a difference linear system. To do so, they attach to a linear difference system a differential Picard-Vessiot ring, a sort of differential splitting ring, and therefore a differential Galois group, in the sense of Kolchin. Very roughly, that is a matrix group defined as the zero set of algebraic differential equations. In [HS08], both the differential Picard-Vessiot ring and the differential Galois group are proven to be well defined under the assumption that the difference operator and the derivations commutes each other and that the field of constant for the difference operator is differentially closed. The differential closure of a differential field $K$ is an enormous field that contains a solution of any algebraic differential equation with coefficients in $K$, that has a solution
in some differential extension of $K$. When one works with $q$-difference equations, the field of constants for the homothetic $x \mapsto qx$ in the field of meromorphic functions over $\mathbb{C}^*$ is the field of elliptic functions; its differential closure is a very big field. The same happens for the shift $x \mapsto x + 1$, whose field of constants are periodic functions. In the applications, C. Hardouin and M. Singer prove that one can always descend with an ad hoc argument the differential Galois group. Here we prove that in the case of one derivation we can actually suppose that the field of constants is algebraically closed and that the differential Galois group descend from a differentially closed field to an algebraically closed one (see Proposition 1.22). As a corollary we obtain that also the criteria used in the applications descend to an algebraically closed field, namely (see Remark 1.24):

- the differential transcendence degree of an extension generated by a fundamental solution matrix of the difference equation is equal to the dimension of the differential Galois group (see Proposition 1.3);
- the sufficient and necessary condition for solutions of rank 1 difference equations to be differentially transcendent (see Proposition 1.10);
- the sufficient and necessary condition for a difference system to admit a linear differential system totally integrable with the difference system (see Proposition 1.11).

The proof of the descent (see subsection 1.2) is based on an idea of M. Wibmer which he used in a parallel difference setting (see [Wib10], [Coh65]). The differential counterpart of his method is the differential prolongation of ideals, which goes back at least to E. Cartan, E. Vessiot and E.R. Kolchin, but has never been exploited in this framework. The prolongation of a system of nonlinear partial differential equations consists in the equations obtained by deriving the initial system. For instance, B. Malgrange proves a statement by E. Cartan, saying that a system of partial differential equation is involutive i.e. the iteration of the process of prolongation of the system is stationary (see [Mal98], [Mal01], [Mal05]). This result is one of the starting point of the Galois theory of nonlinear differential equations (see [Mal98], [Mal01], [Mal05]). In [Gra], A. Granier adapted the ideas of B. Malgrange to build a Galois theory for nonlinear $q$-difference equations. She attached to a nonlinear $q$-difference equation $Y(qx) = F(x, Y(x))$ a $D$-groupoid, which, by construction, contains the prolongations of the algebraic relations satisfied by the dynamic of the nonlinear $q$-difference system. In [DVH10a], it is proved using a $q$-analogue of the Grothendieck conjecture that for a linear $q$-difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in GL_n(\mathbb{C}(x))$, the $D$-groupoid of A. Granier essentially coincides with a generic form of the differential Galois $x \frac{d}{dx}$-group of $Y(qx) = A(x)Y(x)$ introduced in [HS08]. Below we will show how the constructions by differential prolongation of A. Granier are intimately related to the Picard-Vessiot constructions in [HS08] and how the differential prolongation of algebraic ideals leads to a descent process.

In the second part of the paper we show that the theory applies to inquire the differential relations with respect to the derivations $\frac{d}{dx}$ and $\frac{d}{dq}$ among solutions of $q$-difference equations over a field $K(q, x)$ of rational functions with coefficients in a field $K$ of zero characteristic. For instance one can consider the Jacobi theta function $\theta_q(x) = \sum_{n \in \mathbb{Z}} q^{n(n-1)/2}x^n$, which is solution of the $q$-difference equation $y(qx) = qxy(x)$: in this specific case we show below that the differential Galois group can be considered as the galoisian counterpart of the heat equation. We show that for solutions of rank one equations $y(qx) = a(x)y(x)$ with $a(x) \in k(q, x)$ the following facts are equivalent (see Proposition 2.6):

- $a(x) = \mu x^r \frac{g(qx)}{g(x)}$, for some $r \in \mathbb{Z}$, $g \in k(q, x)$ and $\mu \in k(q)$;
- satisfying an algebraic $\frac{d}{dx}$-relation (over some field we are going to specify in the text below);
- satisfying a relation with respect to the derivation $\ell_q(x) x \frac{d}{dx} + q \frac{d}{dq}$.

In the higher rank case, we deduce from a general statement a necessary and sufficient condition on the Galois group so that a system $Y(qx) = A(x)Y(x)$ can be completed in a compatible (i.e. integrable) way, by two linear differential systems in $\frac{d}{dx}$ and $x \ell_q \frac{d}{dx} + q \frac{d}{dq}$ (see Proposition 2.9). The condition consists in the property of the group of being, up to a conjugation, contained in the subgroup of constants of $GL$, that is the differential subgroup of $GL$ whose points have coordinates that are annihilated by the derivations. A result of P. Cassidy [Cas72] says that for a proper Zariski dense differential subgroup of a simple algebraic group this is always the case.

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1 Differential Galois theory for difference equations

1.1 Introduction to differential Galois theory for difference equations

In this section, we recall briefly some results of \[\text{HS08}\]. Let \(F\) be a field equipped with an automorphism \(\sigma\). We denote by \(K = F^\sigma\) the subfield of \(F\) of all elements fixed by \(\sigma\).

**Definition 1.1.** A \((\sigma-)\)difference module \(M = (M, \Sigma)\) over \(F\) (also called \(\sigma\)-module over \(F\) or a \(F,\sigma\)-module, for short) is a finite dimensional \(F\)-vector space \(M\) together with a \(\sigma\)-semilinear bijection \(\Sigma : M \to M\) i.e. a bijection \(\Sigma\) such that \(\Sigma(\lambda m) = \sigma(\lambda)\Sigma(m)\) for all \((\lambda, m) \in F \times M\).

One can attach to a \(\sigma\)-difference system

\[
\sigma(Y) = AY, \quad \text{with } A \in GL_\nu(F), \quad \text{for some } \nu \in \mathbb{Z}_{>0},
\]

a \(\sigma\)-difference module \(\mathcal{M}_A := (F^\nu, \Sigma_A)\), with \(\Sigma_A : F^\nu \to F^\nu, \quad Y \mapsto A^{-1}\sigma(Y)\), so that the horizontal (i.e. invariant) vectors with respect to \(\Sigma_A\) correspond to the solutions of \(\sigma(Y) = AY\). Conversely, the choice of an \(F\)-basis \(\xi\) of a difference module \(M\) leads to a \(\sigma\)-difference system \(\sigma(Y) = AY\), with \(A \in GL_\nu(F)\), which corresponds to the equation of the horizontal vectors of \(M\) with respect to \(\Sigma_A\) in the chosen basis.

We define a morphism of \((\sigma-)\)difference modules over \(F\) to be an \(F\)-linear map between the underlying \(F\)-vector spaces, commuting to the \(\sigma\)-semilinear operators. As defined above, the \((\sigma-)\)difference modules over \(F\) form a tannakian category (see \[Del90\]), i.e. a category equivalent over the algebraic closure \(\overline{K}\) of \(K\) to the category of finite dimensional representations of an affine group scheme. The affine group scheme corresponding to the subtannakian category generated by the \((\sigma-)\)difference module \(\mathcal{M}_A\), whose nontannakian construction we are going to sketch below, is called Picard-Vessiot group of \((\ref{1.1})\). Its structure measures the algebraic relations satisfied by the solutions of \((\ref{1.1})\).

Let \(\Lambda := \{\partial_1, ..., \partial_n\}\) be a set of commuting derivations of \(F\) such that for all \(i = 1, ..., n\) we have \(\sigma \circ \partial_i = \partial_i \circ \sigma\).

