ALGEBRAIC AND DIFFERENTIAL GENERIC GALOIS GROUPS FOR q-DIFFERENCE EQUATIONS

LUCIA DI VIZIO AND CHARLOTTE HARDOUIN

followed by the appendix

The Galois D-groupoid of a q-difference system by Anne Granier

Abstract. In the present paper, we give a complete answer to the analogue of Grothendieck conjecture on p-curvatures for q-difference equations defined over the field of rational function $K(x)$, where K is any finitely generated extension of $\mathbb{Q}$ and $q \in K$ can be either a transcendental or an algebraic number. This generalizes the results in [DV02], proved under the assumption that K is a number field and q an algebraic number. The results also hold for a field K which is a finite extension of a purely transcendental extension $k(q)$ of a perfect field k. In particular, if k is a number field and q is transcendental parameter, one can either reduce the equation modulo a finite place of k or specialize the parameter q, or both. In particular for $q = 1$, we obtain a differential equation defined over a number field or in positive characteristic.

In Part II, we consider two kinds of Galois groups (the second one only under the assumption that k has zero characteristic) attached to a q-difference module $\mathcal{M}$ over $K(x)$:
- the generic (also called intrinsic) Galois group in the sense of [Kat82] and [DV02], which is an algebraic group over $K(x)$;
- the generic differential Galois group, which is a differential algebraic group in the sense of Kolchin, associated to the smallest differential tannakian category generated by $\mathcal{M}$, equipped with the forgetful functor.
The results in the first part of the paper lead to an arithmetic description of the algebraic (resp. differential) generic Galois group. Although no general Galois correspondence holds in this setting, in the case of postive characteristic, we can prove some devissage.

There are many Galois theories for q-difference equations defined over fields such as $\mathbb{C}$, the field of elliptic functions, or the differential closure of $\mathbb{C}$. We prove the comparisons among them. In Part III, we show that the Malgrange-Granier D-groupoid of a nonlinear q-difference system generalizes the generic differential Galois group introduced in Part II, in the sense that in the linear case the two notions essentially coincide. In Part IV we give some comparison results between the two generic Galois groups above and the other Galois groups for linear q-difference equations in the literature. In particular we compare the generic differential Galois group with the differential Galois group introduced in [HS08]. This allows us to relate the dimension of the generic differential Galois group to the differential relations among the meromorphic solutions of a given q-difference equation and to compare the differential group in [HS08] with the Malgrange-Granier D-groupoid (problem strictly related to the question in [Mal09, page 2]). Moreover we compare the generic, algebraic and differential, Galois groups to the generic Galois groups of the modules obtained by specialization of q or by reduction to positive characteristic. In particular, by specialization of the generic Galois group at $q = 1$ we obtain an upper bound for the generic Galois group of the differential equation obtained by specialization.

Date: March 25, 2011.

2000 *Mathematics Subject Classification.* 39A13, 12H10.

Key words and phrases. Generic Galois group; intrinsic Galois group; q-difference equations; differential Tannakian categories; Kolchin differential groups; Grothendieck conjecture on p-curvatures; D-groupoid.

Work partially supported by ANR-06-JCJC-0028.

Contents

Introduction

The question of the algebraicity of the solutions of differential or difference equations goes back at least to Schwarz, who established in 1872 an exhaustive list of hypergeometric differential equations having a full set of algebraic solutions. Galois theory of linear differential equations, and more recently Galois theory of linear difference equations, have been developed to investigate the existence of algebraic relations between the solutions of linear functional equations via the computation of a linear algebraic group called the Galois group of the equation. In particular, the existence of a basis of algebraic solutions is essentially equivalent to having a finite Galois group. The computation of these Galois groups, as linear algebraic groups, thus provides a powerful tool to study the algebraicity of special functions. The direct problem in differential Galois theory (*i.e.* for differential equations) was solved by Hrushovski in [Hru02]. Although he actually has a computational algorithm, the calculations of the Galois group of a differential equation is still a very difficult problem, most of the time, out of reach. For difference Galois theory, the existence of a general computational algorithm is still an open question.

Grothendieck-Katz conjecture on p-curvatures conjugates these two aspects of the theory: determining whether a differential equation has a full basis of algebraic solutions and solving the direct problem. In fact, thanks to Grothendieck's conjecture on p-curvatures we have a (necessary and) sufficient conjectural condition to test whether the solutions of a differential equation are algebraic or not. More precisely, one can reduce a differential equation

$$\mathcal{L}y = a_\mu(x)\frac{d^\mu y}{dx^\mu} + a_{\mu-1}(x)\frac{d^{\mu-1}y}{dx^{\mu-1}} + \cdots + a_0(x)y = 0,$$

with coefficients in the field $\mathbb{Q}(x)$, modulo p for almost all primes $p \in \mathbb{Z}$. Then Grothendieck's conjecture, which remains open in full generality (*cf.* [And04]) predicts:

Conjecture 1 (Grothendieck's conjecture on p-curvatures)**.** *The equation $\mathcal{L}y = 0$ has a full set of algebraic solutions if (and only if)*[1] *for almost all primes $p \in \mathbb{Z}$ the reduction modulo p of $\mathcal{L}y = 0$ has a full set of solutions in $\mathbb{F}_p(x)$.*

Following [Kat82], this is equivalent to a conjectural arithmetic description of the generic Galois group of a differential equation, which gives a conjectural positive answer to the direct problem in differential Galois theory:

Conjecture 2 (Katz's conjectural description of the generic Galois group)**.** *The Lie algebra of the generic Galois group $Gal(\mathcal{M})$ of a differential module $\mathcal{M} = (M, \nabla)$ is the smallest algebraic Lie subalgebra of $\mathrm{End}_{\mathbb{Q}(x)}(M)$ whose reduction modulo p contains the p-curvature ψ_p for almost all p.*

Let us briefly explain the last statement. Let $\mathcal{M} = (M, \nabla)$ be a $\mathbb{Q}(x)$-vector space with a $\mathbb{Q}(x)/\mathbb{Q}$-connection ∇. The generic Galois group $Gal(\mathcal{M})$ of $\mathcal{M}$ is the algebraic subgroup of $Gl(M)$, which is the stabilizer of all the subquotients of the mixed tensor spaces $\oplus_{i,j}(\mathcal{M}^{\otimes^i} \otimes_{\mathbb{Q}(x)} (\mathcal{M}^*)^{\otimes^j})$, where $\mathcal{M}^*$ is the dual of $\mathcal{M}$. We can consider a lattice $\widetilde{M}$ of M over a finite type algebra over $\mathbb{Z}$, stable under the connection, and we can reduce $\widetilde{M}$ modulo p, for almost all primes p. The operator $\psi_p = \nabla\left(\frac{d}{dx}\right)^p$ acting over $\widetilde{M} \otimes_{\mathbb{Z}} \mathbb{F}_p$, is called the *$p$-curvature*. One can give a precise meaning to the fact that the reduction modulo p for almost all p of the Lie algebra of $Gal(\mathcal{M})$ contains the p-curvatures (*cf.* [Kat82]).

[1] This part of the implication is easy to prove.

In [DV02], the first author has proved an analogue of the conjecture above for q-difference equations. More precisely, let $q \neq 0, 1$ be a rational number. We consider the q-difference equation

$$\mathcal{L}y = a_\mu(x)y(q^\mu x) + a_{\mu-1}(x)y(q^{\mu-1}x) + \cdots + a_0(x)y(x) = 0 \ , \ a_0(x) \neq 0 \neq a_\mu(x) \ ,$$

with $a_j(x) \in \mathbb{Q}(x)$, for all $j = 0, \ldots, \mu$. For almost all rational primes p the image of q in $\mathbb{F}_p$ is well defined and nonzero and generates a cyclic subgroup of $\mathbb{F}_p^\times$ of order κ_p. Let ℓ_p be a positive integer such that $1 - q^{\kappa_p} = p^{\ell_p}\frac{h}{g}$, with $h, g \in \mathbb{Z}$ prime with respect to p. We consider a $\mathbb{Z}$-algebra $\mathcal{A} = \mathbb{Z}\left[x, \frac{1}{P(q^i x)}, i \geq 0\right]$, with $P(x) \in \mathbb{Z}[x] \smallsetminus \{0\}$, such that $a_j(x) \in \mathbb{Z}\left[x, \frac{1}{P(q^i x)}, i \geq 0\right]$, for all $j = 0, \ldots, \mu$, and denote by $\mathcal{L}_p y = 0$ the reduction of $\mathcal{L}y = 0$ modulo p^{ℓ_p}.

Theorem 3 ([DV02, Thm.7.1.1]). *The q-difference equation $\mathcal{L}y = 0$ has a full set of solutions in $\mathbb{Q}(x)$ if and only if for almost all rational primes p the set of equations $\mathcal{L}_p y = 0$ has a full set of solutions in $\mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}/p^{\ell_p}\mathbb{Z}$.*

Let M be a finite dimensional $\mathbb{Q}(x)$-vector space equipped with a q-difference operator $\Sigma_q : M \to M$, *i.e.*, with a $\mathbb{Q}$-linear invertible morphism such that $\Sigma_q(fm) = f(qx)\Sigma_q(m)$ for all $f(x) \in \mathbb{Q}(x)$ and all $m \in M$. As in the differential case, it is equivalent to consider a q-difference equation or a couple $\mathcal{M} = (M, \Sigma_q)$.

One can attach to $\mathcal{M}$ an algebraic closed subgroup $Gal(\mathcal{M})$ of $Gl(M)$, also called generic Galois group. It is the stabilizer of all q-difference submodules of all finite sums of the form $\oplus_{i,j}(\mathcal{M}^{\otimes^i} \otimes_{\mathbb{Q}(x)} (\mathcal{M}^*)^{\otimes^j})$, equipped with the operator induced by Σ_q. We consider the reduction modulo p^{ℓ_p} of M for almost all p, by reducing a lattice $\widetilde{M}$ of M, defined over a $\mathbb{Z}$-algebra and stable by Σ_q. The algebraic group $Gal(\mathcal{M})$ can also be reduced modulo p^{ℓ_p} for almost all p. Then the theorem above is equivalent to:

Theorem 4 ([DV02, Thm.10.2.1]). *The algebraic group $Gal(\mathcal{M})$ is the smallest algebraic subgroup of $Gl(M)$ whose reduction modulo p^{ℓ_p} contains the reduction of $\Sigma_q^{\kappa_p}$ modulo p^{ℓ_p} for almost all p.*

We have recalled the theorems in [DV02] for a q-difference equation with coefficients in $\mathbb{Q}(x)$, but they are actually proved for q-difference equations with coefficients in a field of rational functions $K(x)$, such that K is a number field, meaning a finite extension of $\mathbb{Q}$.

In the present paper, we give a complete answer to the Grothendieck conjecture for q-difference equation allowing q to be a transcendental parameter. In [DV02], there was no hope of recovering information on the Grothendieck conjecture for differential equations by letting q tends to 1, for lack of an appropriate topology. On the contrary, the parametric version we consider here, could give some new method to tackle the Grothendieck conjecture for differential equations, by confluence and q-deformation of a differential equation. The idea would be to find a suitable q-deformation to translate the arithmetic of the curvatures of the linear differential equation into the q-arithmetic of the curvatures of the q-difference equation obtained by deformation. We also combine the Grothendieck conjecture with the differential approach to difference equations of Hardouin-Singer to obtain an arithmetic characterization of the differential algebraic relations satisfied by the solutions of a q-difference equation. This allows us to build the first path between Kolchin's theory of linear differential algebraic groups and Malgrange's D-groupoid, answering a question of B. Malgrange (see [Mal09, page 2]). In fact A. Granier, following Malgrange's work, has constructed a Galoisian object attached to a non-linear q-difference equation: in the linear case we give an interpretation of such a

D-groupoid in terms of the Grothendieck-Katz conjecture. This should lead to an arithmetic approach of the integrability of a non linear q-difference equation. The main objects of the first part of the paper are:

- *Function field version of the Grothendieck-Katz conjecture for q-difference equations.* First of all we consider a perfect field k and a finite extension K of a field of rational functions $k(q)$. For a q-difference equation $\mathcal{L}y = 0$ with coefficients in $K(x)$ we prove that $\mathcal{L}y = 0$ has a whole basis of solutions in $K(x)$ if and only if for almost all primitive roots of unity ξ in a fixed algebraic closure $\overline{k}$ of k, the ξ-difference equation obtained by specializing q to ξ has a whole basis of solutions in $\overline{k}(x)$ (*cf.* Theorem 3.1).

  Let κ_ξ be the order of ξ as a root of unity and ϕ_ξ be the minimal polynomial of ξ over k. If $\mathcal{M} = (M, \Sigma_q)$ is a q-difference module over $K(x)$, we prove that the generic Galois group of $\mathcal{M}$ is the smallest algebraic subgroup of $Gl(M)$ that contains $\Sigma_q^{\kappa_\xi}$ modulo ϕ_ξ for almost all ξ (*cf.* Theorem 4.5).

  Moreover, we prove that the results in [DV02], cited above, hold for q-difference equations over $K(x)$, where K is a finitely generated (not necessarily algebraic) extension of $\mathbb{Q}$ and q is a nonzero algebraic number, which is not a root of unity (*cf.* Theorems 6.11 and 6.13). As a result we conclude that the generic Galois group of a complex q-difference equation can always be characterized in the style of Grothendieck-Katz conjecture, applying one description or the other, according that q is algebraic or transcendental. This gives an arithmetical answer to the direct problem in q-difference Galois theory and solves completely the Grothendieck-Katz conjecture for those equations.
- *Generic differential Galois groups.* In [HS08], the authors attach to a q-difference equation a differential Galois group *à la Kolchin*, also called a differential algebraic group. This is a subgroup of the group of invertible matrices of a given order, defined by a set of nonlinear algebraic differential equations. The differential dimension of this Galois group measures the hypertranscendence properties of a basis of solutions. We recall that a function f is hypertranscendental over a field F equipped with a derivation ∂ if $F[\partial^n(f), n \geq 0]/F$ is a transcendental extension of infinite degree, or equivalently, if f is not a solution of a nonlinear algebraic differential equation with coefficients in F. The question of hypertranscendence of solutions of functional equations appears in various mathematical domains: in special function theory (see for instance [LY08], [Mar07] for the differential independence ζ and Γ functions), in enumerative combinatorics (see for instance [BMP03] for problems of hypertranscendence and D-finiteness[2] specifically related to q-difference equations[3]), ... The problem of the differential Galois group of Hardouin-Singer is that it is defined over the differential closure of the elliptic functions over $C^*/q^{\mathbb{Z}}$, which is an enormous field. In §5.1 we introduce a differential generic Galois group attached to a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $K(x)$. We prove that it is the smallest differential

[2]A function f is D-finite over a differential field $(\mathcal{F}, \partial)$ if it is solution of a linear differential equation with coefficients in $\mathcal{F}$.

[3]In [BMP03], the authors consider some formal power series generated by enumeration of random walks with constraints. Such generating series are solutions of q-difference equations: one natural step towards their rationality is to establish whether they satisfy an algebraic (maybe nonlinear) differential equation. In fact, as proven by J.-P. Ramis, a formal power series which is solution of a linear differential equation and a linear q-difference equation, both with coefficients in $\mathbb{C}(x)$, is necessarily rational (see [Ram92]). Other examples of q-difference equations for which it would be interesting to establish hypertranscendency are given in [BMF95] and [BM96].

algebraic subgroup of $Gl(M)$ that contains $\Sigma_q^{\kappa_\xi}$ modulo ϕ_ξ, if q is transcendental, or $\Sigma_q^{\kappa_p}$ modulo p^{ℓ_p} if q is algebraic. In this way, we have replaced the group introduced in [HS08] by another group defined over a smaller field, namely $K(x)$, and which admits an arithmetic characterization. As above, combining the two situations we have an arithmetic description of the differential generic Galois group for any complex q-difference module. Notice that the differential generic Galois group is Zariski dense in the generic Galois group.

After constructing and characterizing the algebraic and differential generic Galois groups of a linear q-difference equation, we relate them to the existing theories. This problem was not treated in [DV02]. In fact only lately the Galois theory of q-difference equations has had a ramified and articulated development, so that we have more different approaches. The comparison among those approaches has become a necessity:

- *Malgrange Galois theory for non linear systems.* Malgrange has defined and studied Galois D-groupoids for nonlinear differential equations. In the differential case the Galois D-groupoid has been shown to generalize the Galois group in the sense of Kolchin-Picard-Vessiot by Malgrange himself (*cf.* [Mal01]). Roughly speaking the Galois D-groupoid is a groupoid of local diffeomorphisms of a variety defined by a sheaf of differential ideals in the jet space of the variety. This construction can be generalized to quite general dynamical systems. The idea of considering a groupoid defined by a differential structure is actually quite natural. In fact, it encodes the "linearizations" of the dynamical system along its orbits, *i.e.* the variational equations attached to the dynamical system, also called the linearized equations. One of the two proofs (*cf.* [CR08]) of the analog of Morales-Ramis theorem for q-difference equations, *i.e.* the connection between integrability of a nonlinear system and solvability of the Lie algebra of its Galois D-groupoid, was done under the following conjecture: "for linear (q-)difference systems, the action of Malgrange groupoid on the fibers gives the classical Galois groups" (*cf.* [CR08, §7.3]). As an application of the arithmetic characterization of the differential generic Galois group explained above, we actually show that the D-groupoid of a dynamical system associated with a nonlinear q-difference equation, as introduced by A. Granier, generalizes the notion of differential generic Galois group. In the particular case of linear q-difference systems with constant coefficients we retrieve an algebraic Galois group and the result obtained in [Gra]. In other words, for linear q-difference systems, the Galois D-groupoid essentially coincides with the differential generic Galois group, so it contains more informations than the algebraic Galois group. Thanks to the comparison results in the last part of the paper, we conclude that the Galois D-groupoid of a linear q-difference system allows to recover both the classical Galois group (*cf.* [vdPS97], [Sau04b]) by taking its Zariski closure and extending the base field conveniently, and the Hardouin-Singer differential Galois group (*cf.* [HS08]) only by extending the field.
- *Comparison theorem with other differential Galois theory.* As we have already pointed out, there are many different Galois theories for q-difference equations. We elucidate the comparison of all of them with the generic Galois groups introduced above. This implies in particular that the (differential) dimension of the generic (differential) Galois group over $K(x)$ is equal to the (hyper-)transcendency degree of the extension generated over the field of rational functions with elliptic coefficients over $C^*/q^{\mathbb{Z}}$ by a full

set of solutions of a q-difference equation. A consequence of the comparison results in Part IV and the theory in [HS08] is that, if the generic Galois group is simple, then either the only differential relations among the solutions are the algebraic ones or there exists a connection on the q-difference module compatible with the q-difference structure. We illustrate this situation with two examples in §5, namely the Jacobi Theta function and the logarithm. The latter example also explains how the eventual differential structure over the q-difference module can be nontrivial in spite of the theorem by J.P. Ramis on the rationality of a formal power series simultaneous solution of a differential equation and a q-difference equation, both with rational coefficients (*cf.* [Ram92]).

- *Specialization of the parameter q.* We have proved that when q is a parameter, *i.e.* when it is transcendental over k, independently of the characteristic, the structure of a q-difference equation is totally determined by the structure of the ξ-difference equations obtained specializing q to almost all primitive root of unity ξ.[4] In the last section of the paper, we prove that the specialization of the algebraic (resp. differential) generic Galois group at $q = a$ for any a in the algebraic closure of k, contains the algebraic (resp. differential) generic Galois group of the specialized equation. If k is a number field, this holds also if we reduce the equations to positive characteristic, so that q reduces to a parameter to positive characteristic. So if we have a q-difference equation $Y(qx) = A(q,x)Y(x)$ with coefficients in a field $k(q,x)$ such that $[k : \mathbb{Q}] < \infty$, we can either reduce it to positive characteristic and then specialize q, or specialize q and then reduce to positive characteristic. In particular, letting $q \to 1$ in
$$\frac{Y(qx) - Y(x)}{(q-1)x} = \frac{A(q,x) - 1}{(q-1)x} Y(x)$$
we obtain a differential system. In this way we obtain many q-difference systems that either reduce to a differential system or to its reduction modulo p. All these parameterized families of systems are compatible and the associated generic Galois groups contain the generic Galois group of the differential equation at $q = 1$. On the other hand starting from a linear differential equation, one could express the generic Galois group of the differential equation in terms of the curvatures of a suitable q-deformation of the initial equation.

We give now a more detailed presentation of the content of the present paper.

Grothendieck's conjecture for generic q-difference equation. Let k be a perfect field and K a finite extension of the field of rational functions $k(q)$. We will denote by σ_q the q-difference operator $f(x) \mapsto f(qx)$, acting on any algebra where it make sense to consider it (for instance $K(x)$, $K((x))$ etc.). A q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$ is a $K(x)$-vector space of finite dimension ν equipped with a σ_q-semilinear bijective operator Σ_q:
$$\Sigma_q(fm) = \sigma_q(f)\Sigma_q(m), \text{ for any } m \in M \text{ and } f \in K(x).$$
The coordinates of a vector fixed by Σ_q with respect to a given basis are solution of a linear q-difference system of the form
$$(\mathcal{S}_q) \qquad Y(qx) = A(x)Y(x), \text{ with } A(x) \in Gl_\nu(K(x)).$$

[4] This is a recurrent situation in the literature. For instance in the case of quantum invariant of knots, and the Volume conjecture.

We want to give a sense to considering the specialization of q. Notice that K satisfies the product formula. Let us call $\mathcal{O}_K$ the integral closure of $k[q]$ in K. Then we can always find an $\mathcal{A}$-lattice M of $M_{K(x)}$, stable by Σ_q, where $\mathcal{A}$ is a q-difference algebra of the form:

$$\mathcal{A} = \mathcal{O}_K \left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \frac{1}{P(q^2x)}, \dots \right],$$

for a convenient polynomial $P(x) \in \mathcal{O}_K[x]$. We obtain in this way a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $\mathcal{A}$, that allows to recover $\mathcal{M}_{K(x)}$ by extension of scalars. It turns out that we can reduce $\mathcal{M}$ modulo almost all places of $\mathcal{O}_K$. In particular, let $\mathcal{C}$ be the set of places v of K such that q reduces on a root of unity q_v in $\overline{k}$ and let ϕ_v the the minimal polynomial of q_v over k. We call κ_v the order of q_v as a root of unity. We prove (*cf.* Theorem 3.1 below):

Theorem 5. *The q-difference module $\mathcal{M}_{K(x)}$ is trivial if and only if the operator $\Sigma_q^{\kappa_v}$ acts as the identity on $M \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$ for almost all $v \in \mathcal{C}$.*

The module $\mathcal{M}_{K(x)}$ is said to be trivial if it is isomorphic to $(K^\nu \otimes_K K(x), 1 \otimes \sigma_q)$, for some positive integer ν or, equivalently if an associated linear q-difference equation in a cyclic basis has a full set of rational solutions. The proof of Theorem 5 is divided in two steps:

- We prove that if $\Sigma_q^{\kappa_v}$ has unipotent reduction modulo infinitely many places $v \in \mathcal{C}$, then $\mathcal{M}_{K(x)}$ is regular singular. Then we prove that if there exists an infinite set of positive primes $\wp \subset \mathbb{Z}$ such that $\Sigma_q^{\kappa_v}$ has unipotent reduction modulo all places of K for which $\kappa_v \in \wp$, then $\mathcal{M}_{K(x)}$ has integral exponents. This part of the proof of Theorem 5 is the major difference with the differential case (*cf.* [Kat70]) and the q-difference case over number fields (*cf.* [DV02]). The proof reduces to a rational dynamic characterization of rational functions for which the roots of unity of a given order are periodic orbits (*cf.* Lemma 2.9 and Remark 2.10 below).
- If there exists an infinite set of positive primes $\wp \subset \mathbb{Z}$ such that $\Sigma_q^{\kappa_v}$ acts as the identity modulo ϕ_v for almost all $v \in \mathcal{C}$ such that $\kappa_v \in \wp$, then $\mathcal{M}_{K(x)}$ has integer exponents, as above, and no logarithmic singularities. This implies that the module $\mathcal{M}_{K((x))} = (M \otimes_{\mathcal{A}} K((x)), \Sigma_q)$ is trivial. To conclude the proof we use a function field version of the Borel-Dwork theorem.

Notice that $q^{\kappa_v} = 1$ modulo ϕ_v, therefore $\Sigma_q^{\kappa_v}$ induces an $\mathcal{A}/(\phi_v)$-linear map on $M \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$.

In Part III we generalize Theorem 3, cited above, relaxing the assumption that K is a number field. So we prove the same statement assuming that $K/\mathbb{Q}$ is a finitely generated extension. The proof is not a generalization of the proof of Theorem 3, but rather relies on it by considering a transcendent basis of $K/\mathbb{Q}$ as a set of parameters. Then the argument is based on the properties of the poles and the zeros of Birkhoff matrices, which are are q-elliptic meromorphic matrices connecting the solutions at zero and infinity. In this way, we obtain a triviality criteria in the Grothendieck conjecture style that applies to any q-difference equations in characteristic zero, both for q algebraic (using reduction to positive characteristic) and q transcendental (specializing q to roots of unity).

Generic Galois group. We consider the collection $Constr(\mathcal{M}_{K(x)})$ of $K(x)$-linear algebraic constructions of $\mathcal{M}_{K(x)}$ (direct sums, tensor product, symmetric and antisymmetric product, dual). The operator Σ_q induces a q-difference operator on every element of $Constr(\mathcal{M}_{K(x)})$, that we will still call Σ_q. Then, the generic

Galois group of $\mathcal{M}_{K(x)}$ is defined as:

$$Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{\varphi \in Gl(M_{K(x)}) : \varphi \text{ stabilizes every subset stabilized by } \Sigma_q, \text{ in any construction}\}$$

Of course, $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is associated with a tannakian category. Moreover, Chevalley theorem ensures that it can actually be defined as a stabilizer, inside $Gl(M_{K(x)})$, of a single line $L_{K(x)}$ in a construction $\mathcal{W}_{K(x)} = (W_{K(x)}, \Sigma_q)$ of $\mathcal{M}_{K(x)}$. Notice that the choice of an $\mathcal{A}$-lattice M of $M_{K(x)}$ induces an $\mathcal{A}$-lattice L (resp. W) of $L_{K(x)}$ (resp. $W_{K(x)}$). As in [Kat82], Theorem 5 is equivalent to the following statement (*cf.* Theorem 4.5 below):

Theorem 6. *The generic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest algebraic subgroup of $Gl(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v for almost all $v \in \mathcal{C}$;* i.e.

> $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest algebraic subgroup of $Gl(M_{K(x)})$ such that, if $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the stabilizer inside $Gl(M_{K(x)})$ of some line $L_{K(x)}$ in some construction $\mathcal{W}_{K(x)} = (W_{K(x)}, \Sigma_q)$ of $\mathcal{M}_{K(x)}$, then $\Sigma_q^{\kappa_v}$ stabilizes $L \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$ inside $W \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$ for almost all $v \in \mathcal{C}$.

In the case of positive characteristic, the group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is not necessarily reduced. Although there is no Galois correspondence for generic Galois groups, in the nonreduced case we can prove a sort of devissage. In fact, let $p > 0$ be the characteristic of k and let us consider the short exact sequence associated with the largest reduced subgroup $Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ of the generic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$:

$$1 \longrightarrow Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow \mu_{p^\ell} \longrightarrow 1.$$

Then we have (*cf.* Theorem 4.15 and Corollary 4.18 below):

Theorem 7. *The group $Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest algebraic subgroup of $Gl(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v p^\ell}$ modulo ϕ_v for almost all $v \in \mathcal{C}$.*

Corollary 8. *$Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the generic Galois group of the q^{p^ℓ}-difference module $(M_{K(x)}, \Sigma_q^{p^\ell})$.*

Corollary 9. *Let $\widetilde{K}$ be a finite extension of K containing a p^ℓ-th root q^{1/p^ℓ} of q. The generic Galois group $Gal(\mathcal{M}_{\widetilde{K}(x^{1/p^\ell})}, \eta_{\widetilde{K}(x^{1/p^\ell})})$ is reduced and*

$$Gal(\mathcal{M}_{\widetilde{K}(x^{1/p^\ell})}, \eta_{\widetilde{K}(x^{1/p^\ell})}) \subset Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \widetilde{K}(x^{1/p^\ell}).$$

Differential generic Galois group. We will always speak of differential Galois groups under the assumption that the characteristic of k is 0. Since the field $K(x)$ is endowed with the simultaneous action of two distinct operators, the q-difference operator $\sigma_q : f(x) \mapsto f(qx)$ and the derivation $\partial := x\frac{d}{dx}$, it seems very natural to ask, whether a solution of a q-difference system may be solution of an algebraic nonlinear differential equation. In [HS08], Hardouin and Singer exhibit, for a given q-difference module, a linear differential algebraic group, whose dimension measures the differential relations between the solutions. Thereby they succeed in building a differential Galois theory for difference equations.

The main difficulty of this theory is that, even if they start with q-difference modules defined over $K(x)$, the differential Galois group is defined over the differential closure of K, which is an enormous and complicated field. To avoid such a large field of definition, inspired by the construction of the generic Galois group, we propose to attach to a q-difference module defined over $K(x)$, a linear differential algebraic group, also defined over $K(x)$, and whose dimension will also measure the differential complexity of the q-difference module.

As the notion of generic Galois group is deeply related to the notion of tannakian category, the notion of differential generic Galois group is intrinsicly related to the notion of differential tannakian category developed by A. Ovchinnikov in [Ovc09a]. We show in this paper how the category of q-difference modules over $K(x)$ may be endowed with a prolongation functor F and thus turns out to be a differential tannakian category. Intuitively, if $\mathcal{M}$ is a q-difference module, associated with a q-difference system $\sigma_q(Y) = AY$, the q-difference module $F(\mathcal{M})$ is attached to the q-difference system

$$\sigma_q(Z) = \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix} Z.$$

Notice that if Y verifies $\sigma_q(Y) = AY$, then $Z = \begin{pmatrix} \partial(Y) \\ Y \end{pmatrix}$ is solution of the above system. We consider the constructions of differential algebra $Constr^{\partial}(\mathcal{M}_K(x))$ of $\mathcal{M}_{K(x)}$ as the collection of all algebraic constructions of $\mathcal{M}_{K(x)}$ (direct sums, tensor product, symmetric and antisymmetric product, dual) plus those obtained by iteration of the prolongation functor F. Then the differential generic Galois group of $\mathcal{M}_{K(x)}$ is defined as:

$$Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{\varphi \in Gl(M_{K(x)}) : \ \varphi \text{ stabilizes every } \Sigma_q\text{-stable subset}$$
$$\text{in any construction of differential algebra}\}$$

We can look at $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ as the group of differential automorphisms of the forgetful functor. It is endowed with a structure of linear differential algebraic group (*cf.* [Kol73]). Since there exists a line $L_{K(x)}$ in a construction of differential algebra $\mathcal{W}_{K(x)} = (W_{K(x)}, \Sigma_q)$ of $\mathcal{M}_{K(x)}$ such that $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the stabilizer, inside $Gl(M_{K(x)})$, of the line $L_{K(x)}$, the meaning of the following statement should be intuitive (*cf.* Theorem 6 above and Theorem 5.11 below):

Theorem 10. *The differential generic Galois group $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest differential algebraic subgroup of $Gl(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v for almost all $v \in \mathcal{C}$.*

This implies, for instance, (*cf.* Theorem 6 above and Corollary 5.14 in the text below):

Corollary 11. *The differential generic Galois group $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is a Zariski dense subset of the algebraic generic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$.*

Application to complex q-difference modules. In the third part of the paper we apply the previous results to the characterization of the generic (differential) Galois group of a q-difference module over $\mathbb{C}(x)$, with $q \in \mathbb{C} \smallsetminus \{0, 1\}$. We prove a statement that can be written a little bit informally in the following way:

Theorem 12. *The generic (differential) Galois group of a complex q-difference module $\mathcal{M} = (M, \Sigma_q)$ is the smallest (differential) algebraic subgroup of $Gl(M)$, that contains a cofinite nonempty subset of curvatures.*

This means that there exists a field K, finitely generated over $\mathbb{Q}$ and containing q, and a q-difference module $\mathcal{M}_{K(x)}$ over $K(x)$, such that

$$\begin{cases} \mathcal{M} = \mathcal{M}_{K(x)} \otimes \mathbb{C}(x), \\ Gal(\mathcal{M}, \eta_{\mathbb{C}(x)}) = Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes \mathbb{C}(x), \\ Gal^{\partial}(\mathcal{M}, \eta_{\mathbb{C}(x)}) = Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes \mathbb{C}(x). \end{cases}$$

Then, over a finitely generated extension of $\mathbb{Q}$, we can always give a convenient definition of curvature, according that q is a root of unity, or an algebraic number, not a root of unity, or a transcendental number. So the theorem above is informal in the sense that it includes many different statements, that would require a specific formalism (*cf.* §6.3 below). The theorem above is already known in the following two cases:

- If q is a root of unity, the theorem is proved for algebraic generic Galois groups in [Hen96].
- If q is an algebraic number and K is a number field, it is proved for algebraic generic Galois groups in [DV02].

In Part II of the present paper, the theorem is proved under the assumption that q is transcendental. If q is algebraic and $K/\mathbb{Q}$ is not finite the theorem is proved in Part III.

We obtain an application to the Malgrange-Granier D-groupoid of a q-difference linear system. A. Granier has defined a Galois D-groupoid for nonlinear q-difference equations, in the wake of Malgrange's work. In the case of a linear system $Y(qx) = A(x)Y(x)$, with $A \in Gl_\nu(\mathbb{C}(x))$, the Malgrange-Granier D-groupoid is the D-envelop of the dynamic, *i.e.* it encodes all the partial differential equations over $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$ with analytic coefficients, satisfied by local diffeomorphisms of the form $(x, X) \mapsto (q^k x, A_k(x)X)$ for all $k \in \mathbb{Z}$, where $A_k(x) \in Gl_\nu(\mathbb{C}(x))$ is the matrix obtained by iterating the system $Y(qx) = A(x)Y(x)$ so that:

$$Y(q^k x) = A_k(x)Y(x).$$

Using Theorem 10, we relate this analytic D-groupoid with the more algebraic notion of differential generic Galois group. Precisely, we consider a differential variety $\mathcal{K}ol(A)$ containing the dynamic of $Y(qx) = A(x)Y(x)$, *i.e.* the smallest differential subvariety of $Gl_{\nu+1}(\mathbb{C}(x))$ defined over $\mathbb{C}(x)$, containing the matrices $\begin{pmatrix} q^k & 0 \\ 0 & A_k(x) \end{pmatrix}$, for all $k \in \mathbb{Z}$, and satisfying some other technical properties. Then we build a D-groupoid over $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$ generated by the global equations of $\mathcal{K}ol(A)$ and we show that its solutions coincide with those of the Malgrange-Granier D-groupoid. Precisely their solutions are local diffeomorphisms of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$ of the form

$$(x, X) \mapsto (\alpha x, \beta(x)X),$$

where $\alpha \in \mathbb{C}^*$ and $\beta \in Gl_\nu(\mathbb{C}\{x - x_0\})$ and $diag(\alpha, \beta(x))$ is a local analytic solution of the defining equations of $\mathcal{K}ol(A)$. At last, Theorem 12 implies that the solutions in a neighborhood of $\{x_0\} \times \mathbb{C}^\nu$ of the sub-D-groupoid which fix the transversals in the Malgrange-Garnier D-groupoid are precisely the $\mathbb{C}\{x - x_0\}$-points of the differential generic Galois group.

For systems with constant coefficients, we retrieve the result of A. Granier (*cf.* [Gra, Thm. 2.4]) *i.e.* the evaluation in $x = x_0$ of the solutions of the transversal D-groupoid is the usual Galois group (in that case algebraic and differential Galois groups coincide). The analogous result for differential equations is proved in [Mal01]. Notice that B. Malgrange, in the differential case, and A. Granier, in the q-difference constant case, establish a link between the Galois D-groupoid and the usual Galois group: this is compatible with the results below since in those cases the algebraic generic and differential Galois groups coincide and are deeply linked to the usual Galois group (*cf.* §7.3 below).

Comparison theorems. We then compare the two generic Galois groups we have introduced with the ones constructed in the several Galois theories for q-difference equations. To this purpose, we study the dependence of the notions of generic Galois group with respect to some particular scalar extensions. In particular, we relate the generic Galois groups, differential and algebraic, with the more classical notion of Galois groups: differential, introduced by Hardouin-Singer ([HS08]), and algebraic, introduced by Singer-van der Put ([vdPS97]). Finally, we relate, in Corollary 9.12, the differential dimension of $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ as a differential algebraic group to the differential transcendence degree of the field generated by the meromorphic solutions of $Y(qx) = A(x)Y(x)$ over the differential closure of the field of elliptic functions.

Specialization of the parameter q. To study the specialization of the generic Galois groups, differential and algebraic, we use the language of generalized differential rings and modules, introduced by Y. André (*cf.* [And01]), that allows to treat differential and difference modules in the same setting. It is therefore adapted to our situation where the reductions of $\mathcal{M}$ can be either q-difference modules or differential modules.

Inspired by André's results on the specialization of the usual Galois group, we prove that the specialization of $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ (resp. $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$) gives an upper bound for the generic (resp. differential) Galois group of the reduction of $\mathcal{M}$ modulo almost all places v of K.

When we specialize q to 1, we find a differential module. Going backwards, *i.e.* deforming a differential module, we can deduce from the results above a description of an upper bound of its generic Galois group, defined in [Kat82]. In fact, given a $k(x)/k$-differential module (M, ∇), we can fix a basis $\underline{e}$ of M such that

$$\nabla(\underline{e}) = \underline{e}G(x),$$

so that the horizontal vectors of ∇ are solutions of the system $Y'(x) = -G(x)Y(x)$. We set $M_{K(x)} = M \otimes_{k(x)} K(x)$ and consider a q-difference algebra $\mathcal{A}$ as above, such that the entries of $G(x)$ are contained in $\mathcal{A}$, and a $\mathcal{A}$-lattice M of $M_{K(x)}$. For instance considering the most naïve q-deformation, we have:

Corollary 13. *The generic Galois group of (M, ∇) is contained in the "specialization at $q = 1$" of the smallest algebraic subgroup G of $Gl(M_{k(q,x)})$ that contains the q^{κ_v}-semilinear operators $\Lambda_v : M_{k(q,x)} \rightarrow M_{k(q,x)}$ modulo ϕ_v for almost all $v \in \mathcal{C}$, defined by:*

$$\Lambda_v \underline{e} = \underline{e} \prod_{i=0}^{\kappa_v - 1} \left(1 + (q-1)q^i x G(q^i x)\right).$$

Corollary 14. *Let k be an algebraically closed field. Then a differential module (M, ∇) is trivial over $k(x)$ if and only if there exists a basis $\underline{e}$ such that $\nabla(\underline{e}) = \underline{e}G(x)$ and for almost all roots of unity $\zeta \in k$ the following identity is verified:*

$$\left[\prod_{i=0}^{n-1} \left(1 + (q-1)q^i x G(q^i x)\right)\right]_{q=\zeta} = \textit{identity matrix},$$

where n is the order of ζ.

