REFINEMENTS ON HIGHER ORDER WEIL-OESTERLÉ BOUNDS VIA A SERRE TYPE ARGUMENT

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Abstract. Weil's theorem gives the most standard bound on the largest number of points $N_q(g)$ of a curve of genus g over a finite field $\mathbb{F}_q$. This bound was improved by Ihara and Oesterlé for larger genus. Recently, the first and third authors gave a new point of view on these bounds, that can be obtained by solving a sequence of semi-definite programs, and the two first steps of this hierarchy recover Weil's and Ihara's bounds. On the other hand, by taking into account arithmetic constraints, Serre obtained a refinement on Weil's bound. In this article, we combine these two approaches and propose a strengthening of Ihara's bound, based on an argument similar to Serre's refinement. We obtain a closed formula that generically improves upon Ihara's bound, even in the range where it was the best explicit bound so far. This provides new upper bounds on $N_q(g)$ for infinitely many couples (q, g), twenty of them entering the table `manYPoints` [vdGHLR09]. Finally we discuss possible extensions to higher order Weil-Oesterlé bounds.

Introduction

This work deals with the problem of bounding above the number of rational points of an absolutely, irreducible, smooth, projective curve X, of genus g, defined over the finite field $\mathbb{F}_q$, where q is a prime power. In the 1940s Weil [Wei40, Wei41] proved that this number $\sharp X(\mathbb{F}_q)$ satisfies the inequalities

$$(q+1) - 2g\sqrt{q} \leq \sharp X(\mathbb{F}_q) \leq (q+1) + 2g\sqrt{q}.$$

In 1985, Serre brought a new focus to this issue by giving a course on this topic at Harvard. Gouvea's handwritten notes on these lectures, which have long circulated in the community, have recently been published [Ser20]. It contains lots of ideas to improve Weil's bounds in several directions such as computing the exact values of the constants $N_q(g)$ defined by

$$N_q(g) = \max \left\{ \sharp X(\mathbb{F}_q), \quad \begin{array}{l} X \text{ absolutely, irreducible, smooth, projective} \\ \mathbb{F}_q\text{-curve of genus } g \end{array} \right\}$$

for small values of the genus g, or computing better upper bounds for large genus. This course has opened the way for many new works and developments leading to better and better bounds for different constants $N_q(g)$ (on this topic or some other developments, see for example [AI15, How21, BHLS15, How12, HL03, HL12, HL07, BHLGR25] and the references in [Ser20]). These improvements are now listed on the very useful website `manYPoints` [vdGHLR09], where the best-known lower and upper bounds on $N_q(g)$ for small values of q and g are regularly updated.

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This work is no exception to this rule, and continues the ideas developed in Serre's course, particularly two strategies that we now point out.

First, taking into account the fact that the eigenvalues of the Frobenius are algebraic integers, Serre sharpened Weil's bounds [Ser83] to what is now called the ***Weil-Serre bound*** (see Section 1.2):

$$(q+1) - g\lfloor 2\sqrt{q}\rfloor \leq \sharp X(\mathbb{F}_q) \leq (q+1) + g\lfloor 2\sqrt{q}\rfloor.$$

Of course this is only an improvement for non squares values of q and the gain is in $g\{2\sqrt{q}\}$ where, for $x \in \mathbb{R}$, $\{x\}$ denotes its fractional part.

On the other hand, Ihara [Iha81] noted that the Weil bounds cannot be optimal for large genus and Serre, in his course, developed the so-called *explicit formulæ*. This approach results in an optimization problem whose solution gives an upper bound for $N_q(g)$, for increasing genus. Oesterlé solved this problem, leading to the Oesterlé bounds, see [Ser20, Chapter VI]. In [HP19], the first and third authors gave a new point of view on these bounds, by reproving them in the spirit of the original proof of the Weil bounds using intersection theory in the algebraic surface $X\times X$. In their setting, recalled in section 1.1, it is shown that there exists an explicit strictly increasing sequence $(g_n)_{n\geq 1}$ of non-negative real numbers and a sequence $(\mathbf{N}_n^\star)_{n\geq 1}$ of strictly increasing functions from $[g_n, +\infty)$ to $\mathbb{R}$, such that for any $g \geq g_n$, the value $\mathbf{N}_n^\star(g)$ is an upper bound for $N_q(g)$, that we call the ***Weil-Oesterlé bound*** of order n. Moreover, the bound $\mathbf{N}_{n+1}^\star(g)$ is sharper than $\mathbf{N}_n^\star(g)$ for $g > g_{n+1}$. Furthermore, the bound $\mathbf{N}_1^\star(g)$ is nothing else than Weil's bound, while $\mathbf{N}_2^\star(g)$ is Ihara's bound [Iha81], and one has