In \[HS08\], the authors proved, among other things, that the category of \(\sigma\)-modules carries also a \(\Lambda\)-structure i.e. it is a differential tannakian category as defined by A. Ovchinnikov in \[Ovc09\]. The latter is equivalent to a category of finite dimensional representations of a differential group scheme (see \[Kol73\]), whose structure measures the differential relations satisfied by the solutions of the \(\sigma\)-difference modules. In the next section, we describe the Picard-Vessiot approach to the theory in \[HS08\] (in opposition to the differential tannakian approach), i.e. the construction of minimal rings containing the solutions of \(\sigma(Y) = AY\) and their derivatives with respect to \(\Lambda\), whose automorphism group is a concrete incarnation of the differential group scheme defined by differential tannakian equivalence.

We will implicitly consider the usual Galois theory of \((\sigma-)\)difference equation by allowing \(\Delta\) to be the empty set (cf. for instance \[vdPS97\]): we will informally refer to this theory and the objects considered in it as classical.

### 1.1.1 \(\Delta\)-Picard-Vessiot rings

Let \(F\) be a \((\sigma, \Delta)\)-field as above, i.e. a field equipped with an automorphism \(\sigma\) and a set of commuting derivations \(\Delta = \{\partial_1, ..., \partial_n\}\), such that \(\sigma \circ \partial_i = \partial_i \circ \sigma\) for any \(i = 1, ..., n\). We set \(K = F^\sigma\). The commutativity condition among \(\sigma\) and the derivations of \(\Delta\) implies that \(K\) is a \(\Delta\)-field.

Let us consider \(\sigma\)-difference system

\[
\sigma(Y) = AY,
\]

with \(A \in GL_\nu(F)\), as in \((\ref{1.1})\).

**Definition 1.2** (Def. 6.10 in \[HS08\].) A \((\sigma, \Delta)\)-extension \(\mathcal{R}\) of \(F\) is a \(\Delta\)-Picard-Vessiot extension for \((\ref{1.2})\) if

1. \(\mathcal{R}\) is a simple \((\sigma, \Delta)\)-ring i.e. it has no nontrivial ideal stable under both \(\sigma\) and \(\Delta\);

2. \(\mathcal{R}\) is generated as a \(\Delta\)-ring by \(Z \in GL_\nu(\mathcal{R})\) and \(\frac{1}{\det(\sigma)}\), where \(Z\) is a fundamental solution matrix of \((\ref{1.2})\).

\(^1\)We will use the terms \(\Delta\)-ring, \(\Delta\)-field, \(\partial\)-ring, \(\partial\)-field, for \(\partial \in \Delta\), \(\sigma\)-ring, \(\sigma\)-field, etc. etc. with the evident meaning, analogous to the definition of \((\sigma, \Delta)\)-field.

\(^2\)That is a ring extension \(\mathcal{R}\) of \(F\) equipped with an extension of \(\sigma\) and of the derivations of \(\Delta\), such that the commutativity conditions are preserved.
One can construct formally such an object as follows. We consider the ring of $\Delta$-polynomials $F\{X, \det X^{-1}\}_\Delta$, that we equip with a structure of $\sigma$-difference algebra by setting $\sigma(X) = AX$ and

\begin{align*}
\sigma(\partial^\omega X) := \sum_{i+j=\omega} \left( \frac{\omega_1}{i_1} \right) \cdots \left( \frac{\omega_n}{i_n} \right) \partial^\omega(A)X^\omega,
\end{align*}

for each multi-index $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{N}^n$. Then the quotient $\mathcal{R}$ of $F\{X, \det X^{-1}\}_\Delta$ by a maximal $(\sigma, \Delta)$-ideal obviously satisfies the conditions of the definition above and hence is a $\Delta$-Picard-Vessiot ring. It has moreover the following properties:

**Proposition 1.3** (Prop. 6.14 and 6.16 in [HS08]). If the field $K$ is $\Delta$-closed then:

1. The ring of constants $\mathcal{R}^\sigma$ of a $\Delta$-Picard-Vessiot ring $\mathcal{R}$ for (1.2) is equal to $K$, i.e. there are no new constants compare to $F$.

2. Two $\Delta$-Picard-Vessiot ring for (1.2) are isomorphic as $(\sigma, \Delta)$-ring.

**Remark 1.4.** Although in most applications of [HS08] a descent argument ad hoc proves that one can consider smaller, nondifferentially closed, field of constants, the assumption that $K$ is $\Delta$-closed is quite restrictive. We show in section 1.1.2 that one can adapt an argument by M. Wibmer ([Wib10]), that he used in the case of $\sigma$-rings, and build a $\Delta$-Picard Vessiot ring $\mathcal{R}$ such that $\mathcal{R}^\sigma = K$ under the assumption that $K$ is only algebraically closed.

### 1.1.2 $\Delta$-Picard-Vessiot groups

Until the end of this subsection, we assume that $K$ is a $\Delta$-closed field.

Let $\mathcal{R}$ be a $\Delta$-Picard-Vessiot ring for (1.2). Notice that, as in classical Galois theory for $(\sigma)$-difference equations, the ring $\mathcal{R}$ does not need to be a domain. One can show that it is in fact the direct sum of a finite number of copies of an integral domain, therefore one can consider the ring $L$ of total fractions of $\mathcal{R}$ which is isomorphic to the product of a finite number of copies of one field (cf. [HS08]).

**Definition 1.5.** The group $\text{Gal}^{\Delta}(\mathcal{M}_A)$ (also denoted $\text{Aut}^{\sigma, \Delta}(L/F)$) of the automorphisms of $L$ that fix $F$ and commute to $\sigma$ and $\Delta$ is called the $\Delta$-Picard-Vessiot group of (1.2). We will also call it the Galois $\Delta$-group of (1.2).

**Remark 1.6.** The group $\text{Gal}^{\Delta}(\mathcal{M}_A)$ consists in the $K$-points of a linear algebraic $\Delta$-subgroup of $GL_n(K)$ in the sense of Kolchin. That is a subgroup of $GL_n(K)$ defined by a $\Delta$-ideal of $K \{ Z, \det Z^{-1} \}_\Delta$.

We recall below some fundamental properties of the Galois $\Delta$-group, which are the starting point of proving the Galois correspondence:

**Proposition 1.7** (Lemma 6.19 in [HS08]).

1. The ring $L^{\text{Gal}^{\Delta}(\mathcal{M}_A)}$ of elements of $L$ fixed by $\text{Gal}^{\Delta}(\mathcal{M}_A)$ is $F$.

2. Let $H$ a $\Delta$-subgroup of $\text{Gal}^{\Delta}(\mathcal{M}_A)$. If $L^H = F$, then $H = \text{Gal}^{\Delta}(\mathcal{M}_A)$.

As we have already pointed out, the Galois $\varnothing$-group is an algebraic group defined over $K$ and corresponds to the classical Picard-Vessiot group attached to the $\sigma$-difference system (1.2) (see [vdPS97], [Sau04]):

**Proposition 1.8** (Prop 6.21 in [HS08]). The algebraic $\Delta$-group $\text{Gal}^{\Delta}(\mathcal{M}_A)$ is a Zariski dense subset of $\text{Gal}^\varnothing(\mathcal{M}_A)$.

---

3. A ring of $\Delta$-polynomials is a ring of polynomials in infinitely many variables

\[
F\{X, \det X^{-1}\}_\Delta := F\left\{ X^{(\omega)}_{i,j} : i, j = 1, \ldots, \nu; \omega \in \mathbb{Z}_{\geq 0}^n \right\} \left[ \frac{1}{\det X} \right],
\]

equipped with a differential structure such that $\partial^\omega (X^{(\omega)}_{i,j}) = X^{(\omega + e_k)}_{i,j}$, for any $i, j = 1, \ldots, \nu$, $k = 1, \ldots, n$, $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}_{\geq 0}^n$, with $e_k = (0, \ldots, 0, \underbrace{1}_{k\text{-th}}, 0, \ldots, 0)$. We write $X$ for $(X_{i,j})$, $\partial^\omega (X)$ for $\partial^\omega_1 \cdots \partial^\omega_n (X)$ and det $X$ for $\det(X_{i,j})$. If $\Delta$ is empty, the ring $F\{X, \det X^{-1}\}_\Delta$ is nothing else than the ring of polynomials $F\{X, \det X^{-1}\}$.