Structure of the paper. The main result of Part I, namely Theorem 3.1, is used as a black box in Part II, to prove Theorems 4.5 and 5.11, on the arithmetic characterization of the algebraic generic and differential Galois groups. Part III uses Theorems 3.1 and the analogous results on number field in [DV02], to prove a "curvature" characterization of the algebraic generic and differential Galois groups of a complex q-difference modules. This characterization plays a crucial role in the section on the Malgrange-Granier groupoid. Part IV is a digression on comparison results, to relate the content of the present paper to the many approaches to q-difference Galois theory in the literature, and on the specialization of the parameter q.

Acknowledgements. We would like to thank D. Bertrand, Z. Djadli, C. Favre, M. Florence, A. Granier, D. Harari, F. Heiderich, A. Ovchinnikov, B. Malgrange, J-P. Ramis, J. Sauloy, M. Singer, J. Tapia, H. Umemura and M. Vaquie for the many discussions on different points of this paper, and the organizers of the seminars of the universities of Grenoble I, Montpellier, Rennes II, Caen, Toulouse and Bordeaux that invited us to present the results below at various stages of our work.

We would like to thank the ANR projet Diophante that has made possible a few reciprocal visits, and the Centre International de Rencontres Mathématiques in Luminy for supporting us *via* the Research in pairs program and for providing a nice working atmosphere.

1. Notation and definitions

1.1. The base field. Let us consider the field of rational function $k(q)$ with coefficients in a perfect field k. We fix $d\in]0,1[$ and for any irreducible polynomial $v=v(q)\in k[q]$ we set:

$$|f(q)|_v=d^{\deg_q v(q)\cdot \mathrm{ord}_{v(q)}f(q)},\ \forall f(q)\in k[q].$$

The definition of $|\ |_v$ extends to $k(q)$ by multiplicativity. To this set of norms one has to add the q^{-1}-adic one, defined on $k[q]$ by:

$$|f(q)|_{q^{-1}}=d^{-deg_q f(q)}.$$

Once again, this definition extends by multiplicativity to $k(q)$. Then, the product formula holds:

$$\begin{aligned}\prod_{v\in k[q]\text{ irred.}}\left|\frac{f(q)}{g(q)}\right|_v &= d^{\sum_v \deg_q v(q)\ \left(\mathrm{ord}_{v(q)}f(q)-\mathrm{ord}_{v(q)}g(q)\right)}\\ &= d^{deg_q f(q)-\deg_q g(q)}\\ &= \left|\frac{f(q)}{g(q)}\right|_{q^{-1}}^{-1}.\end{aligned}$$

For any finite extension K of $k(q)$, we consider the family $\mathcal{P}$ of ultrametric norms, that extends the norms defined above, up to equivalence. We suppose that the norms in $\mathcal{P}$ are normalized so that the product formula still holds. We consider the following partition of $\mathcal{P}$:

- the set $\mathcal{P}_\infty$ of places of K such that the associated norms extend, up to equivalence, either $|\ |_q$ or $|\ |_{q^{-1}}$;
- the set $\mathcal{P}_f$ of places of K such that the associated norms extend, up to equivalence, one of the norms $|\ |_v$ for an irreducible $v=v(q)\in k[q]$, $v(q)\neq q$.[5]

Moreover we consider the set $\mathcal{C}$ of places $v\in\mathcal{P}_f$ such that v divides a valuation of $k(q)$ having as uniformizer a factor of a cyclotomic polynomial, other than $q-1$. Equivalently, $\mathcal{C}$ is the set of places $v\in\mathcal{P}_f$ such that q reduces to a root of unity modulo v of order strictly greater than 1. We will call $v\in\mathcal{C}$ a cyclotomic place.

Sometimes we will write $\mathcal{P}_K$, $\mathcal{P}_{K,f}$, $\mathcal{P}_{K,\infty}$ and $\mathcal{C}_K$, to stress out the choice of the base field.

---

[5]The notation $\mathcal{P}_f$, $\mathcal{P}_\infty$ is only psychological, since all the norms involved here are ultrametric. Nevertheless, there exists a fundamental difference between the two sets, in fact for any $v\in\mathcal{P}_\infty$ one has $|q|_v\neq 1$, while for any $v\in\mathcal{P}_f$ the v-adic norm of q is 1. Therefore, from a v-adic analytic point of view, a q-difference equation has a totally different nature with respect to the norms in the sets $\mathcal{P}_f$ or $\mathcal{P}_\infty$.

1.2. q-difference modules. The field $K(x)$ is naturally a q-difference algebra, *i.e.* is equipped with the operator

$$\begin{array}{cccc}\sigma_q: & K(x) & \longrightarrow & K(x) \\ & f(x) & \longmapsto & f(qx)\end{array}.$$

The field $K(x)$ is also equipped with the q-derivation

$$d_q(f)(x)=\frac{f(qx)-f(x)}{(q-1)x},$$

satisfying a q-Leibniz formula:

$$d_q(fg)(x)=f(qx)d_q(g)(x)+d_q(f)(x)g(x),$$

for any $f,g\in K(x)$.

More generally, we will consider a field K, with a fixed element $q\neq 0$, and an extension $\mathcal{F}$ of $K(x)$ equipped with a q-difference operator, still called σ_q, extending the action of σ_q, and with the skew derivation $d_q:=\frac{\sigma_q-1}{(q-1)x}$. Typically, in the sequel, we will consider the fields $K(x)$ or $K((x))$.

A q-difference module over $\mathcal{F}$ (of rank ν) is a finite dimensional $\mathcal{F}$-vector space $M_{\mathcal{F}}$ (of dimension ν) equipped with an invertible σ_q-semilinear operator, *i.e.*

$$\Sigma_q(fm)=\sigma_q(f)\Sigma_q(m),\ \text{for any } f\in\mathcal{F} \text{ and } m\in M_{\mathcal{F}}.$$

A morphism of q-difference modules over $\mathcal{F}$ is a morphism of $\mathcal{F}$-vector spaces, commuting with the q-difference structures (for more generalities on the topic, *cf.* [vdPS97], [DV02, Part I] or [DVRSZ03]). We denote by $Diff(\mathcal{F},\sigma_q)$ the category of q-difference modules over $\mathcal{F}$.

Let $\mathcal{M}_{\mathcal{F}}=(M_{\mathcal{F}},\Sigma_q)$ be a q-difference module over $\mathcal{F}$ of rank ν. We fix a basis $\underline{e}$ of $M_{\mathcal{F}}$ over $\mathcal{F}$ and we set:

$$\Sigma_q\underline{e}=\underline{e}A,$$

with $A\in Gl_\nu(\mathcal{F})$. A horizontal vector $\vec{y}\in\mathcal{F}^\nu$ with respect to the basis $\underline{e}$ for the operator Σ_q is a vector that verifies $\Sigma_q(\underline{e}\vec{y})=\underline{e}\vec{y}$, *i.e.* $\vec{y}=A\sigma_q(\vec{y})$. Therefore we call

$$\sigma_q(Y)=A_1Y,\ \text{with } A_1=A^{-1},$$

the (q-difference) system associated to $\mathcal{M}_{\mathcal{F}}$ with respect to the basis $\underline{e}$. Recursively, we obtain a family of higher order q-difference systems:

$$\sigma_q^n(Y)=A_nY \text{ and } d_q^nY=G_nY,$$

with $A_n\in Gl_\nu(\mathcal{F})$ and $G_n\in M_\nu(\mathcal{F})$. Notice that:

$$A_{n+1}=\sigma_q(A_n)A_1,\ G_1=\frac{A_1-1}{(q-1)x} \text{ and } G_{n+1}=\sigma_q(G_n)G_1(x)+d_qG_n.$$

It is convenient to set $A_0=G_0=1$. Moreover we set $[n]_q=\frac{q^n-1}{q-1}$, $[n]_q^!=[n]_q[n-1]_q\cdots[1]_q$, $[0]_q^!=1$ and $G_{[n]}=\frac{G_n}{[n]_q^!}$ for any $n\geq 1$.

1.3. Reduction modulo places of K. In the sequel, we will deal with an arithmetic situation, in the following sense. We consider the ring of integers $\mathcal{O}_K$ of K, *i.e.* the integral closure of $k[q]$ in K, and a q-difference algebra of the form

$$\mathcal{A}=\mathcal{O}_K\left[x,\frac{1}{P(x)},\frac{1}{P(qx)},\frac{1}{P(q^2x)},\ldots\right], \tag{1.1}$$

for some $P(x) \in \mathcal{O}_K[x]$. Then $\mathcal{A}$ is stable by the action of σ_q and we can consider a free $\mathcal{A}$-module M equipped with a semilinear invertible operator[6] Σ_q. Notice that $\mathcal{M}_{K(x)} = (M_{K(x)} = M \otimes_{\mathcal{A}} K(x), \Sigma_q \otimes \sigma_q)$ is a q-difference module over $K(x)$. We will call $\mathcal{M} = (M, \Sigma_q)$ a q-difference module over $\mathcal{A}$.

Any q-difference module over $K(x)$ comes from a q-difference module over $\mathcal{A}$, for a convenient choice of $\mathcal{A}$. The reason for considering q-difference modules over $\mathcal{A}$ rather than over $K(x)$, is that we want to reduce our q-difference modules with respect to the places of K, and, in particular, with respect to the cyclotomic places of K.

We denote by k_v the residue field of K with respect to a place $v \in \mathcal{P}$, π_v the uniformizer of v and q_v the image of q in k_v, which is defined for all places $v \in \mathcal{P}$. For almost all $v \in \mathcal{P}_f$ we can consider the $k_v(x)$-vector space $M_{k_v(x)} = M \otimes_{\mathcal{A}} k_v(x)$, with the structure induced by Σ_q. In this way, for almost all $v \in \mathcal{P}$, we obtain a q_v-difference module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$ over $k_v(x)$,

In particular, for almost all $v \in \mathcal{C}$, we obtain a q_v-difference module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$ over $k_v(x)$, having the particularity that q_v is a root of unity, say of order κ_v. This means that $\sigma_{q_v}^{\kappa_v} = 1$ and that $\Sigma_{q_v}^{\kappa_v}$ is a $k_v(x)$-linear operator. The results in [DV02, §2] apply to this situation. We recall some of them. Since we have:

$$\sigma_{q_v}^{\kappa_v} = 1 + (q-1)^{\kappa_v} x^{\kappa_v} d_{q_v}^{\kappa_v} \quad \text{and} \quad \Sigma_{q_v}^{\kappa_v} = 1 + (q-1)^{\kappa_v} x^{\kappa_v} \Delta_{q_v}^{\kappa_v},$$

where $\Delta_{q_v} = \frac{\Sigma_{q_v} - 1}{(q_v - 1)x}$, the following facts are equivalent:

1. $\Sigma_{q_v}^{\kappa_v}$ is the identity;
2. $\Delta_{q_v}^{\kappa_v}$ is zero;
3. the reduction of A_{κ_v} modulo π_v is the identity matrix;
4. the reduction of G_{κ_v} modulo π_v is zero.

Definition 1.4. If the conditions above are satisfied we say that $\mathcal{M}$ has *zero κ_v-curvature (modulo π_v)*. We say that $\mathcal{M}$ has *nilpotent κ_v-curvature (modulo π_v)* or *has nilpotent reduction*, if $\Delta_{q_v}^{\kappa_v}$ is a nilpotent operator or equivalently if $\Sigma_{q_v}^{\kappa_v}$ is a unipotent operator.

We will use this notion in §2, while in §3 we will need the following stronger notion.

1.5. κ_v-curvatures (modulo ϕ_v). We denote by ϕ_v the uniformizer of the cyclotomic place of $k(q)$ induced by $v \in \mathcal{C}_K$. The ring $\mathcal{A} \otimes_{\mathcal{O}_K} \mathcal{O}_K/(\phi_v)$ is not reduced in general, nevertheless it has a q-difference algebra structure and the results in [DV02, §2] apply again. Therefore we set:

Definition 1.6. A q-difference module $\mathcal{M}$ has *zero κ_v-curvature (modulo ϕ_v)* if the operator $\Sigma_q^{\kappa_v}$ induces the identity (or equivalently if the operator $\Delta_q^{\kappa_v}$ induces the zero operator) on the module $M \otimes_{\mathcal{O}_K} \mathcal{O}_K/(\phi_v)$.

Remark 1.7. The rational function ϕ_v is, up to a multiplicative constant, a factor of a cyclotomic polynomial for almost all v. It is a divisor of $[\kappa_v]_q$ and $|\phi_v|_v = |[\kappa_v]_q|_v = |[\kappa_v]_q^!|_v$.

We recall the definition of the Gauss norm associated to an ultrametric norm $v \in \mathcal{P}$:

$$\text{for any } \frac{\sum a_i x^i}{\sum b_j x^j} \in K(x), \quad \left| \frac{\sum a_i x^i}{\sum b_j x^j} \right|_{v,Gauss} = \frac{\sup |a_i|_v}{\sup |b_j|_v}.$$

[6] We could have asked that Σ_q is only injective, but then, enlarging the scalar to a q-difference algebra $\mathcal{A}' \subset K(x)$, of the same form as (1.1), we would have obtained an invertible operator. Since we are interested in the reduction of $\mathcal{M}$ modulo almost all places of K, we can suppose without loss of generality that Σ_q is invertible.

Proposition 1.8. *Let $v \in \mathcal{C}_K$. We assume that $|G_1(x)|_{v,Gauss} \leq 1$. Then the following assertions are equivalent:*

(1) *The module $\mathcal{M} = (M, \Sigma_q)$ has zero κ_v-curvature modulo ϕ_v.*
(2) *For any positive integer n, we have $\left|G_{[n]}\right|_{v,Gauss} \leq 1$,* i.e. *the operator $\frac{\Delta_q^n}{[n]_q^!}$ induces a well defined operator on $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$.*

Remark 1.9. Even if q_v is a root of unity, the family of operators $\frac{d_{q_v}^n}{[n]_{q_v}^!}$ acting on $k_v(x)$ is well defined. This remark is the starting point for the theory of iterated q-difference modules constructed in [Har10, §3]. Then the second assertion of the proposition above can be rewritten as:

$\mathcal{M}_{k_v(x)}$ *has a natural structure of iterated q_v-difference module.*

Proof. The only nontrivial implication is "1 ⇒ 2" whose proof is quite similar to [DV02, Lemma 5.1.2]. The Leibniz Formula for d_q and Δ_q implies that:

$$G_{(n+1)\kappa_v} = \sum_{i=0}^{\kappa_v} \binom{\kappa_v}{i}_q \sigma_q^{\kappa_v - i}(d_q^i\,(G_{n\kappa_v}))G_{\kappa_v - i},$$

where $\binom{n}{i}_q = \frac{[n]_q^!}{[i]_q^![n-i]_q^!}$ for any $n \geq i \geq 0$. If $\mathcal{M}$ has zero κ_v-curvature modulo ϕ_v then $|G_{\kappa_v}|_{v,Gauss} \leq |\phi_v|_v$. One obtains recursively that $|G_m|_{v,Gauss} \leq |\phi_v|_v^{\left[\frac{m}{\kappa_v}\right]}$, where we have denoted by $[a]$ the integral part of $a \in \mathbb{R}$, *i.e.* $[a] = max\{n \in \mathbb{Z} : n \leq a\}$. Since $|[\kappa_v]_q|_v = |\phi_v|_v$ and $|[m]_q^!|_v = |\phi_v|_v^{\left[\frac{m}{\kappa_v}\right]}$, we conclude that:

$$\left|\frac{G_m}{[m]_q^!}\right|_{v,Gauss} \leq 1. \tag{1.2}$$

□

Part I. Function field analogue of Grothendieck conjecture on p-curvatures for q-difference equations

In the first part of the paper we are going to prove the following theorem. In the notation above we consider a linear q-difference equation

$$\mathcal{L}y(x) := a_\nu(x)y(q^\nu x) + \cdots + a_1(x)y(qx) + a_0(x)y(x) = 0$$

with coefficients in an algebra $\mathcal{A}$ of the form (1.1). Then:

Theorem 1.10. *The equation $\mathcal{L}y(x) = 0$ has a full set of solutions in $K(x)$, linearly independent over K, if and only if for almost all $v \in \mathcal{C}$ the equation $\mathcal{L}y(x) = 0$ has a full set of solutions in $\mathcal{A}/(\phi_v)$, linearly independent over $\mathcal{O}_K/(\phi_v)$.*

In the sequel we are rather working with q-difference modules than linear q-difference equations. The cyclic vector lemma easily allows to deduce the theorem above from Theorem 3.1.

2. "Global" nilpotence

In this section, we are going to prove that a q-difference module is regular singular and has integral exponents if it has nilpotent reduction for sufficiently many cyclotomic places. In this setting, and in particular if the characteristic of k is zero, speaking of global nilpotence is a little bit abusive. Nevertheless, it is the terminology used in arithmetic differential equations and we think that it is evocative of the ideas that have inspired what follows.

Definition 2.1. A q-difference module (M, Σ_q) over $\mathcal{A}$ (or another sub-q-difference algebra of $K((x))$) is said to be regular singular at 0 if there exists a basis $\underline{e}$ of $(M \otimes_{\mathcal{A}} K((x)), \Sigma_q \otimes \sigma_q)$ over $K((x))$ such that the action of $\Sigma_q \otimes \sigma_q$ over $\underline{e}$ is represented by a constant matrix $A \in Gl_\nu(K)$.

Remark 2.2. It follows from the Frobenius algorithm[7], that a q-difference module $M_{K(x)}$ over $K(x)$ is regular singular if and only if there exists a basis $\underline{e}$ such that $\Sigma_q \underline{e} = \underline{e} A(x)$ with $A(x) \in Gl_\nu(K(x)) \cap Gl_n(K[[x]])$.

The eigenvalues of $A(0)$ are called the exponents of $\mathcal{M}$ at zero. They are well defined modulo $q^{\mathbb{Z}}$. The q-difference module $\mathcal{M}$ is said to be regular singular *tout court* if it is regular singular both at 0 and at ∞, *i.e.* after a variable change of the form $x = 1/t$.

In the notation of the previous section, we prove the following result, which is actually an analogue of [Kat70, §13] (*cf.* also [DV02] for a q-difference version over a number field):

Theorem 2.3.

(1) *If a q-difference module $\mathcal{M}$ over $\mathcal{A}$ has nilpotent κ_v-curvature modulo π_v for infinitely many $v \in \mathcal{C}$ then it is regular singular.*
(2) *Let $\mathcal{M}$ be a q-difference module over $\mathcal{A}$. If there exists an infinite set of positive primes $\wp \subset \mathbb{Z}$ such that $\mathcal{M}$ has nilpotent κ_v-curvature modulo π_v for all $v \in \mathcal{C}$ such that $\kappa_v \in \wp$, then $\mathcal{M}$ is regular singular and its exponents (at zero and at ∞) are all in $q^{\mathbb{Z}}$.*

Remark 2.4. The proof of the first part of Theorem 2.3 is inspired by [Kat70, 13.1] and therefore is quite similar to [DV02, §6]. On the other hand, the proof of the triviality of the exponents (*cf.* Proposition 2.7 below) has significant differences with respect to the analogous results on number fields. In fact in the differential case the proof is based on Chebotarev density theorem. In [DV02] it is a consequence on some considerations on Kummer extensions and Chebotarev density theorem, while in this setting it is a consequence of Lemma 2.9 below, which can be interpreted as a statement in rational dynamic.

The proof of Theorem 2.3 is the object of the following two subsections.

2.1. **Regularity.** We prove the first part of Theorem 2.3. It is enough to prove that 0 is a regular singular point for $\mathcal{M}$, the proof at ∞ being completely analogous.

Let $r \in \mathbb{N}$ be a divisor of $\nu!$ where ν is the dimension of $M_{K(x)}$ over $K(x)$ and let L be a finite extension of K containing an element $\widetilde{q}$ such that $\widetilde{q}^r = q$. We consider the field extension $K(x) \hookrightarrow L(t)$, $x \mapsto t^r$. The field $L(t)$ has a natural structure of $\widetilde{q}$-difference algebra extending the q-difference structure of $K(x)$.

Lemma 2.5. *The q-difference module $\mathcal{M}$ is regular singular at $x = 0$ if and only if the $\widetilde{q}$-difference module $\mathcal{M}_{L(t)} := (M \otimes_{\mathcal{A}} L(t), \Sigma_{\widetilde{q}} := \Sigma_q \otimes \sigma_{\widetilde{q}})$ over $L(t)$ is regular singular at $t = 0$.*

Proof. It is enough to notice that if $\underline{e}$ is a basis for $\mathcal{M}$, then $\underline{e} \otimes 1$ is a basis for $\mathcal{M}_{L(t)}$ and $\Sigma_{\widetilde{q}}(\underline{e} \otimes 1) = \Sigma_q(\underline{e}) \otimes 1$. The other implication is a consequence of the Frobenius algorithm (*cf.* [vdPS97] or [Sau00]). □

The next lemma can be deduced from the formal classification of q-difference modules (*cf.* [Pra83, Cor. 9 and §9, 3)], [Sau04c, Thm. 3.1.7]):

---

[7] *cf.* [vdPS97] or [Sau00, §1.1]. The algorithm is briefly summarized also in [Sau04b, §1.2.2] and [DVRSZ03].

Lemma 2.6. *There exist an extension $L(t)/K(x)$ as above, a basis $\underline{f}$ of the $\widetilde{q}$-difference module $\mathcal{M}_{L(t)}$ and an integer ℓ such that $\Sigma_{\widetilde{q}}\underline{f} = \underline{f}B(t)$, with $B(t) \in Gl_\nu(L(t))$ of the following form:*

$$\text{(2.1)}\qquad \begin{cases} B(t) = \frac{B_\ell}{t^\ell} + \frac{B_{\ell-1}}{t^{\ell-1}} + \dots, \text{ as an element of } Gl_\nu(L((t))); \\ B_\ell \text{ is a constant nonnilpotent matrix.} \end{cases}$$

Proof of the first part of Theorem 2.3. Let $\mathcal{B} \subset L(t)$ be a $\widetilde{q}$-difference algebra over the ring of integers $\mathcal{O}_L$ of L, of the same form as (1.1), containing the entries of $B(t)$. Then there exists a $\mathcal{B}$-lattice $\mathcal{N}$ of $\mathcal{M}_{L(t)}$ inheriting the $\widetilde{q}$-difference module structure from $\mathcal{M}_{L(t)}$ and having the following properties:
1. $\mathcal{N}$ has nilpotent reduction modulo infinitely many cyclotomic places of L;
2. there exists a basis $\underline{f}$ of $\mathcal{N}$ over $\mathcal{A}$ such that $\Sigma_{\widetilde{q}}\underline{f} = \underline{f}B(t)$ and $B(t)$ verifies (2.1).

Iterating the operator $\Sigma_{\widetilde{q}}$ we obtain:

$$\Sigma_{\widetilde{q}}^m(\underline{f}) = \underline{f}B(t)B(\widetilde{q}t)\cdots B(\widetilde{q}^{m-1}t) = \underline{f}\left(\frac{B_\ell^m}{\widetilde{q}^{\frac{\ell m(\ell m-1)}{2}}t^{m\ell}} + h.o.t.\right).$$

We know that for infinitely many cyclotomic places w of L, the matrix $B(t)$ verifies

$$\text{(2.2)}\qquad \big(B(t)B(\widetilde{q}t)\cdots B(\widetilde{q}^{\kappa_w-1}t) - 1\big)^{n(w)} \equiv 0 \mod \pi_w,$$

where π_w is an uniformizer of the place w, κ_w is the order $\widetilde{q}$ modulo π_w and $n(w)$ is a convenient positive integer. Suppose that $\ell \neq 0$. Then $B_\ell^{\kappa_w} \equiv 0$ modulo π_w, for infinitely many w, and hence B_ℓ is a nilpotent matrix, in contradiction with Lemma 2.6. So necessarily $\ell = 0$.

Finally we have $\Sigma_{\widetilde{q}}(\underline{f}) = \underline{f}\,(B_0 + h.o.t)$. It follows from (2.2) that B_0 is actually invertible, which implies that $\mathcal{M}_{L(t)}$ is regular singular at 0. Lemma 2.5 allows to conclude. □

2.2. **Triviality of the exponents.** Let us prove the second part of Theorem 2.3. We have already proved that 0 is a regular singularity for $\mathcal{M}$. This means that there exists a basis $\underline{e}$ of $\mathcal{M}$ over $K(x)$ such that $\Sigma_q\underline{e} = \underline{e}A(x)$, with $A(x) \in Gl_\nu(K[[x]]) \cap Gl_\nu(K(x))$.

The Frobenius algorithm (*cf.* [Sau00, §1.1.1]) implies that there exists a shearing transformation $S \in Gl_\nu(K[x, 1/x])$, such that $S(qx)A(x)S(x)^{-1} \in Gl_\nu(K[[x]]) \cap Gl_\nu(K(x))$ and that the constant term A_0 of $S(x)^{-1}A(x)S(qx)$ has the following properties: if α and β are eigenvalues of A_0 and $\alpha\beta^{-1} \in q^{\mathbb{Z}}$, then $\alpha = \beta$. So choosing the basis $\underline{e}S(x)$ instead of $\underline{e}$, we can assume that $A_0 = A(0)$ has this last property.

Always following the Frobenius algorithm (*cf.* [Sau00, §1.1.3]), one constructs recursively the entries of a matrix $F(x) \in Gl_\nu(K[[x]]))$, with $F(0) = 1$, such that we have $F(x)^{-1}A(x)F(qx) = A_0$. This means that there exists a basis $\underline{f}$ of $\mathcal{M}_{K((x))}$ such that $\Sigma_q\underline{f} = \underline{f}A_0$.

The matrix A_0 can be written as the product of a semi-simple matrix D_0 and a unipotent matrix N_0. Since $\mathcal{M}$ has nilpotent reduction, we deduce from §1.3 that the reduction of $A_{\kappa_v} = A_0^{\kappa_v}$ modulo π_v is the identity matrix. Then D_0 verifies:

$$\text{(2.3)}\qquad \text{for all } v \in \mathcal{C} \text{ such that } \kappa_v \in \wp, \text{ we have } D_0^{\kappa_v} \equiv 1 \text{ modulo } \pi_v.$$

Let $\widetilde{K}$ be a finite extension of K in which we can find all the eigenvalues of D_0. Then any eigenvalue $\alpha \in \widetilde{K}$ of A_0 has the property that $\alpha^{\kappa_v} = 1$ modulo w, for all $w \in \mathcal{C}_{\widetilde{K}}$, $w|v$ and v satisfies (2.3). In other words, the reduction modulo w of an eigenvalue α of A_0 belongs to the multiplicative cyclic group generated by the reduction of q modulo π_v.

To end the proof, we have to prove that $\alpha \in q^{\mathbb{Z}}$. So we are reduced to prove the proposition below.

Proposition 2.7. *Let $K/k(q)$ be a finite extension and $\wp \subset \mathbb{Z}$ be an infinite set of positive primes. For any $v \in \mathcal{C}$, let κ_v be the order of q modulo π_v, as a root of unity.*

If $\alpha \in K$ is such that $\alpha^{\kappa_v} \equiv 1$ modulo π_v for all $v \in \mathcal{C}$ such that $\kappa_v \in \wp$, then $\alpha \in q^{\mathbb{Z}}$.

Remark 2.8. Let $K = \mathbb{Q}(\widetilde{q})$, with $\widetilde{q}^r = q$, for some integer $r > 1$. If $\widetilde{q}$ is an eigenvalue of A_0 we would be asking that for infinitely many positive primes $\ell \in \mathbb{Z}$ there exists a primitive root of unity $\zeta_{r\ell}$ of order $r\ell$, which is also a root of unity of order ℓ. Of course, this cannot be true, unless $r = 1$.

Notice that Proposition 2.7 can be rewritten in the language of rational dynamic. In fact, the following assertions are equivalent:

(1) $f(q) \in k(q)$ satisfies the assumptions of Lemma 2.9.
(2) There exist infinitely many $\ell \in \mathbb{N}$ such that the group μ_ℓ of roots of unity of order ℓ verifies $f(\mu_\ell) \subset \mu_\ell$.
(3) $f(q) \in q^{\mathbb{Z}}$.
(4) The Julia set of f is the unit circle.

As it was pointed out to us by C. Favre, the equivalence between the last two assumptions is a particular case of [Zdu97], while the equivalence between the second and the fourth assumption can be deduced from [FRL06] or [Aut01].

2.3. **Proof of Proposition 2.7.** We denote by k_0 either the field of rational numbers $\mathbb{Q}$, if the characteristic of k is zero, or the field with p elements $\mathbb{F}_p$, if the characteristic of k is $p > 0$. First of all, let us suppose that k is a finite perfect extension of k_0 of degree d and fix an embedding $k \hookrightarrow \overline{k}$ of k in its algebraic closure $\overline{k}$. In the case of a rational function $f \in k(q)$, Proposition 2.7 is a consequence of the following lemma:

Lemma 2.9. *Let $[k : k_0] = d < \infty$ and let $f(q) \in k(q)$ be nonzero rational function. If there exists an infinite set of positive primes $\wp \subset \mathbb{Z}$ with the following property:*

> *for any $\ell \in \wp$ there exists a primitive root of unity ζ_ℓ of order ℓ such that $f(\zeta_\ell)$ is a root of unity of order ℓ,*

then $f(q) \in q^{\mathbb{Z}}$.

Remark 2.10. If $k = \mathbb{C}$ and $y - f(q)$ is irreducible in $\mathbb{C}[q, y]$, the result can be deduced from [Lan83, Ch.8, Thm.6.1], whose proof uses Bézout theorem. We give here a totally elementary proof, that holds also in positive characteristic.

Proof. We denote by μ_ℓ the group of root of unity of order ℓ. Let $f(q) = \frac{P(q)}{Q(q)}$, with $P = \sum_{i=0}^{D} a_i q^i, Q = \sum_{i=0}^{D} b_i q^i \in k[q]$ coprime polynomials of degree less equal to D, and let ℓ be a prime such that:

- $f(\zeta_\ell) \in \mu_\ell$;
- $2D < \ell - 1$.

Moreover we can chose $\ell >> 0$ so that the extensions k and $k_0(\mu_\ell)$ are linearly disjoint over k_0. Since k is perfect, this implies that the minimal polynomial of the primitive ℓ-th root of unity ζ_ℓ over k is $\chi(X) = 1 + X + ... + X^{\ell-1}$. Now let $\kappa \in \{0, \ldots, \ell - 1\}$ be such that $f(\zeta_\ell) = \zeta_\ell^\kappa$, *i.e.*

$$\sum_{i=0}^{D} a_i \zeta_\ell^i = \sum_{i=0}^{D} b_i \zeta_\ell^{i+\kappa}.$$

We consider the polynomial $H(q)=\sum_{i=0}^{D} a_iq^i - \sum_{j=\kappa}^{D+\kappa} b_{j-\kappa}q^j$ and distinguish three cases:

(1) If $D+\kappa<\ell-1$, then $H(q)$ has ζ_ℓ as a zero and has degree strictly inferior to $\ell-1$. Necessarily $H(q)=0$. Thus we have

$$a_0=a_1=...=a_{\kappa-1}=b_{D+1-\kappa}=...=b_D=0 \quad \text{and} \quad a_i=b_{i-\kappa} \text{ for } i=\kappa,\dots,D,$$

which implies $f(q)=q^\kappa$.

(2) If $D+\kappa=\ell-1$ then $H(q)$ is a k-multiple of $\chi(q)$ and therefore all the coefficients of $H(q)$ are equal. Notice that the inequality $D+\kappa\geq\ell-1$ forces κ to be strictly bigger than D, in fact otherwise one would have $\kappa+D\leq 2D<\ell-1$. For this reason the coefficients of $H(q)$ of the monomials $q^{D+1},\dots,q^\kappa$ are all equal to zero. Thus

$$a_0=a_1=...=a_D=b_0=...=b_D=0$$

and therefore $f=0$ against the assumptions. So the case $D+\kappa=l-1$ cannot occur.

(3) If $D+\kappa>\ell-1$, then $\kappa>D>D+\kappa-\ell$, since $\kappa>D$ and $\kappa-\ell<0$. In this case we shall rather consider the polynomial $\widetilde{H}(q)$ defined by:

$$\widetilde{H}(q)=\sum_{i=0}^{D} a_iq^i-\sum_{i=\kappa}^{\ell-1} b_{i-\kappa}q^i-\sum_{i=0}^{D+\kappa-\ell} b_{i+\ell-\kappa}q^i.$$

Notice that $H(\zeta_\ell)=\widetilde{H}(\zeta_\ell)=0$ and that $\widetilde{H}(q)$ has degree smaller or equal than $\ell-1$. As in the previous case, $\widetilde{H}(q)$ is a k-multiple of $\chi(q)$. We get

$$b_j=0 \text{ for } j=0,...,\ell-1-\kappa$$

and

$$a_0-b_{\ell-\kappa}=...=a_{D+\kappa-\ell}-b_D=a_{D+\kappa-\ell+1}=...=a_D=0.$$

We conclude that $f(q)=q^{\kappa-\ell}$.

This ends the proof. □

We are going to deduce Proposition 2.7 from Lemma 2.9 in two steps: first of all we are going to show that we can drop the assumption that $[k:k_0]$ is finite and then that one can always reduce to the case of a rational function.

Lemma 2.11. *Lemma 2.9 holds if k/k_0 is a finitely generated (not necessarily algebraic) extension.*

Remark 2.12. Since $f(q)\in k(q)$, replacing k by the field generated by the coefficients of f over k_0, we can always assume that k/k_0 is finitely generated.

Proof. Let $\widetilde{k}$ be the algebraic closure of k_0 in k and let k' be an intermediate field of $k/\widetilde{k}$, such that $f(q)\in k'(q)\subset k(q)$ and that $k'/\widetilde{k}$ has minimal transcendence degree ι. We suppose that $\iota>0$. So let $a_1,\dots,a_\iota$ be transcendent algebraically independent elements of $k'/\widetilde{k}$ and let $k''=\widetilde{k}(a_1,\dots,a_\iota)$. If $k'/\widetilde{k}$ is purely transcendental, *i.e.* if $k'=k''$, then $f(q)=P(q)/Q(q)$, where $P(q)$ and $Q(q)$ can be written in the form:

$$P(q)=\sum_i\sum_{\underline{j}}\alpha_{\underline{j}}^{(i)}a_{\underline{j}}q^i \quad \text{and} \quad Q(q)=\sum_i\sum_{\underline{j}}\beta_{\underline{j}}^{(i)}a_{\underline{j}}q^i,$$

with $\underline{j}=(j_1,\dots,j_\iota)\in\mathbb{Z}_{\geq 0}^\iota$, $a_{\underline{j}}=a_{j_1}\cdots a_{j_\iota}$ and $\alpha_{\underline{j}}^{(i)},\beta_{\underline{j}}^{(i)}\in\widetilde{k}$. If we reorganize the terms of P and Q so that

$$P(q)=\sum_{\underline{j}}a_{\underline{j}}D_{\underline{j}}(q) \quad \text{and} \quad Q(q)=\sum_{\underline{j}}a_{\underline{j}}C_{\underline{j}}(q),$$

we conclude that the assumption $f(\zeta_\ell) \subset \mu_\ell$ for infinitely many primes ℓ implies that $f_{\underline{j}} = \frac{D_{\underline{j}}}{C_{\underline{j}}}$ is a rational function with coefficients in $\widetilde{k}$ satisfying the assumptions of Lemma 2.9. Moreover, since the f_j's take the same values at infinitely many roots of unity, they are all equal. Finally, we conclude that $f_{\underline{j}}(q) = q^d$ for any $\underline{j}$ and hence that $f = q^d \frac{\sum \alpha_{\underline{j}}}{\sum \alpha_{\underline{j}}} = q^d$.

Now let us suppose that $k' = k''(b)$ for some primitive element b, algebraic over k'', of degree e. Then once again we write $f(q) = P(q)/Q(q)$, with:

$$P(q) = \sum_i \sum_{h=0}^{e-1} \alpha_{i,h} b^h q^i \quad \text{and} \quad Q(q) = \sum_i \sum_{h=0}^{e-1} \beta_{i,h} b^h q^i,$$

with $\alpha_{i,h}, \beta_{i,h} \in k''$. Again we conclude that $\frac{\sum_i \alpha_{i,h} q^i}{\sum_i \beta_{i,h} q^i} = q^d$ for any $h = 0, \dots, e-1$, and hence that $f(q) = q^d$. $\square$

End of the proof of Proposition 2.7. Let $\widetilde{K} = k(q, f) \subset K$. If the characteristic of k is p, replacing f by a p^n-th power of f, we can suppose that $\widetilde{K}/k(q)$ is a Galois extension. So we set:

$$y = \prod_{\varphi \in Gal(\widetilde{K}/k(q))} f^\varphi \in k(q).$$

For infinitely many $v \in \mathcal{C}_{k(q)}$ such that κ_v is a prime, we have $f^{\kappa_v} \equiv 1$ modulo w, for any $w|v$. Since $Gal(\widetilde{K}/K)$ acts transitively over the set of places $w \in \mathcal{C}_{\widetilde{K}}$ such that $w|v$, this implies that $y^{\kappa_v} \equiv 1$ modulo π_v. Then Lemmas 2.11 and 2.9 allow us to conclude that $y \in q^{\mathbb{Z}}$. This proves that we are in the following situation: f is an algebraic function such that $|f|_w = 1$ for any $w \in \mathcal{P}_{\widetilde{K},f}$ and that $|f|_w \neq 1$ for any $w \in \mathcal{P}_{\widetilde{K},\infty}$. We conclude that $f = cq^{s/r}$ for some nonzero integers s, r and some constant c in a finite extension of k. Since $f^{\kappa_v} \equiv 1$ modulo w for all $w \in \mathcal{C}_{\widetilde{K}}$ such that $\kappa_v \in \wp$, we finally obtain that $r = 1$ and $c = 1$. $\square$

3. Triviality criteria: a function field q-analogue of the Grothendieck conjecture

In this section we are proving a statement in the wake of the Grothendieck conjecture on p-curvatures. Roughly speaking, we are going to prove that a q-difference module is trivial if and only if its reduction modulo almost all cyclotomic places is trivial.