$$g_1 = 0, \qquad g_2 = \frac{\sqrt{q}\left(\sqrt{q}-1\right)}{2}, \qquad g_3 = \frac{\sqrt{q}\,(q-1)}{\sqrt{2}}.$$

The main contribution of this article is contained in Section 2. We combine the two previous strategies to obtain an improvement of Ihara's bound following Serre's improvement on Weil's bound. This leads us to a new upper bound for $N_q(g)$ (Theorem 2.5) which works for every q, and the gain compared to Ihara's bound is explicit. Like Serre, this gain depends on the fractional part of a quantity which is a little bit more difficult to analyse than for the Weil-Serre bound. Section 2.5 is devoted to this analysis, in several respects. First in Section 2.5.1, we fix q and let g go to infinity. We prove that, except for $q = 3$, our gain has an asymptote in g with positive sloape, and thus tends to infinity. Second we focus on the Ihara range $[g_2, g_3]$. It is fair to compare our bound with Ihara's therein, because $\mathbf{N}_2^\star(g)$ does not get improved by Weil-Oesterlé bounds of higher order. In this interval, the fractional part in our bound makes more difficult the analysis of the gain. However, we use numerical experiments, for several values of q, to compare our bound with Ihara's (see Section 2.5.2), and provide an explicit infinite sequence of couples (q, g_q) with $g_q \in [g_2, g_3]$ such that this gain is the best we can hope for and goes to infinity with q, see Section 2.5.3. Also note that sometimes we cannot expect to improve upon Ihara's bound, because this bound can be sharp for several curves, such as Suzuki curves, see [AHM25]. Last in Section 2.5.4 we compare our bound with the entries in `manYPoints`, namely for $q < 100$ and $g \leq 50$. In the Ihara range, we find more than 150 couples (q, g) for which our bound improves upon Ihara's by 1. Among them, we recover the current record in more than 130 cases (where Ihara's bound was already improved by other techniques, from several works [Sav03, SV86, Rig10, KÖ3, LJP01, How12, HL03, HL12]), and improve this record for 20 couples (q, g). We summarise the techniques that provide the current best bounds in the Ihara range in Table 2.

We end by a more prospective Section 3. It is natural to ask whether or not Weil-Oesterlé bounds of higher order could be improved in the same way. We give some experiments for the Weil-Oesterlé bound of order 3 which lead to new records, but we discuss why our method fails to be easily generalized in greater order.

1. Weil-Oesterlé bounds and Serre's trick to improve Weil's bound

Let X be an absolutely irreducible smooth projective curve of genus g defined over $\mathbb{F}_q$. Weil proved the existence of algebraic integers $\omega_1, \ldots, \omega_g$ of modulus $\sqrt{q}$ such that the number N_k of $\mathbb{F}_{q^k}$-points of X satisfies

$$N_k = \left(q^k + 1\right) - \sum_{j=1}^{g} \omega_j^k + \overline{\omega_j}^k.$$

Moreover, the family $\{\omega_1, \ldots, \omega_g, \overline{\omega_1}, \ldots, \overline{\omega_g}\}$ is stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, they are nothing else than the eigenvalues of the Frobenius. This results immediately implies bounds on N_k. Indeed, writing $\omega_j = \sqrt{q}e^{i\theta_j}$, one gets

$$\left|N_k - (1+q^k)\right| \leq 2q^{k/2} \sum_{j=1}^{g} |\cos(k\theta_j)| \leq 2gq^{k/2}.$$

In particular, for $k = 1$, one gets the celebrated Weil inequality

$$|N_1 - (1+q)| \leq 2g\sqrt{q}. \tag{1}$$

More generally, if we introduce

$$t_k = \sum_{j=1}^{g} \omega_j^k + \overline{\omega_j}^k = 1 + q^k - N_k, \tag{2}$$

any lower bound on t_k gives an upper bound on N_k.