4. The following formula is justified by the commutativity relation $\sigma(\partial^\omega X) = \partial^\omega (\sigma(X)) = \partial^\omega (AX)$, and the application of the Leibniz rule.

5. By maximal $(\sigma, \Delta)$-ideal we mean an ideal which is invariant by both $\sigma$ and the derivations in $\Delta$, and is maximal for this property.

6. This means that each algebraic differential equations in $\partial_1, \ldots, \partial_n$ with coefficients in $K$, having a solution in an extension of $K$, has a solution in $K$. 

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4
1.1.3 Differential dependency and total integrability

The \( \Delta \)-Picard-Vessiot ring \( \mathcal{R} \) of \( \{1,2\} \) is a \( \text{Gal}^{\Delta}(\mathcal{M}_A) \)-torsor in the sense of Kolchin. This implies in particular that the \( \Delta \)-relations satisfied by a fundamental solution of the \( \sigma \)-difference system \( \{1,2\} \) are entirely determined by \( \text{Gal}^{\Delta}(\mathcal{M}_A) \):

**Proposition 1.9 (Prop 6.29 in [HS08]).** The \( \Delta \)-transcendence degree of \( \mathcal{R} \) over \( F \) is equal to the \( \Delta \)-dimension of \( \text{Gal}(\mathcal{M}_A) \).

Since the \( \Delta \)-subgroups of \( \mathbb{G}_a^n \) coincide with the zero set of an homogeneous linear \( \Delta \)-polynomial \( L(Y_1, \ldots, Y_n) \) (cf. [Cas73]), we have:

**Proposition 1.10.** Let \( a_1, \ldots, a_n \in F \) and let \( S \) be a \( (\sigma, \Delta) \)-extension of \( F \) such that \( S^\sigma = K \). If \( z_1, \ldots, z_n \in S \) satisfy \( \sigma(z_i) = z_i = a_i \) for \( i = 1, \ldots, n \), then \( z_1, \ldots, z_n \in S \) satisfy a nontrivial \( \Delta \)-relation over \( F \) if and only if there exists a nonzero homogeneous linear differential polynomial \( L(Y_1, \ldots, Y_n) \) with coefficients in \( K \) and an element \( f \in F \) such that \( L(a_1, \ldots, a_n) = \sigma(f) - f \).

**Proof.** If \( \Delta = \{ \partial_1 \} \) the proposition coincide with Prop 3.1 in [HS08]. The proof in the case of many derivations is a straightforward generalization of their argument.

The following proposition relates the structural properties (cf. the remark immediately below for the definition of constant \( \Delta \)-group) of the differential Galois group with the holonomy of a \( \sigma \)-difference system:

**Proposition 1.11.** The following facts are equivalent:

1. The \( \Delta \)-Galois group \( \text{Gal}^{\Delta}(\mathcal{M}_A) \) is conjugated over \( K \) to a constant \( \Delta \)-group.

2. For all \( i = 1, \ldots, n \), there exists a \( B_i \in M_n(F) \) such that \( \sigma(B_i) = AB_iA^{-1} + \partial_i(A)A^{-1} \) and the linear systems

\[
\begin{align*}
\sigma(Y) &= AY \\
\partial_1 Y &= B_1 Y \\
\cdots & \cdots \\
\partial_n Y &= B_n Y
\end{align*}
\]

are compatible.

**Proof.** The proof is a straightforward generalization of Prop 2.9 in [HS08] to the case of several derivations.

**Remark 1.12.** Let \( K \) be a \( \Delta \)-field and \( C \) its subfield of \( \Delta \)-constants. A linear \( \Delta \)-group \( G \subset GL_\nu \) defined over \( K \) is said to be a constant \( \Delta \)-group (or \( \Delta \)-constant, for short) if one of the following equivalent facts hold:

- the set of differential polynomials \( \partial_i(X_{i,j}) \), for \( h = 1, \ldots, n \) and \( i, j = 1, \ldots, \nu \), belong to the ideal of definition of \( G \) in the differential Hopf algebra \( K\{X_{i,j}, \frac{1}{\det_{\Delta}(X_{i,j})}\} \Delta \) of \( GL_\nu \) over \( K \);

- the differential Hopf algebra of \( G \) over \( K \) is an extension of scalars of a finitely generated Hopf algebra over \( C \);

- the points of \( G \) in \( K \) (which is \( \Delta \)-closed!) coincide the \( C \)-points of an algebraic group defined over \( C \).

For instance, let \( \mathbb{G}_m \) be the multiplicative group defined over \( K \). Its differential Hopf algebra is \( K\{x, \frac{1}{x}\} \Delta \), i.e. the differential polynomial ring generated by \( x \) and \( \frac{1}{x} \). The \( \Delta \)-constant group \( \mathbb{G}_m(C) \) corresponds to the differential Hopf algebra

\[
\frac{K\{x, \frac{1}{x}\} \Delta}{(\partial_h(x); h = 1, \ldots, n)} \cong C\left[ x, \frac{1}{x} \right] \otimes_C K,
\]

where \( C\left[ x, \frac{1}{x} \right] \) is a \( \Delta \)-ring with the trivial action of the derivations in \( \Delta \).

A Zariski dense \( \Delta \)-subgroup of a simple algebraic group \( G \) is either conjugated to a constant \( \Delta \)-subgroup of \( G \) or equal to the whole group \( G \) (see [Cas72]), therefore:

**Corollary 1.13.** If \( \text{Gal}^{\emptyset}(\mathcal{M}_A) \) is simple, we are either in the situation of the proposition above or there are no \( \Delta \)-relations among the solutions of \( \sigma(Y) = AY \).

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7We informally call a \( \Delta \)-relation an element of the \( \Delta \)-ideal of \( F\{X, \det X^{-1}\} \Delta \) of all differential polynomials that have the chosen fundamental solution of \( \sigma(Y) = AY \) as solution.

8Here compatible means that the matrixes \( B_i \) and \( A \) have to satisfy the functional equations deduced from the commutativity of the operators, i.e. \( \partial_i(B_j) + B_jB_i = \partial_j(B_i) + B_iB_j \), \( \sigma(B_i)A = \partial_j(A) + AB_j \), for any \( i, j = 1, \ldots, n \).
1.2 Descent over an algebraically closed field

In this subsection, we consider a $(\sigma, \Delta)$-field $F$, where $\sigma$ is an automorphism of $F$ and $\Delta = \{\partial\}$ is a set containing only one derivation. Moreover we suppose that $\sigma$ commutes with $\partial$.

We show that if the $\sigma$-constants $K$ of $F$ form an algebraically closed field, we can construct a $\partial$-Picard-Vessiot ring, whose ring of $\sigma$-constants coincides with $K$, which allows us to descent the group introduced in the previous section from a $\partial$-closed field to an algebraically closed field. This kind of results were first obtained in [Wib09] with model theoretic arguments. Here we develop a differential analogue of the ideas of [Wib09] for the difference parameter case (see Lemma 2.16 in loc. cit.).

Let $\Theta$ be the semigroup generated by $\partial$ and for all $k \in \mathbb{N}$ let $\Theta_{\leq k}$ be the elements of $\Theta$ of order less or equal to $k$. We endow the differential polynomial ring $S := K\{X, \frac{1}{\det(X)}\}_\partial$, where $X = (X_{i,j})$, with the graduation associated to the usual ranking i.e. we consider for all $k \in \mathbb{N}$ the polynomial ring $S_k := K\left[\beta(X), \frac{1}{\det(X)} : \forall \beta \in \Theta_{\leq k}\right]$. Of course, we have $\partial(S_k) \subset S_{k+1}$.