We say that the q-difference module $\mathcal{M} = (M, \Sigma_q)$ of rank ν over a q-difference field $\mathcal{F}$ is trivial if there exists a basis $\underline{f}$ of M over $\mathcal{F}$ such that $\Sigma_q \underline{f} = \underline{f}$. This is equivalent to ask that the q-difference system associated to $\mathcal{M}$ with respect to a basis (and hence any basis) $\underline{e}$ has a fundamental solution in $Gl_\nu(\mathcal{F})$. We say that a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $\mathcal{A}$ becomes trivial over a q-difference field $\mathcal{F}$ over $\mathcal{A}$ if the q-difference module $(M \otimes_{\mathcal{A}} \mathcal{F}, \Sigma_q \otimes \sigma_q)$ is trivial.

Theorem 3.1. *A q-difference module $\mathcal{M}$ over $\mathcal{A}$ has zero κ_v-curvature modulo ϕ_v for almost all $v \in \mathcal{C}$ if and only if $\mathcal{M}$ becomes trivial over $K(x)$.*

Remark 3.2. As proved in [DV02, Prop.2.1.2], if $\Sigma_q^{\kappa_v}$ is the identity modulo ϕ_v then the q_v-difference module $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$ is trivial.

Theorem 3.1 is equivalent to the following statements, which are a q-analog of the conjecture stated at the very end of [MvdP03]:

Corollary 3.3. *For a q-difference module $\mathcal{M}$ over $\mathcal{A}$ the following statement are equivalent:*

(1) *The q-difference module $\mathcal{M}$ over $\mathcal{A}$ becomes trivial over $K(x)$;*
(2) *It induces an iterative q_v-difference structure over $\mathcal{M}_{k_v(x)}$ for almost all $v \in \mathcal{C}$;*
(3) *It induces a trivial iterative q_v-difference structure over $\mathcal{M}_{k_v(x)}$ for almost all $v \in \mathcal{C}$.*

Remark 3.4. The first assertion is equivalent to the fact that the Galois group of $\mathcal{M}_{K(x)}$ is trivial, while the fourth assertion is equivalent to the fact that the iterative Galois group of $\mathcal{M}_{k_v(x)}$ over $k_v(x)$ is 1 for almost all $v \in \mathcal{C}$.

Proof. The equivalence $1 \Leftrightarrow 2$ is a consequence of Proposition 1.8 and Theorem 3.1, while the implication $3 \Rightarrow 2$ is tautological.

Let us prove that $1 \Rightarrow 3$. If the q-difference module $\mathcal{M}$ becomes trivial over $K(x)$, then there exist an $\mathcal{A}$-algebra $\mathcal{A}'$, of the form (1.1), obtained from $\mathcal{A}$ inverting a polynomial and its q-iterates, and a basis $\underline{e}$ of $M \otimes_{\mathcal{A}} \mathcal{A}'$ over $\mathcal{A}'$, such that the associated q-difference system is $\sigma_q(Y) = Y$. Therefore, for almost all $v \in \mathcal{C}$, $\mathcal{M}$ induces an iterative q_v-difference module $\mathcal{M}_{k_v(x)}$ whose iterative q_v-difference equations are given by $\frac{d_{q_v}^{\kappa_v}}{[\kappa_v]_{q_v}^!}(Y) = 0$ for all $n \in \mathbb{N}$ (*cf.* [Har10, Prop.3.17]). □

As far as the proof of Theorem 3.1 is regarded, one implication is trivial. The proof of the other is divided into steps. So let us suppose that the q-difference module $\mathcal{M}$ over $\mathcal{A}$ has zero κ_v-curvature modulo ϕ_v for almost all $v \in \mathcal{C}$, then:

Step 1. The q-difference module $\mathcal{M}$ becomes trivial over $K((x))$, meaning that the module $\mathcal{M}_{K((x))} = (M \otimes_{\mathcal{A}} K((x)), \Sigma_q \otimes \sigma_q)$ is trivial (*cf.* Corollary 3.6 below).
Step 2. There exists a basis $\underline{e}$ of $\mathcal{M}_{K(x)}$, such that the associated q-difference system has a fundamental matrix of solution $Y(x)$ in $Gl(K[[x]])$ whose entries are Taylor expansions of rational functions (*cf.* Proposition 3.7 below).

Remark 3.5. Theorem 3.1 is the function field analogue of the main result of [DV02]. Step 1 is inspired by [Kat70, 13.1] (*cf.* also [DV02, §6] for q-difference equations over number fields). The main difference is Proposition 2.7 proved above. Step 2 is closed to [DV02, §8] and uses the Borel-Dwork criteria (*cf.* [And89, VIII, 1.2]).

3.1. **Step 1: triviality over $K((x))$.** The triviality over $K((x))$ is a consequence of Theorem 2.3:

Corollary 3.6. *If there exists an infinite set of positive primes $\wp \subset \mathbb{Z}$ such that the q-difference module $\mathcal{M}$ over $\mathcal{A}$ has zero κ_v-curvature modulo π_v (and* a fortiori *modulo ϕ_v) for all $v \in \mathcal{C}$ with $\kappa_v \in \wp$, then $\mathcal{M}$ becomes trivial over the field of formal Laurent series $K((x))$.*

Proof. If $\mathcal{M}$ has zero κ_v-curvature modulo π_v then (*cf.* (2.3) for notation) we actually have:

$$\text{for all } v \in \mathcal{C} \text{ such that } \kappa_v \in \wp,\ D_0^{\kappa_v} \equiv 1 \text{ and } N_0^{\kappa_v} \equiv 1 \text{ modulo } \pi_v,$$

where $\Sigma_q \underline{e} = \underline{e} A_0$, for a chosen basis $\underline{e}$ of $\mathcal{M}_{K((x))}$ and a constant matrix $A_0 = D_0 N_0 \in Gl_\nu(K)$. This immediately implies, because of Proposition 2.7, that all the exponents are in $q^{\mathbb{Z}} \subset k(q) \subset K$ and that the matrix A_0 of $\mathcal{M}$, w.r.t. the $K(x)$-basis $\underline{e}$, is diagonalisable. Therefore there exist a diagonal matrix D with coefficients in $\mathbb{Z}$ and a matrix $C \in Gl_\nu(K)$ such that the basis $\underline{e}' = \underline{e} C x^D$ of $\mathcal{M}_{K((x))}$ is invariant under the action of Σ_q. □

3.2. Step 2: rationality of solutions.

Proposition 3.7. *If a q-difference module $\mathcal{M}$ over $\mathcal{A}$ has zero κ_v-curvature modulo ϕ_v for almost all $v \in \mathcal{C}$ then there exists a basis $\underline{e}$ of $M_{K(x)}$ over $K(x)$ such that the associated q-difference system has a formal fundamental solution $Y(x) \in Gl_\nu(K((x)))$, which is the Taylor expansion at 0 of a matrix in $Gl_\nu(K(x))$,* i.e. *$\mathcal{M}$ becomes trivial over $K(x)$.*

Remark 3.8. This is the only part of the proof of Theorem 3.1 where we need to suppose that the κ_v-curvature are zero modulo ϕ_v *for almost all v*.

Proof. (*cf.* [DV02, Prop.8.2.1]) Let $\underline{e}$ be a basis of M over $K(x)$. Because of Corollary 3.6, applying a basis change with coefficients in $K\left[x, \frac{1}{x}\right]$, we can actually suppose that $\Sigma_q \underline{e} = \underline{e} A(x)$, where $A(x) \in Gl_\nu(K(x))$ has no pole at 0 and $A(0)$ is the identity matrix. In the notation of §1.2, the recursive relation defining the matrices $G_n(x)$ implies that they have no pole at 0. This means that $Y(x) := \sum_{n \geq 0} G_{[n]}(0) x^n$ is a fundamental solution of the q-difference system associated to $\mathcal{M}_{K(x)}$ with respect to the basis $\underline{e}$, whose entries verify the following properties:

- For any $v \in \mathcal{P}_\infty$, the matrix $Y(x)$ has infinite v-adic radius of meromorphy. This assertion is a general fact about regular singular q-difference systems with $|q|_v \neq 1$. The proof is based on the estimate of the growth of the q-factorials compared to the growth of $G_n(0)$, which gives the analyticity at 0, and on the fact that the q-difference system itself gives a meromorphic continuation of the solution.
- Since $|[n]_q|_{v,Gauss} = 1$ for any noncyclotomic place $v \in \mathcal{P}_f$, we have $\left|G_{[m]}(x)\right|_{v,Gauss} \leq 1$ for almost all $v \in \mathcal{P}_f \setminus \mathcal{C}$. For the finitely many $v \in \mathcal{P}_f$ such that $|G_1(x)|_{v,Gauss} > 1$, there exists a constant $C > 0$ such that $\left|G_{[m]}(x)\right|_{v,Gauss} \leq C^m$, for any positive integer m.
- For almost all $v \in \mathcal{C}$ and all positive integer m, $\left|G_{[m]}(x)\right|_{v,Gauss} \leq 1$ (*cf.* Proposition 1.8), while for the remaining finitely many $v \in \mathcal{C}$ there exists a constant $C > 0$ such that $\left|G_{[m]}(x)\right|_{v,Gauss} \leq C^m$ for any positive integer m.

This implies that:

$$\limsup_{m \to \infty} \frac{1}{m} \sum_{v \in \mathcal{P}} \log^+ \left|G_{[m]}(x)\right|_{v,Gauss} = \limsup_{m \to \infty} \frac{1}{m} \sum_{v \in \mathcal{C}} \log^+ \left|G_{[m]}(x)\right|_{v,Gauss} < \infty.$$

To conclude that $Y(x)$ is the expansion at zero of a matrix with rational entries we apply a simplified form of the Borel-Dwork criteria for function fields, which says exactly that a formal power series having positive radius of convergence for almost all places and infinite radius of meromorphy at one fixed place is the expansion of a rational function. The proof in this case is a slight simplification of [DV02, Prop. 8.4.1][8], which is itself a simplification of the more general criteria [And04, Thm. 5.4.3]. We are omitting the details. □

Part II. Algebraic generic and differential Galois groups

In this section we are going to use Theorem 3.1 to give an arithmetic characterization of the generic Galois group of a q-difference module using the v-curvatures introduced in the first part, following [Kat82]. Since we have made no assumption on the characteristic of the base field k, non reduced generic Galois groups may occur: in this case we will prove some devissage of the group, also based on a v-curvature description. In a second moment, under the assumption that k has zero

[8]The simplification comes from the fact that there are no archimedean norms in this setting.

characteristic, we introduce the differential generic Galois group of a q-difference module, for which we prove the same kind of arithmetic characterization, based on Theorem 3.1. In §8 we will give a tannakian definition of such a group, while here we stick to an elementary definition. Moreover in §9 we will compare the differential generic Galois group to the differential Galois group introduced in [HS08] and therefore we will prove that its differential dimension measures the differential complexity of the q-difference module. We conclude the section making some explicit calculation in the case of the Jacobi Theta function.

4. Generic Galois groups

Let $\mathcal{M} = (M, \Sigma_q)$ be a q-difference module of rank ν over $\mathcal{A}$, as in the previous sections. Since $M_{K(x)} = (M_{K(x)}, \Sigma_q)$ is a q-difference module over $K(x)$, we can consider the collection $Constr_{K(x)}(\mathcal{M}_{K(x)})$ of all q-difference modules obtained from $\mathcal{M}_{K(x)}$ by algebraic construction. This means that we consider the family of q-difference modules containing $\mathcal{M}_{K(x)}$ and closed under direct sum, tensor product, dual, symmetric and antisymmetric products. For the reader convenience, we remind the definition of the duality and the tensor product, from which we can deduce all the other algebraic constructions:

- The q-difference structure on the dual $M^*_{K(x)}$ of $M_{K(x)}$ is defined by:
$$\langle \Sigma_q^*(m^*), m\rangle = \sigma_q\left(\langle m^*, \Sigma_q^{-1}(m)\rangle\right),$$
for any $m^* \in M^*_{K(x)}$ and any $m \in M_{K(x)}$.
- If $\mathcal{N}_{K(x)} = (N_{K(x)}, \Sigma_q)$, the q-difference structure on the tensor product $M_{K(x)} \otimes_{K(x)} N_{K(x)}$ is defined by
$$\Sigma_q(m \otimes n) = \Sigma_q(m) \otimes \Sigma_q(n),$$
for any $m \in M_{K(x)}$ and any $n \in N_{K(x)}$ (*cf.* for instance [DV02, §9.1] or [Sau04c, §2.1.6]).

We will denote $Constr_{K(x)}(M_{K(x)})$ the collection of algebraic constructions of the $K(x)$-vector space $M_{K(x)}$, *i.e.* the collection of underlying vector spaces of the family $Constr_{K(x)}(\mathcal{M}_{K(x)})$. Notice that $Gl(M_{K(x)})$ acts naturally, by functoriality, on any element of $Constr_{K(x)}(M_{K(x)})$.

Definition 4.1. The *generic Galois group*[9] $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ *of* $\mathcal{M}_{K(x)}$ is the subgroup of $Gl(M_{K(x)})$ which is the stabiliser of all the q-difference submodules over $K(x)$ of any object in $Constr_{K(x)}(\mathcal{M}_{K(x)})$.

The group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is a tannakian object. In fact, the full tensor category $\langle \mathcal{M}_{K(x)}\rangle^{\otimes}$ generated by $\mathcal{M}_{K(x)}$ in $Diff(K(x), \sigma_q)$ is naturally a tannakian category, when equipped with the forgetful functor
$$\eta : \langle \mathcal{M}_{K(x)}\rangle^{\otimes} \longrightarrow \{K(x)\text{-vector spaces}\}.$$
The functor $Aut^{\otimes}(\eta)$ defined over the category of $K(x)$-algebras is representable by the algebraic group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$.

Notice that in positive characteristic p, the group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is not necessarily reduced. An easy example is given by the equation $y(qx) = q^{1/p}y(x)$, whose generic Galois group is μ_p (*cf.* [vdPR07, §7]).

Remark 4.2. We recall that the Chevalley theorem, that holds also for nonreduced groups (*cf.* [DG70, II, §2, n.3, Cor.3.5]), ensures that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ can be defined as the stabilizer of a rank one submodule (which is not necessarily a q-difference module) of a q-difference module contained in an algebraic construction

[9] In [And01] it is called the *intrinsic* Galois group of $\mathcal{M}_{K(x)}$.

of $\mathcal{M}_{K(x)}$. Nevertheless, it is possible to find a line that defines $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ as the stabilizer and that is also a q-difference module. In fact the noetherianity of $Gl(M_{K(x)})$ implies that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is defined as the stabilizer of a finite family of q-difference submodules $\mathcal{W}^{(i)}_{K(x)} = (W^{(i)}_{K(x)}, \Sigma_q)$ contained in some objects $\mathcal{M}^{(i)}_{K(x)}$ of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes}$. It follows that the line

$$L_{K(x)} = \wedge^{\dim \oplus_i W^{(i)}_{K(x)}} \left(\bigoplus_i W^{(i)}_{K(x)} \right) \subset \wedge^{\dim \oplus_i W^{(i)}_{K(x)}} \left(\bigoplus_i M^{(i)}_{K(x)} \right)$$

is a q-difference module and defines $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ as a stabilizer (*cf.* [Kat82, proof of Prop.9]).

In the sequel, we will use the notation $Stab(W^{(i)}_{K(x)}, i)$ to say that a group is the stabilizer of the set of vector spaces $\{W^{(i)}_{K(x)}\}_i$.

Let G be a closed algebraic subgroup of $Gl(M_{K(x)})$ such that $G = Stab(L_{K(x)})$, for some line $L_{K(x)}$ contained in an object $\mathcal{W}_{K(x)}$ of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes}$. The $\mathcal{A}$-lattice M of $M_{K(x)}$ determines an $\mathcal{A}$-lattice L of $L_{K(x)}$ and an $\mathcal{A}$-lattice W of $W_{K(x)}$. The latter is the underlying space of a q-difference module $\mathcal{W} = (W, \Sigma_q)$ over $\mathcal{A}$.

Definition 4.3. Let $\widetilde{\mathcal{C}}$ be a cofinite subset of $\mathcal{C}_K$ and $(\Lambda_v)_{v \in \widetilde{\mathcal{C}}}$ be a family of $\mathcal{A}/(\phi_v)$-linear operators acting on $M \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$. We say that *the algebraic group* $G \subset Gl(M_{K(x)})$ *contains the operators* Λ_v *modulo* ϕ_v *for almost all* $v \in \mathcal{C}_K$ if for almost all $v \in \widetilde{\mathcal{C}}$ the operator Λ_v stabilizes $L \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$ inside $W \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$:

$$\Lambda_v \in Stab_{\mathcal{A}/(\phi_v)}(L \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)).$$

Remark 4.4. As in [DV02, 10.1.2], one can prove that the definition above is independent of the choice of $\mathcal{A}$, M and $L_{K(x)}$.

The main result of this section is the following:

Theorem 4.5. *The algebraic group* $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ *is the smallest closed algebraic subgroup of* $Gl(M_{K(x)})$ *that contains the operators* $\Sigma_q^{\kappa_v}$ *modulo* ϕ_v*, for almost all* $v \in \mathcal{C}$.

Remark 4.6. The noetherianity of $Gl(M_{K(x)})$ implies that the smallest closed algebraic subgroup of $Gl(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v \in \mathcal{C}$, is well-defined.

A part of Theorem 4.5 is easy to prove:

Lemma 4.7. *The algebraic group* $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ *contains the operators* $\Sigma_q^{\kappa_v}$ *modulo* ϕ_v *for almost all* $v \in \mathcal{C}_K$.

Proof. The statement follows immediately from the fact that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ can be defined as the stabilizer of a rank one q-difference module in $\langle \mathcal{M}_{K(x)} \rangle^{\otimes}$, which is *a fortiori* stable by the action of $\Sigma_q^{\kappa_v}$. □

Corollary 4.8. $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{1\}$ *if and only if* $\mathcal{M}_{K(x)}$ *is a trivial* q*-difference module.*

Proof. Because of the lemma above, if $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{1\}$ is the trivial group, then $\Sigma_q^{\kappa_v}$ induces the identity on $M \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$. Therefore Theorem 3.1 implies that $\mathcal{M}_{K(x)}$ is trivial. On the other hand, if $\mathcal{M}_{K(x)}$ is trivial, then it is isomorphic to the q-difference module $(K^\nu \otimes_K K(x), 1 \otimes \sigma_q)$. It follows that the generic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is forced to stabilize all the lines generated by vectors of the type $v \otimes 1$, with $v \in K^\nu$. Therefore it is the trivial group. □

Now we are ready to prove Theorem 4.5. The argument follows from [DV02, §10.3], which is itself inspired by [Kat82, §X].

Proof of Theorem 4.5. Lemma 4.7 says that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ contains the smallest subgroup G of $Gl(M_{K(x)})$ that contains the operator $\Sigma_q^{\kappa_v}$ modulo ϕ_v for almost all $v \in \mathcal{C}_K$. Let $L_{K(x)}$ be a line contained in some object of the category $\langle \mathcal{M}_{K(x)} \rangle^{\otimes}$, that defines G as a stabilizer. Then there exists a smaller q-difference module $\mathcal{W}_{K(x)}$ over $K(x)$ that contains $L_{K(x)}$. Let L and $\mathcal{W} = (W, \Sigma_q)$ be the associated $\mathcal{A}$-modules. Any generator m of L as an $\mathcal{A}$-module is a cyclic vector for $\mathcal{W}$ and the operator $\Sigma_q^{\kappa_v}$ acts on $W \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$ with respect to the basis induced by the cyclic basis generated by m via a diagonal matrix. By the very definition of the q-difference structure on the dual module $\mathcal{W}^*$ of $\mathcal{W}$, the group G can be define as the subgroup of $Gl(M_{K(x)})$ that fixes a line L' in $W^* \otimes W$, *i.e.* such that $\Sigma_q^{\kappa_v}$ acts as the identity on $L' \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$, for almost all cyclotomic places v. It follows from Theorem 3.1 that the minimal submodule $\mathcal{W}'$ that contains L' becomes trivial over $K(x)$. Since the tensor category generated by $\mathcal{W}'_{K(x)}$ is contained in $\langle \mathcal{M}_{K(x)} \rangle^{\otimes}$, we have a functorial surjective group morphism

$$Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow Gal(\mathcal{W}'_{K(x)}, \eta_{K(x)}) = \{1\}.$$

We conclude that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ acts trivially over $\mathcal{W}'_{K(x)}$, and therefore that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is contained in G. □

Corollary 4.9. *Theorem 3.1 and Theorem 4.5 are equivalent.*

Proof. We have seen in the proof above that Theorem 3.1 implies Theorem 4.5. Corollary 4.8 gives the opposite implication. □

4.1. **Calculation of generic Galois groups.** The following corollary anticipates a little bit on the §10. Anyway we state it here because it is useful in the calculation of the generic Galois group and gives a sense to Definition 4.3. In fact, in the notation of Theorem 4.5, we know that $\Sigma_q^{\kappa_v} \in\subset Stab_{\mathcal{A}/(\phi_v)}(L \otimes \mathcal{O}_K/(\phi_v))$. We can actually say a little bit more:

Corollary 4.10. *In the notation of Theorem 4.5, let $L_{K(x)}$ be some line in some algebraic construction of $\mathcal{M}_{K(x)}$ such that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) = Stab(L_{K(x)})$. Then for almost all $v \in \mathcal{C}$ we have:*

$$\Sigma_q^{\kappa_v} \in Stab_{\mathcal{A}}(L) \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v),$$

where $Stab_{\mathcal{A}}(L)$ is the stabilizer of the $\mathcal{A}$-lattice L of $L_{K(x)}$ in the group $Gl(\mathcal{M})$ of $\mathcal{A}$-linear automorphisms of $\mathcal{M}$, and we have identified $\Sigma_q^{\kappa_v}$ with its reduction modulo ϕ_v.

Proof. Let $Gal(\mathcal{M}, \eta_{\mathcal{A}})$ be the generic Galois group associated to the forgetful functor on the tannakian category generated by $\mathcal{M} = (M, \Sigma_q)$, inside the category of q-difference modules over $\mathcal{A}$ (*cf.* §10.1 below). Since we can choose $L_{K(x)}$ to be a q-difference module and therefore L to be a q-difference module over $\mathcal{A}$, we have $Gal(\mathcal{M}, \eta_{\mathcal{A}}) \subset Stab_{\mathcal{A}}(L)$. Therefore we obtain

$$\begin{array}{rcl}\Sigma_q^{\kappa_v} & \in & Gal(\mathcal{M} \otimes \mathcal{O}_K/(\phi_v), \eta_{\mathcal{A}/(\phi_v)}) \\ & \subset & Gal(\mathcal{M}, \eta_{\mathcal{A}}) \otimes \mathcal{O}_K/(\phi_v) \\ & \subset & Stab_{\mathcal{A}}(L) \otimes \mathcal{O}_K/(\phi_v).\end{array}$$

□

Remark 4.11. The statement above says that the generic Galois group is the smallest such that when we reduce module Φ_v the generators of its ideal of definition we find a group that contains the curvature of $\mathcal{M}_{K(x)}$ modulo Φ_v, for almost all cyclotomic places v.

4.2. **Finite generic Galois groups.** We deduce from Theorem 4.5 the following description of a finite generic Galois group:

Corollary 4.12. *The following facts are equivalent:*

(1) *There exists a positive integer r such that the q-difference module $\mathcal{M} = (M, \Sigma_q)$ becomes trivial as a $\widetilde{q}$-difference module over $K(\widetilde{q}, t)$, with $\widetilde{q}^r = q$, $t^r = x$.*
(2) *There exists a positive integer r such that for almost all $v \in \mathcal{C}$ the morphism $\Sigma_q^{\kappa_v r}$ induces the identity on $M \otimes_{\mathcal{O}_K} \mathcal{O}_K/(\phi_v)$.*
(3) *There exists a q-difference field extension $\mathcal{F}/K(x)$ of finite degree such that $\mathcal{M}$ becomes trivial over $\mathcal{F}$.*
(4) *The (generic) Galois group of $\mathcal{M}$ is finite.*

In particular, if $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is finite, it is necessarily cyclic (of order r, if one chooses r minimal in the assertions above).

Proof. The equivalence "1 $\Leftrightarrow$ 2" follows from Theorem 3.1 applied to the $\widetilde{q}$-difference module $(M \otimes K(\widetilde{q}, t), \Sigma_q \otimes \sigma_{\widetilde{q}})$, over the field $K(\widetilde{q}, t)$.

The equivalence "2 $\Leftrightarrow$ 4" follows from Corollary 4.10 above. In fact, if the generic Galois group is finite, the reduction modulo ϕ_v of $\Sigma_q^{\kappa_v}$ must be a cyclic operator of order dividing the cardinality of $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$. On the other hand, assertion 2 implies that there exists a basis of $M_{K(x)}$ such that the representation of $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is given by the group of diagonal matrices, whose diagonal entries are r-th roots of unity.

Of course, assertion 1 implies assertion 3. The inverse implication follows from the proposition below, applied to a cyclic vector of $\mathcal{M}_{K(x)}$. □

Lemma 4.13. *Let K be a field and q an element of K which is not a root of unity. We suppose that there exists a norm $|\ |$ over K such that $|q| \neq 1$[10] and we consider a linear q-difference equations*

$$a_\nu(x)y(q^\nu x) + a_{\nu-1}(x)y(q^{\nu-1}x) + \cdots + a_0(x)y(x) = 0 \tag{4.1}$$

with coefficients in $K(x)$. If there exists an algebraic q-difference extension $\mathcal{F}$ of $K(x)$ containing a solution f of (4.1)*, then f is contained in an extension of $K(x)$ isomorphic to $K(\widetilde{q}, t)$, with $\widetilde{q}^r = 1$ and $t^r = x$.*

Proof. Let us look at (4.1) as an equation with coefficients in $K((x))$. Then the algebraic solution f of (4.1) can be identified to a Laurent series in $\overline{K}((t))$, where $\overline{K}$ is the algebraic closure of K and $t^r = x$, for a convenient positive integer r. Let $\widetilde{q}$ be an element of $\overline{K}$ such that $\widetilde{q}^r = q$ and that $\sigma_q(f) = f(\widetilde{q}t)$. We can look at (4.1) as a $\widetilde{q}$-difference equation with coefficients in $K(\widetilde{q}, t)$. Then the recurrence relation induced by (4.1) over the coefficients of a formal solution shows that there exist $f_1, \dots, f_s$ solutions of (4.1) in $K(\widetilde{q})((t))$ such that $f \in \sum_i \overline{K} f_i$. It follows that there exists a finite extension $\widetilde{K}$ of $K(\widetilde{q})$ such that $f \in \widetilde{K}((t))$.

We fix an extension of $|\ |$ to $\widetilde{K}$, that we still call $|\ |$. Since f is algebraic, it is a germ of meromorphic function at 0. Since $|\widetilde{q}| \neq 1$, the functional equation (4.1) itself allows to show that f is actually a meromorphic function with infinite radius of meromorphy. Finally, f can have at worst a pole at $t = \infty$, since it is an algebraic function, which actually implies that f is the Laurent expansion of a rational function in $K(\widetilde{q}, t)$. □

[10] This assumption is always verified if K is a finite extension of a field of rational functions $k(q)$, as in this paper, or if there exists an immersion of K in $\mathbb{C}$.

4.3. **Generic Galois groups defined over K.** For further reference (in §7), we point out that:

Proposition 4.14. *For a q-difference module $\mathcal{M}=(M,\Sigma_q)$ over $K(x)$, the following facts are equivalent:*

(1) *There exists a basis $\underline{e}$ over $K(x)$ such that $\Sigma_q\underline{e}=\underline{e}A$ for some $A\in Gl_\nu(K)$.*
(2) *$\mathcal{M}\cong(V\otimes_K K(x),\varphi\otimes\sigma_q)$, where V is a K-vector space and $\varphi\in Gl(V)$.*

If the conditions above are satisfied then the generic Galois group of $\mathcal{M}$ over $K(x)$ is defined over K.

Proof. The equivalence between 1. and 2. is straightforward. We assume 2. The Galois group $Gal(\mathcal{M},\eta_{K(x)})$ is defined as the stabilizer of a line $L_{K(x)}$ in a convenient construction $\mathcal{W}$ of $\mathcal{M}$. We can assume that the line $L_{K(x)}$ is a q-difference module, therefore stable by a morphism of the form $\widetilde{\varphi}\otimes\sigma_q$, where $\widetilde{\varphi}$ is the morphism induced by φ on the corresponding construction of V. Therefore $L_{K(x)}$ is defined over K and so does $Gal(\mathcal{M},\eta_{K(x)})$. □

4.4. **Devissage of nonreduced generic Galois groups.** Independently of the characteristic of the base field, there is no proper Galois correspondence for generic Galois groups. If $\mathcal{N}=(N,\Sigma_q)$ is an object of $\langle\mathcal{M}_{K(x)}\rangle^{\otimes}$, then there exists a normal subgroup H of $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ such that H acts as the identity on $N_{K(x)}$ and

$$Gal(\mathcal{N}_{K(x)},\eta_{K(x)})\cong\frac{Gal(\mathcal{M}_{K(x)},\eta_{K(x)})}{H}. \tag{4.2}$$

In fact, the category $\langle\mathcal{N}_{K(x)}\rangle^{\otimes}$ is a full subcategory of $\langle\mathcal{M}_{K(x)}\rangle^{\otimes}$ and therefore there exists a surjective functorial morphism

$$Gal(\mathcal{M}_{K(x)},\eta_{K(x)})\longrightarrow Gal(\mathcal{N}_{K(x)},\eta_{K(x)}).$$

The kernel of such morphism is the normal subgroup of $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ that acts as the identity of $\mathcal{N}_{K(x)}$. On the other hand, if H is a normal subgroup of $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$, it is not always possible to find an object $\mathcal{N}_{K(x)}=(N_{K(x)},\Sigma_q)$ of $\langle\mathcal{M}_{K(x)}\rangle^{\otimes}$ such that we have (4.2). This happens because the generic Galois group $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ stabilizes all the sub-q-difference modules of the constructions on $\mathcal{M}_{K(x)}$ but also other submodules, which are not stable by Σ_q. So, if $H=Stab(L_{K(x)})$, for some line $L_{K(x)}$ in some algebraic construction of $\mathcal{M}_{K(x)}$, the orbit of $L_{K(x)}$ with respect to $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ could be a q-difference module, allowing to establish (4.2), but in general it won't be.

In spite of the fact that in this setting we do not have a Galois correspondence, we can establish some devissage of $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$, when it is not reduced. So let us suppose that the group $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ is nonreduced, and therefore that the characteristic of k is $p>0$. Then there exists a maximal reduced subgroup $Gal_{red}(\mathcal{M}_{K(x)},\eta_{K(x)})$ of $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ and a short exact sequence of groups:

$$1\longrightarrow Gal_{red}(\mathcal{M}_{K(x)},\eta_{K(x)})\longrightarrow Gal(\mathcal{M}_{K(x)},\eta_{K(x)})\longrightarrow\mu_{p^\ell}\longrightarrow 1, \tag{4.3}$$

for some positive integer ℓ, uniquely determined by the above short exact sequence. We remind that the subgroup $Gal_{red}(\mathcal{M}_{K(x)},\eta_{K(x)})$ of $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ is normal.

Theorem 4.15. *The subgroup $Gal_{red}(\mathcal{M}_{K(x)},\eta_{K(x)})$ of $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ is the smallest algebraic subgroup of $Gl(M_{K(x)})$ whose reduction modulo ϕ_v contains the operators $\Sigma_q^{\kappa_v p^\ell}$ for almost all $v\in\mathcal{C}_K$.*

We first prove two lemmas.

Lemma 4.16. *The group* $Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ *is contained in the smallest algebraic subgroup of* $Gl(M_{K(x)})$ *whose reduction modulo* ϕ_v *contains the operators* $\Sigma_q^{\kappa_v p^\ell}$ *for almost all* $v \in \mathcal{C}_K$.

Proof. Let H be the smallest algebraic subgroup of $Gl(M_{K(x)})$ whose reduction modulo ϕ_v contains the operators $\Sigma_q^{\kappa_v p^\ell}$ for almost all $v \in \mathcal{C}_K$. We know that $H = Stab(L_{K(x)})$ for some line $L_{K(x)}$ contained in some object of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes}$. Once again, as in the proof of Theorem 4.5, we can find another line $L'_{K(x)}$, that defines H as a stabilizer and which is actually fixed by H. It follows that L' generates a q-difference module $\mathcal{W}'$ over $\mathcal{A}$, that satisfies the hypothesis of Corollary 4.12. We conclude that there exists a nonnegative integer $\ell' \leq \ell$ such that H is contained in the kernel of the surjective map:

$$Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow Gal(\mathcal{W}'_{K(x)}, \eta_{K(x)}) = \mu_{p^{\ell'}}, \tag{4.4}$$

and therefore that $Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \subset H$. □

Lemma 4.17. *Let* $q^{(\ell)} = q^{p^\ell}$. *We consider the* $q^{(\ell)}$*-difference module* $\mathcal{M}^{(\ell)}_{K(x)}$ *obtained from* $\mathcal{M}_{K(x)}$ *iterating* Σ_q, i.e. $\mathcal{M}^{(\ell)}_{K(x)} = (M_{K(x)}, \Sigma_{q^{(\ell)}})$, *with* $\Sigma_{q^{(\ell)}} = \Sigma_q^{p^\ell}$. *Then* $Gal(\mathcal{M}^{(\ell)}_{K(x)}, \eta_{K(x)})$ *is the smallest algebraic subgroup of* $Gl(M_{K(x)})$ *whose reduction modulo* ϕ_v *contains the operators* $\Sigma_q^{\kappa_v p^\ell}$ *for almost all* $v \in \mathcal{C}_K$.

Proof. Since the characteristic of k is $p > 0$, the order κ_v of q_v in the residue field k_v is a divisor of $p^n - 1$ for some positive integer n. It follows that the order of $q^{(\ell)}$ modulo v is equal to κ_v for almost all $v \in \mathcal{C}_K$. Theorem 4.5 allows to conclude, since $\Sigma_{q^{(\ell)}}^{\kappa_v} = \Sigma_q^{\kappa_v p^\ell}$. □

Proof of Theorem 4.15. We will prove the statement by induction on $\ell \geq 0$, in the short exact sequence (4.3). The statement is trivial for $\ell = 0$, since in this case $Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$. Let us suppose that $\ell > 0$ and that the statement is proved for any $\ell' < \ell$. In the notation of the lemmas above, we have:

$$Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \subset H.$$

We suppose that the inclusion is strict, *i.e.* that $\ell' > 0$ in (4.4), otherwise there would be nothing to prove.

We claim that H is the smallest subgroup that contains $\Sigma_q^{\kappa_v p^{\ell'}}$ modulo ϕ_v for almost all v and therefore that $H = Gal(\mathcal{M}^{(l')}_{K(x)}, \eta_{K(x)})$, because of Lemma 4.17. In fact the smallest subgroup that contains $\Sigma_q^{\kappa_v p^{\ell'}}$ modulo ϕ_v for almost all v is contained in H by definition, while morphism (4.4) proves that $\Sigma_q^{\kappa_v p^{\ell'}}$ stabilizes the line $L_{K(x)}$, considered in Lemma 4.16, modulo ϕ_v. Then Lemma 4.17 implies that $H = Gal(\mathcal{M}^{(\ell')}_{K(x)}, \eta_{K(x)})$.

Since $Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = Gal_{red}(\mathcal{M}^{(\ell')}_{K(x)}, \eta_{K(x)}) \subset H$, we have a short exact sequence:

$$1 \longrightarrow Gal_{red}(\mathcal{M}^{(\ell')}_{K(x)}, \eta_{K(x)}) \longrightarrow Gal(\mathcal{M}^{(\ell')}_{K(x)}, \eta_{K(x)}) \longrightarrow \mu_{p^{\ell-\ell'}} \longrightarrow 1.$$

The inductive hypotheses implies that $Gal_{red}(\mathcal{M}^{(\ell')}_{K(x)}, \eta_{K(x)})$ is the smallest subgroup of $Gl(M_{K(x)})$ containing the operators $\Sigma_{q^{(\ell)}}^{\kappa_v p^{\ell-\ell'}} = \Sigma_q^{\kappa_v p^\ell}$. This ends the proof. □

We obtain the following corollary:

Corollary 4.18. *In the notation of the theorem above:*

- $Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = Gal(\mathcal{M}^{(\ell)}_{K(x)}, \eta_{K(x)})$.
- *Let* $\widetilde{K}$ *be a finite extension of* K *containing a* p^ℓ*-th root* q^{1/p^ℓ} *of* q*. Then the generic Galois group* $Gal(\mathcal{M}_{\widetilde{K}(x^{1/p^\ell})}, \eta_{\widetilde{K}(x^{1/p^\ell})})$ *of the* q^{1/p^ℓ}*-difference module* $\mathcal{M}_{\widetilde{K}(x^{1/p^\ell})}$ *is reduced and*

$$Gal(\mathcal{M}_{\widetilde{K}(x^{1/p^\ell})}, \eta_{\widetilde{K}(x^{1/p^\ell})}) \subset Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \widetilde{K}(x^{1/p^\ell}).$$

Proof. The first statement is a rewriting of Lemma 4.17. We have to prove the second statement. If $\underline{e}$ is a basis of $M_{K(x)}$ such that $\Sigma_q \underline{e} = \underline{e}A(x)$, then in $\mathcal{M}_{K(x^{1/p^\ell})} = (M_{K(x^{1/p^\ell})}, \Sigma_{q^{1/p^\ell}} := \Sigma_q \otimes \sigma_{q^{1/p^\ell}})$ we have:

$$\Sigma_{q^{1/p^\ell}}(\underline{e} \otimes 1) = (\underline{e} \otimes 1)A(x).$$

It follows that the generic Galois group $Gal(\mathcal{M}_{K(x^{1/p^\ell})}, \eta_{K(x^{1/p^\ell})})$ is the smallest algebraic subgroup of $Gl(M_{K(x^{1/p^\ell})})$ that contains the operators $\Sigma^{\kappa p^\ell}_{q^{1/p^\ell}} = \Sigma_q^{\kappa_v p^\ell} \otimes 1$. This proves that

$$Gal(\mathcal{M}_{\widetilde{K}(x^{1/p^\ell})}, \eta_{\widetilde{K}(x^{1/p^\ell})}) \subset Gal_{red}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \widetilde{K}(x^{1/p^\ell}).$$

□

5. Differential generic Galois groups of q-difference equations

In this section and whenever we consider algebraic differential groups, we will assume that the characteristic of k *is* 0*. So, as before,* $k(q)$ *is the field of rational functions with coefficients in a fixed field* k *of zero characteristic and* K *is a finite extension of* $k(q)$*.*

5.1. **Differential generic Galois group.**

Let $\mathcal{F}$ be a q-difference-differential field of zero characteristic, that is, an extension of $K(x)$ equipped with an extension of the q-difference operator σ_q and a derivation ∂ commuting with σ_q (*cf.* [Har08, §1.2]). For instance, later on we will consider the q-difference-differential field $(K(x), \sigma_q, \partial := x\frac{d}{dx})$.

