UNIPOTENT RADICALS OF TANNAKIAN GALOIS GROUPS IN POSITIVE CHARACTERISTIC

by

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Abstract. — Let $\mathcal{T}$ be a Tannakian category over a field C of strictly positive characteristic. We show in this note how one can characterize the unipotent radical of the Tannakian Galois group of an object $\mathcal{U}$, extension of the unit object $\mathbf{1}$ by a completely reducible object $\mathcal{Y}$ in terms of the group $Ext^1(\mathbf{1}, \mathcal{Y})$ of isomorphism classes of extension of $\mathbf{1}$ by $\mathcal{Y}$. We deduce from our Theorem that, under certain hypothesis, the Tannakian Galois group of a direct sum of extensions is entirely determined by the relations of linear dependence satisfied by these extensions in $Ext^1(\mathbf{1}, \mathcal{Y})$. This corollary reduces the computation of an algebraic group to a question of linear algebra. As an application, we show how it gives an alternative proof of the algebraic independence of the Carlitz logarithms of M. Papanikolas ([**14**]).

Contents

Introduction. —

Computation of Tannakian Galois groups. — The theory of Tannakian categories gives a precise answer to the question: "when is a category equivalent to the category Rep_G of finite dimensional representations of an affine group scheme?" By definition, a Tannakian category $\mathcal{T}$ over a field C is a rigid abelian tensor category (see [**7**, §2.8]). It is said to be neutral if there exists a functor $\omega : \mathcal{T} \to Vect_C$ from $\mathcal{T}$ into the category of finite dimensional C-vector spaces, called "fiber functor", that is C-linear, faithful, exact and tensor compatible (see [**7**, §1.9]). For instance, the category of differential

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modules over the differential field $(\mathbb{C}(x), \partial := \frac{d}{dx})$ is a Tannakian category (see [**18**, §2.2]). The choice of a basis of a differential module $\mathcal{M}$ yields to a differential system $\partial Y = AY$ with $A \in Gl_\nu(\mathbb{C}(x))$. Then, a basis of solutions of this last system provides a fiber functor for the full Tannakian sub-category generated by $\mathcal{M}$. The category of differential modules over a differential field of characteristic zero, the category of iterative differential modules over a field of positive characteristic (see [**13**]), the category of q-difference modules (see [**19**]), the category of Frobenius modules (see [**12**]), which includes isocrystals in the p-adic case and t-motives (see [**14**]) in the t-adic case are just other examples of Tannakian categories. The fundamental theorem for Tannakian categories is the following

Theorem 1 **(see Theorem 1.12 of [7]).** — *Let $\mathcal{T}$ be a neutral Tannakian category over a field C together with a fiber functor $\omega : \mathcal{T} \rightarrow Vect_C$. Then, the functor $Aut^{\otimes}(\omega)$ of tensor compatible automorphisms of ω is representable by an affine group scheme G defined over C and ω induces an equivalence of categories between $\mathcal{T}$ and the category Rep_G of finite dimensional representations of G.*

Then, the Galois group of an object $\mathcal{M}$ of a Tannakian category is defined as follows.

Definition 1 **(see Theorem 3.2.1.1 of [3]).** — *Let $(\mathcal{T}, \omega)$ be a neutral Tannakian category defined over a field C. We denote by $\langle \mathcal{M} \rangle^{\otimes}$ the full Tannakian sub-category generated by $\mathcal{M}$ in $\mathcal{T}$. Then,*

- *there exists an affine group scheme $G_{\mathcal{M}}$ defined over C, together with a closed immersion ι from $G_{\mathcal{M}}$ into $Gl(\omega(\mathcal{M}))$, such that $\omega_{|\mathcal{M}} : \langle \mathcal{M} \rangle^{\otimes} \rightarrow Vect_C$ induces a $\otimes$-tensor equivalence of categories between $\langle \mathcal{M} \rangle^{\otimes}$ and the category $Rep_{G_{\mathcal{M}}}$;*
- *the image of ι is the closed sub-group of $Gl(\omega(\mathcal{M}))$ which stabilizes all the sub-objects $\mathcal{N}$ contained in any finite sum $\bigoplus_{i,j} (\mathcal{M}^{\otimes i} \otimes (\mathcal{M}^*)^{\otimes j})$, where $\mathcal{M}^*$ denotes the dual of $\mathcal{M}$.*

We call $G_{\mathcal{M}}$ the Galois group of $\mathcal{M}$.

For a differential module $\mathcal{M}$ over $\mathbb{C}(x)$, the linear algebraic group $G_{\mathcal{M}}$ is defined over $\mathbb{C}$ and isomorphic to the differential Galois group attached to $\mathcal{M}$ by Picard-Vessiot constructions (see [**18**, Definition 1.25]). Its dimension over $\mathbb{C}$ is equal to the transcendence degree of the field generated over $\mathbb{C}(x)$ by a basis of solutions of a differential system attached to $\mathcal{M}$. In [**3**, Theorem 3.4.2.3], it is shown that, for an object $\mathcal{M}$ in a neutral Tannakian category of either differential or difference modules over C, there is a one to one correspondence between fiber functors over $\langle \mathcal{M} \rangle^{\otimes}$, Picard-Vessiot extensions of $\mathcal{M}$ (roughly, C-algebras generated by a basis of solution of $\mathcal{M}$ plus some minimality conditions) and $G_{\mathcal{M}}$-torsors. Specifically, this implies that the algebraic relations between the solutions of $\mathcal{M}$ are controlled by the Galois group of $\mathcal{M}$. The Tannakian categories are thus strongly related to questions of functional transcendence and the computation of Tannakian Galois groups is a powerful tool since it reduces these questions to the computation of a linear algebraic group.

There exist some algorithms to compute Galois groups of Tannakian objects but they are, most of the time, specific to the Tannakian category considered. For instance, in [**10**], E. Hrushovski proves that one can compute the Galois group of a linear differential equation over $\overline{Q}(x)$. In the first part of this note, we present some theorems of computation for Tannakian Galois groups, which generalize those mentioned in [**16**]. Even if they requires some technical hypothesis, these theorems are valid for any Tannakian category in positive characteristic and they may thus apply, for instance, as well for iterative differential equations as for Frobenius difference equations.

We detail below the results of the first section of this note. Let C be a field and let $(\mathcal{T},\omega)$ be a neutral Tannakian category over C. Let $\mathbf{1}$ be the unit object for the tensor product. Let $\mathcal{Y}$ be a completely reducible object of $\mathcal{T}$, *i.e.* $\mathcal{Y}$ is a direct sum of finitely many irreducible objects. We say that $\mathcal{U}$, an object of $\mathcal{T}$, is an extension of $\mathbf{1}$ by $\mathcal{Y}$ if there exists an exact sequence in $\mathcal{T}$ such that

$$0 \rightarrow \mathcal{Y} \rightarrow \mathcal{U} \rightarrow \mathbf{1} \rightarrow 0.$$

To consider extensions of $\mathbf{1}$ by $\mathcal{Y}$ is a way to build "logarithms of the solutions" of $\mathcal{Y}$. For instance, if $\mathcal{Y}$ is a differential module over $(\mathbb{C}(x), \frac{d}{dx})$ associated to the differential system $\frac{d}{dx}Y(x) = A(x)Y(x)$ with $A(x) \in Gl_\nu(\mathbb{C}(x))$, then an extension of $\mathbf{1}$ by $\mathcal{Y}$ corresponds to a differential system of the form $\frac{d}{dx}Z(x) = A(x)Z(x) + B(x)$ with $B(x) \in (\mathbb{C}(x))^\nu$.