**Definition 1.14** (cf. §3 in [Lan70]). The prolongation of an ideal $\mathcal{I}_k$ of $S_k$ is the ideal $\pi_1(\mathcal{I}_k)$ of $S_{k+1}$ generated by $\Theta_{\leq 1}(\mathcal{I}_k)$. We say that a prime ideal $\mathcal{I}_k$ of $S_k$ is a differential Kernel of length $k$ if the prime ideal $\mathcal{I}_{k-1} := \mathcal{I}_k \cap S_{k-1}$ of $S_{k-1}$ is such that $\pi_1(\mathcal{I}_{k-1}) \subset \mathcal{I}_k$.

**Remark 1.15.** In [Lan70], the author defines a differential Kernel as a finitely generated field extension $F(A_0, A_1, ..., A_k)/F$, together with an extension of $\partial$ to a derivation of $F(A_0, A_1, ..., A_k)$ into $F(A_0, A_1, ..., A_k)$ such that $\partial(A_i) = A_i + 1$ for $i = 0, ..., k - 1$. We can recover $\mathcal{I}_{k+1}$ as the kernel of the $F$-morphism $\mathcal{S}_k = F[X, ..., \partial^k(X)] \rightarrow F(A_0, A_1, ..., A_k)$, with $\partial^i(X) \mapsto A_i$.

**Proposition 1.17** (Prop.1 in [Lan70]). Let $\mathcal{I}_k$ be a differential kernel of $S_k$. There exists a differential kernel $\mathcal{I}_{k+1}$ of $S_{k+1}$ such that $\mathcal{I}_k = \mathcal{I}_{k+1} \cap S_k$.

Let $\mathcal{S}_k := \mathcal{I}_k \cap S_k$ is a maximal $\sigma$-ideal of $S_k$ endowed with the $\sigma$-structure induced by $\sigma(X) = AX$, which implies that $\mathcal{S}_k$ itself is a $\sigma$-maximal ideal of $S$. In order to descend the $\partial$-Picard-Vessiot ring $\mathcal{S}$ we are going to proceed somehow in the opposite way. Without any assumption on $K$, we will construct a sequence $(\mathcal{S}_k)_{k \in \mathbb{Z}}$ of $\sigma$-maximal ideals of $S_k$ such that $\bigcup_{k \in \mathbb{Z}} \mathcal{S}_k$ is a $\sigma$-maximal ideal of $S$ stable by $\partial$. Such an ideal will provide us with a $\partial$-Picard-Vessiot ring $\mathcal{R}$ which will be a simple $\sigma$-ring. If moreover $K$ is algebraically closed we will be able to compare its group of automorphisms and the Galois $\partial$-group of the previous section.

**Proposition 1.18.** Let $A \in GL_\sigma(F)$. Then there exists a $(\sigma, \partial)$-extension $\mathcal{R}$ of $F$ such that:

1. $\mathcal{R}$ is generated over $F$ as a $\partial$-ring by $Z \in GL_\sigma(\mathcal{R})$ and $\frac{1}{\det(Z)}$, for some matrix $Z$ satisfying $\sigma(Z) = AZ$;
2. $\mathcal{R}$ is a simple $\sigma$-ring, i.e. it has no nontrivial ideal stable under $\sigma$.

**Remark 1.19.** Of course, a simple $\sigma$-ring carrying a structure of $\partial$-ring is a simple $(\sigma, \partial)$-ring and thus $\mathcal{R}$ is a $\partial$-Picard-Vessiot ring in the sense of Definition 1.12.

**Proof.** Let $S = \mathbf{K}\left\{X, \frac{1}{\det(X)}\right\}_\partial$ be the differential polynomial ring in the variables $X = (X_{i,j})$ and let $S_k, k \in \mathbb{N}$ be as above. We define a $\sigma$-ring structure on $S$ as in (1.3), so that, in particular, $\sigma(X) = AX$. We will prove by induction on $k \geq 0$ that there exists a maximal $\sigma$-ideal $\mathcal{I}_k$ of $S_k$ such that $\mathcal{S}_k$ is a differential Kernel of length $k$.
For $k = 0$ we can take $\mathcal{I}_0$ to be any $\sigma$-maximal ideal of $\mathcal{S}_0$. Then, the $\sigma$-ring $\mathcal{S}_0/\mathcal{I}_0$ is a classical Picard-Vessiot ring for $\sigma(Y) = AY$, in the sense of [vdPS97]. Now, let us construct $\mathcal{I}_{k+1}$ starting from $\mathcal{I}_{k-1}$ and $\mathcal{I}_k$. By [vdPS97, Corollary 1.16], both $\mathcal{I}_{k-1}$ and $\mathcal{I}_k$ can be written as intersections of the form:

$$\mathcal{I}_{k-1} = \bigcap_{i=0}^{t_k-1} \mathfrak{a}_1^{k-1} \bigg( \text{resp. } \mathcal{I}_k = \bigcap_{i=0}^{t_k} J_i^k \bigg),$$

where the $\mathfrak{a}_1^{k-1}$ (resp. $J_i^k$) are prime ideals of $\mathcal{S}_{k-1}$ (resp. $\mathcal{S}_k$). We shall assume that these representations are minimal and so unique. Then,

1. the prime ideals $\mathfrak{a}_1^{k-1}$ (resp. $J_i^k$) are permuted by $\sigma$;
2. for any $i = 1, \ldots, t_k$ there exists $j \in \{1, \ldots, t_{k-1}\}$ such that $\mathfrak{a}_1^{k-1} \cap \mathcal{S}_{k-1} = J_j^{k-1}$.

The last assertion means that for all $i = 0, \ldots, t_k$ the prime ideal $\mathfrak{a}_1^k$ is a differential kernel of $\mathcal{S}_k$. Proposition 1.17 implies that $\pi_1(\mathfrak{a}_1^k)$, and hence $\cap_{i=0}^{t_k} \pi_1(\mathfrak{a}_1^k)$, is a proper $\sigma$-ideal of $\mathcal{S}_{k+1}$. Therefore there exists a $\sigma$-maximal ideal $\mathcal{I}_{k+1}$ of $\mathcal{S}_{k+1}$ containing $\cap_{i=0}^{t_k} \pi_1(\mathfrak{a}_1^k)$. Moreover, the $\sigma$-maximality of $\mathcal{I}_k$ and the inclusion $\mathcal{I}_k \subset \mathcal{I}_{k+1} \subset \mathcal{S}_k$ imply that $\mathcal{I}_k = \mathcal{I}_{k+1} \cap \mathcal{S}_k$, which ends the recursive argument. The ideal $\mathcal{I} = \bigcup_{k \in \mathbb{N}} \mathcal{I}_k$ is clearly $\sigma$-maximal in $\mathcal{S}$ and $\partial$-stable. Then, $\mathcal{R}^\# := \mathcal{S}/\mathcal{I}$ satisfies the requirements.

**Remark 1.20.** By Lemma 6.8 in [HS08], where the assumption that the field $K$ is $\partial$-closed plays no role, we have that there exists a set of idempotents $e_0, \ldots, e_r \in \mathcal{R}^\#$ such that $\mathcal{R}^\# = R_0 \oplus \cdots \oplus R_r$ where $R_i = e_i \mathcal{R}^\#$ is an integral domain and $\sigma$ permutes the set $\{R_0, \ldots, R_r\}$. Let $L_i$ be the fraction field of $R_i$. The total field of fraction $L^\#$ of $\mathcal{R}^\#$ is equal to $L_0 \oplus \cdots \oplus L_r$.

**Corollary 1.21.** If $K = F^\sigma$ is an algebraically closed field, the set of $\sigma$-constants of $\mathcal{R}^\#$ is equal to $K$.

**Proposition 1.22.** Let $\sigma(Y) = AY$ be a linear $\sigma$-difference system with coefficients in $F$ and let $\mathcal{R}^\#$ be the $\partial$-Picard-Vessiot ring constructed in Proposition 1.18.