We denote by $Diff(\mathcal{F}, \sigma_q)$ the tannakian category of q-difference modules over $\mathcal{F}$ (*cf.* §1.2, [SR72, III.3.2]) and define an action of the derivation ∂ on the category $Diff(\mathcal{F}, \sigma_q)$, twisting the q-difference modules with the $\mathcal{F}$-vector space $\mathcal{F}[\partial]_{\leq 1}$ of differential operators of order less or equal than one. We recall that the structure of right $\mathcal{F}$-module on $\mathcal{F}[\partial]_{\leq 1}$ is defined via the Leibniz rule, *i.e.* $\partial\lambda = \lambda\partial + \partial(\lambda)$. Let V be an $\mathcal{F}$-vector space. We denote by $V^{(1)}$ the tensor product of the right $\mathcal{F}$-module $\mathcal{F}[\partial]_{\leq 1}$ by the left $\mathcal{F}$-module V. We will write simply v for $1 \otimes v \in V^{(1)}$ and $\partial(v)$ for $\partial \otimes v \in V^{(1)}$, so that $av + b\partial(v) := (a + b\partial) \otimes v$, for any $v \in V$ and $a + b\partial \in \mathcal{F}[\partial]_{\leq 1}$. Notice that, similarly to the constructions of [GM93, Prop.16] for $\mathcal{D}$-modules, we have endowed $V^{(1)}$ with a left $\mathcal{F}$-module structure such that if $\lambda \in \mathcal{F}$:

$$\lambda\partial(v) = \partial(\lambda v) - \partial(\lambda)v, \text{ for all } v \in V.$$

In other words, this construction comes out of the Leibniz rule $\partial(\lambda v) = \lambda\partial(v) + \partial(\lambda)v$, which justifies the notation introduced above.

Definition 5.1. The prolongation functor F from the category $Diff(\mathcal{F}, \sigma_q)$ to itself is defined as follow:

(1) If $\mathcal{M}_{\mathcal{F}} := (M_{\mathcal{F}}, \Sigma_q)$ is an object of $Diff(\mathcal{F}, \sigma_q)$ then $F(\mathcal{M}_{\mathcal{F}})$, is the q-difference module, whose underlying $\mathcal{F}$-vector space is $M^{(1)}_{\mathcal{F}} = \mathcal{F}[\partial]_{\leq 1} \otimes M_{\mathcal{F}}$, equipped with the q-invertible σ_q-semilinear operator $\Sigma_q(\partial^k(m)) := \partial^k(\Sigma_q(m))$ for $0 \leq k \leq 1$.

(2) If $f \in Hom(M_{\mathcal{F}}, N)$ then we define

$$F(f) : M_{\mathcal{F}}^{(1)} \to N^{(1)}, f(\partial^k(m)) := \partial^k(f(m)) \text{ for } 0 \leq k \leq 1.$$

Remark 5.2. This formal definition comes from a simple and concrete idea. Let $\mathcal{M}_{\mathcal{F}}$ be an object of $Diff(\mathcal{F}, \sigma_q)$. We fix a basis $\underline{e}$ of $\mathcal{M}_{\mathcal{F}}$ over $\mathcal{F}$ such that $\Sigma_q \underline{e} = \underline{e} A$. Then $(\underline{e}, \partial(\underline{e}))$ is a basis of $M_{\mathcal{F}}^{(1)}$ and

$$\Sigma_q(\underline{e}, \partial(\underline{e})) = (\underline{e}, \partial(\underline{e})) \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix}.$$

In other terms, if $\sigma_q(Y) = A^{-1}Y$ is a q-difference system associated to $\mathcal{M}_{\mathcal{F}}$ with respect to a fixed basis, the object $M_{\mathcal{F}}^{(1)}$ is attached to the q-difference system

$$\sigma_q(Z) = \begin{pmatrix} A^{-1} & \partial(A^{-1}) \\ 0 & A^{-1} \end{pmatrix} Z = \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix}^{-1} Z.$$

If Y is a solution of $\sigma_q(Y) = A^{-1}Y$ in some q-difference-differential extension of $\mathcal{F}$ then we have:

$$\sigma_q \begin{pmatrix} \partial Y \\ Y \end{pmatrix} = \begin{pmatrix} A^{-1} & \partial(A^{-1}) \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} \partial Y \\ Y \end{pmatrix},$$

in fact the commutation of σ_q and ∂ implies:

$$\sigma_q(\partial Y) = \partial(\sigma_q Y) = \partial(A^{-1}\, Y) = A^{-1}\, \partial Y + \partial(A^{-1})\, Y.$$

The definition of the prolongation functor F is actually independent of the q-difference structure, in fact we have defined it on the category $Vect_{\mathcal{F}}$ of $\mathcal{F}$-vector spaces, in the first place. We will call this functor F or sometimes $F_{Vect_{\mathcal{F}}}$. Let V be a finite dimensional $\mathcal{F}$-vector space. We denote by $Constr_{\mathcal{F}}^{\partial}(V)$ the set of finite dimensional $\mathcal{F}$-vector spaces obtained by applying the constructions of linear algebra (*i.e.* direct sums, tensor product, symmetric and antisymmetric product, dual) and the functor F. We will say that an element $Constr_{\mathcal{F}}^{\partial}(V)$ is a construction of differential linear algebra of V. By functoriality, the linear algebraic group $Gl(V)$ operates on $Constr_{\mathcal{F}}^{\partial}(V)$. For example $g \in Gl(V)$ acts on $V^{(1)} := F(V)$ through $g(\partial^s(v)) = \partial^s(g(v)), 0 \leq s \leq 1$. As already noticed in the previous section, if we start with a q-difference module $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$ over $\mathcal{F}$, then every object of $Constr_{\mathcal{F}}^{\partial}(M_{\mathcal{F}})$ has a natural structure of q-difference module. We will denote $Constr_{\mathcal{F}}^{\partial}(\mathcal{M}_{\mathcal{F}})$ the family of q-difference modules obtained in this way.

Definition 5.3. We call *differential generic Galois group* of an object $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$ of $Diff(\mathcal{F}, \sigma_q)$ the group defined by

$$Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}}) := \Big\{ g \in Gl(M_{\mathcal{F}}) : g(N) \subset N \text{ for all sub-}q\text{-difference module } (N, \Sigma_q) \text{ contained in an object of } Constr_{\mathcal{F}}^{\partial}(\mathcal{M}_{\mathcal{F}}) \Big\} \subset Gl(M_{\mathcal{F}}).$$

For further reference, we recall (a particular case of) the Ritt-Raudenbush theorem (*cf.* [Kap57, Thm.7.1]):

Theorem 5.4. *Let $(\mathcal{F}, \partial)$ be a differential field of zero characteristic. If R is a finitely generated $\mathcal{F}$-algebra equipped with a derivation ∂, extending the derivation ∂ of $\mathcal{F}$, then R is ∂-noetherian.*

This means that any ascending chain of radical differential ideals (*i.e.* radical ∂-stable ideals) is stationary or equivalently that every radical ∂-ideal is ∂-finitely generated (which in general does not mean that it is a finitely generated ideal).

Theorem 5.4 asserts that $Gl_n(\mathcal{F})$ is a ∂-noetherian differential variety in the sense that its algebra of differential rational functions[11] over $Gl_n(\mathcal{F})$ is ∂-noetherian.

Proposition 5.5. *The group $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is a linear algebraic differential $\mathcal{F}$-subgroup of $Gl(M_{\mathcal{F}})$.*

Proof. Let $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$ be an object of $Diff(\mathcal{F}, \sigma_q)$. Following [Ovc08, Section 2], we look at linear differential algebraic groups defined over $\mathcal{F}$ as representable functors from the category of ∂-$\mathcal{F}$-differential algebras, *i.e.* commutative associative $\mathcal{F}$-algebras A with unit, equipped with a derivation $\partial : A \rightarrow A$ extending the one of $\mathcal{F}$, to the category of groups. Now, the functor $\underline{Stab}$ that associates to a ∂-$\mathcal{F}$-differential algebra A, the stabilizer, inside $Gl(M_{\mathcal{F}})(A)$, of $N_{\mathcal{F}} \otimes A$ for all sub-q-difference module $\mathcal{N}_{\mathcal{F}} = (N_{\mathcal{F}}, \Sigma_q)$ contained in an object of $Constr^{\partial}_{\mathcal{F}}(\mathcal{M}_{\mathcal{F}})$, is representable by a linear differential algebraic group. It is the differential analogous of [DG70, II.1.36].

□

Following [Ovc09a, Def. page 3057], we denote by $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ the full abelian tensor subcategory of $Diff(\mathcal{F}, \sigma_q)$ generated by $\mathcal{M}_{\mathcal{F}}$ and closed under the prolongation functor. A noetherianity argument already used in Remark 4.2 proves the following:

Corollary 5.6. *The differential generic Galois group $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ can be defined as the differential stabilizer of a line in a construction of differential algebra of $\mathcal{M}_{\mathcal{F}}$, which is also a q-difference module in the category $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$.*

Proof. Let us consider a descending chain of differential algebraic subgroups $\mathcal{G}_h = Stab(\mathcal{W}^{(i)}; i \in I_h)$, *i.e.* such that $\left\{\mathcal{W}^{(i)}; i \in I_h\right\}$ are an ascending chain of finite set of q-difference submodules contained in some elements of $Constr^{\partial}(\mathcal{M}_{K(x)})$. Then the ascending chain of the radical differential ideals of the differential rational functions that annihilates $\mathcal{G}_h$ is stationary and so does the chain of differential groups $\mathcal{G}_h$. This proves that $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is the stabilizer of a finite number of q-difference submodules $\mathcal{W}^{(i)}$, $i \in I$, contained in some elements of $Constr^{\partial}(\mathcal{M}_{K(x)})$. We conclude using a standard argument (*cf.* Remark 4.2). □

We have the following inclusion, that we will characterize in a more precise way in the next pages:

Lemma 5.7. *Let $\mathcal{M}_{\mathcal{F}}$ be an object of $Diff(\mathcal{F}, \sigma_q)$. The following inclusion of algebraic differential groups holds*

$$Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}}) \subset Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}}).$$

Remark 5.8. We would like to put the accent on the fact that differential algebraic groups are not algebraic groups, while algebraic groups are differential algebraic groups (whose "equations" do not contain "derivatives"). In particular, the differential generic Galois group is not an algebraic subgroup of the generic Galois group but only an algebraic differential subgroup.

---

[11] We denote by $\mathcal{F}\{Y\}_{\partial}$ the ring of differential polynomials in the ∂-differential indeterminates $Y = \{y_{i,j} : i, j = 1, \dots, \nu\}$. This means that $\mathcal{F}\{Y\}_{\partial}$ is isomorphic as a differential $\mathcal{F}$-algebra to a polynomial ring in infinite indeterminates $\mathcal{F}[\widetilde{y}^k_{i,j}; i, j = 1, \dots, \nu, k \geq 0]$, equipped with a derivation ∂ extending the derivation of $\mathcal{F}$ and such that $\partial \widetilde{y}^k_{i,j} = \widetilde{y}^{k+1}_{i,j}$, via the map

$$\partial^k y_{i,j} \longmapsto \widetilde{y}^k_{i,j}.$$

The differential Hopf-algebra $\mathcal{F}\left\{Y, \frac{1}{\det Y}\right\}_{\partial}$ of $Gl_{\nu}(\mathcal{F})$ is obtained from $\mathcal{F}\{Y\}_{\partial}$ by inverting $\det Y$.

Proof. We recall, that the algebraic group $Gal(\mathcal{M}_{\mathcal{F}},\eta_{\mathcal{F}})$ is defined as the stabilizer in $Gl(M_{\mathcal{F}})$ of all the subobjects contained in a construction of linear algebra of $\mathcal{M}$. Because the list of subobjects contained in a construction of differential linear algebra of $\mathcal{M}$ includes those contained in a construction of linear algebra of $\mathcal{M}_{\mathcal{F}}$, we get the claimed inclusion. □

5.2. **Arithmetic characterization of the differential generic Galois group.** We go back to the special case $\mathcal{F}=K(x)$ where K is a finite field extension of $k(q)$ and keep the notations of §4. We endow $K(x)$ with a structure of q-difference-differential field by setting $\partial := x\frac{d}{dx}$ and $\sigma_q(x) := qx$. In this section, we are going to deduce a characterization of $Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})$ from Theorem 3.1.

Let $\mathcal{M}=(M,\Sigma_q)$ be a q-difference module of rank ν over $\mathcal{A}$ (*cf.* (1.1)) and $\mathcal{M}_{K(x)}$ be the q-difference module obtained by scalar extension to $K(x)$. Notice that the $\mathcal{O}_K$-algebra $\mathcal{A}$ is stable under the action of the derivation ∂. The differential version of Chevalley's theorem (*cf.* [Cas72, Prop.14], [MO10, Thm.5.1]) implies that any closed differential algebraic subgroup G of $Gl(M_{K(x)})$ can be defined as the stabilizer of some line $L_{K(x)}$ contained in an object $\mathcal{W}_{K(x)}$ of $\langle\mathcal{M}_{K(x)}\rangle^{\otimes,\partial}$. Because the derivation does not modify the set of poles of a rational function, the lattice $\mathcal{M}$ of $\mathcal{M}_{K(x)}$ determines a Σ_q-stable $\mathcal{A}$-lattice of all the objects of $\langle\mathcal{M}_{K(x)}\rangle^{\otimes,\partial}$. In particular, the $\mathcal{A}$-lattice M of $M_{K(x)}$ determines an $\mathcal{A}$-lattice L of $L_{K(x)}$ and an $\mathcal{A}$-lattice W of $W_{K(x)}$. The latter is the underlying space of a q-difference module $\mathcal{W}=(W,\Sigma_q)$ over $\mathcal{A}$.

Definition 5.9. Let $\widetilde{\mathcal{C}}$ be a cofinite subset of $\mathcal{C}_K$ and $(\Lambda_v)_{v\in\widetilde{\mathcal{C}}}$ be a family of $\mathcal{A}/(\phi_v)$-linear operators acting on $M\otimes_{\mathcal{A}}\mathcal{A}/(\phi_v)$. We say that *the differential group G contains the operators Λ_v modulo ϕ_v for almost all $v\in\mathcal{C}_K$* if for almost all $v\in\widetilde{\mathcal{C}}$ the operator Λ_v stabilizes $L\otimes_{\mathcal{A}}\mathcal{A}/(\phi_v)$ inside $W\otimes_{\mathcal{A}}\mathcal{A}/(\phi_v)$:

$$\Lambda_v\in Stab_{\mathcal{A}/(\phi_v)}(L\otimes_{\mathcal{A}}\mathcal{O}_K/(\phi_v)).$$

Remark 5.10. The differential Chevalley's theorem and the ∂-noetherianity of $Gl(M_{K(x)})$ imply that the notion of a differential algebraic group containing the operators Λ_v modulo ϕ_v for almost all $v\in\mathcal{C}_K$ and the smallest closed differential algebraic subgroup of $Gl(M_{K(x)})$ containing the operators Λ_v modulo ϕ_v for almost all $v\in\mathcal{C}_K$ are well defined. In particular they are independent of the choice of $\mathcal{A}$, $\mathcal{M}$ and $L_{K(x)}$.

The main result of this section is the following:

Theorem 5.11. *The differential algebraic group $Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})$ is the smallest closed differential algebraic subgroup of $Gl(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v\in\mathcal{C}$.*

The two statements below are preliminary to the proof of Theorem 5.11.

Lemma 5.12. *The differential algebraic group $Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})$ contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v for almost all $v\in\mathcal{C}_K$.*

Proof. The statement follows immediately from the fact that $Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})$ can be defined as the stabilizer of one rank one q-difference module in $\langle\mathcal{M}_{K(x)}\rangle^{\otimes,\partial}$, which is *a fortiori* stable under the action of $\Sigma_q^{\kappa_v}$. □

Lemma 5.13. $Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})=\{1\}$ *if and only if $\mathcal{M}_{K(x)}$ is a trivial q-difference module.*

Proof. The proof is analogous to the proof of Corollary 4.8. Just replace $\langle\mathcal{M}_{K(x)}\rangle^{\otimes}$ with $\langle\mathcal{M}_{K(x)}\rangle^{\otimes,\partial}$. □

The lemmas above plus the differential Chevalley theorem allow to prove Theorem 5.11 in exactly the same way as Theorem 4.5. We obtain the following:

Corollary 5.14. *The differential generic Galois group* $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ *is a Zariski dense subset of the algebraic generic Galois group* $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$.

Proof. We have seen in Lemma 5.7 that $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is a subgroup of $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$. By Theorem 4.5 (resp. Theorem 5.11) we have that the generic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ (resp. $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$) is the smallest closed algebraic (resp. differential) subgroup of $Gl(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v \in \mathcal{C}$. This observation concludes the proof. □

The following corollary is a differential analogue of Corollary 4.10 and can be deduced from Theorem 5.11 in the same way as Corollary 4.10 is deduced from Theorem 4.5:

Corollary 5.15. *In the notation of Theorem 5.11, let* $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = Stab(L_{K(x)})$ *for some line* $L_{K(x)}$ *in some differential construction. Then for almost all* $v \in \mathcal{C}$ *we have:*

$$\Sigma_q^{\kappa_v} \in Stab_{\mathcal{A}}(L) \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v),$$

where $Stab_{\mathcal{A}}(L)$ *is the differential stabilizer of the* $\mathcal{A}$*-lattice* L *of* $L_{K(x)}$ *in the group* $Gl(\mathcal{M})$ *of* $\mathcal{A}$*-linear automorphisms of* $\mathcal{M}$*, and we have identified* $\Sigma_q^{\kappa_v}$ *with its reduction modulo* ϕ_v*.*

In the last part of the paper we will prove some comparison results between the differential generic Galois group and the differential Galois group introduced in [HS08]. Supposing that K comes equipped with a norm such that $|q| \neq 1$ and replacing K by a finitely generated extension, we will prove (*cf.* Corollary 9.9) that the differential dimension of $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ over $K(x)$ is equal to the differential transcendence degree of the extension generated by a meromorphic fundamental solution matrix of a system associated to $\mathcal{M}_{K(x)}$ over the field $\widetilde{C}_E(x)$ of rational functions with coefficients in the differential closure of the field C_E of elliptic functions over $\mathbb{C}^*/q^{\mathbb{Z}}$. In particular, if the differential generic Galois group is conjugated to a constant subgroup of $Gl(\mathcal{M}_{K(x)})$ with respect to ∂, then there exists a connection acting on $\mathcal{M}_{K(x)}$, compatible with the q-difference structure (*cf.* [HS08, Prop.2.9]). If the algebraic generic group is simple we are either in the previous situation or the differential generic Galois group is equal to $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$, which means that there are no differential relations, except perhaps some algebraic ones, among the solutions of the q-difference system.

Finally let us notice that the result by Ramis (*cf.* [Ram92]), which states that a formal power series that is simultaneously solution of a q-difference and a differential equation, both with complex polynomial coefficients, is actually a rational function, does not implies that a $K(x)$-vector space equipped with both a connection and a q-difference structure is trivial (*cf.* next example).

Example 5.16. The logarithm verifies both a q-difference and a differential system:

$$Y(qx) = \begin{pmatrix} 1 & \log q \\ 0 & 1 \end{pmatrix} Y(x), \qquad \partial Y(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y(x).$$

It is easy to verify that the two systems are integrable in the sense that $\partial \sigma_q Y(x) = \sigma_q \partial Y(x)$ (and the induced condition on the matrices of the systems is verified). Nonetheless, the q-difference module and the differential module associated with the systems above are nontrivial. Moreover, Proposition 4.14 implies that the differential generic Galois group of the q-difference module associated with the system $Y(qx) = \begin{pmatrix} 1 & \log q \\ 0 & 1 \end{pmatrix} Y(x)$ is defined over the constants, compatibly with the results recalled above.

5.3. **The example of the Jacobi Theta function.** Consider the Jacobi Theta function

$$\Theta(x) = \sum_{n\in\mathbb{Z}} q^{-n(n-1)/2} x^n,$$

which is solution of the q-difference equation

$$\Theta(qx) = qx\Theta(x).$$

Iterating the equation, one proves that Θ satisfies $y(q^n x) = q^{n(n+1)/2} x^n y(x)$, therefore we immediately deduce that the generic Galois group of the rank one q-difference module $\mathcal{M}_\Theta = (K(x).\Theta, \Sigma_q)$, with

$$\begin{array}{rccc} \Sigma_q & : K(x).\Theta & \longrightarrow & K(x).\Theta \\ & f(x)\Theta & \longmapsto & f(qx)qx\Theta \end{array},$$

is the whole multiplicative group $\mathbb{G}_{m,K(x)}$. As far as the differential Galois group is concerned we have:

Proposition 5.17. *The differential generic Galois group* $Gal^\partial\left(\mathcal{M}_\Theta, \eta_{K(x)}\right)$ *is defined by* $\partial(\partial(y)/y) = 0$.

Proof. For almost any $v \in \mathcal{C}$, the reduction modulo ϕ_v of $q^{\kappa_v(\kappa_v+1)/2} x^{\kappa_v}$ is the monomial x^{κ_v}, which satisfies the equation $\partial\left(\frac{\partial x^{\kappa_v}}{x^{\kappa_v}}\right) = 0$. This means that differential generic Galois group $Gal^\partial\left(\mathcal{M}_\Theta, \eta_{K(x)}\right)$ is a subgroup of the differential group defined by $\partial\left(\frac{\partial y}{y}\right) = 0$. In other words, the logarithmic derivative

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_a \\ y & \longmapsto & \frac{\partial y}{y} \end{array}$$

sends $Gal^\partial\left(\mathcal{M}_\Theta, \eta_{K(x)}\right)$ to a subgroup of the additive group $\mathbb{G}_a(K)$ defined over the constants K. Since $Gal^\partial\left(\mathcal{M}_\Theta, \eta_{K(x)}\right)$ is not finite, it must be the whole group $\mathbb{G}_a(K)$. □

Let us consider a norm $|\ |$ on K such that $|q| \neq 1$. The differential dimension of the subgroup $\partial\left(\frac{\partial y}{y}\right) = 0$ is zero. We will show in §9 (*cf.* Corollary 9.12) that this means that Θ is differentially algebraic over the field of rational functions $\widetilde{C}_E(x)$ with coefficients in the differential closure $\widetilde{C}_E$ of the elliptic function over $K^*/q^{\mathbb{Z}}$. In fact, the function Θ satisfies

$$\sigma_q\left(\frac{\partial\Theta}{\Theta}\right) = \frac{\partial\Theta}{\Theta} + 1,$$

which implies that $\partial\left(\frac{\partial\Theta}{\Theta}\right)$ is an elliptic function. Since the Weierstrass function is differentially algebraic over $K(x)$, the Jacobi Theta function is also differentially algebraic over $K(x)$.

Part III. Complex q-difference modules, with $q \neq 0, 1$

6. Grothendieck conjecture for q-difference modules in characteristic zero

Let K be a a finitely generated extension of $\mathbb{Q}$ and $q \in K \smallsetminus \{0, 1\}$. The previous results, combined with an improved version of [DV02], give a "curvature" characterization of the generic (differential) Galois group of a q-difference module over $K(x)$. We will constantly distinguish three cases:

- q is a root of unity;
- q is transcendental over $\mathbb{Q}$;
- q is algebraic over $\mathbb{Q}$, but is not a root of unity.

6.1. **Curvature criteria for triviality.** If q it a primitive root of unity of order κ, it is not difficult to prove that:

Proposition 6.1 ([DV02, Prop. 2.1.2])**.** *A q-difference module $\mathcal{M}_{K(x)}$ over $K(x)$ is trivial if and only if Σ_q^κ is the identity.*

If q is transcendental over $\mathbb{Q}$, we can always find an intermediate field k of $K/\mathbb{Q}$ such that K is a finite extension of $k(q)$. We are in the situation of Theorem 3.1, that we can rephrase as follows:

Theorem 6.2. *A q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$ is trivial if and only if there exists a k-algebra $\mathcal{A}$ (as in (1.1)) and a Σ_q-stable $\mathcal{A}$-lattice M of $M_{K(x)}$ such that for almost all cyclotomic places $v \in \mathcal{C}$ the v-curvature*

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$$

is the identity.

Finally if q is algebraic, but not a root of unity, we are in the following situation. We call Q the algebraic closure of $\mathbb{Q}$ inside K and $\mathcal{O}_Q$ the ring of integer of Q. For almost all finite places v of Q, let κ_v be the order as a root of unity of q modulo v, π_v a v-adic uniformizer and ϕ_v an integer power of π_v such that $\phi_v^{-1}(1 - q^{\kappa_v})$ is a unit of $\mathcal{O}_Q$. The field K has the form $Q(\underline{a}, b)$, where $\underline{a} = (a_1, \dots, a_r)$ is a transcendent basis of K/Q and b is a primitive element of the algebraic extension $K/Q(\underline{a})$. Choosing conveniently the set of generators $\underline{a}, b$, we can always find an algebra $\mathcal{A}$ of the form:

$$\mathcal{A} = \mathcal{O}_Q \left[\underline{a}, b, x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots\right], \tag{6.1}$$

for some $P(x) \in \mathcal{O}_Q[\underline{a}, b, x]$, and a Σ_q-stable $\mathcal{A}$-lattice M of $\mathcal{M}_{K(x)}$, so that we can consider the linear operator

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{O}_Q/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{O}_Q/(\phi_v),$$

that we will call the v-curvature of $\mathcal{M}_{K(x)}$-modulo ϕ_v. Notice that $\mathcal{O}_Q/(\phi_v)$ is not an integral domain in general. We are going to prove the following:

Theorem 6.3. *A q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$ is trivial if and only if there exists a k-algebra $\mathcal{A}$ as above and a Σ_q-stable $\mathcal{A}$-lattice M of $M_{K(x)}$ such that for almost all finite places v of Q the v-curvature*

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{O}_K/(\phi_v)$$

is the identity.

In order to give a unified statement for the three theorems above we introduce the following notation:

- if q is a root of unity, we can take $\mathcal{C}$ to be the set containing only the trivial valuation v on K, $\mathcal{A}$ to be a σ_q-stable extension of $K[x]$ obtained inverting a convenient polynomial, $(\phi_v) = (0)$ and $\kappa_v = \kappa$;
- if q is transcendental the notation is already defined;
- if q is algebraic, not a root of unity, we set $\mathcal{C}$ to be the set of finite places of Q.

Therefore we have:

Theorem 6.4. *A q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$ is trivial if and only if there exists a k-algebra $\mathcal{A}$ as above and a Σ_q-stable $\mathcal{A}$-lattice M of $M_{K(x)}$ such that for any v in a cofinite nonempty subset of $\mathcal{C}$, the v-curvature*

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$$

is the identity.

We only need to prove Theorem 6.3 under the assumption that K is not a number field. The proof (*cf.* the two subsections below) will repose on [DV02, Thm.7.1.1], which is exactly the same statement plus the extra assumption that K is a number field.

6.1.1. *Global nilpotence.* In this and in the following subsection we assume that:
$(\mathcal{H})$
K is a transcendental finite type extension of $\mathbb{Q}$ and q is an algebraic number.

Proposition 6.5. *Under the hypothesis $(\mathcal{H})$, for a q-difference module $\mathcal{M}_{K(x)} = (M_K(x), \Sigma_q)$ we have:*

1. *If $\Sigma_q^{\kappa_v}$ induces a unipotent linear morphism on $M_{\mathcal{A}} \otimes_{\mathcal{O}_Q} \mathcal{O}_Q/(\pi_v)$ for infinitely many finite places v of Q, then the q-difference module $\mathcal{M}_{K(x)}$ is regular singular.*
2. *If there exists a set of finite places v of Q of Dirichlet density 1 such that $\Sigma_q^{\kappa_v}$ induces a unipotent linear morphism on $M_{\mathcal{A}} \otimes_{\mathcal{O}_Q} \mathcal{O}_Q/(\pi_v)$, then the q-difference module $\mathcal{M}_{K(x)}$ is regular singular and its exponents at 0 and ∞ are in $q^{\mathbb{Z}}$.*
3. *If $\Sigma_q^{\kappa_v}$ induces the identity on $M_{\mathcal{A}} \otimes_{\mathcal{O}_Q} \mathcal{O}_Q/(\pi_v)$ for almost all finite places v of Q in a set of Dirichlet density 1, then the q-difference modules $\mathcal{M}_{K((x))}$ and $\mathcal{M}_{K((1/x))}$ are trivial.*

We recall that a subset S of the set of finite places $\mathcal{C}$ of Q has Dirichlet density 1 if

$$\limsup_{s\to 1^+} \frac{\sum_{v\in S, v|p} p^{-sf_v}}{\sum_{v\in S_f, v|p} p^{-sf_v}} = 1, \tag{6.2}$$

where f_v is the degree of the residue field of v over $\mathbb{F}_p$.

Proof. The proof is the same as [DV02, Thm.6.2.2 and Prop.6.2.3] (*cf.* also Theorem 2.3 and Corollary 3.6 above). The idea is that one has to choose a basis $\underline{e}$ of $M_{\mathcal{A}}$ such that $\Sigma_q \underline{e} = \underline{e}A(x)$ for some $A(x) \in Gl_\nu(\mathcal{A})$. Then the hypothesis on the reduction of $\Sigma_q^{\kappa_v}$ modulo π_v forces $A(x)$ not to have poles at 0 and ∞. Moreover we deduce that $A(0), A(\infty) \in Gl_\nu(K)$ are actually semisimple matrices, whose eigenvalues are in $q^{\mathbb{Z}}$. □

6.1.2. *Proof of Theorem 6.3.* We assume $(\mathcal{H})$. We will deduce Theorem 6.3 from the analogous results in [DV02], where K is assumed to be a number field. To do so, we will consider the transcendence basis of K/Q as a set of parameter that we will specialize in the algebraic closure of Q. We will need the following (very easy) lemma:

Lemma 6.6. *Let F be a field and q be an element of F, not a root of unity. We consider a q-difference system $Y(qx) = A_0(x)Y(x)$ such that $A_0(x) \in Gl_\nu(F(x))$, zero is not a pole of $A_0(x)$ and such that $A_0(0)$ is the identity matrix. Then, for any norm $|\ |$ (archimedean or ultrametric) over F such that $|q| > 1$ the formal solution*

$$Z_0(x) = \Big(A_0(q^{-1}x)A_0(q^{-2}x)A_0(q^{-3}x)\dots\Big)$$

of $Y(qx)=A_0(x)Y(x)$ *is a germ of an analytic fundamental solution at zero having infinite radius of meromorphy.*[12]

Proof. Since $|q|>1$ the infinite product defining $Z_0(x)$ is convergent in the neighborhood of zero. The fact that $Z_0(x)$ is a meromorphic function with infinite radius of meromorphy follows from the functional equation $Y(qx)=A_0(x)Y(x)$ itself. □

Proof of Theorem 6.3. One side of the implication in Theorem 6.3 is trivial. So we suppose that $\Sigma_q^{\kappa_v}$ induces the identity on $M_{\mathcal{A}}\otimes_{\mathcal{O}_Q}\mathcal{O}_Q/(\phi_v)$ for almost all finite places v of Q, and we prove that $\mathcal{M}_{\mathcal{A}}$ becomes trivial over $K(x)$. The proof is divided into steps:

Step 0. Reduction to a purely transcendental extension K/Q. Let $\underline{a}$ be a transcendence basis of K/Q and b is a primitive element of $K/Q(\underline{a})$, so that $K=Q(\underline{a},b)$. The q-difference field $K(x)$ can be considered as a trivial q-difference module over the field $Q(\underline{a})(x)$. By restriction of scalars, the module $\mathcal{M}_{K(x)}$ is also a q-difference module over $Q(\underline{a})(x)$. Since the field $K(x)$ is a trivial q-difference module over $Q(\underline{a})(x)$, we have:

- the module $\mathcal{M}_{K(x)}$ is trivial over $K(x)$ if and only if it is trivial over $Q(\underline{a})(x)$;
- under the present hypothesis, there exist an algebra $\mathcal{A}'$ of the form

$$\mathcal{A}'=\mathcal{O}_Q\left[\underline{a},x,\frac{1}{R(x)},\frac{1}{R(qx)},....\right],\ R(x)\in\mathcal{O}_Q[\underline{a},x],\tag{6.3}$$

  and a $\mathcal{A}'$-lattice $\mathcal{M}_{\mathcal{A}'}$ of q-difference module $\mathcal{M}_{K(x)}$ over $Q(\underline{a})(x)$, such that $\mathcal{M}_{\mathcal{A}'}\otimes_{\mathcal{A}'}Q(\underline{a},x)=\mathcal{M}_{K(x)}$ as a q-difference module over $Q(\underline{a},x)$ and $\Sigma_q^{\kappa_v}$ induces the identity on $\mathcal{M}_{\mathcal{A}'}\otimes_{\mathcal{A}}\mathcal{O}_Q/(\phi_v)$, for almost all places v of Q.

For this reason, we can actually assume that K is a purely transcendental extension of Q of degree $d>0$ and that $\mathcal{A}=\mathcal{A}'$. We fix an immersion of $Q\hookrightarrow\overline{\mathbb{Q}}$, so that we will think to the transcendental basis $\underline{a}$ as a set of parameter generically varying in $\overline{\mathbb{Q}}^d$. □

Step 0bis. Initial data. Let $K=Q(\underline{a})$ and q be a nonzero element of Q, which is not a root of unity. We are given a q-difference module $\mathcal{M}_{\mathcal{A}}$ over a convenient algebra $\mathcal{A}$ as above, such that $K(x)$ is the field of fraction of $\mathcal{A}$ and such that $\Sigma_q^{\kappa_v}$ induces the identity on $M_{\mathcal{A}}\otimes\mathcal{O}_q/(\phi_v)$, for almost all finite places v. We fix a basis $\underline{e}$ of $\mathcal{M}_{\mathcal{A}}$, such that $\Sigma_q\underline{e}=\underline{e}A^{-1}(x)$, with $A(x)\in Gl_\nu(\mathcal{A})$. We will rather work with the associated q-difference system:

$$Y(qx)=A(x)Y(x).\tag{6.4}$$

It follows from Proposition 6.5 that $\mathcal{M}_{K(x)}$ is regular singular, with no logarithmic singularities, and that its exponents are in $q^{\mathbb{Z}}$. Enlarging a little bit the algebra $\mathcal{A}$ (more precisely replacing the polynomial R by a multiple of R), we can suppose that both 0 and ∞ are not poles of $A(x)$ and that $A(0),A(\infty)$ are diagonal matrices with eigenvalues in $q^{\mathbb{Z}}$ (*cf.* [Sau00, §2.1]). □

Step 1. Construction of canonical solutions at 0. We construct a fundamental matrix of solutions, applying the Frobenius algorithm to this particular situation (*cf.* [vdPS97] or [Sau00, §1.1]). There exists a shearing transformation $S_0(x)\in Gl_\nu(K[x,x^{-1}])$ such that

$$S_0^{-1}(qx)A(x)S_0(x)=A_0(x)$$

and $A_0(0)$ is the identity matrix. In particular, the matrix $S_0(x)$ can be written as a product of invertible constant matrices and diagonal matrix with integral powers of

[12] In the sense introduced in §8.2, over an algebraically closed complete extension of $F,|\ |$.

x on the diagonal. Once again, up to a finitely generated extension of the algebra $\mathcal{A}$, obtained inverting a convenient polynomial, we can suppose that $S_0(x) \in Gl_\nu(\mathcal{A})$.

Notice that, since q is not a root of unity, there always exists a norm, non necessarily archimedean, on Q such that $|q| > 1$. We can always extend such a norm to K. Then the system

$$Z(qx) = A_0(x)Z(x) \tag{6.5}$$

has a unique convergent solution $Z_0(x)$, as in Lemma 6.6. This implies that $Z_0(x)$ is a germ of a meromorphic function with infinite radius of meromorphy. So we have the following meromorphic solution of $Y(qx) = A(x)Y(x)$:

$$Y_0(x) = \Big(A_0(q^{-1}x)A_0(q^{-2}x)A_0(q^{-3}x)\dots\Big)\,S_0(x).$$

We remind that this formal infinite product represent a meromophic fundamental solution of $Y(qx) = A(x)Y(x)$ for any norm over K such that $|q| > 1$ (*cf.* Lemma 6.6). □

Step 2. Construction of canonical solutions at ∞. In exactly the same way we can construct a solution at ∞ of the form $Y_\infty(x) = Z_\infty(x)S_\infty(x)$, where the matrix S_∞ belongs to $GL_\nu(K[x, x^{-1}]) \cap Gl_\nu(\mathcal{A})$ and has the same form as $S_0(x)$, and $Z_\infty(x)$ is analytic in a neighborhood of ∞, with $Z_\infty(\infty) = 1$:

$$Y_\infty(x) = \Big(A_\infty(x)A_\infty(qx)A_\infty(q^2x)\dots\Big)\,S_\infty(x).$$

□

Step 3. The Birkhoff matrix. To summarize we have constructed two fundamental matrices of solutions, $Y_0(x)$ at zero and $Y_\infty(x)$ at ∞, which are meromorphic over $\mathbb{A}^1_K \smallsetminus \{0\}$ for any norm on K such that $|q| > 1$, and such that their set of poles and zeros is contained in the q-orbits of the set of poles at zeros of $A(x)$. The Birkhoff matrix

$$B(x) = Y_0^{-1}(x)Y_\infty(x) = S_0(x)^{-1}Z_0(x)^{-1}Z_\infty(x)S_\infty(x)$$

is a meromorphic matrix on $\mathbb{A}^1_K \smallsetminus \{0\}$ with elliptic entries: $B(qx) = B(x)$. All the zeros and poles of $B(x)$, other than 0 and ∞, are contained in the q-orbit of zeros and poles of the matrices $A(x)$ and $A(x)^{-1}$. □

Step 4.Rationality of the Birkhoff matrix. Let us choose $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, with α_i in the algebraic closure $\overline{\mathbb{Q}}$ of Q, so that we can specialize $\underline{a}$ to $\underline{\alpha}$ in the coefficients of $A(x), A(x)^{-1}, S_0(x), S_\infty(x)$ and that the specialized matrices are still invertible. Then we obtain a q-difference system with coefficients in $Q(\underline{\alpha})$. It follows from Lemma 6.6 that for any norm on $Q(\underline{\alpha})$ such that $|q| > 1$ we can specialize $Y_0(x), Y_\infty(x)$ and therefore $B(x)$ to matrices with meromorphic entries on $Q(\underline{\alpha})^*$. We will write $A^{(\underline{\alpha})}(x)$, $Y_0^{(\underline{\alpha})}(x)$, etc. for the specialized matrices.