1.1. Weil-Oesterlé bounds.

Using the method of *explicit formulæ* (see [Ser20, Chapter V.3]), any trigonometric polynomial f which is nonnegative on the unit circle and whose coefficients in the cosine expansion are nonnegative gives a lower bound on t_1. Finding the best f possible then becomes a conic optimization problem, and Oesterlé gave an explicit procedure to solve it, see [Ser20, Chapter VI]. In [HP19], the first and the third authors provided the following new point of view on this question. Thanks to the Hodge index Theorem on the surface $X \times X$, the opposite of the intersection pairing defines a scalar product on the orthogonal of the space generated by horizontal and vertical divisors inside the space of divisors of $X \times X$ up to numerical equivalence. In this Euclidean space, if we denote by $p(\Gamma^i)$, $i \geq 0$ (Γ^0 is the diagonal of $X \times X$) the orthogonal projection of Γ^i onto the orthogonal of the vertical and horizontal parts, then one shows using elementary intersection theory that

$$\text{Gram}\left(p(\Gamma^0),\dots,p(\Gamma^n)\right) = \begin{pmatrix} 2g & t_1 & t_2 & \dots & t_n \\ t_1 & 2gq & qt_1 & \ddots & qt_{n-1} \\ t_2 & qt_1 & 2gq^2 & \ddots & q^2t_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ t_n & qt_{n-1} & q^2t_{n-2} & \dots & 2gq^n \end{pmatrix}. \tag{3}$$

Being a Gram matrix, it must be positive semi-definite (PSD). Moreover, because the number of points N_k of points of X over $\mathbb{F}_{q^k}$ can only be greater or equal than N_1, the variables t_k are also constrained, using (2), by the affine inequalities

$$t_k \leq t_1 + q^k - q. \tag{4}$$

Therefore, the vector $(t_1,\dots,t_n)$ must belong to a spectrahedron, and by minimising t_1 over this convex set, one gets a lower bound on t_1 for any curve. In fact, this semi-definite program is closely related to the dual of the optimization problem solved by Oesterlé, as shown in [HP19], but provides a more geometric approach. Moreover, note that for $n = 1$, one gets exactly the bound by Weil (1), while $n = 2$ leads to Ihara's [Iha81]. For this reason, for $n \geq 2$, these bounds can be seen as higher order Weil bounds, and we call this sequence of bounds the *Weil-Oesterlé* hierarchy. Nevertheless, if increasing n leads in theory to better bounds, it was shown in [HP19] that for any field size q, there is an explicit sequence $g_n = g_n(q)$ such that if $g_k \leq g \leq g_{k+1}$, then the bound for q and g does not improve for $n > k$.

While this approach hides the eigenvalues ω_j of the Frobenius, they play a crucial role in the next Section.

1.2. **Serre's improvement on Weil's bound.** Turning back to Weil's bound (1), a simple observation due to Serre leads to the following refinement. Recall that

$$t_1 = 1 + q - N_1 = \sum_{j=1}^{g} \omega_j + \overline{\omega_j},$$

with $|\omega_j| = \sqrt{q}$, and $\{\omega_1,\dots,\omega_g,\overline{\omega_1},\dots,\overline{\omega_g}\}$ stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If we set $\tau_1(\omega) = \omega + \overline{\omega}$, then for every $1 \leq j \leq g$,

$$\tau_1(\omega_j) = 2\sqrt{q}\cos(\theta_j) \in [-2\sqrt{q}, 2\sqrt{q}],$$

so that

$$\tau_1(\omega_j) + \lfloor 2\sqrt{q} \rfloor + 1 > 0,$$

where $\lfloor 2\sqrt{q} \rfloor$ denotes the floor of $2\sqrt{q}$. Now, the arithmetic-geometric mean inequality implies that

$$\frac{1}{g}\sum_{j=1}^{g}(\tau_1(\omega_j) + \lfloor 2\sqrt{q} \rfloor + 1) \geq \left(\prod_{j=1}^{g}(\tau_1(\omega_j) + \lfloor 2\sqrt{q} \rfloor + 1)\right)^{\frac{1}{g}}. \tag{5}$$

Next, observe that the product $\prod_{j=1}^{g}(\tau_1(\omega_j) + \lfloor 2\sqrt{q} \rfloor + 1)$ is

i) a rational number, because it is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,
ii) an algebraic integer, since it is a product of algebraic numbers,
iii) positive, as a product of positive real numbers.