If $K = F^\sigma$ is algebraically closed, the functor

$$Aut^{\sigma, \partial} : K-\sigma$$

is representable by a linear algebraic $\partial$-group scheme $G_A$ defined over $K$. Moreover $G_A$ becomes isomorphic to $Gal^0(\mathcal{M}_A)$ over a differential closure of $K$ (see Definition 1.3 above).

**Proof.** The first assertion is proved exactly as in [HS08, p. 368], where they only use the fact that the constants of the $\partial$-Picard-Vessiot ring do not increase with respect to the base field $F$. The second one is just a consequence of the theory of differential tannakian category (cf. [Ovc09]), which asserts that two differential fiber functors become isomorphic on a common $\partial$-closure of their fields of definition.

**Definition 1.23.** We say that $G_A$ is the $\partial$-group scheme attached to $\sigma(Y) = AY$.

**Remark 1.24.** The main consequence of Proposition 1.22 is that all the results of [1.3] and more precisely Propositions 1.13, 1.14 and 1.11 in the case of $\Delta = \{\partial\}$, remain valid if one only assumes that the $\sigma$-constants $K$ of the base field $F$ form an algebraically closed field and by replacing the differential Picard-Vessiot group $Gal^0(\mathcal{M}_A)$ defined over the differential closure of $K$ by the $\partial$-group scheme $G_A$ defined over $K$.

Since we are working with schemes and not with the points of linear algebraic $\partial$-groups in a $\partial$-closure of $K$, we need to consider functorial definitions of $\partial$-subgroup scheme and invariants (see [Mau10] in the case of iterative differential equations). A $\partial$-subgroup functor $H$ of the functor $G_A$ is a $\partial$-group functor

$$H : \{K-\sigma\} \rightarrow \{\text{Groups}\}$$

10By Ritt-Raudenbush theorem on the $\partial$-noetherianity of $S$ the sequence of integer $(t_k)_{k \in \mathbb{N}}$ will become stationary.

11This is precisely the definition of a linear algebraic $\partial$-group scheme $G_A$ defined over $K$.

12The assumption that $K$ is $\partial$-closed being used to prove this property of $\partial$-Picard-Vessiot rings.
such that for all \(K\)-\(\partial\)-algebra \(S\), the group \(H(S)\) is a subgroup of \(G_A(S)\). So, let \(L^\#\) be the total ring of fraction of \(R^\#\) and let \(H\) be a \(\partial\)-subgroup functor of \(G_A\). We say that \(r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{L}^\#\), with \(a, b \in R^\#, \ b \neq 0\), is an invariant of \(H\) if for all \(K\)-\(\partial\)-algebra \(S\) and all \(h \in H(S)\), we have
\[
h(a \otimes 1), (b \otimes 1) = (a \otimes 1). h(b \otimes 1).
\]
We denote by \((L^\#)^H\) the ring of invariant of \(L^\#\) under the action of \(H\).

The Galois correspondence is proved by classical arguments starting from the following theorem.

**Theorem 1.25.** Let \(H\) be a \(\partial\)-subgroup functor \(H\) of \(G_A\). Then \((L^\#)^H = K\) if and only if \(H = G_A\).

**Proof.** The proof relies on the same arguments than [Mau10, Theorem 11.4] and [HS08, Lemma 6.19]. \(\square\)

## 2 Confluence and \(q\)-dependency

### 2.1 Differential Galois theory for \(q\)-dependency

Let \(k\) be a characteristic zero field, \(k(q)\) the field of rational functions in \(q\) with coefficients in \(k\) and \(K\) a finite extension of \(k(q)\). We fix an extension \(|\ |\) to \(K\) of the \(q^{-1}\)-adic valuation on \(k(q)\)[13]. By definition we have \(|q| > 1\) and therefore it makes sense to consider elliptic functions with respect to \(|\ |\). So let \((C, |\ |)\) be the smallest valued extension of \((K, |\ |)\) which is both complete and algebraically closed, \(\text{Mer}(C^*)\) the field of meromorphic functions over \(C^* := C \setminus \{0\}\) with respect to \(|\ |\), i.e. the field of fractions of the analytic functions over \(C^*\), and \(C_E\) the field of elliptic functions on the torus \(C^*/q^\mathbb{Z}\), i.e. the subfield of \(\text{Mer}(C^*)\) invariant with respect to the \(q\)-difference operators \(\sigma_q : f(x) \mapsto f(qx)\).

The derivation \(\delta_q = \frac{d}{dq}\) naturally acts of the completion of \(K\) with respect to \(|\ |\), and therefore on the completion of its algebraic closure, which coincides with \(C\) (cf. [Rob00, Chap.3]). It follows that \(\delta_q\) acts on \(\text{Mer}(C^*)\). The fact that \(\delta_x = \frac{d}{dx}\) acts on \(\text{Mer}(C^*)\) is straightforward. We notice that
\[
\delta_x \cdot \sigma_q = \sigma_q \cdot \delta_x;
\delta_q \cdot \sigma_q = \sigma_q \cdot (\delta_x + \delta_q).
\]

This choice of the derivations is not optimal, in the sense that we would like to have two derivations commuting each other and, more important, commuting to \(\sigma_q\). We are going to reduce to this assumption in two steps. First of all, we consider the logarithmic derivative, \(\ell_q(x) = \frac{\delta_x(q)}{\delta_q(q)}\) of the Jacobi Theta function:
\[
\theta_q(x) = \sum_{n \in \mathbb{Z}} q^{n(n-1)/2 x^n}.
\]

We remind that if \(|q| > 1\) the formal series \(\theta_q\) naturally defines a meromorphic function on \(C^*\) and satisfies the the \(q\)-difference equation
\[
\theta_q(qx) = qx \theta_q(x),
\]
so that \(\ell_q(qx) = \ell_q(x) + 1\). This implies that \(\sigma_q \delta_x (\ell_q) = \delta_x (\ell_q)\) and hence that \(\delta_x (\ell_q)\) is an elliptic function.

**Lemma 2.1.** The derivations
\[
\left\{ \begin{array}{l}
\delta_x, \\
\delta = \ell_q(x) \delta_x + \delta_q.
\end{array} \right.
\]

**Proof.** For \(\delta_x\) it is clear. For \(\delta\), we have:
\[
\delta \circ \sigma_q(f(q, x)) = [\ell_q(x) \delta_x + \delta_q] \circ \sigma_q(f(q, x)) = \sigma_q \circ [\ell_q(q^{-1}x) + 1] \delta_x + \delta_q] f(q, x) = \sigma_q(\delta(q^{-1}x) \delta_x + \delta_q) f(q, x) = \sigma_q(\delta f(q, x)).
\]

\[\text{[13]}\text{Actually, }|\ |\text{ is defined on }k[q]\text{ in the following way: there exists }d \in \mathbb{R}, d > 1\text{, such that }|f(q)| = q^{|\text{deg}_q(f)}|.\text{ It extends by multiplicativity to }k(q).]
\[\text{[14]}\text{One can actually prove that }\delta_q\text{ is continuous on }k(q)\text{ with respect to }|\ |\text{ so that it extends to }C.\]
Corollary 2.2.

1. The derivations $\delta_x, \delta$ stabilize $C_E$ in $\text{Mer}(C^*)$.

2. The field of constants $\text{Mer}(C^*)^{x, \delta}$ of $\text{Mer}(C^*)$ with respect to $\delta_x, \delta$ is equal to the algebraic closure $\bar{k}$ of $k$ in $C$.

Proof. The first part of the proof immediately follows from the lemma above. The constants of $\text{Mer}(C^*)$ with respect to $\delta_x$ coincides with $C$. As far as the constants of $C$ with respect to $\delta$ is regarded, we are reduced to determine the constants $C_{\delta q}^{\delta q}$ of $C$ with respect to $\delta_q$. Since the topology induced by $| |$ on $k$ is trivial, one concludes that $C_{\delta q}^{\delta q}$ is the algebraic closure of $k$ in $C$. □

We consider now a $(\delta_x, \delta)$-closure $\bar{C}_E$ of $C_E$ and we extend $\sigma_q$ to the identity of $\bar{C}_E$. The $(\delta_x, \delta)$-field of $\sigma_q$-constants $\bar{C}_E$ almost satisfies the hypothesis of [HS08], apart from the fact that $\delta_x, \delta$ do not commute each other:

Lemma 2.3. There exists $h \in \bar{C}_E$ verifying the differential equation

$$\delta(h) = \delta_x(\ell_q(x))h,$$

such that the derivations

$$\begin{align*}
\partial_1 &= h \delta_x; \\
\partial_2 &= \sigma_q = \ell_q(x) \delta_x + \delta_q,
\end{align*}$$

commute each other and to $\sigma_q$.