Using Levi decomposition, we see that the Galois group $G_\mathcal{Y}$ of an extension $\mathcal{U}$ of $\mathbf{1}$ by $\mathcal{Y}$ a completely reducible object, may be written as the semi-direct product $G_\mathcal{U} = R_u(G_\mathcal{U}) \rtimes G_\mathcal{Y}$ where $R_u(G_\mathcal{U})$ stands for the unipotent radical of $G_\mathcal{U}$. Then, if we assume that $G_\mathcal{Y}$ is given, the computation of $G_\mathcal{U}$ is reduced to the computation of the unipotent radical of $G_\mathcal{U}$. If the characteristic of C is equal to zero, it is proved in [**9**, Theorem 2.1] that the unipotent radical of $G_\mathcal{U}$ is isomorphic to a vectorial subgroup of the fiber $\omega(\mathcal{Y})$, entirely determined by the structure of $Ext^1(\mathbf{1},\mathcal{Y})$, the group of isomorphism classes of extensions of $\mathbf{1}$ by $\mathcal{Y}$ in $\mathbf{T}$. The proof extends and follows closely the kummerian arguments of [**5**] and [**4**]. If the characteristic of C is strictly positive, one has to be more careful ; first of all, it may happen that the Galois groups are not reduced and, secondly, the image of a vectorial group by a group morphism is not necessarily a vectorial group. Then, if $\mathbf{G}_m$ denotes the multiplicative group over C, we have

Theorem 2. — *Let $\mathcal{Y}$ be an object of $\mathbf{T}$, and let $\mathcal{U}$ be an extension of $\mathbf{1}$ by $\mathcal{Y}$. Assume that*

1. *every $G_\mathcal{Y}$-module is completely reducible,*
2. *the center of $G_\mathcal{Y}$ contains $\mathbf{G}_m$,*
3. *the action of $\mathbf{G}_m$ on $\omega(\mathcal{Y})$ is isotypic* (1)*,*
4. *$G_\mathcal{U}$ is reduced.*

(1) We recall that the action of a group G on a module V is isotypic if the module V is the direct sum of irreducible isomorphic G-modules.

Then, there exists a smallest sub-object $\mathcal{V}$ of $\mathcal{Y}$ such that $\mathcal{U}/\mathcal{V}$ is a trivial extension of $\mathbf{1}$ by $\mathcal{Y}/\mathcal{V}$. The unipotent radical of the Galois group $G_{\mathcal{U}}$ is then equal to $\omega(\mathcal{V})$.

First of all, we just want to emphasize the fact that every diagonal C-group scheme satisfy the first hypothesis (see [**11**, p.35]). Secondly, the third hypothesis may be removed if one thinks in terms of weights of the characters of $\mathbf{G}_m$ acting on each isotypical components of $\omega(Y)$. But, for simplicity of exposition, we assume that the action of $\mathbf{G}_m$ is isotypic, *i.e.* involves one single character. As a corollary of Theorem 2, we show

Corollary 1. — *Let $\mathcal{Y}$ be an object of $\mathbf{T}$. Let Δ be the ring $End(\mathcal{Y})$, and let $\mathcal{E}_1, ..., \mathcal{E}_n$ be extensions of $\mathbf{1}$ by $\mathcal{Y}$. Assume that*

1. *every $G_{\mathcal{Y}}$-module is completely reducible,*
2. *the center of $G_{\mathcal{Y}}$ contains $\mathbf{G}_m$,*
3. *the action of $\mathbf{G}_m$ on $\omega(\mathcal{Y})$ is isotypic,*
4. *$G_{\mathcal{E}_1}, ..., G_{\mathcal{E}_n}$ are reduced.*

Then, if $\mathcal{E}_1, ..., \mathcal{E}_n$ are Δ-linearly independent in $Ext^1(\mathbf{1}, \mathcal{Y})$, the unipotent radical of $G_{\mathcal{E}_1 \oplus ... \oplus \mathcal{E}_n}$ is isomorphic to $\omega(\mathcal{Y})^n$.

The meaning of this corollary is the following. Algebraic relations between the extensions occur if and only if the group $G_{\mathcal{E}_1 \oplus ... \oplus \mathcal{E}_n}$ is not as big as possible, i.e. , if and only if its unipotent radical is strictly contained in $\omega(\mathcal{Y})^n$. Corollary 1 then states that algebraic relations are exactly given by the relations of linear dependence. As for Theorem 2, this corollary holds in characteristic zero (see [**9**, Cor. 2.2]). Even if, in full generality, the criteria of linear dependency of the extensions shall seem rather complicated, it reduces most of the time, to a question of existence of a rational solution of a given equation.

An application to the transcendency of periods of Drinfeld module. — In the second section, we show how the computation theorems of the first section may apply to the Tannakian category of t-motives defined by M. Papanikolas in [**14**] and thus to the study of the transcendence properties of some periods of Drinfeld modules.

Let $\mathbf{F}_q$ be the field of q elements, where q is a power of a prime p. Let $k := \mathbf{F}_q(\theta)$, where θ is transcendental over $\mathbf{F}_q$. Define a valuation $|.|_\infty$ at the infinite place of k such that $|\theta|_\infty = q$. Let $k_\infty := \mathbf{F}_q((1/\theta))$ be the ∞-adic completion of k, let $\overline{k_\infty}$ be an algebraic closure of k_∞, let $\mathbf{K}$ be the ∞-adic completion of $\overline{k_\infty}$, and let $\overline{k}$ be the algebraic closure of k in $\mathbf{K}$. One call "numbers " the elements of $\mathbf{K}$. A number which is not in $\overline{k}$ is a transcendent number. Let $\mathbf{K}[\tau]$ be the twisted polynomial ring in τ over $\mathbf{K}$ subject to the relation $\tau c = c^q \tau$ for all $c \in \mathbf{K}$. Now, let t be an independent variable from θ. A Drinfeld $\mathbf{F}_q[t]$-module ρ is an $\mathbf{F}_q$-algebra homorphism $\rho : \mathbf{F}_q[t] \to \mathbf{K}[\tau]$ such that for all $a \in \mathbf{F}_q[t]$, the constant term of $\rho(a)$, as a polynomial in τ, is $a(\theta)$. The rank of a Drinfeld $\mathbf{F}_q[t]$-module ρ is defined as the degree of $\rho(t)$ in τ. The period lattice of a Drinfeld $\mathbf{F}_q[t]$-module ρ of rank r is defined as follows. The exponential function of ρ is the function $exp_\rho(z) = z + \sum_{i=1}^{\infty} \alpha_i z^{q^i}$, with $\alpha_i \in \mathbf{K}$ such that $exp_\rho(\theta z) = \rho(t)(exp_\rho(z))$. It is shown that exp_ρ is entire, $\mathbf{F}_q$-linear. The kernel Λ_ρ of exp_ρ is a discrete, free $\mathbf{F}_q[\theta]$-module of rank r called the period lattice of ρ. They are many analogies between the periods of Drinfeld $\mathbf{F}_q[t]$-modules and the complex numbers $2\pi i, log(n), \zeta(n)$ with $n \geq 2$ (see [**16**]). A famous conjecture in Number Theory affirms that if $u_1, ..., u_n$ are n complex numbers, $\mathbb{Q}$-linearly independent such that $exp(u_i) \in \overline{\mathbb{Q}}$ for all i, then $1, u_1, ..., u_n$ are algebraically independent over $\overline{\mathbb{Q}}$. In a recent work , C. Chang and M. Papanikolas proved the algebraic independence of Drinfeld logarithm, *i.e.*