This product thus has to be a positive integer, and is therefore at least 1, so that (5) gives

$$t_1 \geq -g\lfloor 2\sqrt{q}\rfloor. \tag{6}$$

If it does not improve upon Weil's bound when q is a square, this refinement can lead to substantial improvements for non-square values of q when g grows.

2. Improvement on Ihara's bound

2.1. A generalisation of Serre's argument.

We first describe a general strategy to get an analogue of the trick described in Section 1.2 to higher order Weil-Oesterlé bounds. Recall that for any $k \geq 1$,

$$t_k = \sum_{j=1}^{g} \tau_k(\omega_j) \tag{7}$$

where

$$\tau_k(\omega_j) = \omega_j^k + \overline{\omega_j}^k = 2q^{\frac{k}{2}}\cos(k\theta_j).$$

Serre's argument relies on an affine inequality whose coefficients are integers, and which is satisfied by any $\tau_1(\omega) \in [-2\sqrt{q}, 2\sqrt{q}]$. This idea can be generalized as follows.

Lemma 2.1. *Let $a_0, \dots a_n$ integers. Assume that for every ω with $|\omega| = \sqrt{q}$,*

$$\sum_{k=1}^{n} a_k\tau_k(\omega) + a_0 > 0. \tag{8}$$

Then

$$\sum_{k=1}^{n} a_k t_k + ga_0 \geq g. \tag{9}$$

Proof. Following the method given in Section 1.2, we use the arithmetic-geometric mean, which gives

$$\frac{1}{g}\sum_{j=1}^{g}\left(\sum_{k=1}^{n} a_k\tau_k(\omega_j) + a_0\right) \geq \left(\prod_{j=1}^{g}\left(\sum_{k=1}^{n} a_k\tau_k(\omega_j) + a_0\right)\right)^{\frac{1}{g}}. \tag{10}$$

Because the coefficients a_k are integers, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes the algebraic integers $\sum_{k=1}^{n} a_k\tau_k(\omega_j) + a_0$, for $1 \leq j \leq g$. Therefore the product of these positive real numbers is a positive integer, hence greater than 1. Together with (7), this implies the result. □

When the coefficients a_k are nonnegative for $1 \leq k \leq n$, we can further combine Lemma 2.1 with the conditions (4), to get the following general bound.

Theorem 2.2. *Let $a_0, \dots a_n$ integers such that $a_k \geq 0$ for $1 \leq k \leq n$. Assume that for every ω with $|\omega| = \sqrt{q}$, the affine integral inequality*

$$\sum_{k=1}^{n} a_k\tau_k(\omega) + a_0 > 0 \tag{11}$$

holds. Then

$$t_1 \geq \frac{g(1-a_0) - \sum_{k=1}^n a_k(q^k - q)}{\sum_{k=1}^n a_k}. \tag{12}$$

Proof. The conditions of Lemma 2.1 are satisfied. Then the conditions $t_k \leq t_1 + q^k - q$ from (4) give, because $a_k \geq 0$ for $1 \leq k \leq n$,

$$\left(\sum_{k=1}^n a_k\right) t_1 + \sum_{k=1}^n a_k(q^k - q) \geq g(1-a_0),$$

which proves the result. □

In the following, we focus on the case $n = 2$, explaining how to obtain inequalities of the form (8) that improve upon Ihara's bound.