Since $A_{\kappa_v}(x)$ is the identity modulo ϕ_v, the same holds for $A_{\kappa_v}^{(\underline{\alpha})}(x)$. Therefore the reduced system has zero κ_v-curvature modulo ϕ_v for almost all v. We know from [DV02], that $Y_0^{(\underline{\alpha})}(x)$ and $Y_\infty^{(\underline{\alpha})}(x)$ are the germs at zero of rational functions, and therefore that $B^{(\underline{\alpha})}(x)$ is a constant matrix in $Gl_\nu(Q(\underline{\alpha}))$.

As we have already pointed out, $B(x)$ is q-invariant meromorphic matrix on $\mathbb{P}^1_K \smallsetminus \{0, \infty\}$. The set of its poles and zeros is the union of a finite numbers of q-orbits of the forms $\beta q^{\mathbb{Z}}$, such that β is algebraic over K and is a pole or a zero of $A(x)$ or $A(x)^{-1}$. If β is a pole or a zero of an entry $b(x)$ of $B(x)$ and $h_\beta(x), k_\beta(x) \in Q[\underline{a}, x]$ are the minimal polynomials of β and β^{-1} over K, respectively, then we have:

$$b(x) = \lambda \frac{\prod_\gamma \prod_{n\geq 0} h_\gamma(q^{-n}x) \prod_{n\geq 0} k_\gamma(1/q^n x)}{\prod_\delta \prod_{n\geq 0} h_\delta(q^{-n}x) \prod_{n\geq 0} k_\delta(1/q^n x)},$$

where $\lambda \in K$ and γ and δ vary in a system of representatives of the q-orbits of the zeroes and the poles of $b(x)$, respectively. We have proved that there exists a Zariski open set of $\overline{\mathbb{Q}}^d$ such that the specialization of $b(x)$ at any point of this set is constant. Since the factorization written above must specialize to a convergent factorization of the same form of the corresponding element of $B^{\underline{\alpha}}(x)$, we conclude that $b(x)$, and therefore $B(x)$ is a constant. □

The fact that $B(x) \in Gl(K)$ implies that the solutions $Y_0(x)$ and $Y_\infty(x)$ glue to a meromorphic solution on $\mathbb{P}^1_K$ and ends the proof of Theorem 6.3. □

6.2. **Curvature characterization of the generic (differential) group.** For any field K of zero characteristic, any $q \in K \smallsetminus \{0,1\}$ and q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ we can define as in the previous sections two generic Galois groups: $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ and $Gal^\partial(\mathcal{M}_{K(x)}, \eta_{K(x)})$. If K is a finite type extension of $\mathbb{Q}$, in the notation of Theorem 6.4, we have:

Theorem 6.7. *The generic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest algebraic subgroup of $Gl_\nu(M_{K(x)})$ that contains the v-curvatures of the q-difference module $\mathcal{M}_{K(x)}$ modulo ϕ_v, for all v in a nonempty cofinite subset of $\mathcal{C}$.*

The group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is a stabilizer of a line $L_{K(x)}$ in a construction $\mathcal{W}_{K(x)} = (W_{K(x)}, \Sigma_q)$ of $\mathcal{M}_{K(x)}$. The statement above says that we can find a σ_q-stable algebra $\mathcal{A} \subset K(x)$ of one of the forms described above, and a Σ_q-stable $\mathcal{A}$-lattice M of $M_{K(x)}$ such that M induces an $\mathcal{A}$-lattice L of $L_{K(x)}$ and W of $W_{K(x)}$ with the following properties: the reduction modulo ϕ_v of $\Sigma_q^{\kappa_v}$ stabilizes $L \otimes_K \mathcal{O}_K/(\phi_v)$ inside $W \otimes_K \mathcal{O}_K/(\phi_v)$, for any v in a nonempty cofinite subset of $\mathcal{C}$.

Theorem 6.7 has been proved in [Hen96, Chap.6] when q is a root of unity, in the previous sections when q is transcendental and in [DV02] when q is algebraic and K is a number field. The remaining case (*i.e.* q algebraic and K of transcendental of finite type) is proved exactly as Theorem 4.5 and [DV02, Thm.10.2.1].

We can give an analogous description of the differential generic Galois group:

Theorem 6.8. *The differential generic Galois group $Gal^\partial(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest algebraic differential subgroup of $Gl_\nu(M_{K(x)})$ that contains the v-curvatures of the q-difference module $\mathcal{M}_{K(x)}$ modulo ϕ_v, for all v in a nonempty cofinite subset of $\mathcal{C}$.*

The meaning of the statement above is the same as Theorem 6.7, once one has replaced algebraic group with algebraic differential group, construction of linear algebra with construction of differential algebra, etc etc. The proof follows the proof of Theorem 5.11.

Remark 6.9. We can of course state the analogues of Corollaries 4.10 and 5.15.

6.3. **Generic (differential) Galois group of a q-difference module over $\mathbb{C}(x)$, for $q \neq 0, 1$.** We deduce from the previous section a curvature characterization of the generic (differential) Galois group of a q-difference module over $\mathbb{C}(x)$, for $q \in \mathbb{C} \smallsetminus \{0,1\}$.[13]

Let $\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q)$ be a q-difference module over $\mathbb{C}(x)$. We can consider a finitely generated extension of K of $\mathbb{Q}$ such that there exists a q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ satisfying $\mathcal{M}_{\mathbb{C}(x)} = \mathcal{M}_{K(x)} \otimes_{K(x)} \mathbb{C}(x)$. First of all let us notice that:

[13] All the statements in this subsection remain true if one replace $\mathbb{C}$ with any field of characteristic zero.

Lemma 6.10. *The q-difference module $\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q)$ is trivial if and only if $\mathcal{M}_{K(x)}$ is trivial.*

Proof. If $\mathcal{M}_{K(x)}$ is trivial, then $\mathcal{M}_{\mathbb{C}(x)}$ is of course trivial. The inverse statement is equivalent to the following claim. If a linear q-difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in Gl_\nu(K(x))$, has a fundamental solution $Y(x) \in Gl_\nu(\mathbb{C}(x))$, then $Y(x)$ is actually defined over K. In fact, the system $Y(qx) = A(x)Y(x)$ must be regular singular with exponents in $q^{\mathbb{Z}}$, therefore the Frobenius algorithm allows to construct a solution $\widetilde{Y}(x) \in Gl_\nu(K((x)))$. We can look at $Y(x)$ as an element of $Gl_\nu(\mathbb{C}((x)))$. Then there must exists a constant matrix $C \in Gl_\nu(\mathbb{C})$ such that $Y(x) = C\widetilde{Y}(x)$. This proves that $\widetilde{Y}(x)$ is the expansion of a matrix with entries in $K(x)$. □

With an abuse of language, Theorem 6.4 can be rephrased as:

Theorem 6.11. *The q-difference module $\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q)$ is trivial if and only if there exists a nonempty cofinite set of curvatures of $\mathcal{M}_{K(x)}$, that are all zero.*

We can of course define as in the previous sections two algebraic generic Galois groups, $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ and $Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$, and two differential generic Galois groups, $Gal^\partial(\mathcal{M}_{K(x)}, \eta_{K(x)})$ and $Gal^\partial(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$. A (differential) noetherianity argument, that we have already used several times, on the submodules stabilized by those groups shows the following:

Proposition 6.12. *In the notation above we have:*

$$Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}) \subset Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \mathbb{C}(x)$$

and

$$Gal^\partial(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}) \subset Gal^\partial(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \mathbb{C}(x).$$

Moreover there exists a finitely generated extension K' (resp. K'') of K such that

$$Gal(\mathcal{M}_{K(x)} \otimes_{K(x)} K'(x), \eta_{K'(x)}) \otimes_{K'(x)} \mathbb{C}(x) \cong Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$$

$$\left(\textit{resp.}\quad Gal^\partial(\mathcal{M}_{K(x)} \otimes_{K(x)} K''(x), \eta_{K''(x)}) \otimes_{K''(x)} \mathbb{C}(x) \cong Gal^\partial(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})\right).$$

Choosing K large enough, we can assume that $K = K' = K''$, which we will do implicitly in the following statements. For the generic Galois group we have the following theorem, that we can deduce from Theorem 6.7:

Theorem 6.13. *The generic Galois group $Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$ is the smallest algebraic subgroup of $Gl_\nu(M_{\mathbb{C}(x)})$ that contains a nonempty cofinite set of curvatures of the q-difference module $\mathcal{M}_{K(x)}$.*

We can deduce from Theorem 6.8 an analogous description of the differential generic Galois group:

Theorem 6.14. *The differential generic Galois group $Gal^\partial(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$ is the smallest algebraic differential subgroup of $Gl_\nu(M_{\mathbb{C}(x)})$ that contains a nonempty cofinite set of curvatures of the q-difference module $\mathcal{M}_{K(x)}$.*

Once again, we can state the analogs of Corollaries 4.10 and 5.15.

7. The Kolchin closure of the Dynamic and the Malgrange-Granier groupoid

In this section we prove that for linear q-difference systems the Malgrange-Granier groupoid is “essentially“ the differential generic Galois group. Very roughly speaking, to prove this result we construct an algebraic D-groupoid called $\mathcal{G}al^{alg}(A)$ and we show on one hand that $\mathcal{G}al^{alg}(A)$ and the Malgrange-Granier groupoid $\mathcal{G}al(A)$ have the same solutions, and on the other hand that the solutions of the sub-D-groupoid of $\mathcal{G}al^{alg}(A)$ that fixes the transversals coincides with the solutions of the differential equations defining the differential generic Galois group.

We have been unable to prove that $\mathcal{G}al^{alg}(A)$ and $\mathcal{G}al(A)$ coincide as D-groupoids. This seems to be a particular case of a more general question in Malgrange theory. In fact, in [Mal01] B. Malgrange introduces the notion of D-groupoid in the space of invertible jets $J^*(M,M)$ of an analytic variety M. Since M carries also a structure of algebraic variety $\mathbb{M}$ over $\mathbb{C}$, it is very natural to ask the question of the algebraicity of the Galois D-groupoid. The problem has been tackled in more recent works by B. Malgrange himself.

In the linear case, *i.e.* in the case of a vector bundle $M \to S$ where S is an analytic complex variety (*cf.* [KS72, §I.2]), the algebraic counterpart of $J^*(M,M)$ is the sheaf of principal part of the sheaf of sections of M over the $\mathbb{C}$-scheme $\mathbb{S}$, whose analytization is equal to S (*cf.* [Gro67, 16.7.7.1])). The functoriality of this construction (*cf.* [Gro67, 16.7.10]) and the GAGA theorem should give some hint to compare the analytic and the algebraic setting (see for instance [GM93, §2.1. p 75] in the case of $\mathcal{D}$-modules over $\mathbb{S} = \mathbb{P}^1_{\mathbb{C}}$).

In the non linear case, the algebraic framework is less clear. In [Ume08], H. Umemura defines the scheme of invertible jets[14] $\mathbb{J}^*(\mathbb{M},\mathbb{M})$ of a smooth scheme M of finite type over $\mathbb{S} := \mathbb{C}$. However the comparison of the analytic and the algebraic jet spaces does not appear to be straightforward.

Moreover, in the q-difference setting, a further complication comes into the picture, with respect to the differential case considered by Malgrange. In Malgrange theory, the foliation associated to a nonlinear differential equation over the variety M, which exists due to the Cauchy theorem, plays a central role. There is a true hindrance to define a foliation over $\mathbb{C}$ attached to a linear q-difference system, essentially for two reasons (which are actually not independent): the constants of the q-difference theory are elliptic functions and no Cauchy theorem on the unicity of solutions for a given initial data can be proved. In [Gra], A. Granier defines the Galois D-groupoid of a q-difference system as the D-envelop of the dynamic of the system. We propose here to produce an algebraic D-groupoid whose generating equations are precisely those of the differential generic Galois group and whose solutions coincide with those of the transversal Galois-D-groupoid of a linear q-difference system of A. Granier. These results shall give some hint to compare the algebraic definitions of Morikawa of the Galois group of a nonlinear q-difference equation and the analytic definitions of A.Granier (*cf.* [Mor09], [MU09], [Ume10]).

Let $q \in \mathbb{C}^*$ be not a root of unity and let $A(x) \in Gl_\nu(\mathbb{C}(x))$. We consider the linear q-difference system

$$Y(qx) = A(x)Y(x). \tag{7.1}$$

We set:

$$\begin{aligned} &A_k(x) := A(q^{k-1}x)\dots A(qx)A(x) \text{ for all } k \in \mathbb{Z},\ k > 0;\\ &A_0(x) = Id_\nu\\ &A_k(x) := A(q^k x)^{-1}A(q^{k+1}x)^{-1}\dots A(q^{-1}x)^{-1} \text{ for all } k \in \mathbb{Z},\ k < 0, \end{aligned}$$

Following the Appendix, we denote by M the analytic complex variety $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$, by $\mathcal{G}al(A(x))$ the Galois D-groupoid of the system (7.1) *i.e.* the D-envelop of the dynamic

$$Dyn(A(x)) = \left\{(x,X) \longmapsto (q^k x, A_k(x)X)\ :\ k \in \mathbb{Z}\right\}$$

in the space of jets $J^*(M,M)$. We keep the notation of §A, which is preliminary to the content of this section.

[14] Umemura's jet scheme is not the jet scheme of Nash!

We will generalize the methods used by Malgrange in the case of linear differential sytem (*cf.* [Mal01]) and by Granier in the case of a linear q-difference system with constant coefficients (*cf.* [Gra, §2.1]), to the situation described above.

Warning. Following Malgrange and the convention in §A.1, we say that a D-groupoid $\mathcal{H}$ is contained in a D-groupoid $\mathcal{G}$ if the groupoid of solutions of $\mathcal{H}$ is contained in the groupoid of solutions of $\mathcal{G}$. We will write $sol(\mathcal{H}) \subset sol(\mathcal{G})$ or equivalently $\mathcal{I}_{\mathcal{G}} \subset \mathcal{I}_{\mathcal{H}}$, where $\mathcal{I}_{\mathcal{G}}$ and $\mathcal{I}_{\mathcal{H}}$ are the (sheaves of) ideals of definition of $\mathcal{G}$ and $\mathcal{H}$, respectively.

7.1. **The groupoid $\mathcal{G}al^{alg}(A)$.** Let $\mathbb{C}(x)\left\{T, \frac{1}{\det T}\right\}_\partial$, with $T = (T_{i,j} : i,j = 0,1,\dots,\nu)$, be the algebra of differential rational functions over $Gl_{\nu+1}(\mathbb{C}(x))$ (*cf.* footnote 11). We consider the following morphism of ∂-differential $\mathbb{C}(x)$-algebras

$$\tau : \qquad \mathbb{C}[x]\left\{T, \tfrac{1}{\det T}\right\}_\partial \longrightarrow H^0(M \times_{\mathbb{C}} M, \mathcal{O}_{J^*(M,M)})$$

$$\begin{pmatrix} T_{0,0} & T_{0,1} & \dots & T_{0,n} \\ T_{1,0} & & & \\ \vdots & & (T_{i,j})_{i,j} & \\ T_{\nu,0} & & & \end{pmatrix} \longmapsto \begin{pmatrix} \frac{\partial \overline{x}}{\partial x} & \frac{\partial \overline{x}}{\partial X_1} & \dots & \frac{\partial \overline{x}}{\partial X_\nu} \\ \frac{\partial \overline{X_1}}{\partial x} & & & \\ \vdots & & \left(\frac{\partial \overline{X_i}}{\partial X_j}\right)_{i,j} & \\ \frac{\partial \overline{X_\nu}}{\partial x} & & & \end{pmatrix}$$

from $\mathbb{C}[x]\left\{T, \frac{1}{\det T}\right\}_\partial$ to the global sections $H^0(M \times_{\mathbb{C}} M, \mathcal{O}_{J^*(M,M)})$ of $\mathcal{O}_{J^*(M,M)}$, that can be thought as global partial differential equations. The image by τ of the defining ideal of the linear differential algebraic group

$$\left\{ diag(\alpha, \beta(x)) := \begin{pmatrix} \alpha & 0 \\ 0 & \beta(x) \end{pmatrix} : \text{where } \alpha \in \mathbb{C}^* \text{ and } \beta(x) \in Gl_\nu(\mathbb{C}(x)) \right\}$$

generates a D-groupoid $\mathcal{L}in$ of $J^*(M,M)$ (*cf.* Definition A.1 and Proposition A.2).

Definition 7.1. We call $\mathcal{K}ol(A)$ the smallest differential subvariety of $Gl_{\nu+1}(\mathbb{C}(x))$, defined over $\mathbb{C}(x)$, which contains

$$\left\{ diag(q^k, A_k(x)) := \begin{pmatrix} q^k & 0 \\ 0 & A_k(x) \end{pmatrix} : k \in \mathbb{Z} \right\},$$

and has the following property: if we call $I_{\mathcal{K}ol(A)}$ the differential ideal defining $\mathcal{K}ol(A)$ and $I'_{\mathcal{K}ol(A)} = I_{\mathcal{K}ol(A)} \cap \mathbb{C}[x]\left\{T, \frac{1}{\det T}\right\}_\partial$, then the (sheaf of) differential ideal $\langle \mathcal{I}_{\mathcal{L}in}, \tau(I'_{\mathcal{K}ol(A)})\rangle$ generates a D-groupoid, that we will call $\mathcal{G}al^{alg}(A)$, in the space of jets $J^*(M,M)$.

Remark 7.2. The definition above requires some explanations:

- The phrase "smallest differential algebraic subvariety of $Gl_{\nu+1}(\mathbb{C}(x))$" must be understood in the following way. The ideal of definition of $\mathcal{K}ol(A)$ is the largest differential ideal of $\mathbb{C}(x)\left\{T, \frac{1}{\det T}\right\}_\partial$ which admits the matrices $diag(q^k, A_k(x))$ as solutions for any $k \in \mathbb{Z}$ and verifies the second requirement of the definition. Then $I_{\mathcal{K}ol(A)}$ is radical and the Ritt-Raudenbush theorem (*cf.* Theorem 5.4 above) implies that $I_{\mathcal{K}ol(A)}$ is finitely generated. Of course, the $\mathbb{C}(x)$-rational points of $\mathcal{K}ol(A)$ may give very poor information on its structure, so we would rather speak of solutions in a differential closure of $\mathbb{C}(x)$.
- The structure of D-groupoid has the following consequence on the points of $\mathcal{K}ol(A)$: if $diag(\alpha, \beta(x))$ and $diag(\gamma, \delta(x))$ are two matrices with entries in a differential extension of $\mathbb{C}(x)$ that belong to $\mathcal{K}ol(A)$ then the matrix $diag(\alpha\gamma, \beta(\gamma x)\delta(x))$ belongs to $\mathcal{K}ol(A)$. In other words, the set of local diffeomorphisms $(x, X) \mapsto (\alpha x, \beta(x)X)$ of $M \times M$ such that $diag(\alpha, \beta(x))$ belongs to $\mathcal{K}ol(A)$ forms a set theoretic groupoid. We could have supposed

only that $\mathcal{K}ol(A)$ is a differential variety and the solutions of $\mathcal{K}ol(A)$ form a groupoid in the sense above, but this wouldn't have been enough. In fact, it is not known if a sheaf of differential ideals of $J^*(M, M)$ whose solutions forms a groupoid is actually a D-groupoid (*cf.* Definition A.1, and in particular conditions (ii') and (iii')). B. Malgrange told us that he can only prove this statement for a Lie algebra.

The differential variety $\mathcal{K}ol(A)$ is going to be a bridge between the differential generic Galois group and the Galois D-groupoid $\mathcal{G}al(A)$ defined in the appendix, *via* the following theorem. Let $\mathcal{M}_{\mathbb{C}(x)}^{(A)} := (\mathbb{C}(x)^\nu, \Sigma_q : X \mapsto A^{-1}\sigma_q(X))$ be the q-difference module over $\mathbb{C}(x)$ associated to the system $Y(qx) = A(x)Y(x)$, where $\sigma_q(X)$ is defined componentwise, and $\widetilde{\mathcal{K}ol(A)}$ the differential algebraic group over $\mathbb{C}(x)$ defined by the differential ideal $\langle I_{\mathcal{K}ol(A)}, T_{0,0} - 1\rangle$ in $\mathbb{C}(x)\left\{T, \frac{1}{\det T}\right\}_\partial$. Notice that, as for the Zariski closure, the Kolchin closure does not commute with the intersection, therefore $\widetilde{\mathcal{K}ol(A)}$ is not the Kolchin closure of $\{A_k(x)\}_k$. Then we have:

Theorem 7.3. $Gal^\partial(\mathcal{M}_{\mathbb{C}(x)}^{(A)}, \eta_{\mathbb{C}(x)}) \cong \widetilde{\mathcal{K}ol(A)}$.

Remark 7.4. One can define in exactly the same way an algebraic subvariety $\mathcal{Z}ar(A)$ of $Gl_{\nu+1}(\mathbb{C}(x))$ containing the dynamic of the system and such that

$$\{(x, X) \mapsto (\alpha x, \beta(x)X) : diag(\alpha, \beta(x)) \in \mathcal{Z}ar(A)\}$$

is a subgroupoid of the groupoid of diffeomorphisms of $M \times M$. Then one proves in the same way that $\widetilde{\mathcal{Z}ar(A)}$ coincide with the generic Galois group.

Proof. Let $\mathcal{N} = constr^\partial(\mathcal{M})$ be a construction of differential algebra of $\mathcal{M}$. We can consider:

- The basis denoted by $constr^\partial(\underline{e})$ of $\mathcal{N}$ built from the canonical basis $\underline{e}$ of $\mathbb{C}(x)^\nu$, applying the same constructions of differential algebra.
- For any $\beta \in Gl_\nu(\mathbb{C}(x))$, the matrix $constr^\partial(\beta)$ acting on $\mathcal{N}$ with respect to the basis $constr^\partial(\underline{e})$, obtained from β by functoriality. Its coefficients lies in $\mathbb{C}(x)[\beta, \partial(\beta), ...]$
- Any $\psi = (\alpha, \beta) \in \mathbb{C}^* \times Gl_\nu(\mathbb{C}(x))$ acts semilinearly on $\mathcal{N}$ in the following way: $\psi\underline{e} = (constr^\partial(\beta))^{-1}\underline{e}$ and $\phi(f(x)n) = f(\alpha x)n$, for any $f(x) \in \mathbb{C}(x)$ and $n \in \mathcal{N}$. In particular, $(q^k, A_k(x)) \in \mathbb{C}^* \times Gl_\nu(\mathbb{C}(x))$ acts as Σ_q^k on $\mathcal{N}$.

A sub-q-difference module $\mathcal{E}$ of $\mathcal{N}$ correspond to an invertible matrix $F \in Gl_\nu(\mathbb{C}(x))$ such that

$$F(q^k x)^{-1} constr^\partial(A_k) F(x) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \text{ for any } k \in \mathbb{Z}. \tag{7.2}$$

Now, $(1, \beta) \in \mathbb{C}^* \times Gl_\nu(\mathbb{C}(x))$ stabilizes $\mathcal{E}$ if and only if

$$F(x)^{-1} constr^\partial(\beta) F(x) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}. \tag{7.3}$$

Equation (7.2) corresponds to a differential polynomial $L(T_{0,0}, (T_{i,j})_{i,j\geq 1})$ belonging to $\mathbb{C}(x)\left\{T, \frac{1}{\det T}\right\}_\partial$ and having the property that $L(q^k, (A_k)) = 0$ for all $k \in \mathbb{Z}$. On the other hand (7.3) corresponds to $L(1, (T_{i,j})_{i,j\geq 1}))$. It means that the solutions of the differential ideal $\langle I_{\mathcal{K}ol(A)}, T_{0,0} - 1\rangle \subset \mathbb{C}(x)\left\{T, \frac{1}{\det T}\right\}_\partial$ stabilize all the sub-q-difference modules of all the constructions of differential algebra, and hence that

$$\widetilde{\mathcal{K}ol(A)} \subset Gal^\partial(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}).$$

Let us prove the inverse inclusion. In the notation of Theorem 6.14, there exists a finitely generated extension K of $\mathbb{Q}$ and a σ_q-stable subalgebra $\mathcal{A}$ of $K(x)$ of the forms considered in §6.1 such that:

(1) $A(x) \in Gl_\nu(\mathcal{A})$, so that it defines a q-difference module $\mathcal{M}_{K(x)}$ over $K(x)$;
(2) $Gal^\partial(\mathcal{M}_{K(x)}^{(A)}, \eta_{K(x)}) \otimes_{K(x)} \mathbb{C}(x) \cong Gal^\partial(\mathcal{M}_{\mathbb{C}(x)}^{(A)}, \eta_{\mathbb{C}(x)})$;
(3) $\mathcal{K}ol(A)$ is defined over $\mathcal{A}$, *i.e.* there exists a differential ideal I in the differential ring $\mathcal{A}\{T, \frac{1}{\det(T)}\}_\partial$ such that I generates $I_{\mathcal{K}ol(A)}$ in $\mathbb{C}(x)\left\{T, \frac{1}{\det T}\right\}_\partial$.

For any element $\widetilde{L}$ of the defining ideal of $\widetilde{\mathcal{K}ol(A)}$ over $\mathcal{A}$, there exists

$$L(T_{0,0}; T_{i,j}, i,j = 1,\dots,\nu) \in I \subset \mathcal{A}\left\{T, \frac{1}{\det(T)}\right\}_\partial,$$

such that $\widetilde{L} = L(1; T_{i,j}, i,j = 1,\dots,\nu)$. If q is a root of unity of order κ we simply have $\widetilde{L}(A_\kappa) = L(1, A_\kappa) = L(q^\kappa, A_\kappa) = 0$. If q is an algebraic number, other than a root of unity, then for almost all places v of the algebraic closure of $\mathbb{Q}$ is K we have

$$\widetilde{L}(A_{\kappa_v}) \equiv L(1, A_{\kappa_v}) \equiv L(q^{\kappa_v}, A_{\kappa_v}) \equiv 0 \text{ modulo } \phi_v.$$

On the other hand if q is a transcendental number, for almost all cyclotomic places v of K we have

$$\widetilde{L}(A_{\kappa_v}) \equiv L(1, A_{\kappa_v}) \equiv L(q^{\kappa_v}, A_{\kappa_v}) \equiv 0 \text{ modulo } \phi_v.$$

This shows that $\widetilde{\mathcal{K}ol(A)}$ is a differential algebraic subgroup of $Gl_\nu(\mathbb{C}(x))$ which contains a nonempty cofinite set of v-curvatures, in the sense explained in §6.3. By Theorem 6.14, $\widetilde{\mathcal{K}ol(A)}$ contains the differential generic Galois group of $\mathcal{M}_{\mathbb{C}(x)}^{(A)}$. □

We call $\widetilde{\mathcal{G}al^{alg}}(A)$ the D-groupoid on $M \times_{\mathbb{C}} M$ intersection of $\mathcal{G}al^{alg}(A)$ and $\sqrt{\langle \overline{x} - x\rangle}$. It is not difficult to prove that the D-groupoid $\widetilde{\mathcal{G}al^{alg}}(A)$ is generated by its global equations *i.e.* by $\mathcal{L}in$ and the image of the equations of $\widetilde{\mathcal{K}ol(A)}$ by the morphism τ. Therefore we deduce from Theorem 7.3 the following statement:

Corollary 7.5. *As a D-groupoid, $\widetilde{\mathcal{G}al^{alg}}(A)$ is generated by its global sections, namely the D-groupoid $\mathcal{L}in$ and the image of the equations of $Gal^\partial(\mathcal{M}_{\mathbb{C}(x)}^{(A)}, \eta_{\mathbb{C}(x)})$* via *the morphism τ.*

Remark 7.6. The corollary above says not only that a germ of diffeomorphism $(x, X) \mapsto (x, \beta(x)X)$ of M is solution of $\widetilde{\mathcal{G}al^{alg}}(A)$ if and only if $\beta(x)$ is solution of the differential equations defining the differential generic Galois group of $\mathcal{M}_{\mathbb{C}(x)}^{(A)} = (\mathbb{C}(x)^\nu, X \mapsto A(x)^{-1}\sigma_q(X))$, but also that the two differential defining ideals “coincide”.

The D-groupoid $\widetilde{\mathcal{G}al^{alg}}(A)$ is a differential analogous of the D-groupoid generated by an algebraic group introduced in [Mal01, Proposition 5.3.2] by B. Malgrange.

7.2. **The Galois D-groupoid $\mathcal{G}al(A)$ of a linear q-difference system.** Since $Dyn(A(x))$ is contained in the solutions of $\mathcal{G}al^{alg}(A)$, we have

$$sol(\mathcal{G}al(A(x))) \subset sol(\mathcal{G}al^{alg}(A))$$

and

$$sol(\widetilde{\mathcal{G}al(A(x))}) \subset sol(\widetilde{\mathcal{G}al^{alg}}(A)).$$

Theorem 7.7. *The solutions of the D-groupoid $\widetilde{\mathcal{G}al(A(x))}$ (resp. $\mathcal{G}al(A(x))$) coincide with the solutions of $\widetilde{\mathcal{G}al^{alg}}(A)$ (resp. $\mathcal{G}al^{alg}(A)$).*

Combining the theorem above with Corollary 7.5, we immediately obtain:

Corollary 7.8. *The solutions of the D-groupoid $\widetilde{\mathcal{G}al(A(x))}$ are germs of diffeomorphisms of the form $(x,X)\longmapsto(x,\beta(x)X)$, such that $\beta(x)$ is a solution of the differential equations defining $Gal^{\partial}(\mathcal{M}^{(A)}_{\mathbb{C}(x)},\eta_{\mathbb{C}(x)})$, and* vice versa.

Remark 7.9. The corollary above says that the solutions of $\widetilde{\mathcal{G}al(A)}$ in a neighborhood of a transversal $\{x_0\}\times\mathbb{C}^{\nu}$ (*cf.* Proposition A.7 below), rational over a differential extension $\mathcal{F}$ of $\mathbb{C}(x)$, correspond one-to-one with the solutions $\beta(x)\in Gl_{\nu}(\mathcal{F})$ of the differential equations defining the differential generic Galois group.

It does not say that the two defining differential ideals can be compared. We actually don't prove that $\mathcal{G}al(A)$ is an "algebraic D-groupoid" and therefore that $\mathcal{G}al^{alg}(A)$ and $\mathcal{G}al(A)$ coincide as D-groupoids.

Proof of Theorem 7.7. Let $\mathcal{I}$ be the differential ideal of $\mathcal{G}al(A(x))$ in $\mathcal{O}_{J^*(M,M)}$ and let $\mathcal{I}_r$ be the subideal of $\mathcal{I}$ order r. We consider the morphism of analytic varieties given by

$$\begin{array}{rccc}\iota: & \mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}} & \longrightarrow & M\times_{\mathbb{C}}M\\ & (x,\overline{x}) & \longmapsto & (x,0,\overline{x},0)\end{array}$$

and the inverse image $\mathcal{J}_r:=\iota^{-1}\mathcal{I}_r$ (resp. $\mathcal{J}:=\iota^{-1}\mathcal{I}$) of the sheaf $\mathcal{I}_r$ (resp. $\mathcal{I}$) over $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$. We consider similarly to [Mal01, lemma 5.3.3], the evaluation $ev(\iota^{-1}\mathcal{I})$ at $X=\overline{X}=\frac{\partial^i\overline{X}}{\partial x^i}=0$ of the equations of $\iota^{-1}\mathcal{I}$ and we denote by $ev(\mathcal{I})$ the direct image by ι of the sheaf $ev(\iota^{-1}\mathcal{I})$.

The following lemma is crucial in the proof of the Theorem 7.7:

Lemma 7.10. *A germs of local diffeomorphism $(x,X)\mapsto(\alpha x,\beta(x)X)$ of M is solution of $\mathcal{I}$ if and only if it is solution of $ev(\mathcal{I})$.*

Proof. First of all, we notice that $\mathcal{I}$ is contained in $\mathcal{L}in$. Moreover the solutions of $\mathcal{I}$, that are diffeomorphisms mapping a neighborhood of $(x_0,X_0)\in M$ to a neighborhood of $(\overline{x}_0,\overline{X}_0)$, can be naturally continued to diffeomorphisms of a neighborhood of $x_0\times\mathbb{C}^{\nu}$ to a neighborhood of $\overline{x}_0\times\mathbb{C}^{\nu}$. Therefore it follows from the particular structure of the solutions of $\mathcal{L}in$, that they are also solutions of $ev\,(\mathcal{I})$ (*cf.* Proposition A.2).

Conversely, let the germ of diffeomorphism $(x,X)\mapsto(\alpha x,\beta\,(x)\,X)$ be a solution of $ev\,(\mathcal{I})$ and $E\in\mathcal{I}_r$. It follows from Proposition A.4 that there exists $E_1\in\mathcal{I}$ of order r, only depending on the variables x,X,$\frac{\partial\bar{x}}{\partial x}$, $\frac{\partial\bar{X}}{\partial X}$,$\frac{\partial^2\bar{X}}{\partial x\partial X}$,$\ldots$, $\frac{\partial^r\bar{X}}{\partial x^{r-1}\partial X}$, such that $(x,X)\mapsto(\alpha x,\beta\,(x)\,X)$ is solution of E if and only if it is solution of E_1. So we will focus on equations on the form E_1 and, to simplify notation, we will write E for E_1.

By assumption $(x,X)\mapsto(\alpha x,\beta\,(x)\,X)$ is solution of

$$E\left(x,0,\frac{\partial\bar{x}}{\partial x},\frac{\partial\bar{X}}{\partial X},\frac{\partial^2\bar{X}}{\partial x\partial X},\cdots\frac{\partial^r\bar{X}}{\partial x^{r-1}\partial X}\right)$$

and we have to show that $(x,X)\mapsto(\alpha x,\beta\,(x)\,X)$ is a solution of E. We consider the Taylor expansion of E:

$$E\left(x,X,\frac{\partial\bar{x}}{\partial x},\frac{\partial\bar{X}}{\partial X},\frac{\partial^2\bar{X}}{\partial x\partial X},\cdots\frac{\partial^r\bar{X}}{\partial x^{r-1}\partial X}\right)=\sum_{\alpha}E_{\alpha}\,(x,X)\,\partial^{\alpha},$$

where ∂^{α} is a monomial in the coordinates $\frac{\partial\bar{x}}{\partial x},\frac{\partial\bar{X}}{\partial X},\frac{\partial^2\bar{X}}{\partial x\partial X},\cdots\frac{\partial^r\bar{X}}{\partial x^{r-1}\partial X}$. Developing the $E_{\alpha}\,(x,X)$ with respect to $X=(X_1,\ldots,X_{\nu})$ we obtain:

$$E=\sum\left(\sum_{\alpha}\left(\frac{\partial^{\underline{k}}E_{\alpha}}{\partial X^{\underline{k}}}\right)(x,0)\,\partial^{\alpha}\right)X^{\underline{k}},$$

with $\underline{k} \in (\mathbb{Z}_{\geq 0})^{\nu}$. If we show that for any $\underline{k}$ the germ $(x, X) \mapsto (\alpha x, \beta(x) X)$ verifies the equation

$$B_{\underline{k}} := \sum_{\alpha} \left(\frac{\partial^{\underline{k}} E_{\alpha}}{\partial X^{\underline{k}}} \right) (x, 0)\, \partial^{\alpha}$$

we can conclude. For $\underline{k} = (0, \ldots, 0)$, there is nothing to prove since $B_{\underline{0}} = ev(E)$.

Let D_{X_i} be the derivation of $\mathcal{I}$ corresponding to $\frac{\partial}{\partial X_i}$, The differential equation

$$D_{X_i}(E) = \sum_{\alpha} \left(\frac{\partial E_{\alpha}}{\partial X_i} \right) (x, X)\, \partial^{\alpha} + \sum_{\alpha} E_{\alpha}(x, X)\, D_{X_i}(\partial^{\alpha})$$

is still in $\mathcal{I}$, since $\mathcal{I}$ is a differential ideal. Therefore by assumption $(x, X) \mapsto (\alpha x, \beta(x) X)$ is a solution of

$$ev(D_{X_i} E) = \sum_{\alpha} \left(\frac{\partial E_{\alpha}}{\partial X_i} \right) (x, 0)\, \partial^{\alpha} + \sum_{\alpha} E_{\alpha}(x, 0)\, D_{X_i}(\partial^{\alpha}).$$

Since $D_{X_i}(\partial^{\alpha}) \in \mathcal{L}in$ and $(x, X) \mapsto (\alpha x, \beta(x) X)$ is a solution of $\mathcal{L}in$, we conclude that $(x, X) \mapsto (\alpha x, \beta(x) X)$ is a solution of

$$\sum_{\alpha} \left(\frac{\partial E_{\alpha}}{\partial X} \right) (x, 0)\, \partial^{\alpha}$$

and therefore of B_1. Iterating the argument, one deduce that $(x, X) \mapsto (\alpha x, \beta(x) X)$ is solution of $B_{\underline{k}}$ for any $\underline{k} \in (\mathbb{Z}_{\geq 0})^{\nu}$, which ends the proof. □

We go back to the proof of Theorem 7.7. Lemma 7.10 proves that the solutions of $\mathcal{G}al(A(x))$ coincide with those of the D-groupoid Γ generated by $\mathcal{L}in$ and $ev(\mathcal{I})$, defined on the open neighborhoods of any $x_0 \times \mathbb{C}^{\nu} \in M$. By intersection with the equation $\sqrt{< \overline{x} - x >}$, the same holds for the transversal groupoids $\widetilde{\mathcal{G}al(A(x))}$ and $\widetilde{\Gamma}$.