Theorem 3 **(see Theorem 1.1.1 in [15]).** — *Let ρ be a Drinfeld $\mathbf{F}_q[t]$-module defined over $\overline{k}$. Let $u_1, ..., u_n \in \mathbf{K}$ be satisfyng $exp_\rho(u_i) \in \overline{k}$ for $i = 1, .., n$. If $u_1, ..., u_n$ are linearly independent over K_ρ, the fraction field of the endomorphism ring of ρ, then they are algebraically independent over $\overline{k}$.*

Their strategy of proof relies on the deep connection between Drinfeld modules and Anderson t-motives (see [**2**]). An Anderson-t-motive is defined as follows

Definition 2. — *Let $\overline{k}[t, \sigma]$ be the ring of polynomials in t and σ over $\overline{k}$ subject to the relations $ct = tc, \sigma t = t\sigma, \sigma c = c^{1/q}\sigma, c \in \overline{k}$. An Anderson t-motive is a left $\overline{k}[t, \sigma]$-module M that is free and finitely generated as both a left $\overline{k}[t]$-module and as a left $\overline{k}[\sigma]$-module and that satisfies $(t - \theta)^n M \subset M$ for all n sufficiently large.*

Let $\mathbf{T} := \mathbf{K}\{t\}$ be the Tate algebra of power series in $\mathbf{K}[[t]]$ that are convergent on the closed unit disk in $\mathbf{K}$. For a Laurent series $f = \sum_i a_i t^i \in \mathbf{K}((t))$ and an integer $n \in \mathbb{Z}$, we set $\sigma^{-n}(f) := f^{(n)} := \sum_i a_i^{q^n} t^i$. If M is an Anderson t-motive and m is a $\overline{k}[t]$-basis of M, there is a matrix Φ with coefficient in $\overline{k}[t]$ representing the action of σ on M in the basis m, i.e., $\sigma m = \Phi m$, such that $det(\Phi) = c(t - \theta)^s$ for some $c \in \overline{k}^*$ and $s \geq 1$. The Anderson t-motive is "rigid analytically trivial" if there is a matrix $\Psi \in Gl_r(\mathbf{T})$ so that $\Psi^{(-1)} = \Phi\Psi$. In [**2**], it is proved that the category of rigid analytically trivial Anderson t-motives is equivalent to the category of uniformizable abelian Drinfeld $\mathbf{F}_q[t]$-modules defined over $\overline{k}$. There are also explicit connections between the k-linear combination of entries of $\Psi(\theta)^{-1}$ for a given Anderson t-motive M and the periods of its corresponding Drinfeld $\mathbf{F}_q[t]$-module. In [**14**, §3.4], Papanikolas proved that the category of rigid analytically trivial Anderson t-motives up to isogeny embeds as a full category of a neutral Tannakian category $\mathbf{T}$ over $\mathbf{F}_q(t)$ whose object are called t-motives. By Tannakian equivalence, Papanikolas succeed in attaching to a given Anderson t-motive M a Galois group G_M defined over $\mathbf{F}_q(t)$, which, roughly, corresponds to the Galois group of a Frobenius-difference system. Thanks to a linear independence criterion developed by Anderson, Brownawell and Papanikolas in [**1**], M. Papanikolas was able to prove the following theorem

Theorem 4 **(see Theorem 1.1.7 in [14]).** — *Let M be a t-motive and let G_M be its Galois group. Suppose that $\Phi \in Gl_r(\overline{k}[t])$ represents multiplication by σ on M and that $det(\Phi) = c(t - \theta)^s, c \in k^*$. Let Ψ a rigid analytic trivialization of $\Phi \in Gl_r(\mathbf{T})$. Finally, let L be the subfield of $\overline{k_\infty}$ generated over $\overline{k}$ by the entries of $\Psi(\theta)$. Then,*

$$trdeg_{\overline{k}}(L) = dim G_M.$$

This theorem is crucial in the proof since it makes a bridge between the special values at $t=\theta$ of the solutions of the t-motive, which interpolate the periods of the associated Drinfeld module, and the computation of a Tannakian Galois group.

In [**14**], Papanikolas gives, as an illustration of his Tannakian theory of t-motive, the first proof of the algebraic independence of Drinfeld logarithms in the case of the Carlitz module $\mathfrak{C}$. It is the Drinfeld $\mathbf{F}_q[t]$-module of rank 1, which is associated to the homomorphism $\mathbf{F}_q[t] \rightarrow \mathbf{K}[\tau]$ defined by $t \rightarrow \theta + \tau$. The exponential function of the Carlitz module is $exp_{\mathcal{C}}(z) := z + \sum_{i=1} \frac{z^{q^i}}{(\theta^{q^i}-\theta)(\theta^{q^i}-\theta^q)\dots(\theta^{q^i}-\theta^{q^{i-1}})}$. Theorem 3 in the case of the Carlitz module becomes

***Theorem 5* (Theorem 1.2.6 in [14]).** — *Let $u_1, \dots, u_n \in \mathbf{K}$ be satisfying $exp_{\mathcal{C}}(u_i) \in \overline{k}$ for $i = 1, .., n$. If $u_1, \dots, u_n$ are linearly independent over k, , then they are algebraically independent over $\overline{k}$.*

The method of M. Papanikolas for proving Theorem 5 is to compute the Galois group $G_{\mathcal{X}}$ of a certain t-motive $\mathcal{X}$ (see [**14**, §6.1]). This is the content of Theorem 6.3.2 of [**14**], where $G_{\mathcal{X}}$ is denoted by Γ_X. Note, however, that the paragraph following §6.4 needs some clarification, since Γ_X is not a linear subspace. In this note, we give a Tannakian version of the computation of $G_{\mathcal{X}}$ based on the theorems of the first section, which while settling this point, actually simplifies the proof of [**14**]. An old version of Theorem 2 and Corollary 2 was also partially used in the computation of the Galois groups occurring in the proof of Theorem 3 in [**15**] and we think that our theorems could now, in their full generality, perhaps simplify the computations of [**15**]. We also think that they also could apply to more complicated t-motives, involving periods of second and third kind of Drinfeld module, such as, for instance, the ones related to the Carlitz-zeta values (see [**6**]).

1. Computation of Galois groups in Tannakian categories in positive characteristic

1.1. Notations and proof of Corollary 1. — We recall here some notations of the introduction. Let p be a prime number. Let C be a field of characteristic p. Let $(\mathcal{T}, \omega)$ be a neutral Tannakian category over C. Let $\mathbf{1}$ denote the unit object for the tensor product of $\mathcal{T}$, so that $C = End_{\mathcal{T}}(\mathbf{1})$. For any object $\mathcal{X}$ in $\mathcal{T}$, we denote by $\langle \mathcal{X} \rangle^{\otimes}$ the full tensor sub-category generated by $\mathcal{X}$ in $\mathcal{T}$. Let $G_{\mathcal{X}}$ be the linear algebraic group scheme defined over C, which represents the functor $Aut^{\otimes}(\omega|_{\langle \mathcal{X} \rangle})$ of tensor automorphisms of $\omega|_{\langle \mathcal{X} \rangle}$ (see [**7**]). Furthermore, we identify C-vector spaces, such as $\omega(\mathcal{X})$, to *vectorial groups* over C.