2.2. **Ihara's bound as the Weil-Oesterlé bound of order 2.** We first review how to obtain Ihara's bound with the approach from [HP19]. For $n = 2$, the matrix in (3) is

$$\operatorname{Gram}\left(p(\Gamma^0), p(\Gamma^1), p(\Gamma^2)\right) = \begin{pmatrix} 2g & t_1 & t_2 \\ t_1 & 2gq & qt_1 \\ t_2 & qt_1 & 2gq^2 \end{pmatrix}.$$

According to [HP19], a convenient change of basis make the computation easier:

$$\operatorname{Gram}\left(\frac{qp(\Gamma^0) + p(\Gamma^2)}{\sqrt{2q}}, \frac{p(\Gamma^1)}{\sqrt{2q}}, \frac{qp(\Gamma^0) - p(\Gamma^2)}{\sqrt{2q}}\right) = \begin{pmatrix} 2gq + t_2 & t_1 & 0 \\ t_1 & g & 0 \\ 0 & 0 & 2gq - t_2 \end{pmatrix}. \tag{13}$$

This matrix is positive semi-definite if and only if the couple (t_1, t_2) satisfies both the affine constraint $t_2 \leq 2qg$ and the quadratic constraint $t_2 \geq t_1^2/g - 2qg$, namely (t_1, t_2) belongs to the convex set defined by the parabola and the horizontal line depicted in blue in Figure 1.

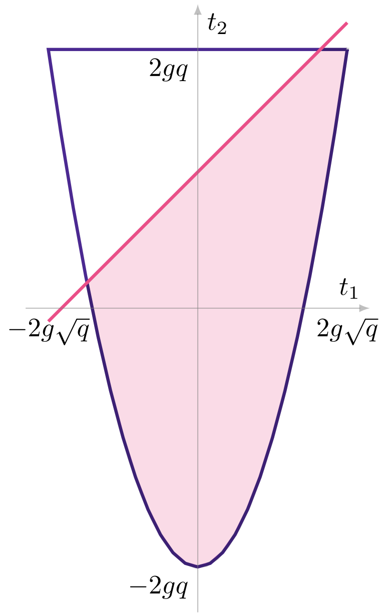

FIGURE 1. The Weil domain for $n = 2$.

The only additional affine constraint from (4) is $t_2 \leq t_1 + q^2 - q$. When $g < \frac{\sqrt{q}(\sqrt{q}-1)}{2}$, the corresponding line does not meet this region, and one recovers the Weil-Oesterlé bound of first order, namely Weil's bound. When $g \geq \frac{\sqrt{q}(\sqrt{q}-1)}{2}$, this line restricts the feasible domain to the convex set depicted in Figure 1, that we call the *Weil domain* of order 2. The minimal t_1 then occurs at the smallest intersection between this line and the parabola given by $t_2 = t_1^2/g - 2qg$. In other words, Ihara's bound is given by the smallest root of the polynomial $t^2/g - t - 2qg - q^2 + q$ and therefore

$$t_1 \geq g\left(\frac{1 - \sqrt{1 + 8q + 4(q^2 - q)/g}}{2}\right). \tag{14}$$

Furthermore, following [HP19], this bound does not get better for higher n while $g \leq \frac{\sqrt{q}(q-1)}{\sqrt{2}}$.

2.3. **Getting affine integral inequalities.** In order to apply Lemma 2.1, we need affine inequalities satisfied by every $\tau_1(\omega)$ and $\tau_2(\omega)$. Here we provide a way to obtain such inequalities. Denote by M the first block of the Gram matrix (13). It can be decomposed as

$$M = \begin{pmatrix} 2gq + t_2 & t_1 \\ t_1 & g \end{pmatrix} = \sum_{j=1}^{g} M(\omega_j), \tag{15}$$

where

$$M(\omega) = \begin{pmatrix} 2q + \tau_2(\omega) & \tau_1(\omega) \\ \tau_1(\omega) & 1 \end{pmatrix}. \tag{16}$$

For every ω with $|\omega| = \sqrt{q}$, this is a PSD matrix of rank 1, because

$$\tau_1(\omega)^2 = (2\sqrt{q}\cos(\theta))^2 = 4q\frac{1+\cos(2\theta)}{2} = 2q + \tau_2(\omega).$$

By duality, it follows that for every PSD matrix A, the trace inner product $\langle M(\omega), A\rangle = \mathrm{Tr}(M(\omega)A)$ is nonnegative for every ω with $|\omega| = \sqrt{q}$, and if moreover A is definite positive, then the inequality becomes strict. This provides a generic way to get inequalities that can be used in Lemma 2.1.