Proof. Since $\ell_q(qx) = \ell_q(x) + 1$, we have $\delta_x(\ell_q(x)) \in C_E$. Therefore we are looking for a solution $h$ of a linear differential equation of order 1, with coefficients in $C_E$. Let us suppose that $h \in \bar{C}_E$ exists. Then since $\sigma_q h = h$, the identity $\sigma_q \circ \partial_i = \partial_i \circ \sigma_q$ follows from Lemma 2.1 for any $i = 1, 2$. The verification of the fact that $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$ is straightforward and, therefore, left to the reader.

We now prove the existence of $h$. Consider the differential polynomial ring $S := C_E\{y, \frac{1}{y}\}$. We endow $S$ with an extension of $\delta$ as follow:

$$\delta(\delta^+_n(y)) = \delta^+_2(\ell_q)y$$

for all $n \geq 0$. Now let $\mathfrak{M}$ be a maximal $(\delta_x, \delta)$-maximal ideal of $S$. Then, the ring $S/\mathfrak{M}$ is a simple $(\delta_x, \delta)$-$C_E$-algebra. By [vdPS03, Lemma 1.17], it is also an integral domain. Let $L$ be the quotient field of $S/\mathfrak{M}$. The field $L$ is a $(\delta_x, \delta)$-field extension of $C_E$ which contains a solution of the equation $\delta(y) = \delta_x(\ell_q(x))y$. Since $\bar{C}_E$ is the $(\delta_x, \delta)$-closure of $C_E$, there exists $h \in \bar{C}_E$ verifying the differential equation $\delta(h) = \delta_x(\ell_q(x))h$. □

Let $\Delta = \{\partial_1, \partial_2\}$. Notice that, since $\ell_q(qx) = \ell_q(x) + 1$ and $\sigma_q$ commutes to $\Delta$, we have:

$$\sigma_q \partial_1(\ell_q(x)) = \partial_1(\ell_q(x)),$$

and therefore $\partial_i(\ell_q(x)) \in C_E$, for $i = 1, 2$. We conclude that the subfield $C_E(x, \ell_q(x))$ of $\text{Mer}(C^*)$ is actually a $(\sigma_q, \Delta)$-field. Moreover, extending the action of $\sigma_q$ trivially to $\bar{C}_E$, we can consider the $(\sigma_q, \Delta)$-field $\bar{C}_E(x, \ell_q(x))$. Since the fields $C_E(x, \ell_q)$ and $\bar{C}_E$ are linearly disjoint over $C_E$ (see [HS08, Lemma 6.11]), $\bar{C}_E(x, \ell_q(x))$ has a $\Delta$-closed field of constants, which coincide with $\bar{C}_E$.

2.2 Galois $\Delta$-group and $q$-dependency

The subsection above shows that one can attach to a $q$-difference system $\sigma_q(Y) = A(x)Y$ with $A \in GL_n(C(x))$ two linear differential algebraic groups:

1. The group $GAL^\Delta(\mathcal{M}_A)$ which corresponds to Definition 1.5 applied to the $(\sigma_q, \Delta)$-field $\bar{C}_E(x, \ell_q)$. This group is defined over of $\bar{C}_E$ and measures all the differential relations satisfied by the solutions of the $q$-difference equation with respect to $\delta_x$ and $\delta_q$. However its computation may be a little difficult. Indeed, since the derivations of $\Delta$ are themselves defined above $\bar{C}_E$, there is no hope of a general descent argument. Nonetheless, in some special cases, one can use the linear disjunction of the field $\bar{C}_E^\Delta$ of $\Delta$-constants and $C_E(x, \ell_q)$ above $\bar{C}_E$ to simplify the computations.

\footnote{Notice that $\delta_x \circ \delta = \delta_x(\ell_q(x)) \delta_x + \delta \circ \delta_x$.}
2. The Galois $\partial_2$-group $Gal^{\partial_2}(\mathcal{M}_A)$ which corresponds to Definition 1.5 applied to the $(\sigma_q, \partial_2)$-field $C_E(x, \ell_q)$. Let us consider the $q$-difference system

\begin{equation}
Y(qx) = A(x)Y(x), \text{ with } A \in GL_\nu(C_E(x, \ell_q)).
\end{equation}

A priori, the Galois $\partial_2$-group $Gal^{\partial_2}(\mathcal{M}_A)$ attached to (2.1) is defined above the algebraic closure $\overline{C}_E$ of $C_E$ (see §1.2). We will prove below that actually it descents to $C_E$ and thus reduce all the computations to calculus over the field of elliptic functions.

The field $C_E(x, \ell_q)$ is a subfield of the field of meromorphic function over $C^*$, therefore (2.1) has a fundamental solution matrix $U \in GL_\nu(Mer(C^*))$ (see [Pra86]). The $(\sigma_q, \partial_2)$-ring $\mathcal{R}_{\text{Mer}} := C_E(x, \ell_q)(U, \det U^{-1})\partial_2 \subset \text{Mer}(C^*)$ is generated as a $(\sigma_q, \partial_2)$-ring by a fundamental solution $U$ of (2.1) and by $\det U^{-1}$ and has the property that $\mathcal{R}_{\text{Mer}}^{\sigma_q} = C_E$, in fact:

$$C_E \subset \mathcal{R}_q \subset \text{Mer}(C^*)^{\sigma_q} = C_E.$$ 

Notice that $\mathcal{R}_{\text{Mer}}$ needs not to be a simple $(\sigma_q, \partial_2)$-ring. For this reason we call it a weak $\partial_2$-Picard-Vessiot ring. We have:

\begin{proposition}
$Aut_{\mathcal{R}_q}(\mathcal{R}_{\text{Mer}}/C_E(x, \ell_q))$ consists of the $C_E$-points of a linear algebraic $\partial_2$-group $G_{C_E}$ defined over $C_E$ such that $G_{C_E} \otimes_{C_E} \overline{C}_E \cong Gal^{\partial_2}(\mathcal{M}_A)$.
\end{proposition}

\begin{proof}
See the proof of Theorem 9.1 in [DVH10a], which gives an analogous statement for the derivation $x \frac{d}{dx}$.
\end{proof}

Therefore, as in Proposition 1.11 one can prove:

\begin{corollary}
Let $a_1, \ldots, a_n \in C_E(x, \ell_q)$ and let $S$ be a $(\sigma_q, \partial_2)$-extension of $C_E(x, \ell_q)$ such that $S^\sigma = C_E$. If $z_1, \ldots, z_n \in S$ satisfy $\sigma(z_i) - z_i = a_i$ for $i = 1, \ldots, n$, then $z_1, \ldots, z_n \in S$ satisfy a nontrivial $\partial_2$-relation over $C_E(x, \ell_q)$ if and only if there exists a nonzero homogeneous linear differential polynomial $L(Y_1, \ldots, Y_n)$ with coefficients in $C_E$ and an element $f \in C_E(x, \ell_q)$ such that $L(a_1, \ldots, a_n) = \sigma(f) - f$.
\end{corollary}

2.3 Galoisian approach to heat equation

We want to show how the computation of the Galois $\partial_2$-group of the $q$-difference equation $y(qx) = qx y(x)$ leads to the heat equation. We remind that the Jacobi theta function verifies $\theta_q(qx) = qx \theta_q(x)$. Corollary 2.5 applied to this equation becomes: the function $\theta_q$ do not satisfy a $\partial_2$-relation with coefficients in $C_E(x, \ell_q)$ if and only if there do not exist $a_1, \ldots, a_m \in C_E$ and $f \in C_E(x, \ell_q)$ such that

$$\sum_{i=0}^{m} a_i \partial^2_2 \left( \frac{\partial^2_2(qx)}{qx} \right) = \sigma_q(f) - f.$$