Let $\mathcal{Y}$ be an object of $\mathcal{T}$. We endow the group $Ext^1_{\mathcal{T}}(\mathbf{1}, \mathcal{Y})$ with a structure of $End_{\mathcal{T}}(\mathcal{Y})$-module as follows: for any extension $\mathcal{E}$ of $\mathbf{1}$ by $\mathcal{Y}$, and for any $\alpha \in \Delta$, we denote by $\alpha_*(\mathcal{E})$, the pushout of $\mathcal{E}$ by α, i.e., the extension of $\mathbf{1}$ by $\mathcal{Y}$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{Y} \arrow[r] \arrow[d, "\alpha"] & \mathcal{E} \arrow[r] \arrow[d] & \mathbf{1} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{Y} \arrow[r] & \alpha_*(\mathcal{E}) \arrow[r] & \mathbf{1} \arrow[r] & 0 \end{tikzcd}$$

The proof of Theorem 2 is the object of section 1.2. We first show how one can easily deduce Corollary 1 from Theorem 2

Proof of Corollary 1. — We first note that the direct sum $\mathcal{Y}^n$ admits $G_{\mathcal{Y}^n} = G_{\mathcal{Y}}$ as Galois group, and that $\mathbb{G}_m$ again acts on $\omega(\mathcal{Y}^n) = \omega(\mathcal{Y})^n$ through an isotypic representation. On the other hand, the extension $\mathcal{E}_1 \oplus ... \oplus \mathcal{E}_n$ of $\mathbf{1}^n$ by $\mathcal{Y}^n$ and its pull-back $\mathcal{E} \in Ext^1_{\mathcal{T}}(\mathbf{1}, \mathcal{Y}^n)$ by the diagonal map from $\mathbf{1}$ to $\mathbf{1}^n$ generate in $\mathcal{T}$ the same sub-Tannakian category. Therefore, their Galois groups $G_{\mathcal{E}_1 \oplus ... \oplus \mathcal{E}_n}$ and $G_{\mathcal{E}}$ are equal, and reduced in view of our hypothesis. Let us assume that the unipotent radical R_u of $G_{\mathcal{E}}$ do not fill up $\omega(\mathcal{Y}^n) = \omega(\mathcal{Y})^n$.

By Theorem 2, R_u is equal to the C-vectorial group $\omega(\mathcal{V})$ where $\mathcal{V} \in \mathcal{T}$ is the smallest sub-object of $\mathcal{Y}^n$ such that the quotient by $\mathcal{V}$ of the extension $\mathcal{E}$ of $\mathbf{1}$ by $\mathcal{Y}^n$ is trivial in the category $\mathcal{T}$. If $\mathcal{V}$ is not equal to $\mathcal{Y}^n$, then $\omega(\mathcal{V}) \subsetneq \omega(\mathcal{Y}^n)$. Because $\omega(\mathcal{V})$ is a sub-representation of the representation $\omega(\mathcal{Y}^n)$ of $G_{\mathcal{Y}}$, it lies in the kernel H of a non trivial $G_{\mathcal{Y}}$-equivariant homomorphism ϕ from $\omega(\mathcal{Y}^n)$ to $\omega(\mathcal{Y})$. By Tannakian equivalence of categories, then there exists a non trivial morphism $\Phi \in Hom_{\mathbf{T}}(\mathcal{Y}^n, \mathcal{Y})$ such that $\mathcal{V} \subset \mathrm{Ker}(\Phi)$.

Now, consider the following diagram:

$$\begin{tikzcd} \mathcal{Y}^n \arrow[d] \arrow[dr, "\Phi"] & \\ \mathcal{Y}^n/\mathcal{V} \arrow[r] & \mathcal{Y}^n/\mathrm{Ker}(\Phi) \simeq \mathcal{Y}. \end{tikzcd}$$

Since $\Phi \in Hom_{\mathbf{T}}(\mathcal{Y}^n, \mathcal{Y})$, we can write $\Phi(X_1, ..., X_n) = \alpha_1 X_1 + ... + \alpha_n X_n$, with $\alpha_i \in End_{\mathbf{T}}(\mathcal{Y})$. Then $\Phi_*(\mathcal{E}) = \alpha_{1*}(\mathcal{E}_1) + \alpha_{2*}(\mathcal{E}_2) + ... + \alpha_{n*}(\mathcal{E}_n)$ is a quotient of $\mathcal{E}/\mathcal{V}$, hence a trivial extension of $\mathbf{1}$ by $\mathcal{Y}$ in $\mathcal{T}$. In conclusion, the extension $\alpha_1\mathcal{E}_1 + ... + \alpha_n\mathcal{E}_n \in Ext^1(\mathbf{1}, \mathcal{Y})$ is trivial. But this contradicts the Δ-linearly independence in $Ext^1_{\mathcal{T}}(\mathbf{1}, \mathcal{Y})$ of the extensions $\mathcal{E}_1, ..., \mathcal{E}_n$. □

As noticed in the introduction, the meaning of Corollary 1 is that, under certain hypothesis, the algebraic relations between the "solutions" of extensions of $\mathbf{1}$ by $\mathcal{Y}$ are given by the relations of linear dependence between the extensions. One reduce an algebraic study to a simple question of linear algebra. If $\mathcal{T}$ is a Tannakian category over a field C of characteristic zero, Corollary 1 holds for any completely reducible object $\mathcal{Y}$. For instance, let $\mathcal{T}$ be the Tannakian category of differential modules with coefficients in the field $\mathbb{C}(x)$ of rational functions over $\mathbb{C}$ and let $\mathcal{Y}$ be an irreducible differential module. Let $L \in \mathbb{C}(x)[d/dx]$ be an irreducible linear differential equation attached to $\mathcal{Y}$ and let $b_1, ..., b_n$ be n elements of $\mathbb{C}(x)$ such that the extension $\mathcal{E}_i$ corresponds to the equation $L(y) = b_i$. Then, $\mathcal{E}_1, ..., \mathcal{E}_n$ are $End(\mathcal{Y})$-linearly dependent if and only if there exists $(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ non equal to zero and a rational function $f \in \mathbb{C}(x)$ such that $\lambda_1 b_1 + ... + \lambda_n b_n = L(f) - f$. Corollary 1 is in that case an analogue of the Kolchin Ostrowski theorem.

1.2. Proof of Theorem 2. — By Tannaka theorem (see [**7**]), there is an equivalence of categories between $\langle \mathcal{Y} \rangle$ and the category $Rep_{G_{\mathcal{Y}}}$ of $G_{\mathcal{Y}}$-modules of finite dimension over C. Then, it is clear that an object $\mathcal{M}$ in $\langle \mathcal{Y} \rangle$ is completely reducible in $\mathbf{T}$ if and only if $\omega(\mathcal{M})$ is a completely reducible $G_{\mathcal{Y}}$-module.