Lemma 2.3. *Let* $A = \begin{pmatrix} d & a \\ a & b \end{pmatrix}$ *be a positive definite matrix. Then for every* ω *with* $|\omega| = \sqrt{q}$,

$$d\tau_2(\omega) + 2a\tau_1(\omega) + 2qd + b > 0.$$

When A runs through all possible positive definite matrices, Lemma 2.1 gives affine equations that have to be satisfied by (t_1, t_2).

Then, going from Lemma 2.1 to Theorem 2.2 corresponds to taking into consideration the additional constraint $t_2 \leq t_1 + q^2 - q$, which gives the following.

Theorem 2.4. *Let* $A = \begin{pmatrix} d & a \\ a & b \end{pmatrix}$ *be a positive definite matrix. Assume that* $d, b \in \mathbb{N}$ *and* $2a \in \mathbb{N}$. *Then*

$$t_1 \geq \frac{g(1 - 2qd - b) - d(q^2 - q)}{d + 2a}.$$

Figure 2 sums up the situation so far: from every matrix A that fulfills the conditions of Theorem 2.4 we obtain an affine constraint on (t_1, t_2), that might exclude some region from the Weil domain.

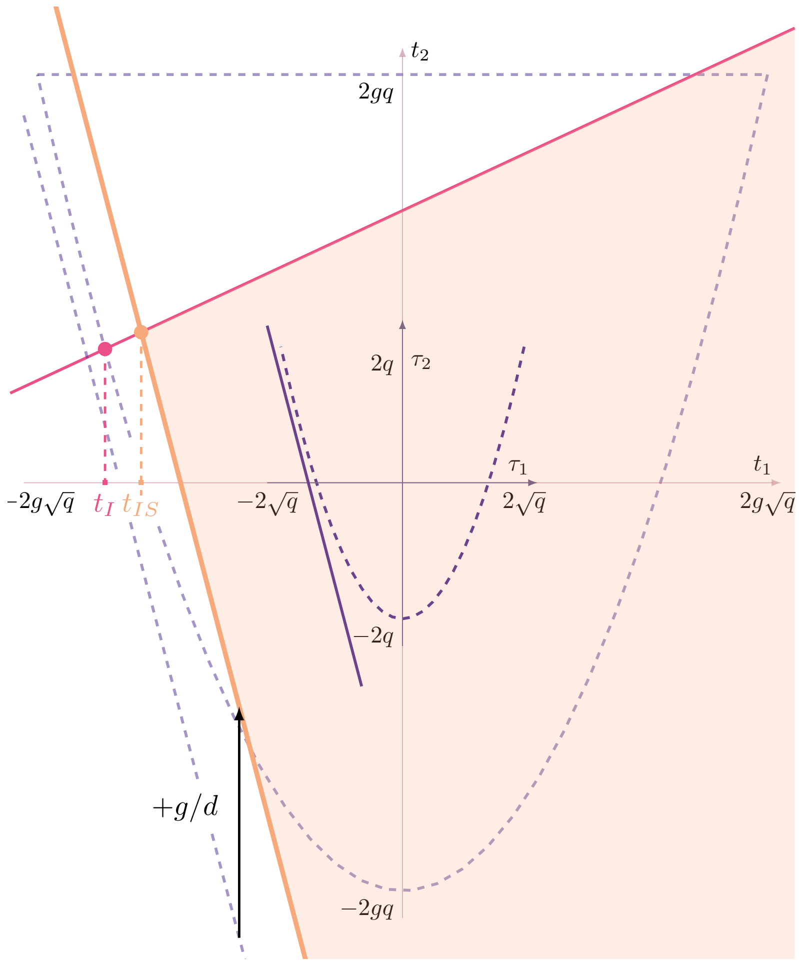

Figure 2. For every j, the couple $(\tau_1(\omega_j), \tau_2(\omega_j))$ lies on the interior parabola. Every matrix A satisfying the assumptions of Theorem 2.4 first gives an affine constraint on $(\tau_1(\omega_j), \tau_2(\omega_j))$, such as the one represented by the plain interior blue line. By summing over the ω_j's, the point (t_1, t_2) is thus constrained by the dashed exterior blue line. However, the Serre type Lemma 2.1 ensures that (t_1, t_2) is in fact above the shifted orange line. If we can find a matrix A such that this line intersects the magenta line given by the additional constraint $t_2 \leq t_1 + q^2 - q$ in the interior of the Weil domain, we get a better lower bound t_{IS} on t_1.