A simple computation leads to

$$\frac{\partial^2_2(qx)}{qx} = \ell_q + 1 = \sigma_q \left( \frac{1}{2} \left( \ell_q^2 + \ell_q \right) \right) - \left( \frac{1}{2} \left( \ell_q^2 + \ell_q \right) \right)$$

and therefore to

$$\sigma_q \left( 2 \frac{\partial^2_2(\theta_q)}{\theta_q} - (\ell_q^2 + \ell_q) \right) = 2 \frac{\partial^2_2(\theta_q)}{\theta_q} - (\ell_q^2 + \ell_q).$$

The last identity is equivalent to the fact that

\begin{equation}
2 \frac{\partial^2_2(\theta_q)}{\theta_q} - (\ell_q^2 + \ell_q) = 2 \frac{\partial^2_2(\theta_q)}{\theta_q} + \ell_q^2 - \ell_q
\end{equation}

is an elliptic function and implies that

$$Gal^{\partial_2}(\mathcal{M}_{q^2}) \subset \left\{ \frac{\partial^2_2(\theta_q)}{\theta_q} \right\} = 0.$$
The heat equation:\[2.3\]
\[2\delta_q\theta_q = -\delta_x^2\theta_q + \delta_x\theta_q\]
can be rewritten as:
\[2\delta_q\theta_q + l^2 - l_q = -\delta_x(l_q).

Since \(\delta_x(l_q)\) is an elliptic function, taking into account \(2.3\), we see the Galois \(\partial_2\)-group of \(\theta_q\) is somehow a galoisian counterpart of the heat equation.

### 2.4 \(q\)-hyperttranscendency of rank 1 \(q\)-difference equations

We want to study the \(q\)-dependency of the solutions of a \(q\)-difference equation of the form \(y(qx) = a(x)y(x)\), where \(a(x) \in k(q, x), a(x) \neq 0\).

**Theorem 2.6.** Let \(u\) be a nonzero meromorphic solution of \(y(qx) = a(x)y(x)\) (in the sense of the previous subsections). The following facts are equivalent:

1. \(a(x) = \mu x^r \frac{a(qx)}{g(x)}\), for some \(r \in \mathbb{Z}, g \in k(q, x)\) and \(\mu \in k(q)\).
2. \(u\) is solution of a nontrivial algebraic \(\partial_2\)-relation, with coefficients in \(C_E(x, \ell_q)\) (and therefore in \(C(x)\)).
3. \(u\) is solution of a nontrivial algebraic \(\partial_2\)-relation, with coefficients in \(C_E(x, \ell_q)\).

We remark, first of all that the equivalence between 1. and 2. is the object of Theorem 1.1 in [Har08] (replacing \(C\) by \(k(q)\) is not an obstacle in the proof). Moreover being \(\partial_2\)-algebraic over \(C_E(x, \ell_q)\) or over \(C(x)\) is equivalent, since \(C_E(\ell_q)\) is \(\delta_x\)-algebraic over \(C(x)\). The equivalence between 1. and 3. is the object of the proposition below:

**Proposition 2.7.** Let \(u\) be a nonzero meromorphic solution of \(y(qx) = a(x)y(x)\). Then \(u\) satisfies an algebraic differential equation with respect to \(\partial_2\) with coefficients in \(C_E(x, \ell_q)\) if and only if \(a(x) = \mu x^r \frac{a(qx)}{g(x)}\), for some \(r \in \mathbb{Z}, g \in k(q, x)\) and \(\mu \in k(q)\).

**Proof.** By Lemma 3.3 in [Har08], there exists \(f(x) \in k(q, x)\) such that \(a(x) = \tilde{a}(x)\frac{f(qx)}{f(x)}\) and \(\tilde{a}(x)\) has the property that if \(\alpha\) is a zero (resp. pole) of \(\tilde{a}(x)\), then \(q^n\alpha\) is neither a zero nor a pole of \(\tilde{a}(x)\), for any \(n \in \mathbb{Z} \setminus \{0\}\). Replacing \(u\) by \(\frac{u}{f(x)}\), we can suppose that \(a(x) = \tilde{a}(x)\) and we can write \(a(x)\) in the form:

\[a(x) = \mu x^r \prod_{i=1}^s (x - \alpha_i)^{l_i}\]

where \(\mu \in k(q), r \in \mathbb{Z}, l_1, ..., l_s \in \mathbb{Z}\) and for all \(i = 1, ..., s\) the \(\alpha_i\)’s are nonzero elements of a fixed algebraic closure of \(k(q)\), such that \(q^2\alpha_i \cap q^2\alpha_j = \emptyset\) if \(i \neq j\). By Corollary 2.5, the solutions of \(y(qx) = a(x)y(x)\) will satisfy a nontrivial algebraic differential equation in \(\partial_2\) if and only if there exists \(f \in C_E(x, \ell_q), a_1, ..., a_m \in C_E\) such that

\[\sum_{i=0}^m a_i \partial_2^i \left( \frac{\partial_2(a(x))}{a(x)} \right) = f(qx) - f(x).\]

We can suppose that \(a_m = 1\). Notice that

\[\frac{\partial_2(a(x)}{a(x)} = \frac{\partial_2(x^r)}{x^r} + \frac{\delta_q(\mu)}{x^r} + \sum_{i=1}^s \frac{\partial_2(x - \alpha_i)^{l_i}}{(x - \alpha_i)^l},\]

where

\[\frac{\partial_2(x^r)}{x^r} = r\ell_q(x) = \left( \frac{r}{2} (\ell^2_q - \ell_q) \right)(x) = \left( \frac{r}{2} (\ell^2_q - \ell_q) \right)(x),\]

with \(\ell^2_q - \ell_q \in C_E(x, \ell_q)\), and

\[\frac{\delta_q(\mu)}{\mu} = \left( \frac{\delta_q(\mu)}{2\mu} \right)(x) = \left( \frac{\delta_q(\mu)}{2\mu} \right)(x),\]

with \(\delta_q(\mu) = \delta_q(\mu) \in C_E(x, \ell_q)\).

\(^{16}\)One deduces from \(2.3\) that \(\tilde{\theta}(q, x) := \theta(q^{1/2}x)\) satisfies the equation \(2\delta_q \tilde{\theta} = \delta_q^2 \tilde{\theta}\). Then to recover the classical form of the heat equation it is enough make the variable change \(q = \exp(-2i\pi r)\) and \(x = \exp(2i\pi z)\).
We are reduced to show that a solution of \( y(qx) = a(x)y(x) \) satisfies a nontrivial differential equation in \( \partial_2 \) if and only if there exists \( h \in C_E(x, \ell_q) \) such that

\[
(2.4) \quad \sum_{j=0}^{m} a_j \partial_2^j \left( \frac{\Sigma_i l_i(\alpha_i \ell_q(x) - \delta_q(\alpha_i))}{(x - \alpha_i)} \right) = h(qx) - h(x).
\]

If we prove that (2.4) never holds, we can conclude that a solution of \( y(qx) = a(x)y(x) \) satisfies a nontrivial algebraic differential equation in \( \partial_2 \) with coefficients in \( C_E(x, \ell_q) \) if and only if \( a(x) = \mu x^r \) (module the reduction done at the beginning of the proof). For all \( i = 1, \ldots, s \) and \( j = 0, \ldots, m \), the fact that \( \partial_2 \ell_q(x) \in C_E \) allows to prove inductively that:

\[
(2.5) \quad \partial_2^j \left( \frac{l_i(\alpha_i \ell_q(x) - \delta_q(\alpha_i))}{(x - \alpha_i)} \right) = \frac{l_i(-1)^j j!(\alpha_i \ell_q(x) - \delta_q(\alpha_i))^{j+1}}{(x - \alpha_i)^{j+1}} + \frac{h_{i,j}}{(x - \alpha_i)^j},
\]

for some \( h_{i,j} \in C_E[\ell_q] \).