1.2.1. Existence of the smallest sub-object. — Let us denote by $\mathbf{V}$ the set of sub-objects $\mathcal{W}$ of $\mathcal{Y}$ such that $\mathcal{U}/\mathcal{W}$ is a trivial extension of $\mathbf{1}$ by $\mathcal{Y}/\mathcal{W}$. The set $\mathbf{V}$ is not empty since $\mathcal{Y}$ is trivially in $\mathbf{V}$. It is enough to prove that if $\mathcal{V}_1$ and $\mathcal{V}_2$ are in $\mathbf{V}$, their intersection $\mathcal{W}$ lies in $\mathbf{V}$. Because $\mathcal{Y}$ is completely reducible, there exist three sub-objects $\mathcal{V}'$, $\mathcal{W}_1'$, $\mathcal{W}_2'$ of $\mathcal{Y}$ such that :

1. $\mathcal{V}_1 = \mathcal{W} \oplus \mathcal{W}_1'$, $\mathcal{V}_2 = \mathcal{W} \oplus \mathcal{W}_2'$.
2. $\mathcal{Y} = \mathcal{V}_1 \oplus \mathcal{W}_2' \oplus \mathcal{V}' = \mathcal{V}_2 \oplus \mathcal{W}_1' \oplus \mathcal{V}' = \mathcal{W} \oplus \mathcal{W}_2' \oplus \mathcal{W}_1' \oplus \mathcal{V}$

We have :

$$Ext^1(\mathbf{1}, \mathcal{Y}) \simeq Ext^1(\mathbf{1}, \mathcal{V}_1) \times Ext^1(\mathbf{1}, \mathcal{W}_2' \oplus \mathcal{V}') \text{ et } Ext^1(\mathbf{1}, \mathcal{Y}) \simeq Ext^1(\mathbf{1}, \mathcal{V}_2) \times Ext^1(\mathbf{1}, \mathcal{W}_1' \oplus \mathcal{V}').$$

Because $\mathcal{V}_1$ and $\mathcal{V}_2$ are in $\mathbf{V}$, the projection of $\mathcal{U}$ is trivial on $Ext^1(\mathbf{1}, \mathcal{W}_2' \oplus \mathcal{V}')$ and on $Ext^1(\mathbf{1}, \mathcal{W}_1' \oplus \mathcal{V}')$. Then the projection of $\mathcal{U}$ is also trivial on $Ext^1(\mathbf{1}, \mathcal{W}_2' \oplus \mathcal{W}_1' \oplus \mathcal{V}')$ and thus $\mathcal{W}$ is in $\mathbf{V}$.

1.2.2. Computation of the unipotent radical R_u of the Galois group $G_{\mathcal{U}}$ of $\mathcal{U}$. — By assumption, $\mathcal{U}$ lies in an exact sequence:

$$0 \longrightarrow \mathcal{Y} \xrightarrow{i} \mathcal{U} \xrightarrow{p} \mathbf{1} \longrightarrow 0 .$$

Let R be a C-algebra. Since the categories $\langle \mathcal{U} \rangle$ and $Rep_{G_{\mathcal{U}}}$ are equivalent, $\omega(\mathcal{U}) \otimes R$ is an extension of the unit representation $1_R := \mathbf{1} \otimes R$ by $\omega(\mathcal{Y}) \otimes R$ in the category $Rep_{G_{\mathcal{U}}(R)}$ of $G_{\mathcal{U}}(R)$-modules of finite rank over R. Consider the exact sequence of free R-modules :

$$\begin{tikzcd} 0 \arrow[r] & \omega(\mathcal{Y}) \otimes_C R \arrow[r, "\omega(i)^R"] & \omega(\mathcal{U}) \otimes_C R \arrow[r, "\omega(p)^R"] & \mathbf{1}_R \arrow[l, bend left, dotted, "s^R"] \arrow[r] & 0 , \end{tikzcd}$$

fix a section s of the underlying exact sequence of C-vector spaces, and put $f = s^R(1_R) \in \omega(\mathcal{U}) \otimes \mathbf{1}_R$, where $s^R = s \otimes 1$.
Let us consider the morphism of C-schemes $\zeta^R_{\omega(\mathcal{U})} : G_{\mathcal{U}}(R) \to \omega(\mathcal{Y}) \otimes R$ defined by the relation :

$$\forall \sigma \in G_{\mathcal{U}}(R),\ \zeta^R_{\omega(\mathcal{U})}(\sigma) = (\sigma - 1)f.$$

This defines a morphism of schemes $\zeta_{\omega(\mathcal{U})}$ over C from $G_{\mathcal{U}}$ with value in the C-vector group $\omega(\mathcal{Y})$, whose restriction to R_u is an immersion of algebraic group-schemes over

C from R_u to the C-vectorial group $\omega(\mathcal{Y})$. Since $G_{\mathcal{U}}$ is reduced, its scheme theoretic image is again reduced, and we have,

Lemma 1.1 **(see Lemma 2.8 in [9] in the case of** $char(C) = 0$ **and Prop. 6.2.3 in [14] for a particular case with** $char(C) > 0$**)**
The image W of R_u under $\zeta_{\omega(\mathcal{U})}$ is a C-vectorial subgroup of the C vectorial group $\omega(\mathcal{Y})$.

Proof. — Since W is reduced, it suffices to check this on points in the algebraic closure of C. For all $\sigma_1 \in G_{\mathcal{Y}}$ and $\sigma_2 \in R_u$, we have

$$\zeta_{\omega(\mathcal{U})}(\sigma_1\sigma_2{\sigma_1}^{-1}) = \sigma_1(\zeta_{\omega(\mathcal{U})}(\sigma_2)).$$

Indeed, we have :

$$\sigma_1\zeta_{\omega(\mathcal{U})}({\sigma_1}^{-1}) = (1-\sigma_1)f = -\zeta_{\omega(\mathcal{U})}(\sigma_1), \tag{1}$$

and

$$\zeta_{\omega(\mathcal{U})}(\sigma_1\sigma_2{\sigma_1}^{-1}) = \sigma_1(\zeta_{\omega(\mathcal{U})}(\sigma_2{\sigma_1}^{-1}))+\zeta_{\omega(\mathcal{U})}(\sigma_1) = \sigma_1(\sigma_2(\zeta_{\omega(\mathcal{U})}({\sigma_1}^{-1}))+\zeta_{\omega(\mathcal{U})}(\sigma_2))+\zeta_{\omega(\mathcal{U})}(\sigma_1).$$

From (1), we deduce that : $\sigma_1(\sigma_2(\zeta_{\omega(\mathcal{U})}({\sigma_1}^{-1}))) = -\sigma_1\sigma_2{\sigma_1}^{-1}(\zeta_{\omega(\mathcal{U})}(\sigma_1))$. But $\sigma_1\sigma_2{\sigma_1}^{-1}$ is an element of R_u and $\zeta_{\omega(\mathcal{U})}(\sigma_1)$ lies in $\omega(\mathcal{Y})$. Then, $\sigma_1(\sigma_2(\zeta_{\omega(\mathcal{U})}({\sigma_1}^{-1}))) = -\zeta_{\omega(\mathcal{U})}(\sigma_1)$. Therefore $\sigma_1(\zeta_{\omega(\mathcal{U})}(\sigma_2)) = \zeta_{\omega(\mathcal{U})}(\sigma_1\sigma_2{\sigma_1}^{-1})$ belongs to W. □

In other words, W is an algebraic subgroup over C of $\omega(\mathcal{Y})$ which is stable under the action of $G_{\mathcal{Y}}$. Now, $\mathbb{G}_m$ is contained in $G_{\mathcal{Y}}$ and acts on $\omega(\mathcal{Y})$ through an *isotypic* representation. This implies that W is a C-vectorial subgroup of the C-vectorial group $\omega(\mathcal{Y})$.

Lemma 1.2 **(see Prop. 2.9 in [9] in the case of** $char(C) = 0$ **)**
The image under ω of the smallest sub-object of $\mathbf{V}$ is equal to W.

Proof. — Let us denote by $\mathcal{V}$ the minimal object of $\mathbf{V}$, and by V its image under ω. Then, $G_{\mathcal{U}}$ acts on $\omega(\mathcal{U}/\mathcal{V})$ through $G_{\mathcal{Y}}$ (because $\mathcal{U}/\mathcal{V}$ is a trivial extension of $\mathbf{1}$ by a quotient of $\mathcal{Y}$ in the category $\mathbf{T}$). Thus the projection of $f^C = s(1)$ in $\omega(\mathcal{U})/V$ is invariant under the action of R_u, and the orbit $\{\sigma f^C - f^C;\ \sigma \in R_u\}$ lies in V. Therefore $\zeta_{\omega(\mathcal{U})}(R_u) := W \subset V$.