Since $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and $M \times_{\mathbb{C}} M$ are locally compact and $\mathcal{I}_r$ is a coherent sheaf over $M \times_{\mathbb{C}} M$, the sheaf $\mathcal{J}_r$ is an analytic coherent sheaf over $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and so is its quotient $ev(\iota^{-1}(\mathcal{I}_r))$. By [Ser56, Theorem 3], there exists an algebraic coherent sheaf $\mathbb{J}_r$ over the projective variety $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ such that the analyzation of $\mathbb{J}_r$ coincides with $ev(\iota^{-1}(\mathcal{I}_r))$. This implies that $ev(\mathcal{I})$ is generated by algebraic differential equations which by definition have the dynamic for solutions.

We thus have that the $sol(\Gamma) = sol(\mathcal{G}al(A)) \subset sol(\mathcal{G}al^{alg}(A))$. Since both Γ and $\mathcal{G}al^{alg}(A)$ are algebraic, the minimality of the variety $\mathcal{K}ol(A)$ implies that $sol(\mathcal{G}al^{alg}(A)) \subset sol(\Gamma)$. We conclude that the solutions of $\mathcal{G}al(A)$ coincide with those $\mathcal{G}al^{alg}(A)$. The same hold for $\widetilde{\mathcal{G}al(A)}$, $\widetilde{\Gamma}$ and $\widetilde{\mathcal{G}al^{alg}(A)})$. This concludes the proof. □

7.3. **Comparison with known results in** [Mal01] **and** [Gra]**.** In [Mal01], B. Malgrange proves that the Galois-D-groupoid of a linear differential equation allows to recover the usual differential Galois group over $\mathbb{C}$. This is not in contradiction with the result above, since:

- due to the fact that local solutions of a linear differential equation form a $\mathbb{C}$-vector space (rather than a vector space on the field of elliptic functions!), [Kat82, Prop.4.1] shows that the generic Galois group and the classical Galois group in the differential setting become isomorphic above a certain extension of the local ring. For more details on the relation between the generic Galois group and the usual Galois group see [Pil02, Cor.3.3].
- in the differential setting both the classical and the generic Galois group and differential Galois group, in the sense of the differential tannakian category, coincide (*cf.* Remark 10.11).

Therefore B. Malgrange actually finds a differential generic Galois group, which is hidden in his construction. The steps of the proof above are the same as in his proof, apart that, to compensate the lack of good local solutions, we are obliged to use Theorem 5.11. Anyway, the application of Theorem 5.11 appears to be very natural, if one considers how close the definition of the dynamic of a linear q-difference system and the definition of the curvatures are.

In [Gra], A. Granier shows that in the case of a q-difference system with constant coefficients the groupoid that fixes the transversals in $\mathcal{G}al(A)$ is the usual Galois group, *i.e.* an algebraic group defined over $\mathbb{C}$. Once again, this is not in contradiction with our results. In fact, under this assumption, we know from Proposition 4.14 that the differential generic Galois group is defined over $\mathbb{C}$. Moreover the algebraic generic and differential Galois groups coincide, in fact if $\mathcal{M}$ is a q-difference module over $\mathbb{C}(x)$ associated with a constant q-difference system, it is easy to prove that the prolongation functor F acts trivially on $\mathcal{M}$, namely $F(\mathcal{M}) \cong \mathcal{M} \oplus \mathcal{M}$. Finally, to conclude that the generic Galois group coincide with the usual one, it is enough to notice that they are associated with the same fiber functor, or equivalently that they stabilize exactly the same objects.

Because of these results, G. Casale and J. Roques have conjectured that "for linear (q-)difference systems, the action of Malgrange groupoid on the fibers gives the classical Galois groups" (*cf.* [CR08]). In *loc. cit.*, they give two proofs of their main integrability result: one of the them relies on the conjecture. Here we have proved that the Galois-D-groupoid allows to recover exactly the differential generic Galois group. By taking the Zariski closure one can also recovers the algebraic generic Galois group. The comparisons theorem in the last part of the paper imply that we can also recover the usual Galois group (*cf.* [vdPS97], [Sau04b]) performing a Zariski closure and a convenient field extension and the differential Galois group (*cf.* [HS08]) performing a field extension.

Part IV. Comparison among Galois theories

As the definition of generic Galois group is related to a tannakian category framework, we define the notion of differential generic Galois group with the help of the differential tannakian category framework developed by A. Ovchinnikov in [Ovc09a].

First we recall some basic facts about tannakian and differential tannakian category and show how the groups previously defined are actually a tannakian objects. Finally, in view of the comparison results of §9, we give a differential tannakian version of the differential Galois theory of Hardouin-Singer. For this purpose we construct a meromorphic basis of solutions for the q^{-1}-adic norms of K, whose differential relations are encoded in the differential generic Galois group.

We remind that, while speaking of differential Galois group, we will always assume implicitly that the characteristic of k is zero. On the other hand, notice that we don't need any assumption on the characteristic to consider meromorphic and elliptic functions. We only need a norm on K for which $|q| \neq 1$.

8. The differential tannakian category of q-difference modules

The aim of the tannakian formalism is to characterize the categories equivalent to a category of representations of a linear algebraic group. Similarly, the aim of the differential tannakian formalism is to characterize the categories equivalent to a category of representations of a linear differential algebraic group. In the construction of A. Ovchinnikov, the axioms defining a differential tannakian category are exactly the classical ones plus those induced by the prolongation functor (*cf.*

§5.1). We won't say more about his definition and we will refer to the original work [Ovc09a].

Let $(\mathcal{F}, \sigma_q, \partial)$ be a q-difference-differential field.

Proposition 8.1. *The category $Diff(\mathcal{F}, \sigma_q)$, endowed with the prolongation functor F, is a differential tannakian category in the sense of* [Ovc09a, Def.3].

We are skipping the proof of this proposition, which is long but has no real difficulties.

Let us denote by $\eta_{\mathcal{F}} : Diff(\mathcal{F}, \sigma_q) \rightarrow Vect_{\mathcal{F}}$, the forgetful functor from the category of q-difference modules over $\mathcal{F}$ to the category of $\mathcal{F}$-vector space. The forgetful functor commutes with the prolongation functor F:

$$F_{Vect_{\mathcal{F}}} \circ \eta_{\mathcal{F}} = \eta_{\mathcal{F}} \circ F_{Diff(\mathcal{F},\sigma_q)},$$

where the subscripts $Vect_{\mathcal{F}^{\sigma_q}}$ and $Diff(\mathcal{F}, \sigma_q)$ emphasize on which category the prolongation functor acts. We could have defined the differential generic Galois group as the group of differential tensor automorphisms of the forgetful functor.

Since we want to build an equivalence of category between $Diff(\mathcal{F}, \sigma_q)$ (or a differential tannakian subcategory $\mathcal{C}$ of $Diff(\mathcal{F}, \sigma_q)$) with the category of differential representations of a linear differential algebraic group, we are interested with a special kind of functors: the differential fiber functors (*cf.* [Ovc09a, Def.4.1] for the general definition):

Definition 8.2. Let $\omega : \mathcal{C} \rightarrow Vect_{\mathcal{F}^{\sigma_q}}$ be a $\mathcal{F}^{\sigma_q}$-linear functor. We say that ω is a differential fiber functor for $\mathcal{C}$ if

(1) ω is a fiber functor in the sense of [SR72, 3.2.1.2];
(2) $F_{Vect_{\mathcal{F}^{\sigma_q}}} \circ \omega = \omega \circ F_{Diff(\mathcal{C})}$.

Then, the category $\mathcal{C}$ is equivalent to the category of differential representations of the linear differential algebraic group $Aut^{\otimes,\partial}(\omega)$. If $\mathcal{C} = \langle \mathcal{M} \rangle^{\otimes,\partial}$, for some $\mathcal{M} \in Diff(\mathcal{F}, \sigma_q)$, then we write $Aut^{\otimes,\partial}(\mathcal{M}, \omega)$ and $Aut^{\otimes}(\mathcal{M}, \omega)$ for the group of tensor automorphisms of the restriction of ω to the usual tannakian category $\langle \mathcal{M} \rangle^{\otimes}$.

Similarly to the tannakian case, if $\mathcal{F}^{\sigma_q}$ is differentially closed (*cf.* [CS06, Sect.9.1] for definition and references), one can always construct a differential fiber functor (*cf.* [Ovc09a, Thm.16]) and two differential fiber functors are isomorphic. Notice that this is very much in the spirit of the tannakian formalism. In fact in [Del90, §7], P. Deligne proves that, if $\mathcal{F}^{\sigma_q}$ is an algebraically closed field, the category $Diff(\mathcal{F}, \sigma_q)$ admits a fiber functor ω into the category $Vect_{\mathcal{F}^{\sigma_q}}$ of finite dimensional $\mathcal{F}^{\sigma_q}$-vector spaces.

To construct explicitly a differential fiber functor, we need to construct a fundamental solution matrix of a q-difference system associated to the q-difference module, with respect to some basis. The first approach is to make an abstract construction of an algebra containing a basis of abstract solutions of the q-difference module and all their derivatives (*cf.* [HS08, Definition 6.10]). We detail this approach in the next subsection. The major disadvantage of this construction is that it requires that the σ_q-constants of the base field form a differentially closed field, *i.e.* an enormous field. For this reason we will rather consider a differential fiber functor ω_E defined by meromorphic solutions of the module (*cf.* 8.2 below). Then, we will establish some comparison results between the differential generic Galois group, the group of differential tensor automorphism of ω_E and the Hardouin-Singer differential Galois group (*cf.* §9 below).

8.1. **Formal differential fiber functor.** Let $(\mathcal{F}, \sigma_q, \partial)$ be a q-difference-differential field. In [HS08], the authors attach to a differential equation $\sigma_q(Y) = AY$ with $A \in Gl_\nu(\mathcal{F})$, a (σ_q, ∂)-Picard-Vessiot ring: a simple (σ_q, ∂)-ring generated over $\mathcal{F}$ by a fundamental solutions matrix of the system and all its derivatives w.r.t. ∂. Here simple means with no non trivial ideal invariant under σ_q and ∂ (*cf.* [HS08, Definition 2.3]). Such (σ_q, ∂)-Picard-Vessiot rings always exist. A basic construction is to consider the ring of differential polynomials $S = \mathcal{F}\{Y, \frac{1}{detY}\}_\partial$, where Y is a matrix of differential indeterminates over $\mathcal{F}$ of order ν , and to endow it with a q-difference operator compatible with the differential structure, *i.e.* such that $\sigma_q(Y) = AY$, $\sigma_q(\partial Y) = A\partial Y + \partial AY, \dots$ Any quotient of the ring S by a maximal (σ_q, ∂)-ideal is a (σ_q, ∂)-Picard-Vessiot ring. If the σ_q-constants of a (σ_q, ∂)-Picard-Vessiot ring coincide with $\mathcal{F}^{\sigma_q}$, we say that this ring is neutral. The connection between neutral (σ_q, ∂)-Picard-Vessiot ring and differential fiber functor for $\mathcal{M}$ is given by the following theorem which is the differential analogue of [And01, Theorem 3.4.2.3].

Theorem 8.3. *Let $\mathcal{M} \in Diff(\mathcal{F}, \sigma_q)$. If the differential tannakian category $\langle\mathcal{M}\rangle^{\otimes,\partial}$ admits a differential fiber functor over $\mathcal{F}^{\sigma_q}$, we have an equivalence of quasi-inverse categories:*

$$\{\textit{differential fiber functor over } \mathcal{F}^{\sigma_q}\} \leftrightarrow \{\textit{neutral } (\sigma_q, \partial) - \textit{Picard-Vessiot ring}\}.$$

Proof. We only give a sketch of proof and refer to [Del90, Section 9] and to [And01, Theorem 3.4.2.3] for the algebraic proof. We consider the forgetful functor $\eta_\mathcal{F} : \langle\mathcal{M}\rangle^{\otimes,\partial} \mapsto \mathcal{F}$-modules of finite type. If ω is a neutral differential fiber functor for $\langle\mathcal{M}\rangle^{\otimes,\partial}$, the functor $Isom^{\otimes,\partial}(\omega \otimes \mathbf{1}_\mathcal{F}, \eta_\mathcal{F})$ over the differential commutative $\mathcal{F}$-algebras, is representable by a differential $\mathcal{F}$-variety $\Sigma^\partial(\mathcal{M}, \omega)$. It is a $Aut^{\otimes,\partial}(\mathcal{M}, \omega)$-torsor. The ring of regular functions $\mathcal{O}(\Sigma^\partial(\mathcal{M}, \omega))$, in the sense of Kolchin, of $\Sigma^\partial(\mathcal{M}, \omega)$, is a neutral (σ_q, ∂)-Picard-Vessiot extension for $\mathcal{M}$ over $\mathcal{F}$. Conversely, let A be a neutral (σ_q, ∂)-Picard-Vessiot ring for $\mathcal{M}$. The functor $\omega_A : \langle\mathcal{M}\rangle^{\otimes,\partial} \mapsto Vect_{\mathcal{F}^{\sigma_q}}$ defined as follow, $\omega_A(\mathcal{N}) := Ker(\Sigma_q - Id, A \otimes \mathcal{N})$, is a neutral differential fiber functor. The functors $\omega \mapsto \mathcal{O}(\Sigma^\partial(\mathcal{M}, \omega))$ and $A \mapsto \omega_A$ are quasi-inverse. □

As a corollary, we get that the differential tannakian category $\langle\mathcal{M}\rangle^{\otimes,\partial}$ admits a differential fiber functor over $\mathcal{F}^{\sigma_q}$ if and only if there exists a neutral (σ_q, ∂)-Picard-Vessiot ring for $\mathcal{M}$. We state below some consequences of Theorem 8.3.

Theorem 8.4. *Let $(\mathcal{F}, \sigma_q, \partial)$ be a q-difference-differential field. Let $\mathcal{M}$ be an object of $Diff(\mathcal{F}, \sigma_q)$ and let R be a neutral (σ_q, ∂)-Picard-Vessiot ring for $\mathcal{M}$. Then,*

(1) *the group of (σ_q, ∂)-$\mathcal{F}$-automorphisms G_R^∂ of R coincides with the $\mathcal{F}^{\sigma_q}$-points of the linear differential algebraic group $Aut^{\otimes,\partial}(\mathcal{M}, \omega_R)$;*
(2) *the differential dimension of $Aut^{\otimes,\partial}(\mathcal{M}, \omega_R)$ over $\mathcal{F}^{\sigma_q}$ is equal to the differential transcendence degree of R over $\mathcal{F}$;*[15]
(3) *the linear differential algebraic group $Aut^{\otimes,\partial}(\mathcal{M}, \omega_R)$ is a Zariski dense subset in the linear algebraic group $Aut^{\otimes}(\mathcal{M}, \omega_R)$.*

Two neutral (σ_q, ∂)-Picard-Vessiot rings for $\mathcal{M}$ become isomorphic over a differential closure of $\mathcal{F}^{\sigma_q}$. The same holds for two differential fiber functors.

Proof. See [Ovc09a] or [HS08, Prop. 6.18 and 6.26]. □

[15] A (σ_q, ∂)-Picard-Vessiot ring R is a direct sum of copies of an integral domain S. By differential transcendence degree of R over $\mathcal{F}$, we mean the differential transcendence degree of the fraction field of S over $\mathcal{F}$.

As in the classical case, a sufficient condition to ensure the existence of a differential fiber functor or equivalently of a neutral (σ_q, ∂)-Picard-Vessiot, is that the field of σ_q-constants $\mathcal{F}^{\sigma_q}$ is differentially closed. This assumption is very strong, since differentially closed field are enormous. We show in the next section, how, for q-difference equations over $K(x)$, one could weaken this assumption by loosing the simplicity of the Picard-Vessiot ring but by requiring the neutrality. We will speak, in that case, of *weak differential Picard-Vessiot ring.* The corresponding algebraic notion was introduced in [CHS08, Definition 2.1].

8.2. **Differential fiber functor associated with a basis of meromorphic solutions.** For a fixed complex number q with $|q| \neq 1$, Praagman proves in [Pra86] that every q-difference equation with meromorphic coefficients over $\mathbb{C}^*$ admits a basis of solutions, meromorphic over $\mathbb{C}^*$, linearly independent over the field of elliptic function C_E, *i.e.* the field of meromorphic functions over the elliptic curve $E := \mathbb{C}^*/q^{\mathbb{Z}}$. The reformulation of his theorem in the tannakian language is that the category of q-difference modules over the field of meromorphic functions on the punctured plane $\mathbb{C}^*$ is a neutral tannakian category over C_E, *i.e.* admits a fiber functor into $Vect_{C_E}$. We give below the generic analogue of this theorem.

Let $K(x)$ be a q-difference field, $\partial = x\frac{d}{dx}$, $|\ |$ a norm on K such that $|q| > 1$ and C an algebraically closed field extension of K, complete w.r.t. $|\ |$.[16] Here are a few examples to keep in mind:

- K is a subfield of $\mathbb{C}$ equipped with the norm induced by $\mathbb{C}$ and $C = \mathbb{C}$;
- K is finite extension of $k(q)$, equipped with the q^{-1}-adic norm;
- K is a finitely generated extension of $\mathbb{Q}$ and q is an algebraic number, nor a root of unity: in this case there always exists a norm on the algebraic closure Q of $\mathbb{Q}$ in K such that $|q| > 1$, that can be extended to K. The field C is equal to $\mathbb{C}$ if the norm is archimedean.

We call *holomorphic function over* C^* a power series $f = \sum_{n=-\infty}^{\infty} a_n x^n$ with coefficients in C that satisfies

$$\lim_{n\to\infty} |a_n|\rho^n = 0 \text{ and } \lim_{n\to-\infty} |a_n|\rho^n = 0 \text{ for all } \rho > 0.$$

The holomorphic functions on C^* form a ring $\mathcal{H}ol(C^*)$. Its fraction field $\mathcal{M}er(C^*)$ is the field of meromorphic functions over C^*.

Remark 8.5. Both $\mathcal{H}ol(C^*)$ and $\mathcal{M}er(C^*)$ are stable under the action of σ_q and ∂.

Proposition 8.6. *Every q-difference system $\sigma_q(Y) = AY$ with $A \in Gl_\nu(K(x))$ admits a fundamental solution matrix with coefficients in $\mathcal{M}er(C^*)$,* i.e. *an invertible matrix $U \in Gl_\nu(\mathcal{M}er(C^*))$, such that $\sigma_q(U) = AU$.*

Remark 8.7. The proposition above is equivalent to the global triviality of the pull back over C^* of the fiber bundles on elliptic curves.

Proof. We are only sketching the proof. The Jacobi theta function

$$\Theta_q(x) = \sum_{n\in\mathbb{Z}} q^{-n(n-1)/2} x^n,$$

is an element of $\mathcal{M}er(C^*)$. It is solution of the q-difference equation

$$y(qx) = qx\, y(x).$$

We follow [Sau00]. Since

[16] What follows is of course valid also for the norms for which $|q| < 1$ and can be deduced by transforming the q-difference system $\sigma_q(Y) = AY$ in the q^{-1}-difference system $\sigma_{q^{-1}}(Y) = \sigma_{q^{-1}}(A^{-1})Y$.

- for any $c \in C^*$, the meromorphic function $\Theta(cx)/\Theta_q(x)$ is solution of $y(qx) = cy(x)$;
- the meromorphic function $x\Theta_q'(x)/\Theta_q(x)$ is solution of the equation $y(qx) = y(x) + 1$;

we can write a meromorphic fundamental solution to any fuchsian system, and, more generally, of any system whose Newton polygon has only one slope (*cf.* for instance [Sau00], [DVRSZ03] or [Sau04b, §1.2.2]). For the "pieces" of solutions linked to the Stokes phenomenon, all the technics of q-summation in the case $q \in \mathbb{C}$, $|q| > 1$, apply in a straightforward way to our situation (*cf.* [Sau04a, §2, §3]) and give a fundamental solution meromorphic over C^*. □

The field of σ_q-constants of $\mathcal{M}er(C^*)$ is the field C_E of elliptic functions over the torus $E = C^*/q^{\mathbb{Z}}$. Because σ_q and ∂ commute, the derivation ∂ stabilizes C_E inside $\mathcal{M}er(C^*)$, so that C_E is naturally endowed with a structure of q-difference-differential field. Let $\widetilde{C}_E$ be a differential closure of C_E with respect to ∂ (*cf.* [CS06, §9.1]).[17] We still denote by ∂ the derivation of $\widetilde{C}_E$ and we extend the action of σ_q to $\widetilde{C}_E$ by setting $\sigma_q|_{\widetilde{C}_E} = id$. Let $C_E(x)$ (resp. $\widetilde{C}_E(x)$) denote the field $C(x)(C_E)$ (resp. $C(x)(\widetilde{C}_E)$)[18].

We consider a q-difference module $\mathcal{M}_{K(x)}$ defined over $K(x)$ and the object $\mathcal{M}_{C_E(x)} := \mathcal{M}_{K(x)} \otimes_{K(x)} C_E(x)$ of $Diff(C_E(x), \sigma_q)$ obtained by scalar extension. Proposition 8.6 produces a fundamental matrix of solution $U \in Gl_\nu(\mathcal{M}er(C^*))$ of the q-difference system associated to $\mathcal{M}_{K(x)}$ with respect to a given basis $\underline{e}$ of $\mathcal{M}_{K(x)}$ over $K(x)$. The (σ_q, ∂)-ring R_M generated over $C_E(x)$ by the entries of U and $1/\det(U)$ (*cf.* [HS08, Def.2.1]), *i.e.* the minimal q-difference-differential ring over $C_E(x)$ that contains U, $1/\det(U)$ and all its derivatives, is a subring of $\mathcal{M}er(C^*)$. It has the following properties:

Lemma 8.8. *The ring R_M is a (σ_q, ∂)-weak Picard-Vessiot ring for $\mathcal{M}_{C_E(x)}$ over $C_E(x)$,* i.e. *it is a (σ_q, ∂)-ring generated over $C_E(x)$ by a fundamental solutions matrix of the system associated to $\mathcal{M}_{C_E(x)}$, whose ring of σ_q-constants is equal to C_E. Moreover, it is an integral domain.*

Let $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes,\partial}$ be the full differential tannakian subcategory generated by $\mathcal{M}_{C_E(x)}$ in $Diff(C_E(x), \sigma_q)$. For any object $\mathcal{N}$ of $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes,\partial}$, we set

$$\omega_E(\mathcal{N}) := Ker(\Sigma_q - Id, R_M \otimes \mathcal{N}) \tag{8.1}$$

Proposition 8.9. *The category $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes,\partial}$ equipped with the differential fiber functor (*cf. [Ovc09a, §4.1]*)*

$$\omega_E : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes,\partial} \to Vect_{C_E}$$

is a neutral differential tannakian category.

Proof. One has to check that the axioms of the definition in [Ovc09a] are verified. The verification is long but straightforward and the exact analogous of [CHS08, Proposition 3.6]. □

Corollary 8.10. *The group of differential automorphisms $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_E)$ of ω_E is a linear differential algebraic group defined over C_E (*cf. [Ovc09b, Def.8 and Thm.1]*).*

---

[17]The differential closure of a field $\mathcal{F}$ equipped with a derivation ∂ is a field $\widetilde{\mathcal{F}}$ equipped with a derivation extending ∂, with the property that any system of differential equations with coefficients in $\mathcal{F}$, having a solution in a differential extension of $\mathcal{F}$, has a solution in $\widetilde{\mathcal{F}}$.

[18]The field $\widetilde{C}_E(x)$ is the generic analogue of the field $\mathcal{G}(x)$ in [HS08, p. 340].

Definition 8.11. We call $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)},\omega_E)$ the differential Galois group of $\mathcal{M}_{C_E(x)}$.

Since R_M is not a (σ_q,∂)-Picard-Vessiot ring, one can not conclude, as in Theorem 8.4, that the group of (σ_q,∂)-automorphisms of R_M over $C_E(x)$ coincides with the group of C_E-points of $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)},\omega_E)$ and that the differential dimension of $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)},\omega_E)$ over C_E is equal to the differential transcendence degree of F_M, the fraction field of R_M over $C_E(x)$. We have to extend the scalars to the differential closure $\widetilde{C}_E$ of C_E in order to compare R_M with a (σ_q,∂)-Picard-Vessiot ring of $\mathcal{M}_{\widetilde{C}_E(x)}$ or, equivalently, ω_E with a differential fiber functor $\widetilde{\omega}_E$ for $\mathcal{M}_{\widetilde{C}_E(x)}$. As a motivation we anticipate the following consequence of the comparison results that we will show in §9 (more precisely *cf.* Corollary 9.9):

Corollary 8.12. *Let $\mathcal{M}_{K(x)}$ be a q-difference module defined over $K(x)$. Let $U\in Gl_\nu(\mathcal{M}er(C^*))$ be a fundamental solution matrix of $\mathcal{M}_{K(x)}$. Then, there exists a finitely generated extension K'/K such that the differential dimension of the differential field generated by the entries of U over $\widetilde{C}_E(x)$ is equal to the differential dimension of $Gal^\partial(\mathcal{M}_{K(x)}\otimes_{K(x)}K'(x),\eta_{K'(x)})$.*[19]

We recall that roughly speaking the ∂-differential dimension of F_M over $C_E(x)$ is equal to the maximal number of elements of F_M that are differentially independent over $C_E(x)$. So the differential dimension of $Gal^\partial(\mathcal{M}_{K(x)}\otimes_{K(x)}K'(x),\eta_{K'(x)})$ gives information on the number of solutions of a q-difference equations that *do not* have any differential relation among them: it measures their hypertranscendence properties.

9. Comparison of Galois groups

Let K be a field and $|\ |$ a norm on K such that $|q|>1$. We will be dealing with groups defined over the following fields:
C = smallest algebraically closed and complete extension of the normed field $(K,|\ |)$;
C_E = field of constants with respect to σ_q of $\mathcal{M}er(C^*)$;
$\overline{C}_E$ = algebraic closure of C_E;
$\widetilde{C}_E$ = differential closure of C_E.
We remind that any q-difference system $Y(qx)=A(x)Y(x)$, with $A(x)\in Gl_\nu(K(x))$ has a fundamental solution in $\mathcal{M}er(C^*)$ (*cf.* Proposition 8.6).

Let $\mathcal{M}_{K(x)}$ be a q-difference module over $K(x)$. For any q-difference field extension $\mathcal{F}/K(x)$ we will denote by $\mathcal{M}_\mathcal{F}$ the q-difference module over $\mathcal{F}$ obtained from $\mathcal{M}_{K(x)}$ by scalar extension. We can attach to $\mathcal{M}_{K(x)}$ a collection of fiber and differential fiber functors defined upon the above field extensions. As explained in Theorem 8.4, the groups of tensor or differential tensor automorphisms attached to these functors correspond to classical notions of Galois groups of a q-difference equation, namely, the Picard-Vessiot groups. Their definition rely on adapted notion of admissible solutions and their dimension measure the algebraic and differential, when it make sense, behavior of these solutions. We give a precise description of some of these Picard-Vessiot groups below.

In [vdPS97, §1.1], Singer and van der Put attached to the q-difference module $\mathcal{M}_{C(x)}:=\mathcal{M}_{K(x)}\otimes C(x)$ a Picard-Vessiot ring R which is a q-difference extension of $C(x)$, containing abstract solutions of the module. This means that the q-difference module $\mathcal{M}_{C(x)}\otimes R$ is trivial. Therefore, the functor ω_C from the subcategory $\langle\mathcal{M}_{C(x)}\rangle^\otimes$ of $Diff(C(x),\sigma_q)$ into $Vect_C$ defined by

$$\omega_C(\mathcal{N}):=Ker(\Sigma_q-Id,R\otimes_{C(x)}\mathcal{N})$$

[19] *cf.* [HS08, p. 337] for definition and references.

is a fiber functor. Since $R \otimes_C C_E$ is a *weak* Picard-Vessiot ring (*cf.* [CHS08, Def.2.1]), we can also introduce the functor ω_{C_E} from the subcategory $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes}$ of $Diff(C_E(x), \sigma_q)$ into $Vect_{C_E}$:

$$\omega_{C_E}(\mathcal{N}) := Ker(\Sigma_q - Id, (R \otimes_C C_E) \otimes_{C_E(x)} \mathcal{N}).$$

One can prove that ω_{C_E} is actually a fiber functor (*cf.* [CHS08, Prop.3.6]).

To summarize, following the construction in §8.2, we have considered the four fiber functors

(1) $\omega_C : \langle \mathcal{M}_{C(x)} \rangle^{\otimes} \longrightarrow Vect_C$;
(2) $\omega_{C_E} : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow Vect_{C_E}$;
(3) $\omega_E : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow Vect_{C_E}$ (defined in (8.1)[20]);
(4) $\widetilde{\omega}_E : \langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes} \longrightarrow Vect_{\widetilde{C}_E}$ any differential fiber functor for $\mathcal{M}_{\widetilde{C}_E(x)}$;

two differential fiber functors induced by the fiber functor with the same name above:

(1) $\omega_E : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial} \longrightarrow Vect_{C_E}$;
(2) $\widetilde{\omega}_E : \langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes, \partial} \longrightarrow Vect_{\widetilde{C}_E}$;

and four forgetful functors:

(1) $\eta_{K(x)} : \langle \mathcal{M}_{K(x)} \rangle^{\otimes} \longrightarrow Vect_{K(x)}$ and its extension to $\langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial}$;
(2) $\eta_{C(x)} : \langle \mathcal{M}_{C(x)} \rangle^{\otimes} \longrightarrow Vect_{C(x)}$ and its extension to $\langle \mathcal{M}_{C(x)} \rangle^{\otimes, \partial}$;
(3) $\eta_{C_E(x)} : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow Vect_{C_E(x)}$ and its extension to $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial}$;
(4) $\eta_{\widetilde{C}_E(x)} \langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes} \longrightarrow Vect_{\widetilde{C}_E}$ and its extension to $\langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes, \partial}$.

The group of tensor automorphisms of ω_C corresponds to the "classical" Picard-Vessiot group of a q-difference equation attached to $\mathcal{M}_{K(x)}$, defined in [vdPS97, §1.2]. It can be identified to the group of ring automorphims of R stabilizing $C(x)$ and commuting with σ_q. Its dimension as a linear algebraic group is equal to the "transcendence degree" of the total ring of quotients of R over $C(x)$, *i.e.* it measures the algebraic relations between the formal solutions introduced by Singer and van der Put over $C(x)$.

The group of tensor automorphisms of ω_E corresponds to another Picard-Vessiot group attached to $\mathcal{M}_{K(x)}$. Its dimension as a linear algebraic group is equal to the transcendence degree of the fraction field F_M of R_M over $C_E(x)$. In other words, $Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_E)$ measures the algebraic relations between the meromorphic solutions, we have introduced in §8.2. One of the main results of [CHS08] is

Theorem 9.1. *The linear algebraic groups* $Aut^{\otimes}(\mathcal{M}_{C(x)}, \omega_C)$, $Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_{C_E})$, $Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_E)$ *and* $Aut^{\otimes}(\mathcal{M}_{\tilde{C}_E(x)}, \tilde{\omega}_E)$ *become isomorphic over* $\widetilde{C}_E$.

The goal of the next sections is to relate the generic (differential) Galois group of $\mathcal{M}_{K(x)}$ with the algebraic and differential behavior of the meromorphic solutions of $\mathcal{M}_{K(x)}$. In a first place, we prove a differential analogous of Theorem 9.1. To conclude, we show how the curvature criteria lead to the comparison between the differential generic Galois group over $C(x)$ and the differential tannakian group induced by $\widetilde{\omega}_E$.

[20]Notice that $\omega_E(\mathcal{N}) = Ker(\Sigma_q - Id, R_M \otimes \mathcal{N})$, where $R_M = C_E(x)\{U, \det U^{-1}\}$ is the smallest ∂-ring containing $C_E(x)$, the entries of U and $\det U^{-1}$. To define ω_E over $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes}$ we should have considered the classical Picard-Vessiot extension $C_E(x)[U, \det U^{-1}]$. Anyway, since $\mathcal{M}$ is trivialized both on R_M and $C_E(x)[U, \det U^{-1}]$ and $R_M^{\sigma_q} = C_E(x)[U, \det U^{-1}]^{\sigma_q} = C_E$, the q-analogue of the wronskian lemma implies that $Ker(\Sigma_q - Id, R_M \otimes \mathcal{M}) = Ker(\Sigma_q - Id, C_E(x)[U, \det U^{-1}] \otimes \mathcal{M})$, as C_E-vector spaces. The same holds for any object of the category $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes}$.

9.1. **Differential Picard-Vessiot groups over the elliptic functions.** In this section, we adapt the technics of [CHS08, Section 2] to a differential framework, in order to compare the distinct (σ_q, ∂)- Picard-Vessiot rings, neutral and weak, attached to $\mathcal{M}_{K(x)}$ over C_E and $\widetilde{C}_E$ in §8.2. For a model theoretic approach of these questions, we refer to [PN09].

Let $R_M = C_E(x)\{U, \frac{1}{detU}\}_\partial$ be the weak (σ_q, ∂)-Picard-Vessiot ring attached to $\mathcal{M}_{C_E(x)}$, with $U \in Gl_\nu(\mathcal{M}er(C^*))$ a fundamental solutions matrix of $\sigma_q(Y) = AY$, a q-difference system attached to $\mathcal{M}_{K(x)}$ with $A \in Gl_v(K(x))$. The differential fiber functor ω_E, attached to R_M, is defined as in (8.1). By Theorem 8.3, there exists a neutral (σ_q, ∂)-Picard-Vessiot ring R'_M such that $\omega_{R'_M} = \omega_E$ (*cf.* Theorem 8.3). Adapting to a differential context [CHS08, Proposition 2.7], we have

Proposition 9.2. *Let $F_M = C_E(x)\langle U\rangle_\partial$ be the fraction field of R_M,* i.e. *the field extension of $C_E(x)$ differentially generated by U. There exists a (σ_q, ∂)-$C_E(x)$-embedding $\rho : R'_M \to F_M \otimes \widetilde{C}_E$, where σ_q acts on $F_M \otimes \widetilde{C}_E$ via $\sigma_q(f \otimes c) = \sigma_q(f) \otimes c$.*

Proof. Let $Y = (Y_{(i,j)})$ be a $\nu \times \nu$-matrix of differential indeterminates over F_M. We have $S = C_E(x)\{Y, \frac{1}{detY}\}_\partial \subset F_M\{Y, \frac{1}{detY}\}_\partial$. As in §8.1, we endow $F_M\{Y, \frac{1}{detY}\}_\partial$ with a q-difference structure compatible with the differential structure by setting $\sigma_q(Y) = AY$. One may assume that $R'_M = S/\mathfrak{M}$ where $\mathfrak{M}$ be a maximal (σ_q, ∂)-ideal of S. Put $X = U^{-1}Y$ in $F_M\{Y, \frac{1}{detY}\}_\partial$. One has $\sigma_q(X) = X$ and $F_M\{Y, \frac{1}{detY}\}_\partial = F_M\{X, \frac{1}{detX}\}_\partial$. Let $S' = C_E\{X, \frac{1}{detX}\}_\partial$. The ideal $\mathfrak{M}$ generates a proper (σ_q, ∂)-ideal $(\mathfrak{M})$ in $F_M\{Y, \frac{1}{detY}\}_\partial$. By [HS08, Lemma 6.12], the map $I \mapsto I \cap S'$ induces a bijective correspondence from the set of (σ_q, ∂)-ideals of $F_M\{X, \frac{1}{detX}\}_\partial$ and the set of ∂-ideals of $C_E\{X, \frac{1}{detX}\}_\partial$. We let $\widetilde{\mathfrak{M}} = (\mathfrak{M}) \cap S'$ and $\mathfrak{P}$ is a maximal differential ideal of S' containing $\widetilde{\mathfrak{M}}$. The differential ring $S'/\mathfrak{P}$ is an integral domain and its fraction field is a finitely generated constrained extension of C_E (*cf.* [Kol74, p.143]). By [Kol74, Corollary 3], there exists a differential homomorphism $S'/\mathfrak{P} \to \widetilde{C}_E$. We then have

$$S' \to S'/\mathfrak{P} \to \widetilde{C}_E.$$

One can extend this differential homomorphism into a (σ_q, ∂)-homomorphism

$$\phi : F_M\{Y, \frac{1}{detY}\}_\partial = F_M \otimes_{C_E} S' \to F_M \otimes_{C_E} \widetilde{C}_E.$$

The kernel of the restriction of ϕ to S contains $\mathfrak{M}$. Since $\mathfrak{M}$ is a maximal (σ_q, ∂)-ideal, this kernel is equal to $\mathfrak{M}$. Then, ϕ induces an embedding $R'_M \to F_M \otimes_{C_E} \widetilde{C}_E$. □

As in Theorem 8.4, let $G^\partial_{R_M}$ (resp. $G^\partial_{R'_M}$) be the group of (σ_q, ∂)-$C_E(x)$-automorphisms of R_M (resp. R'_M). Similarly to [CHS08, Proposition 2.2], one can prove that these groups are the C_E-points of linear differential algebraic groups defined over C_E.