Since \( x \) is transcendent over \( C_E(\ell_q) \), we can consider \( f(x) = \sum_{j=0}^{m} a_j \partial_2^j \left( \sum_{i=1}^{s} \frac{l_i(\alpha_i \ell_q(x) - \delta_q(\alpha_i))}{(x - \alpha_i)} \right) \) as a rational function in \( x \) with coefficients in \( C_E(\ell_q) \). In the partial fraction decomposition of \( f(x) \), the polar term in \( \frac{1}{(x - \alpha_i)} \) of highest order is \( l_i(-1)^m(m!) (\alpha_i \ell_q(x) - \delta_q(\alpha_i))^{m+1} \). By the partial fraction decomposition theorem, identities (2.4) and (2.5) implies that this last term appears either in the decomposition of \( h(x) \) or in the decomposition of \( h(qx) \). In both cases, there exists \( s \in \mathbb{Z}^* \) such that the term \( \frac{1}{(q^{s}x - \alpha_i)^{m+1}} \) appears in the partial fraction decomposition of \( h(x) - h(x) \). This is in contradiction with the assumption that the poles \( \alpha_i \) of \( a(x) \) satisfy \( q^2 \alpha_i \cap q^2 \alpha_j = \emptyset \) if \( i \neq j \).

**Remark 2.8.** The Jacobi theta function is an illustration of the theorem above.

### 2.5 q-confluence

Let \( Y(qx) = AY(x) \) be a \( q \)-difference system with \( A \in GL_{sc}(C(x, \ell_q)) \) and \( \Delta = \{ \partial_1, \partial_2 \} \) as in Lemma 2.3.

**Proposition 2.9.** The Galois \( \Delta \)-group \( Gal^\Delta(M_A) \) of \( Y(qx) = AY(x) \) is \( \Delta \)-constant (see Remark 1.12) if and only if there exist two square matrices \( B_1, B_2 \in M_{sc}(C_E(x, \ell_q)) \) such that the system

\[
\begin{align*}
&\sigma_\Delta(Y) = AY(x) \\
&\delta_x Y = B_1 Y \\
&\partial Y = B_2 Y
\end{align*}
\]

is totally compatible. This is equivalent to say that:

\[
\begin{align*}
&\sigma(B_1) = AB_1 A^{-1} + \delta_x(A).A^{-1}, \\
&\sigma(B_2) = AB_2 A^{-1} + \partial_2(A).A^{-1}, \\
&\partial_2(B_1) + B_1 B_2 + \delta_x(\ell_q) B_1 = \delta_x(B_2) + B_2 B_1.
\end{align*}
\]

**Proof.** One implication is a particular case of Proposition 1.11. So let us suppose that \( Gal^\Delta(M_A) \) is \( \Delta \)-constant. We deduce from Proposition 1.11 applied to \( F = C_E^\Delta(x, \ell_q) \) and \( \Delta = \{ \partial_1, \partial_2 \} \), that there exist \( B_1, B_2 \in M_{sc}(C_E) \) such that the system

\[
\begin{align*}
&\sigma(Y) = AY \\
&\partial Y = B_1 Y \\
&\partial_2 Y = B_2 Y
\end{align*}
\]

is compatible. The commutativity condition (2.7) is verified by construction, while we have to replace \( B_1 \) by \( B_1/h \) and \( \partial_1 \) by \( \delta_x \) to obtain (2.4) and (2.8).

Now we show that one can find another matrix \( B'_1 \in M_{sc}(C_E(x, \ell_q)) \) satisfying (2.4), (2.7) and (2.8) (the proof being the same for \( B_2 \)). We have \( \delta_x(A) = B_1(qx)A - AB_1(x) \). Since \( x, \ell_q \) are transcendental over \( C_E \), we can replace the coefficients in \( B_1 \) before the powers of \( x \) and \( \ell_q \) by indeterminates. We obtain a system \( \delta_x(A) = \delta_x(B_1(x))A - AB_1(x) \) with indeterminate coefficients which possess a specialization in \( C_E \). By clearing denominators and equaling like powers of \( x, \ell_q \), we see that this indeterminate system is equivalent to a finite set of polynomial equations with coefficients in \( C_E \). Since \( C_E \) is algebraically closed and the system has a solution in the extension \( C_E \) of \( C_E \), it must have a solution in \( C_E \), which yields to \( B'_1 \).
Example 2.10. We want to study the $q$-dependency of the $q$-difference system

\begin{equation}
Y(qx) = \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} Y(x),
\end{equation}

where $\lambda, \eta \in k(q, x)$. First of all, the solutions of the equation $y(qx) = \lambda y(x)$ will admit a $\Delta$-relation if and only if $\lambda = \mu x^r g(qx)$, for some $r \in \mathbb{Z}$, $g \in k(q, x)$ and $\mu \in k(q)$ (cf. Theorem 2.6). To simplify the exposition we can suppose that $\lambda = \mu x^r$. Then clearly the Galois $\Delta$-group of $y(qx) = \lambda y(x)$ is $\Delta$-constant if and only if $\mu \in k(q)$ is a root of a power of $q$, i.e. if and only if $\mu \in q^2$. A solution of $y(qx) = \lambda y(x)$ is given by

$$y(x) = \frac{\theta_q(\mu x) \theta_q(x)^r}{\theta_q(x)}.$$ 

To obtain an integrable system

$$y(qx) = \mu x^r y(x), \quad \delta_x y = b_1 y, \quad \partial_2 y = b_2 y,$$

satisfying (2.7), (2.8) and (2.9), it is enough to take:

$$b_1 = \frac{\delta_x(y)}{y}, \quad b_2 = \frac{\partial_2(y)}{y}.$$ 

If $\mu \in q^2$ then $b_1 = \mu \ell_q(\mu x) + (r - 1) \ell_q(x) \in C_E(x, \ell_q(x))$. The same hypothesis on $\mu$ assures that $b_2 \in C_E(x, \ell_q(x))$ in fact we have:

$$\sigma_q \left( \frac{\partial_2(\theta_q(x))}{\theta_q(x)} \right) = (\ell_q(x) + 1) + \frac{\partial_2(\theta_q(x))}{\theta_q(x)},$$

which implies that

$$\frac{\partial_2(\theta_q(x))}{\theta_q(x)} = \frac{\ell_q(x)(\ell_q(x) + 1)}{2} + e(x), \text{ for some } e(x) \in C_E.$$ 

To go back to the initial system, we have to find

$$B_1 = \begin{pmatrix} b_1 & \alpha(x) \\ 0 & b_1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_2 & \beta(x) \\ 0 & b_2 \end{pmatrix} \in M_2(C(x, \ell_q)),$$

satisfying (2.4), (2.7) and (2.8). We find that the $q$-difference system (2.4) has constant Galois $\Delta$-group if and only if there exist $\alpha(x), \beta(x) \in C_E(x, \ell_q)$ such that

- $\delta_x(\eta) = r\eta + (\alpha(qx) - \alpha(x))\mu x^r$;
- $\delta_q(\eta) = \frac{\delta_x(\eta)}{x} + (\beta(qx) - \beta(x) + \ell_q(\alpha(qx) - \alpha(x))) \mu x^r$.
- $\partial_2(\alpha(x)) + \delta_x(\ell_q)\alpha(x) = \delta_x(\beta(x))$.

Remark 2.11. In [Pul06], Pulita shows that given a $p$-adic differential equations $\frac{dY}{dx} = GY$ with coefficients in some classical algebras of functions, or a $q$-difference equations $\sigma_q(Y) = AY$, one can always complete it in an integrable system:

$$\begin{cases} 
\sigma_q(Y) = AY \\
\frac{dY}{dx} = GY \\
\frac{dY}{dY} = 0
\end{cases}.$$

Proposition 2.9 says that this is not always the case in the complex framework.

References