Conversely, the image W of R_u under $\zeta_{\omega(\mathcal{U})}$ is, by Lemma 1.1, a C-vector-space stable under the action of $G_{\mathcal{Y}}$ in $\omega(\mathcal{Y})$. Then, by equivalence of category, there exists a sub-object $\mathcal{W}$ of $\mathcal{Y}$ in $\mathbf{T}$ such that $\omega(\mathcal{W}) = W$. Let us show that $\mathcal{W}$ is an element of $\mathbf{V}$. Since W is the image of R_u, $G_{\mathcal{U}}$ acts on $\omega(\mathcal{U})/W$ through its quotient $G_{\mathcal{U}}/R_u = G_{\mathcal{Y}}$. Therefore, $\omega(\mathcal{U})/W(C)$ is an extension of C by $\omega(\mathcal{Y})/W(C)$ in the category $Rep_{G_{\mathcal{Y}}(C)}$. Since by hypothesis, every $G_{\mathcal{Y}}$-module is completely reducible, this extension is trivial in the category $Rep_{G_{\mathcal{Y}}}$. By the Tannakian equivalence of category, the extension $\mathcal{U}/\mathcal{W}$ is also trivial in $Ext_{\mathbf{T}}(\mathbf{1}, \mathcal{Y}/\mathcal{W})$, and $\mathcal{W} \in \mathbf{V}$. Then $\mathcal{V} \subset \mathcal{W}$ by minimality. This concludes the proof of Lemma 1.2, hence of Theorem 2. □

2. An application to logarithms of t-motives

2.1. A brief overview of the category of t-motives. — We recall here some notations of the introduction. Let $\mathbf{F}_q$ be the field of q elements, where q is a power of a prime p. Let $k := \mathbf{F}_q(\theta)$, where θ is transcendental over $\mathbf{F}_q$. Define a valuation $|.|_\infty$ at the infinite place of k such that $|\theta|_\infty = q$. Let $k_\infty := \mathbf{F}_q((1/\theta))$ be the ∞-adic completion of k, let $\overline{k_\infty}$ be an algebraic closure, let $\mathbf{K}$ be the ∞-adic completion of $\overline{k_\infty}$, and let $\overline{k}$ be the algebraic closure of k in $\mathbf{K}$. Let $\mathbf{T}$ be the Tate algebra of power series in $\mathbf{K}[[t]]$ that are convergent on the closed unit disk in $\mathbf{K}$ and let $\mathbf{L}$ be its fraction field in $\mathbf{K}((t))$. For a Laurent series $f = \sum_i a_i t^i \in \mathbf{K}((t))$ and an integer $n \in \mathbb{Z}$, we set $\sigma^{-n}(f) := f^{(n)} := \sum_i a_i^{q^n} t^i$. The ring $\overline{k}(t)[\sigma, \sigma^{-1}]$ is the non commutative ring of Laurent polynomials in the variable σ with coefficients in $\overline{k}(t)$, subject to the relation

$$\sigma f = \sigma(f)\sigma = f^{(-1)}\sigma,$$

for all $f \in \overline{k}(t)$.

2.1.1. The category of t-motives. — In [**14**, §3.2.1], M. Papanikolas defines the category $\mathcal{P}$ of pre-t-motives as follows:

Definition 2.1. — The objects of $\mathcal{P}$ are the left $\overline{k}(t)[\sigma, \sigma^{-1}]$-modules that are finite dimensional over $\overline{k}(t)$. The morphism in $\mathcal{P}$ are left $\overline{k}(t)[\sigma, \sigma^{-1}]$-modules homomorphisms.

The category $\mathcal{P}$ is the category of σ-difference modules over $\overline{k}(t)$ (see [**17**]). The category $\mathcal{P}$ is a rigid abelian $\mathbf{F}_q(t)$-linear tensor category. To ensure the existence of a fiber functor defined over $\mathbf{F}_q(t)$, Papanikolas consider a sub-category $\mathcal{R}$ of $\mathcal{P}$, defined as follows,

Definition 2.2. — A pre-t-motive M is rigid analytically trivial if there exists $\Psi \in Gl_r(\mathbf{L})$ such that $\Psi^{(-1)} = \Phi\Psi$ where Φ represents the multiplication by σ in some $\overline{k}(t)$-basis of M. Then, the category $\mathcal{R}$ is the full sub-category of $\mathcal{P}$ formed by the rigid analytically trivial pre-t-motives.

Then,

***Theorem 2.3* (see Theorem 3.3.15 in [14]).** — *The category $\mathcal{R}$ is a neutral Tannakian category over $\mathbf{F}_q(t)$ with fiber functor*

$$\mathcal{R} \to Vect_{\mathbf{F}_q(t)}, M \mapsto M^B := \{\mu \in \mathbf{L} \otimes_{\overline{k}(t)} M | \sigma(\mu) = \mu\}.$$

Remark 2.4. — In characteristic 0, [**7**, Theorem 7.1] states that a rigid abelian tensor category over k, whose objects have finite dimension, admits a fiber functor defined on the algebraic closure of k. But nothing ensure that there exists a fiber functor defined over k. The idea to impose some convergence properties on the solutions in order to build a fiber functor defined over a smaller field of constants, was for instance used by P. Etingov in [**8**] to attach to a q-difference equation over $\mathbb{C}(x)$ a Galois group defined over $\mathbb{C}$. The analytic solutions of a q-difference equation with complex solutions are meromorphic functions over $\mathbb{C}^*$ and the constants of the field of meromorphic functions over $\mathbb{C}^*$ are elliptic functions w.r.t. the elliptic curve

$\mathbb{C}^*/q^{\mathbb{Z}}$. In fact, one could build directly a fiber functor over $\mathbb{C}$ but it involves symbolic solutions (see [**17**])

Now,

***Definition 2.5* (see Definition 3.4.8 in [14]).** — We define the category $\mathcal{A}^I$ of Anderson t-motives up to isogeny as follows:

- objects of $\mathcal{A}^I$: Anderson t-motives;
- Morphism of $\mathcal{A}^I$: for Anderson t-motives M and N,

$$Hom_{\mathcal{A}^I}(M,N) := Hom_{\overline{k}[t,\sigma]}(M,N) \otimes_{\mathbf{F}_q[t]} \mathbf{F}_q(t).$$

Then, $\mathcal{AR}^I$ denotes the full sub-category of rigid analytically trivial Anderson t-motives up to isogeny by restriction.

We have,

***Theorem 2.6* (see Theorem 3.4.9 in [14]).** — *The functor $\mathcal{AR}^I \to \mathcal{R}, M \mapsto M$ is fully faithful.*

We are now able to define the category of t-motives and the Galois group of a t-motive.

***Definition 2.7* (see §3.4.10 in [14]).** — The category $\mathcal{T}$ of t-motives is the strictly full Tannakian sub-category generated by the essential image of the functor $\mathcal{AR}^I \to \mathcal{R}, M \mapsto M$. The functor $\omega : \mathcal{T} \to Vect_{\mathbf{F}_q(t)}, M \mapsto M^B$ is a fiber functor of $\mathcal{T}$. For every object P of $\mathcal{T}$, we denote by G_P the Galois group of P w.r.t. ω.