Corollary 9.3. *Let R_M, F_M, R'_M be as above. The morphism ρ maps $R'_M \otimes_{C_E} \widetilde{C}_E$ isomorphically on $R_M \otimes_{C_E} \widetilde{C}_E$. Therefore, the linear differential algebraic groups $G^\partial_{R_M}$ and $G^\partial_{R'_M}$ are isomorphic over $\widetilde{C}_E$.*

Proof. A differential analogous of [CHS08, Corollary 2.8] and Theorem 8.4 give the result. □

It remains to compare the neutral (σ_q, ∂)-Picard-Vessiot ring R'_M and the (σ_q, ∂)-Picard-Vessiot ring corresponding to a neutral differential fiber functor $\widetilde{\omega}_E$ over

$\widetilde{C}_E$[21]. The differential analogous of [CHS08, Proposition 2.4 and Corollary 2.5] gives

Proposition 9.4. *The ring $\widetilde{R}_M := R'_M \otimes_{C_E(x)} \widetilde{C}_E(x)$ is a (σ_q, ∂)-Picard-Vessiot ring for $\mathcal{M}_{\widetilde{C}_E(x)}$. The linear differential algebraic groups $G^{\partial}_{R'_M}$ and $G^{\partial}_{\widetilde{R}_M}$ are isomorphic over $\widetilde{C}_E$.*

Combining the previous results and some generalities about neutral differential fiber functors (*cf.* Theorem 8.4), we find

Theorem 9.5. *Let ω_E be the differential fiber functor for $\mathcal{M}_{C_{E(x)}}$ defined by a fundamental matrix of meromorphic solutions as in (8.1). Let $\widetilde{\omega}_E$ be a differential fiber functor for $\langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes, \partial}$. Then,*

1. *the linear differential algebraic group $Aut^{\otimes,\partial}(\mathcal{M}_{\tilde{C}_E(x)}, \tilde{\omega}_E)$ corresponds to the differential Galois group attached to $\mathcal{M}_{\widetilde{C}_E(x)}$ by* [HS08, Theorem 2.6] *and is isomorphic over $\widetilde{C}_E$ to $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_E)$;*
2. *the differential transcendence degree of the differential field generated over $\widetilde{C}_E(x)$ by a basis of meromorphic solutions of $\mathcal{M}_{K(x)}$ is equal to the differential dimension of $Aut^{\otimes,\partial}(\mathcal{M}_{\tilde{C}_E(x)}, \tilde{\omega}_E)$ over $\widetilde{C}_E$.*

Proof. By Theorem 8.4. 1), the linear algebraic group $Aut^{\otimes,\partial}(\mathcal{M}_{\tilde{C}_E(x)}, \tilde{\omega}_E)$ (resp. $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_E)$) corresponds to the differential Galois group of Hardouin-Singer (resp. to the automorphism group of the neutral Picard-vessiot ring R'_M). Proposition 9.4 combined with Corollary 9.3 yields to the required isomorphism. By Theorem 8.4. 2), the differential dimension of $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_E)$ is equal to the differential transcendence degree of R'_M over C_E. The isomorphism between R'_M and R_M over $\widetilde{C}_E$ ends the proof. □

Remark 9.6. The results of this section are still valid for any q-difference module $\mathcal{M}$ over $K(x)$ with R_M any integral weak (σ_q, ∂)- Picard-Vessiot ring and $\widetilde{R}_M$ a (σ_q, ∂)-Picard-Vessiot ring for $\mathcal{M} \otimes_{K(x)} K(\widetilde{C_K})$ where $\widetilde{C_K}$ is a differential closure of the σ_q-constants of K.

9.2. **Generic Galois groups and base change.** We are now concerned with the generic Galois groups, algebraic and differential. We first relate them with the Picard-Vessiot groups we have studied previously and then we investigate how they behave through certain type of base field extensions.

9.2.1. *Comparison with Picard-Vessiot groups.* Let $\mathcal{M}_{K(x)}$ be a q-difference module defined over $K(x)$. We have attached to $\mathcal{M}_{K(x)}$ the following groups:

[21]We use implicitely the fact that two differential fiber functor over $\widetilde{C}_E$ are isomorphic so that we may choose any of them

group	fiber functor	field of definition
$Aut^{\otimes}(\mathcal{M}_{C(x)}, \omega_C)$	ω_C:$\langle \mathcal{M}_{C(x)} \rangle^{\otimes} \longrightarrow Vect_C$	C
$Gal(\mathcal{M}_{C(x)}, \eta_{C(x)})$	$\eta_{C(x)}$:$\langle \mathcal{M}_{C(x)} \rangle^{\otimes} \longrightarrow Vect_{C(x)}$	$C(x)$
$Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)})$	$\eta_{C(x)}$:$\langle \mathcal{M}_{C(x)} \rangle^{\otimes,\partial} \longrightarrow Vect_{C(x)}$	$C(x)$
$Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_E)$	ω_E:$\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow Vect_{C_E}$	C_E
$Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_E)$	ω_E:$\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes,\partial} \longrightarrow Vect_{C_E}$	C_E
$Gal(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$	$\eta_{C_E(x)}$:$\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow Vect_{C_E}(x)$	$C_E(x)$
$Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$	$\eta_{C_E(x)}$:$\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes,\partial} \longrightarrow Vect_{C_E(x)}$	$C_E(x)$

In order to relate the generic Galois groups and the groups defined by tensor automorphisms of fiber functors, we need to investigate first the structure of the different Picard-Vessiot rings, one can attach to $\mathcal{M}$. So first, let R be the Picard-Vessiot ring over $C(x)$, defined by Singer and van der Put. In general, R is a sum of domains $R = R_0 \oplus ... \oplus R_{t-1}$, where each component R_i is invariant under the action of σ_q^t. The positive integer t corresponds to the number of connected components of the q-difference Galois group $Aut^{\otimes}(\mathcal{M}_{C(x)}, \omega_C)$ of $\mathcal{M}_{C(x)}$. Following [vdPS97, Lemma 1.26], we consider now $\mathcal{M}^t_{C(x)}$, the t-th iterate of $\mathcal{M}_{C(x)}$, which is a q^t-difference module over $C(x)$. Since the Picard-Vessiot ring of $\mathcal{M}^t_{C(x)}$ is isomorphic to one of the components of R, say R_0, its q^t-difference Galois group (resp. its generic Galois group) is equal to the identity component of $Aut^{\otimes}(\mathcal{M}_{C(x)}, \omega_C)$ (resp. $Gal(\mathcal{M}_{C(x)}, \eta_C)$). Then, let R_M be, as in §8.2, the weak Picard-Vessiot ring attached to $\mathcal{M}_{C_E(x)}$ over $C_E(x)$. Since the latter is contained in $\mathcal{M}er(C^*)$, it is an integral domain and its subfield of constants is C_E.

Proposition 9.7. *Let us denote by F_0 and F_M the fractions fields of R_0 and R_M. We have the following isomorphisms of linear algebraic groups:*

(1) $Aut^{\otimes}(\mathcal{M}_{C(x)}, \omega_C)^{\circ} \otimes_K F_0 \simeq Gal(\mathcal{M}, \eta_{C(x)})^{\circ} \otimes_{K(x)} F_0$, *where G° denotes the identity component of a group G*

(2) $Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_E) \otimes_{C_E} F_M \simeq Gal(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \otimes_{C_E(x)} F_M$;

and also the isomorphisms of linear differential algebraic groups:

(3) $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_E) \otimes_{C_E} F_M \simeq Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \otimes_{C_E(x)} F_M$.

Proof. This is an analogue of [Kat82, Proposition 4.1] and we only give a sketch of proof in the case of $\mathcal{M}_{C_E(x)}$. Since R_M is a (σ_q, ∂)-Picard-Vessiot ring, we have an isomorphism of R_M-module between

$$\omega_E(\mathcal{M}_{C_E(x)}) \otimes_{C_E} R_M = Ker(\Sigma_q - Id, R_M \otimes \mathcal{M}_{C_E(x)}) \otimes R_M \simeq \mathcal{M}_{C_E(x)} \otimes_{C_E(x)} R_M.$$

Extending the scalars from R_M to F_M yields to the required isomorphism

$$\omega_E(\mathcal{M}_{C_E(x)}) \otimes_{C_E} F_M \simeq \mathcal{M}_{C_E(x)} \otimes_{C_E(x)} F_M,$$

which in view of its construction is compatible with the constructions of differential linear algebra. In particular, if $\mathcal{W} \subset Constr^{\partial}_{C_E(x)}(\mathcal{M}_{C_E(x)})$ then we have,

$$\omega_E(\mathcal{W}) \otimes_{C_E} F_M \simeq \mathcal{W} \otimes_{C_E(x)} F_M,$$

inside $Constr^{\partial}_{C_E}(\omega_E(\mathcal{M}_{C_E(x)}))\otimes_{C_E}F_M \simeq Constr^{\partial}_{C_E(x)}(\mathcal{M}_{C_E(x)})\otimes_{C_E(x)}F_M$. These canonical identifications give a canonical isomorphism of linear differential algebraic groups over F_M,

$$Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)},\omega_E)\otimes_{C_E}F_M \simeq Gal^{\partial}(\mathcal{M}_{C_E(x)},\eta_{C_E(x)})\otimes_{C_E(x)}F_M.$$

□

Remark 9.8. This proposition expresses the fact that the Picard-Vessiot ring is a bitorsor (differential bitorsor when it makes sense) under the action of the generic (differential) Galois group and the Picard-Vessiot (differential) group.

Since the dimension of a differential algebraic group as well as the differential transcendence degree of a field extension do not vary up to field extension one has proved the following corollary

Corollary 9.9. *Let $\mathcal{M}_{K(x)}$ be a q-difference module defined over $K(x)$. Let R_M be the weak (σ_q,∂)-Picard-Vessiot ring over $C_E(x)$ generated by the meromorphic solutions of $\mathcal{M}_{K(x)}$ and let F_M be its fraction field. Then, the ∂-differential dimension of F_M (*cf. [HS08, p. 337] *for definition and references) over $\widetilde{C}_E(x)$ is equal to the differential dimension of $Gal^{\partial}(\mathcal{M}_{C_E(x)},\eta_{C_E(x)})$.*

Proof. Theorem 9.5 and Proposition 9.7 give the desired equality. □

9.2.2. *Reduction to $C(x)$ and $C_E(x)$.* The following lemma shows how, for any field extension L/K, the differential generic Galois group of $\mathcal{M}_{L(x)}$ is equal, up to scalar extension, to the differential generic Galois group of $\mathcal{M}_{K'(x)}$ for a finitely generated extension K'/K.

Lemma 9.10. *Let L be a field extension of K with $\sigma_q|_L = id$. There exists a finitely generated extension $L/K'/K$ such that*

$$Gal(\mathcal{M}_{L(x)},\eta_{L(x)}) \cong Gal(\mathcal{M}_{K'(x)},\eta_{K'(x)})\otimes_{K'(x)}L(x)$$

and

$$Gal^{\partial}(\mathcal{M}_{L(x)},\eta_{L(x)}) \cong Gal^{\partial}(\mathcal{M}_{K'(x)},\eta_{K'(x)})\otimes_{K'}L(x).$$

These equalities hold then we replace K' by any subfield extension of L containing K'.

Proof. By definition, $Gal^{\partial}(\mathcal{M}_{L(x)},\eta_{L(x)})$ is the stabilizer inside $Gl(M_{L(x)})$ of all $L(x)$-vector spaces of the form $W_{L(x)}$ for $\mathcal{W}$ object of $\langle\mathcal{M}_{L(x)}\rangle^{\otimes,\partial}$. Similarly, for any field extension $L/K'/K$, we have

$$Gal^{\partial}(\mathcal{M}_{K'(x)},\eta_{K'(x)}) = Stab(W_{K'(x)},\mathcal{W}\text{ object of }\langle\mathcal{M}_{K'(x)}\rangle^{\otimes,\partial}).$$

Then,

$$Gal^{\partial}(\mathcal{M}_{L(x)},\eta_{L(x)}) \subset Gal^{\partial}(\mathcal{M}_{K'(x)},\eta_{K'(x)})\otimes L(x).$$

By noetherianity, the (differential) generic Galois group of $\mathcal{M}_{L(x)}$ is defined by a finite family of (differential) polynomial equations, thus we can choose K' more carefully. □

Since K'/K is of finite type, if we can calculate the group $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ (resp. $Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})$) by a curvature procedure. The same holds for the group $Gal(\mathcal{M}_{K'(x)},\eta_{K'(x)})$ (resp. $Gal^{\partial}(\mathcal{M}_{K'(x)},\eta_{K'(x)})$) and thus for $Gal(\mathcal{M}_{L(x)},\eta_{L(x)})$ (resp. $Gal^{\partial}(\mathcal{M}_{L(x)},\eta_{L(x)})$). Applying these considerations to $L = C$ or $L = C_E$, we will forget the field K for a while, keeping in mind that the generic Galois group of $\mathcal{M}$ over $C(x)$ or over $C_E(x)$, may also be computed with the help of curvatures defined over a smaller field.

Proposition 9.11. *The differential linear algebraic group $Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$ is defined over $C(x)$ and we have isomomorphism of linear differential algebraic groups:*

$$Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \tilde{\longrightarrow} Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)}) \otimes C_E(x).$$

The same holds for the generic Galois groups, i.e. *we have an isomorphism of linear algebraic groups*

$$Gal(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \tilde{\longrightarrow} Gal(\mathcal{M}_{C(x)}, \eta_{C(x)}) \otimes C_E(x).$$

Proof. We give the proof in the differential case. The same argument than in the proof of Lemma 9.10 gives the inclusion

$$Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \subset Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)}) \otimes C_E(x).$$

The group $Aut^{\partial}(C_E(x)/C(x))$ of $C(x)$-differential automorphism of $C_E(x)$ acts on $\mathcal{M}_{C_E(x)}$ via the semi-linear action $(\tau \to id \otimes \tau)$. Thus the latter group acts on $Constr^{\partial}_{C_E(x)}(\mathcal{M}_{C_E(x)}) = Constr^{\partial}_{C(x)}(\mathcal{M}_{C(x)}) \otimes C_E$. Since this action commutes with σ_q, it therefore permutes the subobjects of $Diff(C_E(x), \sigma_q)$ contained in $\mathcal{M}_{C_E(x)}$. Since $C_E(x)^{Aut^{\partial}(C_E/C)} = C(x)$ (*cf.* [CHS08, Lemma 3.3]), we obtain that $Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$ is defined over $C(x)$. Putting all together, we have shown that $Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$ is equal to $G \otimes_{C(x)} C_E(x)$ where G is a linear differential subgroup of $Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)})$ defined over $C(x)$. This implies that we can choose a line L in a construction of differential algebra of $\mathcal{M}_{C(x)}$ such that $G = Stab(L)$. By Lemma 9.10, there exists a finitely generated extension K'/K, such that $K' \subset C$ and that:

- $Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)}) \cong Gal^{\partial}(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes C(x)$;
- the line L is defined over $K'(x)$ (and hence so does G).

Since C_E is purely transcendent over the algebraically closed field C, we call also choose a purely transcendental finitely generated extension K''/K', with $K'' \subset C_E$, such that

$$Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \cong Gal^{\partial}(\mathcal{M}_{K''(x)}, \eta_{K''(x)}) \otimes C_E(x).$$

Since $Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) = G \otimes_{C(x)} C_E(x)$, the v-curvatures of $\mathcal{M}_{K''(x)}$ must stabilise L modulo ϕ_v, in the sense of Theorem 6.8. On the other hand, L is $K'(x)$-rational and the v-curvatures of $\mathcal{M}_{K''(x)}$ come from the v-curvatures of $\mathcal{M}_{K'(x)}$ by scalar extensions, therefore L is also stabilized by the v-curvatures of $\mathcal{M}_{K'(x)}$. This proves that $Gal^{\partial}(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes C(x) \subset G = Stab(L)$ and ends the proof. □

Corollary 9.12. *Let $\mathcal{M}_{K(x)}$ be a q-difference module defined over $K(x)$. Let $U \in Gl_{\nu}(\mathcal{M}er(C^*))$ be a fundamental matrix of meromorphic solutions of $\mathcal{M}_{K(x)}$. Then,*

1. *the dimension of $Gal(\mathcal{M}_{C(x)}, \eta_{C(x)})$ is equal to the transcendence degree of the field generated by the entries of U over $C_E(x)$,* i.e. *the algebraic group $Gal(\mathcal{M}_{C(x)}, \eta_{C(x)})$ measures the algebraic relations between the meromorphic solutions of $\mathcal{M}_{C_E(x)}$.*
2. *the ∂-differential dimension of $Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)})$ is equal to the differential transcendence degree of the field generated by the entries of U over $\widetilde{C}_E(x)$,* i.e. *the differential algebraic group $Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)})$ gives an upper bound for the differential algebraic relations between the meromorphic solutions of $\mathcal{M}_{K(x)}$.*
3. *there exists a finitely generated extension K'/K such that the differential transcendence degree of the differential field generated by the entries of U over $\widetilde{C}_E(x)$ is equal to the differential dimension of $Gal^{\partial}(\mathcal{M}_{K'(x)}, \eta_{K'(x)})$.*

Proof. 1. Proposition 9.7 and Proposition 9.11 prove that the dimension of the generic Galois group $Gal(\mathcal{M}_{C(x)}, \eta_{C(x)})$ is equal to the dimension of the group $Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_E)$ over $\widetilde{C}_E(x)$, that is to the transcendence degree of the fraction field of $\widetilde{R}_M$ over $\widetilde{C}_E(x)$.
2. Put together Corollary 9.9 and Proposition 9.11.
3. This is Lemma 9.10.

□

Remark 9.13. [HS08, Proposition 6.18] induces a one-to-one correspondence between the radical (σ_q, ∂)-ideals of the (σ_q, ∂)-Picard-Vessiot ring of the module and the differential subvarieties of the differential Galois group $Aut^{\otimes,\partial}(\mathcal{M}_{\widetilde{C}_E(x)}, \widetilde{\omega}_E)$. The comparison results of this section, show that this correspondence induces a correspondence between the differential subvarieties of the differential generic Galois group of $\mathcal{M}$ and the radical (σ_q, ∂)-ideals of the (σ_q, ∂)-Picard-Vessiot generated by the meromorphic solutions of the module.

10. Specialization of the parameter q

We go back to the notation introduced in §1. So we consider a field K which is a finite extension of a rational function field $k(q)$ of characteristic zero. Let $\mathcal{M} = (M, \Sigma_q)$ be a q-difference module over an algebra $\mathcal{A}$ of the form (1.1). For almost all $v \in \mathcal{C}$ such that $\kappa_v > 1$, we can consider the reduction of $\mathcal{M}$ modulo v, namely the q_v-difference module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$ introduced in §1. For this q_v-difference module, we can define a generic Galois group $Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$ associated to the forgetful functor $\eta_{k_v(x)}$ defined on the tensor category generated by $\mathcal{M}_{k_v(x)}$, with value in the category of $k_v(x)$-vector spaces. The category $Diff(k_v(x), \sigma_{q_v})$ is naturally a differential tannakian category for the derivation $\partial := x\frac{d}{dx}$ acting on $k_v(x)$ (*cf.* Proposition 8.1) and we may define, as in Definition 5.3, the differential generic Galois group $Gal^{\partial}(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$ of the q_v-difference module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$.

It follows from Theorem 6.8 that the generic Galois group $Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$ (resp. $Gal^{\partial}(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$) is the smallest algebraic (resp. differential) subgroup of $Gl(M)$ containing $\Sigma_{q_v}^{\kappa_v}$, *i.e.* the reduction of $\Sigma_q^{\kappa_v}$ modulo v. Theorem 6.7 (resp. 6.8), combined with Corollary 4.10 (resp. Corollary 5.15) implies:

Corollary 10.1. *In the notation of Theorem 4.5 (resp. Theorem 5.11), if the group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ (resp. $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is defined as $Stab(L_{K(x)})$, then for almost all $v \in \mathcal{C}$ such that $\kappa_v > 1$ we have:*

$$Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)}) \hookrightarrow Stab_{\mathcal{A}}(L) \otimes_{\mathcal{A}} k_v(x).$$

$$(\textit{resp. } Gal^{\partial}(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)}) \hookrightarrow Stab_{\mathcal{A}}(L) \otimes_{\mathcal{A}} k_v(x).)$$

We have proved that the reduction modulo $v \in \mathcal{C}$ of the generic Galois group (resp. differential generic Galois group) gives an upper bound for the generic Galois group (resp. differential generic Galois group) of the specialized q_v-difference module.

More generally, for almost all $v \in \mathcal{P}_f$, we can consider the $k_v(x)$-module $M_{k_v(x)}$, endowed with a natural structure of q_v-difference module, if $q_v \neq 1$. If we can specialize modulo $q-1$, then we get a differential module, whose connection is induced by the action of the operator Δ_q on M. We call the module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$ the *specialization* of $\mathcal{M}$ at v. It is naturally equipped with a generic Galois group $Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$, associated to the forgetful functor from the tensor category generated by $\mathcal{M}_{k_v(x)}$ to the category of $k_v(x)$-vector spaces and we can ask how the group of the specialization is related to the specialization of the group of $\mathcal{M}$. For $v \in \mathcal{C}$, Corollary 10.1 shows that the specialization of the group is an upper bound for the group of the specialization whereas Theorem 5.11 proves that one may recover $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ from the knowledge of almost all of generic Galois groups of its specializations at a cyclotomic place.

These problems have been studied by Y. André in [And01] where he shows, among other things, that the Picard-Vessiot groups have a nice behavior w.r.t. the specialization. Some of our results (see Proposition 10.16 for instance) are nothing more than slight adaptation of the results of André to a differential and generic context. However combined with Theorem 5.11, they lead to a description via curvatures of the generic Galois group of a differential equation (see Corollary 10.20).

10.1. **Specialization of the parameter q and localization of the generic Galois group.** Since specializing q we obtain both differential and q-difference modules, the best framework for studying the reduction of generic Galois groups is André's theory of generalized differential rings (*cf.* [And01, 2.1.2.1]). For clarity of exposition, we first recall some definitions and basic facts from [And01]) and then deduce some results on the relation between $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ and $Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$(see Proposition 10.16). Their proof are inspired by analogous results for the local Galois group which can be found in [And01].

10.1.1. *Generalized differential rings.* In §10.1.1 and only in §10.1.1, we adopt a slightly more general notation.

Definition 10.2 (*cf.* [And01, 2.1.2.1])**.** Let R be a commutative ring with unit. A generalized differential ring (A, d) over R is an associative R-algebra A endowed with an R-derivation d from A into a left $A \otimes_R A^{op}$-module Ω^1. The kernel of d, denoted $Const(A)$, is called the set of constants of A.

Example 10.3.

(1) Let k be a field and $k(x)$ be the field of rational functions over k. The ring $(k(x), \delta)$, with

$$\begin{array}{cccc} \delta: & k(x) & \longrightarrow & \Omega^1 := dx.k(x) \\ & f & \longmapsto & dx.x\dfrac{df}{dx} \end{array},$$

is a generalized differential ring over k, associated to the usual derivation $\partial := x\frac{d}{dx}$.

(2) Let $\mathcal{A}$ be the ring defined in (1.1). The ring $(\mathcal{A}, \delta_q)$, with

$$\begin{array}{cccc} \delta_q: & \mathcal{A} & \longrightarrow & \Omega^1 := dx.\mathcal{A} \\ & f & \longmapsto & dx.xd_qf \end{array},$$

is also a generalized differential rings over $\mathcal{O}_K$, associated to the q-difference algebra $(\mathcal{A}, \sigma_q)$.

(3) Let C denote the ring of constants of a generalized differential ring (A, d) and let I be a nontrivial proper prime ideal of C. Then the ring $A_I := A \otimes C/I$ is endowed with a structure of generalized differential ring (*cf.* [And01, 3.2.3.7]). In the notation of the example above, for almost any place $v \in \mathcal{P}_f$ of K, we obtain in this way a generalized differential ring of the form $(\mathcal{A} \otimes_{\mathcal{O}_K} k_v, \delta_{q_v})$.

Definition 10.4 (*cf.* [And01, 2.1.2.3])**.** A morphism of generalized differential rings $(A, d : A \mapsto \Omega^1) \longmapsto (\widetilde{A}, \widetilde{d} : \widetilde{A} \mapsto \widetilde{\Omega}^1)$ is a pair $(u = u^0, u^1)$ where $u^0 : A \mapsto \widetilde{A}$

is a morphism of R-algebras and u^1 is a map from Ω^1 into $\widetilde{\Omega}^1$ satisfying

$$\begin{cases} u^1 \circ d = \widetilde{d} \circ u^0, \\ u^1(a\omega b) = u^0(a)u^1(\omega)u^0(b), \text{ for any } a, b \in A \text{ and any } \omega \in \Omega^1. \end{cases}$$

Example 10.5. In the notation of the Example 10.3, the canonical projection $p : A \mapsto A_I$ induces a morphism u of generalized differential rings from (A, d) into (A_I, d).

Let B be a generalized differential ring. We denote by $Diff_B$ the category of B-modules with connection (*cf.* [And01, 2.2]), *i.e.* left projective B-modules of finite type equipped with a R-linear operator

$$\nabla : M \longrightarrow \Omega^1 \otimes_A M,$$

such that $\nabla(am) = a\nabla(m) + d(a) \otimes m$. The category $Diff_B$ is abelian, $Const(B)$-linear, monoidal symmetric, *cf.* [And01, Theorem 2.4.2.2].

Example 10.6. We consider once again the different cases as in Example 10.3:

(1) If $B = (k(x), \delta)$ then $Diff_B$ is the category of differential modules over $k(x)$.
(2) If $B = (\mathcal{A}, \delta_q)$ then $Diff_B$ is the category of q-difference modules over $\mathcal{A}$. In fact, in the notation of the previous section, it is enough to set $\nabla(m) = dx.\Delta_q(m)$, for any $m \in M$.

Let B be a generalized differential ring. We denote by η_B the forgetful functor from $Diff_B$ into the category of projective B-modules of finite type. For any object M of $Diff_B$, we consider the forgetful functor η_B induced over the full subcategory $\langle M \rangle_B^{\otimes}$ of $Diff_B$ generated by M and the affine B- group-scheme $Gal(M, \eta_B)$ defined over B representing the functor $Aut^{\otimes}(\eta_B|_{\langle M \rangle_B^{\otimes}})$.

Definition 10.7. The B-scheme $Gal(M, \eta_B)$ is called the *generic Galois group* of M.

Let $Constr_B(M)$ be the collection of all constructions of linear algebra of M, *i.e.* of all the objects of $Diff_B$ deduced from M by the following B-linear algebraic constructions: direct sums, tensor products, duals, symmetric and antisymmetric products. Then one can show that $Gal(M, \eta_B)$ is nothing else that the generic Galois group considered in section 4 (*cf.* [And01, 3.2.2.2]):

Proposition 10.8. *Let B be a generalized differential ring and let M be an object of $Diff_B$. The affine groups scheme $Gal(M, \eta_B)$ is the stabilizer inside $Gl(M)$ of all submodules with connection of some algebraic constructions of M.*

This is not the only Galois group one can define. If we assume the existence of a fiber functor ω from $Diff_B$ into the category of $Const(B)$-module of finite type, we can define the Galois group $Aut^{\otimes}(\omega|_{\langle M \rangle_B^{\otimes}})$ of an object M as the group of tensor automorphism of the fiber functor ω restricted to $\langle M \rangle_B^{\otimes}$ (*cf.* [And01, 3.2.1.1]). This group characterizes completely the object M. For further reference, we recall the following property (*cf.* [And01, Theorem 3.2.2.6]):

Proposition 10.9. *The object M is trivial if and only if $Aut^{\otimes}(\omega|_{\langle M \rangle_B^{\otimes}})$ is a trivial group.*

In certain cases, the category $Diff_B$ may be endowed with a differential structure. Since $Diff_B$ is not necessarily defined over a field, we say that a category $\mathcal{C}$ is a *differential tensor* category, if it satisfies all the axioms of [Ovc09a, Definition 3] except the assumption $End(\mathbf{1})$ is a field. We detail below the construction of the prolongation functor associated to $Diff_B$ in some precise cases.

Semi-classic situation. Let us assume that (B, ∂) is a differential subring of the differential field $(L(x), \partial := x\frac{d}{dx})$. Then $Diff_B$ is the category of differential B-modules, equivalently, of left $B[\partial]$-modules M, free and finitely generated over B. We now define a prolongation functor F_B for this category as follows. If $\mathcal{M} = (M, \nabla)$ is an object of $Diff_B$ then $F_B(\mathcal{M}) = (M^{(1)}, \nabla)$ is the differential module defined by $M^{(1)} = B[\partial]_{\leq 1} \otimes M$, where the tensor product rule is the same one as in §5.1 (*i.e.* takes into account the Leibniz rule).

Remark 10.10. This formal definition may be expressed in a very simple and concrete way by using the differential equation attached to the module. If M is an object of $Diff_B$ given by a differential equation $\partial(Y) = AY$, the object $M^{(1)}$ is attached to the differential equation: $\partial(Z) = \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix} Z$.

Mixed situation. Let us assume that B is a generalized differential subring of some q (resp. q_v)-difference differential field $(L(x), \delta_q)$ (resp. $(L(x), \delta_{q_v})$). The category $Diff_B$ is the category of q (resp. q_v)-difference modules. Applying the same constructions than those of Proposition 8.1, we have that $Diff_B$ is a differential tannakian category and we will denote by F_B its prolongation functor.

In both cases, semi-classic and mixed, we may define, as in Definition 5.3, the differential generic Galois group $Gal^{\partial}(\mathcal{M}, \eta_B)$ of an object $\mathcal{M}$ of $Diff_B$.

Remark 10.11. In the semi-classic situation, the differential generic Galois group of a differential module $\mathcal{M}$ is nothing else than the generic Galois group of $\mathcal{M}$. To see this it is enough to notice that there exists a canonical isomorphism:

$$Gal(F(\mathcal{M}), \eta_{K(x)}) \longrightarrow Gal(\mathcal{M}, \eta_{K(x)}).$$

In fact, such an arrow exists since $\mathcal{M}$ is canonically isomorphic to a differential submodule of $F(\mathcal{M})$. Since an element $B \in Gal(\mathcal{M}, \eta_{K(x)})$ acts on $F(\mathcal{M})$ via $\begin{pmatrix} B & \partial B \\ 0 & B \end{pmatrix}$, the arrow is injective. Since an element of $Gal(\mathcal{M}, \eta_{K(x)})$ needs to be sufficiently compatible with the differential structure, it also stabilizes the differential submodules of a construction of $F(\mathcal{M})$. This last argument proves the surjectivity.

The definition below characterizes the morphisms of generalized differential rings compatible with the differential structure. We will need this notion in Lemma 10.15:

Definition 10.12 (*cf.* [And01, 2.2.2])**.** Let $u = (u^0, u^1) : (A, d) \mapsto (A', d')$ be a morphism of generalized differential rings. This morphism induces a tensor-compatible functor denoted by u^* from the category $Diff_A$ into the category $Diff_{A'}$. Moreover, let us assume that $Diff_A$ (resp. $Diff_{A'}$) is a differential category and let us denote by F_A (resp. $F_{A'}$) its prolongation functor. We say that u^* is *differentially compatible* if it commutes with the prolongation functors, *i.e.* $F_{A'} \circ u^* = u^* \circ F_A$.

10.1.2. *Localization and specialization of generic Galois groups.* We go back to the notation of the beginning of §4. We moreover assume that $\mathcal{A}$ (resp. $\mathcal{A}_v := \mathcal{A} \otimes_{\mathcal{O}_K} k_v$) is stable under the action of ∂. As already noticed, the q-difference algebras $\mathcal{A}$ and $\mathcal{A}_v$ are simple generalized differential rings (*cf.* [And01, 2.1.3.4, 2.1.3.6]). Moreover, the fraction field of $\mathcal{A}$ (resp. $\mathcal{A}_v$) is $K(x)$ (resp. $k_v(x)$). If $q_v \neq 1$, the ring $(\mathcal{A}_v, \sigma_{q_v}, \partial := x\frac{d}{dx})$ (resp. the field $(k_v(x), \sigma_{q_v}, \partial := x\frac{d}{dx})$) is a q_v-difference differential ring (resp.field)

The following lemma of *localization* relates the generic (differential) Galois group of a module over the ring $\mathcal{A}$ (resp. over $\mathcal{A}_v$) to the generic (differential) Galois group of its localization over the fraction field $K(x)$ (resp. $k_v(x)$) of $\mathcal{A}$ (resp. $\mathcal{A}_v$). This lemma is a version of [And01, Lemma 3.2.3.6] for (differential) generic Galois groups.

Proposition 10.13. *Let $\mathcal{M},\mathcal{A},v,\mathcal{A}_v$ as above. We have*

(1) $Gal(\mathcal{M},\eta_{\mathcal{A}})\otimes K(x)\simeq Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$;
(2) $Gal^{\partial}(\mathcal{M},\eta_{\mathcal{A}})\otimes K(x)\simeq Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})$
(3) $Gal(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v,\eta_{\mathcal{A}_v})\otimes k_v(x)\simeq Gal(\mathcal{M}_{k_v(x)},\eta_{k_v(x)})$.
(4) $Gal^{\partial}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v,\eta_{\mathcal{A}_v})\otimes k_v(x)\simeq Gal^{\partial}(\mathcal{M}_{k_v(x)},\eta_{k_v(x)})$.

Remark 10.14. In the previous section we have given a description of the generic Galois group $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ *via* the reduction modulo ϕ_v of the operators $\Sigma_q^{\kappa_v}$. We are unable to give a similar description of $Gal(\mathcal{M},\eta_{\mathcal{A}})$, essentially because Chevalley theorem holds only for algebraic groups over a field.

Proof. Because $\mathcal{A}$ (resp. $\mathcal{A}_v$) is a simple differential ring and its fraction field $K(x)$ (resp $k_v(x)$) is semi-simple, we may apply [And01, lemma 3.2.3.6] and [And01, proposition 2.5.1.1]. We obtain that the functor

$$\begin{array}{cccc} Loc: & \langle\mathcal{M}\rangle_{\mathcal{A}}^{\otimes,\partial} & \longrightarrow & \langle\mathcal{M}_{K(x)}\rangle_{K(x)}^{\otimes,\partial} \\ & \mathcal{N} & \longmapsto & \mathcal{N}_{K(x)} \end{array}$$

is an equivalence of monoidal categories. Moreover, Loc commutes with the prolongation functors, *i.e.* $F_{K(x)}\circ Loc=Loc\circ F_{\mathcal{A}}$. To conclude it is enough to remark that Loc also commutes with the forgetful functors. □

So everything works quite well for the localization. Before proving some results concerning the specialization, we state an analogue of [And01, lemma 3.2.3.5] on the compatibility of constructions.

Lemma 10.15. *Let $u:(A,d)\mapsto(B,\tilde{d})$ be a morphism of integral generalized differential rings, such that B is faithfully flat over A. Then for any object M of $Diff_A$ we have*

$$Constr_A(M)\otimes_A B=Constr_B(M\otimes_A B),$$

i.e. *the constructions of linear algebra commute with the base change. If we assume moreover that $Diff_A$ and $Diff_B$ are differential tensor categories and that u^* is differentially compatible, we have*

$$Constr_A^{\partial}(M)\otimes_A B=Constr_B^{\partial}(M\otimes_A B),$$

*where $Constr^{\partial}$ denotes the construction of differential linear algebra (*cf. *§5.1).*

Proof. Because M is a projective A-module of finite type and B is faithfully flat over A, the canonical map $Hom_A(M,A)\otimes B\mapsto Hom_B(M\otimes B,B)$ is bijective. The first statement follows from this remark. The last one follows immediately from the first and from the definition of a differentially compatible functor (*cf.* Definition 10.12). □

Finally, we have:

Proposition 10.16. *Let $(\mathcal{A},\delta_q)$ be the generalized differential ring associated to the q-difference algebra* (1.1)*. Let $v\in\mathcal{P}_{K,f}$. For any $\mathcal{M}$ object of $Diff_{\mathcal{A}}$, we have*

$$Gal(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v,\eta_{\mathcal{A}_v})\subset Gal(\mathcal{M},\eta_{\mathcal{A}})\otimes\mathcal{A}_v$$

and

$$Gal^{\partial}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v,\eta_{\mathcal{A}_v})\subset Gal^{\partial}(\mathcal{M},\eta_{\mathcal{A}})\otimes\mathcal{A}_v.$$

Proof. By definition, $Gal(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v,\eta_{\mathcal{A}_v}) = Aut^{\otimes}(\eta_{\mathcal{A}_v}|_{\langle\mathcal{M}\otimes\mathcal{A}_v\rangle^{\otimes}_{\mathcal{A}_v}})$ is the stabilizer inside $\mathcal{G}l(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v) = \mathcal{G}l(\mathcal{M})\otimes_{\mathcal{A}}\mathcal{A}_v$ of the subobjects W of a construction of linear algebra of $M\otimes_{\mathcal{A}}\mathcal{A}_v$. The group $Gal(\mathcal{M},\eta_{\mathcal{A}})$ admits a similar description. The projection map $p:\mathcal{A}\mapsto\mathcal{A}_v$ is a morphism of generalized differential rings. Since $\mathcal{A}_v$ is faithfully flat over $\mathcal{A}$, we may thus apply the first part of Lemma 10.15 and we conclude that $Constr_{\mathcal{A}}(\mathcal{M})\otimes_{\mathcal{A}}\mathcal{A}_v = Constr_{\mathcal{A}_v}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v)$ and therefore that $Gal(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v,\eta_{\mathcal{A}_v})\subset Gal(\mathcal{M},\eta_{\mathcal{A}})\otimes_{\mathcal{A}}\mathcal{A}_v$. We give now a sketch of proof for the differential part.

If we assume that $\mathcal{A}$ is stable under the action of ∂ then the category $Diff_{\mathcal{A}}$ is a differential tensor category as it is described in the *mixed situation* of §10.1.1 and we denote by $F_{\mathcal{A}}$ its prolongation functor. Moreover, $Diff_{\mathcal{A}_v}$ is also a differential tensor category, either $q_v = 1$ and we are in the *classical situation*, either $q_v \neq 1$ and we are in the *mixed situation*. In both cases, a simple calculation shows that the projection map $p:\mathcal{A}\mapsto\mathcal{A}_v$ induces a differentially compatible functor p^* from $Diff_{\mathcal{A}}$ into $Diff_{\mathcal{A}_v}$. Then Lemma 10.15, the arguments above and the definition of the differential generic Galois group in terms of stabilizer of objets inside the construction of differential algebra give the last inclusion. □

Remark 10.17. Similar results hold for differential equations (*cf.* [Kat90, §2.4] and [And01, §3.3]). In general one cannot obtain any semicontinuity result. In fact, the differential equation $\frac{y'}{y}=\frac{\lambda}{y}$, with λ complex parameter, has differential Galois group equal to $\mathbb{C}^*$. When one specializes the parameter λ on a rational value λ_0, one gets an equation whose differential Galois group is a cyclic group of order the denominator of λ_0. For all other values of the parameter, the Galois group is $\mathbb{C}^*$.