2.4. **Optimisation of the bound.** It remains to prove that we can find a matrix A such that Theorem 2.4 gives a better bound than Ihara's bound. The next theorem, main result of the paper, shows that there is a choice of A that improves upon Ihara's bound, and evaluates the gain between the two bounds.

Theorem 2.5. *Let X be an an absolutely, irreducible, smooth, projective curve, of genus g, defined over the finite field $\mathbb{F}_q$. Suppose that $g \geq \frac{\sqrt{q}(\sqrt{q}-1)}{2}$, let t_I be the*

Ihara trace

$$t_I = g\left(\frac{1-\sqrt{1+8q+4(q^2-q)/g}}{2}\right), \qquad \textit{and set:} \qquad \alpha = -\frac{t_I}{g}.$$

Then, the best upper bound obtained using theorem 2.4 with a matrix having an upper left coefficient equal to 1 *is for*

$$A = \begin{pmatrix} 1 & a \\ a & \lfloor a^2 \rfloor + 1 \end{pmatrix}, \qquad \textit{where} \qquad a = \lfloor \alpha \rfloor + \frac{1}{2} = \left\lfloor -\frac{t_I}{g} \right\rfloor + \frac{1}{2}.$$

The Ihara's upper bound improved à la Serre is

$$\sharp X(\mathbb{F}_q) \leq (q+1) - t_I - g\frac{(\alpha - \lfloor \alpha \rfloor)(\lceil \alpha \rceil - \alpha)}{2\lceil \alpha \rceil}$$

and the gain between the improved bound and the original one equals

$$g\frac{(\alpha - \lfloor \alpha \rfloor)(\lceil \alpha \rceil - \alpha)}{2\lceil \alpha \rceil}. \tag{17}$$

Proof. First note that for a matrix of the form $A = \begin{pmatrix} 1 & a \\ a & b \end{pmatrix}$, the lower bound given by Theorem 2.4 gets only worse when b increases, therefore it is better to take b as the smallest integer such that A is positive definite, namely

$$b = \lfloor a^2 \rfloor + 1.$$

Let us denote by $t_{IS}(a)$, where IS stands for Ihara-Serre, the lower bound on t_1 obtained using Theorem 2.4 with $d = 1$, a such that $2a \in \mathbb{N}$, and $b = \lfloor a^2 \rfloor + 1$. According to Theorem 2.4,

$$t_{IS}(a) = -g\left(\frac{\lfloor a^2 \rfloor + 2q + (q^2-q)/g}{1+2a}\right),$$

and therefore

$$\frac{t_{IS}(a) - t_I}{g} = -\frac{\lfloor a^2 \rfloor + 2q + (q^2-q)/g}{1+2a} + \alpha = -\frac{\lfloor a^2 \rfloor + 2q + (q^2-q)/g - \alpha - 2a\alpha}{1+2a}.$$

We observe that since

$$\alpha = \frac{\sqrt{1+8q+4(q^2-q)/g} - 1}{2},$$

we can rewrite

$$2q + (q^2-q)/g = \alpha(\alpha+1)$$

and thus

$$t_{IS}(a) - t_I = \frac{-g}{1+2a}\left(\alpha^2 - 2a\alpha + \lfloor a^2 \rfloor\right).$$

Since $2a \in \mathbb{N}$, we have two distinct cases.

- If $2a$ is even, then $a \in \mathbb{N}$, and thus $\lfloor a^2 \rfloor = a^2$, and in this case

$$t_{IS}(a) - t_I = \frac{-g}{1+2a}\left(\alpha - a\right)^2 < 0,$$

we always get a weaker bound.