In order to prove some results of transcendence for periods of Drinfeld $\mathbf{F}_q(t)$-module E, on has first to exhibit a rigid analytically trivial Anderson t-motive P, whose special values of a trivialization Ψ at $t = \theta$ will interpolate the periods of the Drinfeld module E. Then, Theorem 4 shows that the dimension of the Galois group G_P of P as t-motive is exactly the transcendence degree of the field generated by the periods of E over $\overline{k}$. The transcendence study is then reduced to the computation of a Galois group. However, the explicit connection between periods of a Drinfeld module and special values of an analytically trivial Anderson t-motives is, to my knowledge, a work in progress of Anderson and Papanikolas.

2.1.2. Exemples of t-motives. — We introduce now the t-motives, whose special values of trivializations are involved in Theorem 5.

2.1.2.1. The unit object $\mathbf{1}$. — Let $\mathbf{1} := \overline{k}(t)$ together with the σ-action defined by $\sigma(f) = \sigma(f) = f^{(-1)}$ for all $f \in \mathbf{1}$. One has $End_{\mathcal{T}}(\mathbf{1}) = \mathbf{F}_q(t)$ and $\omega(\mathbf{1}) = \mathbf{F}_q(t)$ (see [**14**, Lemma 3.3.2]).

2.1.2.2. The Carlitz motive $\mathcal{C}$. — The Carlitz motive is the pre-t-motive whose underlying $\overline{k}(t)$-vector space is $\overline{k}(t)$ with σ-action given by

$$\sigma f := (t-\theta) f^{(-1)}, \text{ for all } f \in \mathcal{C}.$$

One has to show that $\mathcal{C}$ is rigid analytically trivial.
Let Ω be the power series defined as follows

$$\Omega(t) := \zeta_\theta \prod_{i=1}^{\infty}(1 - t/\theta^{(i)}) \in k_\infty(\zeta_\infty)[[t]] \subset \mathbf{K}[[t]],$$

with ζ_∞ is a fixed $(q-1)$-th root of $-\theta$ in $\overline{k_\infty}$. The series $\Omega(t)$ has an infinite radius of convergence and so $\Omega \subset \mathbf{T}$. Since Ω has no zeroes inside the unit disk, $\Omega \in \mathbf{T}^*$. It also satisfies the functional equation

$$\Omega^{(-1)} = (t-\theta)\Omega.$$

The number

$$\tilde{\pi} = -\frac{1}{\Omega(\theta)} = \theta\zeta_\theta \prod_{i=1}^{\infty}(1 - \theta^{1-q^i})^{-1} \in k_\infty(\zeta_\infty)$$

is the *Carlitz period*, i.e., the period of the Carlitz-module $\mathfrak{C}$.

Proposition 2.8. — *The Carlitz motive $\mathcal{C}$ is rigid analytically trivial with fiber $\omega(\mathcal{C}) = \frac{1}{\Omega}\mathbf{F}_q(t)$.*

One can prove the following

Proposition 2.9. —
- *The Galois group of $\mathcal{C}$ is isomorphic to the multiplicative group $\mathbf{G}_m$ over $\mathbf{F}_q(t)$ (see* [**14**, Theorem 3.5.4]*).*
- *Moreover, $End_{\mathcal{T}}(\mathcal{C}) = \mathbf{F}_q(t)$ (see* [**14**, Lemma 3.5.3]*).*

2.1.2.3. The t-motive of Carlitz logarithm. — Let $\alpha_i \in \overline{k}^*$. Set :

$$\Phi(\alpha_i) := \begin{pmatrix} (t-\theta) & 0 \\ \alpha_i^{(-1)}(t-\theta) & 1 \end{pmatrix}.$$

$\Phi(\alpha_i)$ defines a pre-t-motive $\mathcal{X}(\alpha_i)$, which is an extension in the category $\mathcal{T}$ of $\mathbf{1}$ by the Carlitz motive $\mathcal{C}$

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{X}(\alpha_i) \longrightarrow \mathbf{1} \longrightarrow 0\,.$$

For $|\alpha|_\infty < |\theta|_\infty^{q/(q-1)}$, define the power series

$$L_\alpha := \alpha + \sum_{i=1} \frac{\alpha^{q^i}}{(t-\theta^q)(t-\theta^{q^2})...(t-\theta^{q^i})}.$$

One can show that $L_\alpha \in \mathbf{T}$ and that $L_\alpha(z)$ converges for all $z \in \mathbf{K}$ with $|z|_\infty < |\theta|_\infty^q$. It satisfies the functional equation

$$L_{\alpha_i}^{(-1)} = \alpha_i^{(-1)} + \frac{L_{\alpha_i}}{t-\theta}.$$

The special value at θ of L_{α_i} satisfy

$$L_\alpha(\theta) = log_{\mathcal{C}}(\alpha),$$

where $log_{\mathcal{C}}(z) := z + \sum_{i=1} \frac{z^{q^i}}{(\theta-\theta^q)(\theta-\theta^{q^2})...(\theta-\theta^{q^i})}$ is the Carlitz logarithm, the inverse of the exponential of the Carlitz module.

Then

Proposition 2.10 **(see Prop. 6.1.3 in [14]).** — *For $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$, the pre-t-motive $\mathcal{X}(\alpha_i)$ is rigid analytically trivial and its trivialization is given by:*

$$\Psi(\alpha_i) := \begin{pmatrix} \Omega & 0 \\ \Omega L_{\alpha_i} & 1 \end{pmatrix}.$$

2.1.2.4. The multiple Carlitz logarithm motive. — Let $\alpha_1, ..., \alpha_r \in \overline{k}^*$ with $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$. Set :

$$\Phi(\alpha_1, ..., \alpha_r) := \begin{pmatrix} t-\theta & 0 & \cdots & 0 \\ \alpha_1^{(-1)}(t-\theta) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_r^{(-1)}(t-\theta) & 0 & \cdots & 1 \end{pmatrix}.$$

$\Phi(\alpha_1, ..., \alpha_r)$ defines a pre-t-motive $\mathcal{X}(\alpha_1, ..., \alpha_r)$ which is an extension of $\mathbf{1}^r$ by the Carlitz motive $\mathcal{C}$:

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{X}(\alpha_1, ..., \alpha_r) \longrightarrow \mathbf{1}^r \longrightarrow 0 .$$

The pre-t-motive $\mathcal{X}(\alpha_1, ..., \alpha_r)$ is rigid analytically trivial and its trivialization is given by:

$$\Psi(\alpha_1, ..., \alpha_r) := \begin{pmatrix} \Omega & 0 & \cdots & 0 \\ \Omega L_{\alpha_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega L_{\alpha_r} & 0 & \cdots & 1 \end{pmatrix}.$$

2.2. Papanikolas's theorems on algebraic independence of Carlitz logarithms.— First, we present an alternative formulation of Theorem 5. Since the Carlitz period $\tilde{\pi}$ satisfies $exp_{\mathcal{C}}(\tilde{\pi}) = 0$, the logarithmic version of Theorem 5 is the following

Theorem 2.11. — *Let $\alpha_1, ..., \alpha_r \in \overline{k}^*$ with $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$. Assume that $\tilde{\pi}, log_{\mathcal{C}}(\alpha_1), ..., log_{\mathcal{C}}(\alpha_r)$ are linearly independent over k. Then they are algebraically independent over $\overline{k}$.*

By §2.1.2, we have $\tilde{\pi} = -\frac{1}{\Omega(\theta)}, log_{\mathcal{C}}(\alpha_1) = L_{\alpha_1}(\theta), ..., log_{\mathcal{C}}(\alpha_r) = L_{\alpha_r}(\theta)$. Then, the field generated by the periods $\tilde{\pi}, log_{\mathcal{C}}(\alpha_1), ..., log_{\mathcal{C}}(\alpha_r)$ over $\overline{k}$ coincides with the field generated by the special values of $\Psi(\alpha_1, ..., \alpha_r)$ at $t = \theta$ over $\overline{k}$, i.e.