The situation appears to be more rigid for q-difference equations when q is a parameter. In fact, we can consider the q-difference equation $y(qx) = P(q)y(x)$, with $P(q)\in k(q)$. If we specialize q to a root of unity and we find a finite generic Galois group too often (*cf.* Lemma 2.9 and Corollary 4.12), we can conclude that $P(q)\in q^{\mathbb{Z}/r}$, for some positive integer r, and therefore that the generic Galois group of $y(qx) = P(q)y(x)$ over $K(x)$ is finite.

10.2. **Upper bounds for the generic Galois group of a differential equation.** Let us consider a q-difference module $\mathcal{M} = (M,\Sigma_q)$ over $\mathcal{A}$ that admits a reduction modulo the $(q-1)$-adic place of K, *i.e.* such that we can specialize the parameter q to 1. To simplify notation, let us denote by k_1 the residue field of K modulo $q-1$.

In this case the specialized module $\mathcal{M}_{k_1(x)} = (M_{k_1(x)},\Delta_1)$ is a differential module. We can deduce from the results above that:

Corollary 10.18.
$$Gal(\mathcal{M}_{k_1(x)},\eta_{k_1(x)})\subset Gal(\mathcal{M},\eta_{\mathcal{A}})\otimes k_1(x).$$
and
$$Gal^{\partial}(\mathcal{M}_{k_1(x)},\eta_{k_1(x)})\subset Gal^{\partial}(\mathcal{M},\eta_{\mathcal{A}})\otimes k_1(x).$$

Proof. Proposition 10.16 says that:
$$Gal(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}/(q-1),\eta_{\mathcal{A}/(q-1)})\subset Gal(\mathcal{M},\eta_{\mathcal{A}})\otimes\mathcal{A}/(q-1),$$
and
$$Gal^{\partial}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}/(q-1),\eta_{\mathcal{A}/(q-1)})\subset Gal^{\partial}(\mathcal{M},\eta_{\mathcal{A}})\otimes\mathcal{A}/(q-1),$$
We conclude applying Proposition 10.13:
$$Gal(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}/(q-1),\eta_{\mathcal{A}/(q-1)})\otimes_{\mathcal{A}/(q-1)}k_1(x)\cong Gal(\mathcal{M}_{k_1(x)},\eta_{k_1(x)}),$$
and
$$Gal^{\partial}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}/(q-1),\eta_{\mathcal{A}/(q-1)})\otimes_{\mathcal{A}/(q-1)}k_1(x)\cong Gal^{\partial}(\mathcal{M}_{k_1(x)},\eta_{k_1(x)}),$$

remembering that $k_1(x)$ is flat over $\mathcal{A}/(q-1)$. □

Remark 10.19. An example of application of the theorem above is given by the "Schwartz list" for q-difference equations (*cf.* [DV02, Appendix]), where it is proved that the trivial basic q-difference equations are exactly the deformation of the trivial Gauss hypergeometric differential equations.

The Schwartz list for higher order basic hypergeometric equations has been established by J. Roques (*cf.* [Roq09, §8]), and is another example of this phenomenon.

On the other hand, given a $k(x)/k$-differential module (M,∇), we can fix a basis $\underline{e}$ of M such that

$$\nabla(\underline{e}) = \underline{e}G(x),$$

where we have identified ∇ with $\nabla\left(\frac{d}{dx}\right)$. The horizontal vectors for ∇ are solutions of the system $Y'(x) = -G(x)Y(x)$. Then, if $K/k(q)$ is a finite extension, we can define a natural q-difference module structure over $M_{K(x)} = M \otimes_{k(x)} K(x)$ setting

$$\Sigma_q(\underline{e}) = \underline{e}\left(1+(q-1)xG(x)\right),$$

and extending the action of Σ_q to $M_{K(x)}$ by semi-linearity. The definition of Σ_q depends on the choice of $\underline{e}$, so that we should rather write $\Sigma_q^{(\underline{e})}$, which we avoid to not complicate the notation. Thus, starting from a differential module M we may find a q-difference module $M_{K(x)}$ such that M is the specialization of $M_{K(x)}$ at the place of K defined by $q=1$. The q-deformation we have considered here is somehow a little bit trivial and does not correspond for instance to the process used to deform a hypergeometric differential equation into a q-hypergeometric equation. Anyway, we just want to show that a q-deformation combined with our results gives an arithmetic description of the generic Galois group of a differential equation. This description depends obviously of the process of q-deformation and its refinement is strongly related to the sharpness of the q-deformation used.

Using the "trivial" q-deformation, we have the following description

Corollary 10.20. *The generic Galois group of (M,∇) is contained in the "specialization at $q=1$" of the smallest algebraic subgroup of $Gl(M_{K(x)})$ containing the reduction modulo ϕ_v of $\Sigma_q^{\kappa_v}$:*

$$\Sigma_q^{\kappa_v}\underline{e} = \underline{e}\prod_{i=0}^{\kappa_v-1}\left(1+(q-1)q^ixG(q^ix)\right)$$

for almost all $v\in\mathcal{C}_K$.

Corollary 10.21. *Suppose that k is algebraically closed. Then a differential module (M,∇) is trivial over $k(x)$ if and only if there exists a basis $\underline{e}$ such that $\nabla(\underline{e}) = \underline{e}G(x)$ and for almost all primitive roots of unity ζ in a fixed algebraic closure $\overline{k}$ of k we have:*

$$\left[\prod_{i=0}^{n-1}\left(1+(q-1)q^ixG(q^ix)\right)\right]_{q=\zeta} = \textit{identity matrix},$$

where n is the order of ζ.

Proof. If the identity above is verified, then the Galois group of (M,∇) is trivial, which implies that (M,∇) is trivial over $k(x)$. On the other hand, if (M,∇) is trivial over $k(x)$, there exists a basis $\underline{e}$ of M over $k(x)$ such that $\nabla(\underline{e}) = 0$. This ends the proof. □

Appendix A. The Galois D-groupoid of a q-difference system, by Anne Granier

We recall here the definition of the Galois D-groupoid of a q-difference system, and how to recover groups from it in the case of a linear q-difference system. This appendix thus consists in a summary of Chapter 3 of [Gra09].

A.1. **Definitions.** We need to recall first Malgrange's definition of D-groupoids, following [Mal01] but specializing it to the base space $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$ as in [Gra09] and [Gra], and to explain how it allows to define a Galois D-groupoid for q-difference systems.

Fix $\nu \in \mathbb{N}^*$, and denote by M the analytic complex variety $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$. We call *local diffeomorphism of* M any biholomorphism between two open sets of M, and we denote by $Aut(M)$ the set of germs of local diffeomorphisms of M. Essentially, a D-groupoid is a subgroupoid of $Aut(M)$ defined by a system of partial differential equations.

Let us precise what is the object which represents the system of partial differential equations in this rough definition.

A germ of a local diffeomorphism of M is determined by the coordinates denoted by $(x, X) = (x, X_1, \dots, X_\nu)$ of its source point, the coordinates denoted by $(\bar{x}, \bar{X}) = (\bar{x}, \bar{X}_1, \dots, \bar{X}_\nu)$ of its target point, and the coordinates denoted by $\frac{\partial \bar{x}}{\partial x}, \frac{\partial \bar{x}}{\partial X_1}, \dots, \frac{\partial \bar{X}_1}{\partial x}, \dots, \frac{\partial^2 \bar{x}}{\partial x^2}, \dots$ which represent its partial derivatives evaluated at the source point. We also denote by δ the polynomial in the coordinates above, which represents the Jacobian of a germ evaluated at the source point. We will allow us abbreviations for some sets of these coordinates, as for example $\frac{\partial \bar{X}}{\partial X}$ to represent all the coordinates $\frac{\partial \bar{X}_i}{\partial X_j}$ and $\partial \bar{X}$ for all the coordinates $\frac{\partial \bar{X}_i}{\partial x_j}$, $\frac{\partial \bar{X}_i}{\partial \bar{x}_j}$, $\frac{\partial \bar{X}_i}{\partial X_j}$ and $\frac{\partial \bar{X}_i}{\partial \bar{X}_j}$.

We denote by r any positive integer. We call *partial differential equation*, or only *equation*, of order $\leq r$ any fonction $E(x, X, \bar{x}, \bar{X}, \partial \bar{x}, \partial \bar{X}, \dots, \partial^r \bar{x}, \partial^r \bar{X})$ which locally and holomorphically depends on the source and target coordinates, and polynomially on δ^{-1} and on the partial derivative coordinates of order $\leq r$. These equations are endowed with a sheaf structure on $M \times M$ which we denote by $\mathcal{O}_{J_r^*(M,M)}$. We then denote by $\mathcal{O}_{J^*(M,M)}$ the sheaf of all the equations, that is the direct limit of the sheaves $\mathcal{O}_{J_r^*(M,M)}$. It is endowed with natural derivations of the equations with respect to the source coordinates. For example, one has: $D_x . \frac{\partial \bar{X}_i}{\partial X_j} = \frac{\partial^2 \bar{X}_i}{\partial x \partial X_j}$.

We will consider the pseudo-coherent (in the sense of [Mal01]) and differential ideal [22] $\mathcal{I}$ of $\mathcal{O}_{J^*(M,M)}$ as the systems of partial differential equations in the definition of D-groupoid. A *solution* of such an ideal $\mathcal{I}$ is a germ of a local diffeomorphism $g : (M, a) \to (M, g(a))$ such that, for any equation E of the fiber $\mathcal{I}_{(a,g(a))}$, the function defined by $(x, X) \mapsto E((x, X), g(x, X), \partial g(x, X), \dots)$ is null in a neighbourhood of a in M. The set of solutions of $\mathcal{I}$ is denoted by $sol(\mathcal{I})$.

The set $Aut(M)$ is endowed with a groupoid structure for the composition c and the inversion i of the germs of local diffeomorphisms of M. We thus have to characterize, with the comorphisms c^* and i^* defined on $\mathcal{O}_{J^*(M,M)}$, the systems of partial differential equations $\mathcal{I} \subset \mathcal{O}_{J^*(M,M)}$ whose set of solutions $sol(\mathcal{I})$ is a subgroupoid of $Aut(M)$.

[22] We will say everywhere differential ideal for sheaf of differential ideal.

We call *groupoid of order r on M* the subvariety of the space of invertible jets of order r defined by a coherent ideal $\mathcal{I}_r \subset \mathcal{O}_{J_r^*(M,M)}$ such that *(i)*: all the germs of the identity map of M are solutions of $\mathcal{I}_r$, such that *(ii)*: $c^*(\mathcal{I}_r) \subset \mathcal{I}_r \otimes \mathcal{O}_{J_r^*(M,M)} + \mathcal{O}_{J_r^*(M,M)} \otimes \mathcal{I}_r$, and such that *(iii)*: $\iota^*(\mathcal{I}_r) \subset \mathcal{I}_r$. The solutions of such an ideal $\mathcal{I}_r$ form a subgroupoid of $Aut(M)$.

Definition A.1. According to [Mal01], a *D-groupoid* $\mathcal{G}$ on M is a subvariety of the space $(M^2, \mathcal{O}_{J^*(M,M)})$ of invertible jets defined by a reduced, pseudo-coherent and differential ideal $\mathcal{I}_\mathcal{G} \subset \mathcal{O}_{J^*(M,M)}$ such that

(i') all the germs of the identity map of M are solutions of $\mathcal{I}_\mathcal{G}$,

(ii') for any relatively compact open set U of M, there exists a closed complex analytic subvariety Z of U of codimension ≥ 1, and a positive integer $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$ and denoting by $\mathcal{I}_{\mathcal{G},r} = \mathcal{I}_\mathcal{G} \cap \mathcal{O}_{J_r^*(M,M)}$, one has, above $(U \setminus Z)^2$: $c^*(\mathcal{I}_{\mathcal{G},r}) \subset \mathcal{I}_{\mathcal{G},r} \otimes \mathcal{O}_{J_r^*(M,M)} + \mathcal{O}_{J_r^*(M,M)} \otimes \mathcal{I}_{\mathcal{G},r}$,

(iii') $\iota^*(\mathcal{I}_\mathcal{G}) \subset \mathcal{I}_\mathcal{G}$.

The ideal $\mathcal{I}_\mathcal{G}$ totally determines the D-groupoid $\mathcal{G}$, so we will rather focus on the ideal $\mathcal{I}_\mathcal{G}$ than its solution $sol(\mathcal{I}_\mathcal{G})$ in $Aut(M)$. Thanks to the analytic continuation theorem, $sol(\mathcal{I}_\mathcal{G})$ is a subgroupoid of $Aut(M)$.

The flexibility introduced by Malgrange in his definition of D-groupoid allows him to obtain two main results. Theorem 4.4.1 of [Mal01] states that the reduced differential ideal of $\mathcal{O}_{J^*(M,M)}$ generated by a coherent ideal $\mathcal{I}_r \subset \mathcal{O}_{J_r^*(M,M)}$ which satisfies the previous conditions *(i)*,*(ii)*, and *(iii)* defines a D-groupoid on M. Theorem 4.5.1 of [Mal01] states that for any family of D-groupoids on M defined by a family of ideals $\{\mathcal{G}^i\}_{i\in I}$, the ideal $\sqrt{\sum \mathcal{G}^i}$ defines a D-groupoid on M called *intersection*. The terminology is legitimated by the equality: $sol(\sqrt{\sum \mathcal{G}^i}) = \cap_{i\in I} sol(\mathcal{G}^i)$. This last result allows to define the notion of D-envelope of any subgroupoid of $Aut(M)$.

Fix $q \in \mathbb{C}^*$, and let $Y(qx) = F(x, Y(x))$ be a (non linear) q-difference system, with $F(x, X) \in \mathbb{C}(x, X)^\nu$. Consider the set subgroupoid of $Aut(M)$ generated by the germs of the application $(x, X) \mapsto (qx, F(x, X))$ at any point of M where it is well defined and invertible, and denote it by $Dyn(F)$. The Galois D-groupoid, also called the Malgrange-Granier groupoid in §7, of the q-difference system $Y(qx) = F(x, Y(x))$ is the D-enveloppe of $Dyn(F)$, that is the *intersection* of the D-groupoids on M whose set of solutions contains $Dyn(F)$.

A.2. **A bound for the Galois D-groupoid of a linear q-difference system.** For all the following, consider a rational linear q-difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in GL_\nu(\mathbb{C}(x))$. We denote by $\mathcal{G}al(A(x))$ the Galois D-groupoid of this system as defined at the end of the previous section A.1, we denote by $\mathcal{I}_{\mathcal{G}al(A(x))}$ its defining ideal of equations, and by $sol(\mathcal{G}al(A(x)))$ its groupoid of solutions.

The elements of the dynamic $Dyn(A(x))$ of $Y(qx) = A(x)Y(x)$ are the germs of the local diffeomorphisms of M of the form $(x, X) \mapsto (q^k x, A_k(x)X)$, with:

$$A_k(x) = \begin{cases} Id_n & \text{if } k = 0, \\ \prod_{i=0}^{k-1} A(q^i x) & \text{if } k \in \mathbb{N}^*, \\ \prod_{i=k}^{-1} A(q^i x)^{-1} & \text{if } k \in -\mathbb{N}^*. \end{cases}$$

The first component of these diffeomorphisms is independent on the variables X and depends linearly on the variable x, and the second component depends linearly on the variables X. These properties can be expressed in terms of partial differential equations. This gives an *upper bound* for the Galois D-groupoid $\mathcal{G}al(A(x))$ which is defined in the following proposition.

Proposition A.2. *The coherent ideal:*

$$\left\langle \frac{\partial \bar{x}}{\partial X}, \frac{\partial \bar{x}}{\partial x}x - \bar{x}, \partial^2 \bar{x}, \frac{\partial \bar{X}}{\partial X}X - \bar{X}, \frac{\partial^2 \bar{X}}{\partial X^2} \right\rangle \subset \mathcal{O}_{J_2^*(M,M)}$$

satisfies the conditions (i),(ii), and (iii) of A.1. Hence, thanks to Theorem 4.4.1 of [Mal01]*, the reduced differential ideal* $\mathcal{I}_{\mathcal{L}in}$ *it generates defines a* D*-groupoid* $\mathcal{L}in$*. Its solutions* $sol(\mathcal{L}in)$ *are the germs of the local diffeomorphisms of* M *of the form:*

$$(x, X) \mapsto (\alpha x, \beta(x)X),$$

with $\alpha \in \mathbb{C}^*$ *and locally,* $\beta(x) \in GL_\nu(\mathbb{C})$ *for all* x*.*
They contain $Dyn(A(x))$*, and therefore, given the definition of* $\mathcal{G}al(A(x))$*, one has the inclusion*

$$\mathcal{G}al(A(x)) \subset \mathcal{L}in,$$

which means that:

$$\mathcal{I}_{\mathcal{L}in} \subset \mathcal{I}_{\mathcal{G}al(A(x))} \quad \text{and} \quad sol(\mathcal{G}al(A(x))) \subset sol(\mathcal{L}in).$$

Proof. *cf* proof of Proposition 3.2.1 of [Gra09] for more details. □

Remark A.3. Given their shape, the solutions of $\mathcal{L}in$ are naturally defined in neighborhoods of transversals $\{x_a\} \times \mathbb{C}^\nu$ of M. Actually, consider a particular element of $sol(\mathcal{L}in)$, that is precisely a germ at a point $(x_a, X_a) \in M$ of a local diffeomorphism g of M of the form $(x, X) \mapsto (\alpha x, \beta(x)X)$. Consider then a neighborhood Δ of x_a in $P^1\mathbb{C}$ where the matrix $\beta(x)$ is well defined and invertible, consider the "cylinders" $T_s = \Delta \times \mathbb{C}^\nu$ and $T_t = \alpha\Delta \times \mathbb{C}^\nu$ of M, and the diffeomorphism $\tilde{g} : T_s \to T_t$ well defined by $(x, X) \to (\alpha x, \beta(x)X)$. Therefore, according to the previous Proposition A.2, all the germs of $\tilde{g}$ at the points of T_s are in $sol(\mathcal{L}in)$ too.

The defining ideal $\mathcal{I}_{\mathcal{L}in}$ of the bound $\mathcal{L}in$ is generated by very simple equations. This allows to reduce modulo $\mathcal{I}_{\mathcal{L}in}$ the equations of $\mathcal{I}_{\mathcal{G}al(A(x))}$ and obtain some simpler representative equations, in the sense that they only depend on some variables.

Proposition A.4. *Let* $r \geq 2$*. For any equation* $E \in \mathcal{I}_{\mathcal{G}al(A(x))}$ *of order* r*, there exists an invertible element* $u \in \mathcal{O}_{J_r^*(M,M)}$*, an equation* $L \in \mathcal{I}_{\mathcal{L}in}$ *of order* r*, and an equation* $E_1 \in \mathcal{I}_{\mathcal{G}al(A(x))}$ *of order* r *only depending on the variables written below, such that:*

$$uE = L + E_1\left(x, X, \frac{\partial \bar{x}}{\partial x}, \frac{\partial \bar{X}}{\partial X}, \frac{\partial^2 \bar{X}}{\partial x \partial X}, \dots \frac{\partial^r \bar{X}}{\partial x^{r-1} \partial X}\right).$$

Proof. The invertible element u is a good power of δ. The proof consists then in performing the divisions of the equation uE, and then its succesive remainders, by the generators of $\mathcal{I}_{\mathcal{L}in}$. More details are given in the proof of Proposition 3.2.3 of [Gra09]. □

A.3. **Groups from the Galois D-groupoid of a linear q-difference system.** We are going to prove that the solutions of the Galois D-groupoid $\mathcal{G}al(A(x))$ are, like the solutions of the bound $\mathcal{L}in$, naturally defined in neighbourhoods of transversals of M. This property, together with the groupoid structure of $sol(\mathcal{G}al(A(x)))$, allows to exhibit groups from the solutions of $\mathcal{G}al(A(x))$ which fix the transversals.

According to Proposition A.2, an element of $sol(\mathcal{G}al(A(x)))$ is also an element of $sol(\mathcal{L}in)$. Therefore, it is a germ at a point $a = (x_a, X_a) \in M$ of a local diffeomorphism $g:(M,a)\to(M,g(a))$ of the form $(x,X)\mapsto(\alpha x,\beta(x)X)$, such that, for any equation $E\in\mathcal{I}_{\mathcal{G}al(A(x))}$, one has $E((x,X),g(x,X),\partial g(x,X),\ldots)=0$ in a neighbourhood of a in M.

Consider an open connected neighbourhood Δ of x_a in $\mathbb{P}^1_{\mathbb{C}}$ on which the matrix β is well-defined and invertible, that is where β can be prolongated in a matrix $\beta\in GL_\nu(\mathcal{O}(\Delta))$. Consider the "cylinders" $T_s=\Delta\times\mathbb{C}^\nu$ and $T_t=\alpha\Delta\times\mathbb{C}^\nu$ of M, and the diffeomorphism $\tilde{g}:T_s\to T_t$ defined by $(x,X)\to(\alpha x,\beta(x)X)$.

Proposition A.5. *The germs at all points of T_s of the diffeomorphism $\tilde{g}$ are elements of $sol(\mathcal{G}al(A(x)))$.*

Proof. For all $r\in\mathbb{N}$, the ideal $(\mathcal{I}_{\mathcal{G}al(A(x))})_r=\mathcal{I}_{\mathcal{G}al(A(x))}\cap\mathcal{O}_{J_r^*(M,M)}$ is coherent. Thus, for any point $(y_0,\bar{y}_0)\in M^2$, there exists an open neighbourhood Ω of $(y_0,\bar{y}_0)$ in M^2, and equations $E_1^\Omega,\ldots,E_l^\Omega$ of $(\mathcal{I}_{\mathcal{G}al(A(x))})_r$ defined on the open set Ω such that:

$$\big((\mathcal{I}_{\mathcal{G}al(A(x))})_r\big)_{|\Omega}=\big(\mathcal{O}_{J_r^*(M,M)}\big)_{|\Omega}E_1^\Omega+\cdots+\big(\mathcal{O}_{J_r^*(M,M)}\big)_{|\Omega}E_l^\Omega.$$

Let $a_1\in T_s=\Delta\times\mathbb{C}^\nu$. Let $\gamma:[0,1]\to T_s$ be a path in T_s such that $\gamma(0)=a$ and $\gamma(1)=a_1$. Let $\{\Omega_0,\ldots,\Omega_N\}$ be a finite covering of the path $\gamma([0,1])\times\tilde{g}(\gamma([0,1]))$ in $T_s\times T_t$ by connected open sets $\Omega\subset(T_s\times T_t)$ like above, and such that the origin $(\gamma(0),g(\gamma(0)))=(a,g(a))$ belongs to Ω_0.
The germ of g at the point a is an element of $sol(\mathcal{G}al(A(x)))$. Therefore, one has $E_k^{\Omega_0}((x,X),g(x,X),\partial g(x,X),\ldots)\equiv 0$ in a neighbourhood of a for all $1\le l\le k$. Moreover, by analytic continuation, one has also $E_k^{\Omega_0}(x,X,\tilde{g}(x,X),\partial\tilde{g}(x,X),\ldots)\equiv 0$ on the source projection of Ω_0 in M. It means that the germs of $\tilde{g}$ at any point of the source projection of Ω_0 are solutions of $(\mathcal{I}_{\mathcal{G}al(A(x))})_r$.
Then, step by step, one gets that the germs of $\tilde{g}$ at any point of the source projection of Ω_k are solutions of $(\mathcal{I}_{\mathcal{G}al(A(x))})_r$ and, in particular, the germ of $\tilde{g}$ at the point a_1 is also a solution of $(\mathcal{I}_{\mathcal{G}al(A(x))})_r$. □

This Proposition A.5 means that any solution of the Galois D-groupoid $\mathcal{G}al(A(x))$ is naturally defined in a neighbourhood of a transversal of M, above.

Remark A.6. In some sense, the "equations" counterpart of this proposition is Lemma 7.10.

The solutions of $\mathcal{G}al(A(x))$ which fix the transversals of M can be interpreted as solutions of a sub-D-groupoid of $\mathcal{G}al(A(x))$, partly because this property can be interpreted in terms of partial differential equations. Actually, a germ of a diffeomorphism of M fix the transversals of M if and only if it is a solution of the equation $\bar{x}-x$.

The ideal of $\mathcal{O}_{J_0^*(M,M)}$ generated by the equation $\bar{x}-x$ satisfies the conditions *(i)*,*(ii)*, and *(iii)* of A.1. Hence, thanks to Theorem 4.4.1 of [Mal01], the reduced differential ideal $\mathcal{I}_{\mathcal{T}rv}$ it generates defines a D-groupoid $\mathcal{T}rv$. Its solutions $sol(\mathcal{T}rv)$ are the germs of the local diffeomorphisms of M of the form: $(x,X)\mapsto(x,\bar{X}(x,X))$. Thus, consider the *intersection* D-groupoid $\widetilde{\mathcal{G}al(A(x))}=\mathcal{G}al(A(x))\cap\mathcal{T}rv$, in the sense of Theorem 4.5.1 of [Mal01], whose defining ideal of equations $\mathcal{I}_{\widetilde{\mathcal{G}al(A(x))}}$ is generated by $\mathcal{I}_{\mathcal{G}al(A(x))}$ and $\mathcal{I}_{\mathcal{T}rv}$, and whose solutions are $sol(\widetilde{\mathcal{G}al(A(x))})=sol(\mathcal{G}al(A(x)))\cap sol(\mathcal{T}rv)$, that are exactly the solutions of $\mathcal{G}al(A(x))$ of the form $(x,X)\mapsto(x,\beta(x)X)$. They are also naturally defined in neighbourhoods of transversals of M.

Proposition A.7. *Let $x_0\in\mathbb{P}^1_{\mathbb{C}}$. The set of solutions of $\widetilde{\mathcal{G}al(A(x))}$ defined in a neighbourhood of the transversal $\{x_0\}\times\mathbb{C}^\nu$ of M can be identified with a subgroup of $GL_\nu(\mathbb{C}\{x-x_0\})$.*

Proof. The solutions of the D-groupoid $\widetilde{\mathcal{G}al(A(x))}$ defined in a neighbourhood of the transversal $\{x_0\} \times \mathbb{C}^\nu$ can be considered, without loosing any information, only in a neighbourhood of the stable point $(x_0, 0) \in M$. At this point, the groupoid structure of $sol(\widetilde{\mathcal{G}al(A(x))})$ is in fact a group structure because the source and target points are always $(x_0, 0)$. Thus, considering the matrices $\beta(x)$ in the solutions $(x, X) \mapsto (x, \beta(x)X)$ of $\widetilde{\mathcal{G}al(A(x))})$ defined in a neighbourhood of $\{x_0\} \times \mathbb{C}^\nu$, one gets a subgroup of $GL_\nu(\mathbb{C}\{x - x_0\})$. More details are given in the proof of Proposition 3.3.2 of [Gra09]. □

In the particular case of a constant linear q-difference system, that is with $A(x) = A \in GL_\nu(\mathbb{C})$, the solutions of the Galois D-groupoid $\mathcal{G}al(A)$ are in fact global diffeomorphisms of M, and the set of those that fix the transversals of M can be identified with an algebraic subgroup of $GL_\nu(\mathbb{C})$. This can be shown using a better bound than $\mathcal{L}in$ for the Galois D-groupoid of a constant linear q-difference system (*cf* Proposition 3.4.2 of [Gra09]), or computing the D-groupoid $\mathcal{G}al(A)$ directly (*cf* Theorem 2.1 of [Gra] or Theorem 4.2.7 of [Gra09]). Moreover, the explicit computation allows to observe that this subgroup corresponds to the usual q-difference Galois group as described in [Sau04b] of the constant linear q-difference system $X(qx) = AX(x)$ (*cf.* Theorem 4.4.2 of [Gra09] or Theorem 2.4 of [Gra]).

References

[And89] Y. André. *G-functions and geometry.* Friedr. Vieweg & Sohn, Braunschweig, 1989.

[And01] Y. André. Différentielles non commutatives et théorie de Galois différentielle ou aux différences. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 34(5):685–739, 2001.

[And04] Y. André. Sur la conjecture des p-courbures de Grothendieck-Katz et un problème de Dwork. In *Geometric aspects of Dwork theory. Vol. I, II*, pages 55–112. Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[Aut01] P. Autissier. Points entiers sur les surfaces arithmétiques. *Journal für die Reine und Angewandte Mathematik*, 531:201–235, 2001.

[BM96] M. Bousquet-Mélou. A method for the enumeration of various classes of column-convex polygons. *Discrete Mathematics*, 154(1-3):1–25, 1996.

[BMF95] M. Bousquet-Mélou and J.-M. Fédou. The generating function of convex polyominoes: the resolution of a q-differential system. *Discrete Mathematics*, 137(1-3):53–75, 1995.

[BMP03] M. Bousquet-Mélou and M. Petkovšek. Walks confined in a quadrant are not always D-finite. *Theoretical Computer Science*, 307(2):257–276, 2003. Random generation of combinatorial objects and bijective combinatorics.

[Cas72] P.J. Cassidy. Differential algebraic groups. *American Journal of Mathematics*, 94:891–954, 1972.

[CHS08] Z. Chatzidakis, C. Hardouin, and M.F. Singer. On the definitions of difference Galois groups. In *Model Theory with applications to algebra and analysis, I and II*, pages 73–109. Cambridge University Press, 2008.

[CR08] G. Casale and J. Roques. Dynamics of rational symplectic mappings and difference Galois theory. *International Mathematics Research Notices. IMRN*, pages Art. ID rnn 103, 23, 2008.

[CS06] P.J. Cassidy and M.F. Singer. Galois theory of parameterized differential equations and linear differential algebraic groups. In *Differential Equations and Quantum Groups*, volume 9 of *IRMA Lectures in Mathematics and Theoretical Physics*, pages 113–157. 2006.

[Del90] P. Deligne. Catégories tannakiennes. In *in The Grothendieck Festschrift, Vol II*, volume 87 of *Prog.Math.*, pages 111–195. Birkhaäuser, Boston, 1990.

[DG70] M. Demazure and P. Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs.* Masson & Cie, Éditeur, Paris, 1970. Avec un appendice *Corps de classes local* par Michiel Hazewinkel.

[DV02] L. Di Vizio. Arithmetic theory of q-difference equations. The q-analogue of Grothendieck-Katz's conjecture on p-curvatures. *Inventiones Mathematicae*, 150(3):517–578, 2002. arXiv:math.NT/0104178.

[DVH10] L. Di Vizio and C. Hardouin. Courbures, groupes de galois génériques et d-groupoïde de galois d'un système aux q-différences. *Comptes Rendus Mathematique*, 348(17–18):951–954, 2010.

[DVRSZ03] L. Di Vizio, J.-P. Ramis, J. Sauloy, and C. Zhang. Équations aux q-différences. *Gazette des Mathématiciens*, 96:20–49, 2003.

[FRL06] C. Favre and J. Rivera-Letelier. Équidistribution quantitative des points de petite hauteur sur la droite projective. *Mathematische Annalen*, 335(2):311–361, 2006.

[GM93] M. Granger and P. Maisonobe. Differential modules. In *D-modules cohérents et holonomes sous la direction de P. Maisonobe et C. Sabbah*, pages 103–167. Hermann, 1993.

[Gra] A. Granier. A Galois D-groupoid for q-difference equations. *Annales de l'Institut Fourier*. To appear.

[Gra09] A. Granier. *Un groupoïde de Galois pour les équations aux q-différences*. PhD thesis, Université Toulouse III Paul Sabatier, 2009.

[Gra10] A. Granier. Un D-groupoïde de Galois local pour les systèmes aux q-différences fuchsiens. *Comptes Rendus Mathématique. Académie des Sciences. Paris*, 348(5-6):263–265, 2010.

[Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, (32):361, 1967.

[Har08] C. Hardouin. Hypertranscendance des systèmes aux différences diagonaux. *Compositio Mathematica*, 144(3):565–581, 2008.

[Har10] C. Hardouin. Iterative q-difference Galois theory. *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, 644:101–144, 2010.

[Hen96] P. Hendriks. *Algebraic Aspects of Linear Differential and Difference Equations*. PhD thesis, University of Groningen., 1996.

[Hru02] E. Hrushovski. Computing the Galois group of a linear differential equation. In *Differential Galois theory (Bedlewo, 2001)*, volume 58 of *Banach Center Publ.*, pages 97–138. Polish Acad. Sci., Warsaw, 2002.

[HS08] C. Hardouin and M.F. Singer. Differential Galois theory of linear difference equations. *Mathematische Annalen*, 342(2):333–377, 2008.

[Kap57] I. Kaplansky. *An introduction to differential algebra*. Hermann, Paris, 1957.

[Kat70] N. M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, 39:175–232, 1970.

[Kat82] N. M. Katz. A conjecture in the arithmetic theory of differential equations. *Bulletin de la Société Mathématique de France*, 110(2):203–239, 1982.

[Kat90] N. M. Katz. *Exponential sums and differential equations*, volume 124 of *Annals of Mathematics Studies*. Princeton University Press, 1990.

[Kol73] E.R. Kolchin. *Differential algebra and algebraic groups*, volume 54 of *Pure and applied mathematics*. Academic Press, New York and London, 1973.

[Kol74] E. R. Kolchin. Constrained extensions of differential fields. *Advances in Mathematics*, 12:141–170, 1974.

[KS72] A. Kumpera and D. Spencer. *Lie equations. Vol. I: General theory*. Princeton University Press, Princeton, N.J., 1972. Annals of Mathematics Studies, No. 73.

[Lan83] S. Lang. *Fundamentals of Diophantine geometry*. Springer-Verlag, New York, 1983.

[LY08] B.Q. Li and Z. Ye. On differential independence of the Riemann zeta function and the Euler gamma function. *Acta Arithmetica*, 135(4):333–337, 2008.

[Mal01] B. Malgrange. Le groupoïde de Galois d'un feuilletage. In *Essays on geometry and related topics, Vol. 1, 2*, volume 38 of *Monogr. Enseign. Math.*, pages 465–501. 2001.

[Mal09] B. Malgrange. Pseudogroupes de lie et théorie de galois différentielle. preprint, 2009.

[Mar07] L. Markus. Differential independence of Γ and ζ. *Journal of Dynamics and Differential Equations*, 19(1):133–154, 2007.

[MO10] A. Minchenko and A. Ovchinnikov. Zariski Closures of Reductive Linear Differential Algebraic Groups. arXiv:1005.0042, 2010.

[Mor09] S. Morikawa. On a general difference Galois theory I. In *Annales de l'institut Fourier*, volume 59, pages 2709–2732, 2009.

[MU09] S. Morikawa and H. Umemura. On a general difference Galois theory II. In *Annales de l'Institut Fourier*, volume 59, pages 2733–2771, 2009.

[MvdP03] B. Heinrich Matzat and Marius van der Put. Iterative differential equations and the Abhyankar conjecture. *Journal für die Reine und Angewandte Mathematik*, 557:1–52, 2003.

[Ovc08] A. Ovchinnikov. Tannakian approach to linear differential algebraic groups. *Transformation Groups*, 13(2):413–446, 2008.

[Ovc09a] A. Ovchinnikov. Differential tannakian categories. *Journal of Algebra*, 321(10):3043–3062, 2009.

[Ovc09b] A. Ovchinnikov. Tannakian categories, linear differential algebraic groups, and parametrized linear differential equations. *Transformation Groups*, 14(1):195–223, 2009.

[Pil02] A. Pillay. Finite-dimensional differential algebraic groups and the Picard-Vessiot theory. In *Differential Galois theory (Bedlewo, 2001)*, volume 58, pages 189–199. Banach Center Publ., 2002.

[PN09] A. Peón Nieto. On sigma-delta-Picard-Vessiot extensions. To appear in Communications in Algebra, 2009.

[Pra83] C. Praagman. The formal classification of linear difference operators. *Koninklijke Nederlandse Akademie van Wetenschappen. Indagationes Mathematicae*, 45(2):249–261, 1983.

[Pra86] C. Praagman. Fundamental solutions for meromorphic linear difference equations in the complex plane, and related problems. *Journal für die Reine und Angewandte Mathematik*, 369:101–109, 1986.

[Ram92] J.-P. Ramis. About the growth of entire functions solutions of linear algebraic q-difference equations. *Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série 6*, 1(1):53–94, 1992.

[Roq09] J. Roques. Generalized basic hypergeometric equations. *Preprint Université de Grenoble.*, pages 1–35, 2009.

[Sau00] J. Sauloy. Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie. *Annales de l'Institut Fourier*, 50(4):1021–1071, 2000.

[Sau04a] J. Sauloy. Algebraic construction of the Stokes sheaf for irregular linear q-difference equations. *Astérisque*, 296:227–251, 2004. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. I.

[Sau04b] J. Sauloy. Galois theory of Fuchsian q-difference equations. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 36(6):925–968, 2004.

[Sau04c] J. Sauloy. La filtration canonique par les pentes d'un module aux q-différences et le gradué associé. *Annales de l'Institut Fourier*, 54(1):181–210, 2004.

[Ser56] J.-P. Serre. Géométrie algébrique et géométrie analytique. *Annales de l'Institut Fourier*, 6:1–42, 1955–1956.

[SR72] N. Saavedra Rivano. *Catégories tannakiennes*, volume 265 of *Lecture Notes in Mathematics*. Springer, Berlin, 1972.

[Ume08] H Umemura. Sur l'équivalence des théories de Galois différentielles générales. *Comptes Rendus Mathématique. Académie des Sciences. Paris*, 346(21-22):1155–1158, 2008.

[Ume10] H. Umemura. Picard-Vessiot theory in general Galois theory. 2010.

[vdPR07] Marius van der Put and Marc Reversat. Galois theory of q-difference equations. *Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série 6*, 16(3):665–718, 2007.

[vdPS97] M. van der Put and M.F. Singer. *Galois theory of difference equations*. Springer-Verlag, Berlin, 1997.

[Zdu97] A. Zdunik. Harmonic measure on the Julia set for polynomial-like maps. *Inventiones Mathematicae*, 128(2):303–327, 1997.

Lucia DI VIZIO, Institut de Mathématiques de Jussieu, Topologie et géométrie algébriques, Case 7012, 2, place Jussieu, 75251 Paris Cedex 05, France.

E-mail address: `divizio@math.jussieu.fr`

Charlotte HARDOUIN, Institut de Mathématiques de Toulouse, 118 route de Narbonne, 31062 Toulouse Cedex 9, France.

E-mail address: `hardouin@math.univ-toulouse.fr`