$$\overline{k}(\tilde{\pi}, log_{\mathcal{C}}(\alpha_1), ..., log_{\mathcal{C}}(\alpha_r)) = \overline{k}(\Psi(\alpha_1, ..., \alpha_r)(\theta)).$$

Combining this simple remark with Theorem 4 applied to the t-motive $\mathcal{M} = \mathcal{X}(\alpha_1, ..., \alpha_r)$, Papanikolas reduces the proof of Theorem 5 to showing:

Theorem 2.12 **(see Theorem 6.3.2.c in [14]).** — *Let $\alpha_1, ..., \alpha_r \in \overline{k}^*$ with $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$. Assume that $\tilde{\pi}, log_{\mathcal{C}}(\alpha_1), ..., log_{\mathcal{C}}(\alpha_r)$ are linearly independent over*

k. Then, the dimension of the Galois group $G_{\mathcal{X}}$ of the t-motive $\mathcal{X} = \mathcal{X}(\alpha_1, ..., \alpha_r)$ is equal to $r+1$.

2.3. A new proof of the independence of the Carlitz logarithms. — So, we have to compute the dimension of the Galois group attached to the motive $\mathcal{X} = \mathcal{X}(\alpha_1, ..., \alpha_r)$. As in [**9**, Proof of Cor 2.2.], we have:

Lemma 2.13. — *The tannakian sub-category generated by $\mathcal{X}(\alpha_1, ..., \alpha_r)$ in $\mathcal{T}$ is equal to the Tannakian sub-category generated by the motive $\bigoplus_{i=1}^r \mathcal{X}(\alpha_i)$.*

This Lemma implies that the Galois group of $\mathcal{X}$ is equal to the Galois group $G := G_{\bigoplus_{i=1}^r \mathcal{X}(\alpha_i)}$. As in [**14**], 6.2.2, we see that the quotient of G by its unipotent radical is isomorphic to the Galois group of the Carlitz motive $\mathcal{C}$, i.e, to $\mathbb{G}_m$. Therefore, it remains to compute the dimension of the unipotent radical of G, that is the unipotent radical of the Galois group of a sum of extensions of $\mathbf{1}$ by the Carlitz motive $\mathcal{C}$. To compute the latter dimension, we use the Corollary 1.

2.3.1. Application to Theorem 2.12. — We apply Corollary 1 to the category $\mathcal{T}$ of t-motives over $C := End_{\mathcal{T}}(\mathbf{1}) = \mathbb{F}_q(t)$ in order to compute the unipotent radical of the Galois group $G := G_{\bigoplus_{i=1}^r \mathcal{X}(\alpha_i)}$ associated with the extensions $\mathcal{X}(\alpha_i) \in Ext^1_{\mathcal{T}}(\mathbf{1}, \mathcal{C})$ described in §2.1.2. The assumptions of Corollary 1 are satisfied since

- every $G_{\mathcal{C}} = \mathbf{G}_m$-module is completely reducible;
- $G_{\mathcal{C}} = \mathbf{G}_m$ acts on the line $\omega(\mathcal{C})$ through its canonical character;
- the Galois group of t-motives are reduced (see [**14**]);

Then, the dimension of the algebraic group $G = G_{\bigoplus_{i=1}^r \mathcal{X}(\alpha_i)}$ is equal to $1+n$, where n denotes the dimension of the vector space over $\Delta := End_{\mathcal{T}}(\mathcal{C}) = \mathbb{F}_q(t)$ generated by the $\mathcal{X}(\alpha_i)$'s in $Ext^1_{\mathcal{T}}(\mathbf{1}, \mathcal{C})$.

First let us remark that a difference equation $\Psi^{(-1)} = \begin{pmatrix} (t-\theta) & 0 \\ b & 1 \end{pmatrix} \Psi$, where $b \in \overline{k}(t)$ corresponds to a trivial extension extension of $\mathbf{1}$ by $\mathcal{C}$, if and only if, b is in the cokernel of the Carlitz-motive, *i.e.* , if and only if, there exists $f \in \overline{k}(t)$ such that $b = (t-\theta)f^{(-1)} - f$. Secondly, it is easy to see that a $\mathbb{F}_q(t)$- linear combination of the $\mathcal{X}(\alpha_i)$'s in $Ext^1_{\mathcal{T}}(\mathbf{1}, \mathcal{C})$ corresponds to a difference equation

$$\Psi^{(-1)} = \begin{pmatrix} (t-\theta) & 0 \\ \sum_{i=1}^s \mu_i \alpha_i^{(-1)}(t-\theta) & 1 \end{pmatrix} \Psi, \text{ where } \mu_i \in \mathbb{F}_q(t).$$

Thus, we get

$$n = max\{s | \exists f \in \overline{k}(t), (\mu_i)_{i=1}^s \in \mathbb{F}_q(t) \text{ not all zero, such that} (t-\theta)f^{(-1)} - f = \sum_{i=1}^s \mu_i \alpha_i^{(-1)}(t-\theta)\}.$$

By assumption, $\tilde{\pi}, log_{\mathcal{C}}(\alpha_1), ..., log_{\mathcal{C}}(\alpha_r)$ are linearly independent over $k = \mathbb{F}_q(\theta)$. Following [**14**], bottom of p. 171, we now prove that under this hypothesis, n is equal to r.

Suppose that $n < r$. Then, let us consider s such that $\exists f \in \overline{k}(t), (\mu_i)_{i=1}^s \in \mathbb{F}_q(t)$ non all equal to zero such that

$$(t-\theta)f^{(-1)} - f = \sum_{i=1}^{s} \mu_i \alpha_i^{(-1)}(t-\theta). \tag{2}$$

It follows from Equation (2) that f is regular at $t=\theta$: if not, $f^{(-1)}$ must have a pole at $t=\theta^{(-1)}$ which implies that f has a pole at $t=\theta^{(-1)}$. By repeating this argument, we get that if f is singular at $t=\theta$ it is also singular at $t=\theta^{(-i)}$ for all $i \geq 1$, which is impossible. Therefore, f and $f^{(-1)}$ are regular at $t=\theta$.

Considering the form of Equation (2), we then get $f(\theta)=0$. Moreover, the solutions y of (2) are of the following type :

$$y = \mu\frac{1}{\Omega} + \sum_{i=1}^{s} \mu_i L_{\alpha_i}$$

with $\mu \in \mathbb{F}_q(t)$. So, there exists $\mu \in \mathbb{F}_q(t)$, such that :

$$f = \mu\frac{1}{\Omega} + \sum_{i=1}^{s} \mu_i L_{\alpha_i}. \tag{3}$$

By taking $t=\theta$ in (3), we get:

$$0 = \mu(\theta)\tilde{\pi} + \sum_{i=1}^{s} \mu_i(\theta) log_{\mathcal{C}}(\alpha_i).$$

This is a non trivial relation over k between $\tilde{\pi}, log_{\mathcal{C}}(\alpha_1), ..., log_{\mathcal{C}}(\alpha_r)$, which contradicts our assumption.

So, $dimG = r+1$. This concludes the proof of Theorem 2.12, and implies, as recalled in Section 1, that $trdeg_{\overline{k}}\overline{k}(\tilde{\pi}, log_C(\alpha_1), ..., log_C(\alpha_r)) = r+1$, i.e. that $\tilde{\pi}, log_C(\alpha_1), ..., log_C(\alpha_r)$ are algebraically independent over $\overline{k}$.

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