• If $2a$ is odd, then we can write $a = k + 1/2$ with $k \in \mathbb{N}$, and in this case $\lfloor a^2 \rfloor = k(k+1)$. Then the bounds reads

$$\begin{aligned} t_{IS}(a) - t_I &= \frac{-g}{2(1+k)} \left(\alpha^2 - (2k+1)\alpha + k(k+1)\right) \\ &= \frac{-g}{2(1+k)} (\alpha - k)(\alpha - k - 1) \\ &= \frac{g}{2(1+k)} (\alpha - k)(k + 1 - \alpha), \end{aligned}$$

and this difference is positive if and only if $k < \alpha < k+1$. If α is not an integer, this occurs only for $k = \lfloor \alpha \rfloor$. In the very specific case where α is an integer, then we recover Ihara's bound for $k = \alpha$ and $k = \alpha - 1$. □

2.5. **Improvements on previous bounds.** Theorem 2.5 gives an explicit comparison between our bound and Ihara's bound. Because α in the statement depends on g, our gain is harder to analyse than Serre's improvement upon Weil's bound. In this section we comment on this comparison, with several points of view.

2.5.1. *Fixed q, g grows.* First, observe that for a fixed q, when g grows,

$$\alpha = -\frac{t_I}{g} = \frac{\sqrt{1 + 8q + 4(q^2 - q)/g} - 1}{2}$$

goes to

$$\alpha^\infty = \frac{\sqrt{1+8q} - 1}{2}.$$

Thus, when g goes to infinity, our Ihara-Serre bound

$$t_{IS} = t_I + g\frac{(\alpha - \lfloor \alpha \rfloor)(\lceil \alpha \rceil - \alpha)}{2\lceil \alpha \rceil} \tag{18}$$

on t_1 satisfies

$$\frac{t_{IS} - t_I}{g} \to_{g \to \infty} \frac{(\alpha^\infty - \lfloor \alpha^\infty \rfloor)(\lceil \alpha^\infty \rceil - \alpha^\infty)}{2\lceil \alpha^\infty \rceil}.$$

By pushing the analysis further, one finds explicitely that:

Proposition 2.6. *For a fixed q, the gain (17) admits the asymptote*

$$\left(\frac{(\alpha^\infty - \lfloor \alpha^\infty \rfloor)(\lceil \alpha^\infty \rceil - \alpha^\infty)}{2\lceil \alpha^\infty \rceil}\right) g + \frac{(\lfloor \alpha^\infty \rfloor - \alpha^\infty + 1/2)(q^2 - q)}{(\lfloor \alpha^\infty \rfloor + 1)\sqrt{8q+1}}$$

when g goes to infinity.

This has two consequences. First it shows that whenever α^∞ is not an integer (namely whenever $q \neq 3$), the difference between our bound and Ihara's bound goes to infinity. Second, this shows that our method gives an improvement on the upper bound given by Ihara's bound on the constant

$$A(q) = \limsup_{g \to \infty} \frac{N_q(g)}{g}.$$

When Ihara's bound gives $A(q) \leq \alpha^\infty$, our bound gives

$$A(q) \leq \alpha^\infty - \frac{(\alpha^\infty - \lfloor \alpha^\infty \rfloor)(\lceil \alpha^\infty \rceil - \alpha^\infty)}{2\lceil \alpha^\infty \rceil}. \tag{19}$$

Of course our bound, as Ihara's, gets weaker than higher order Weil-Oesterlé bounds when g grows, and it was shown in [HP19] that these bounds recover the

Drinfeld-Vlăduţ [VD83] bound when n grows, namely $A(q) \leq \sqrt{q} - 1$. However, it makes sense to compare our bound with Ihara's, since they correspond to Weil-Oesterlé bounds of the same order.

2.5.2. *The Ihara range.* For every q there is a range $[g_2, g_3]$ for which Ihara's bound is not improved by higher order Weil-Oesterlé bounds. We now focus on this range, in order to show that our bounds gives explicit improvements on Ihara's bound, even when it was the best bound known so far. Recall that $g_2 = \frac{\sqrt{q}(\sqrt{q}-1)}{2}$ and $g_3 = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$.

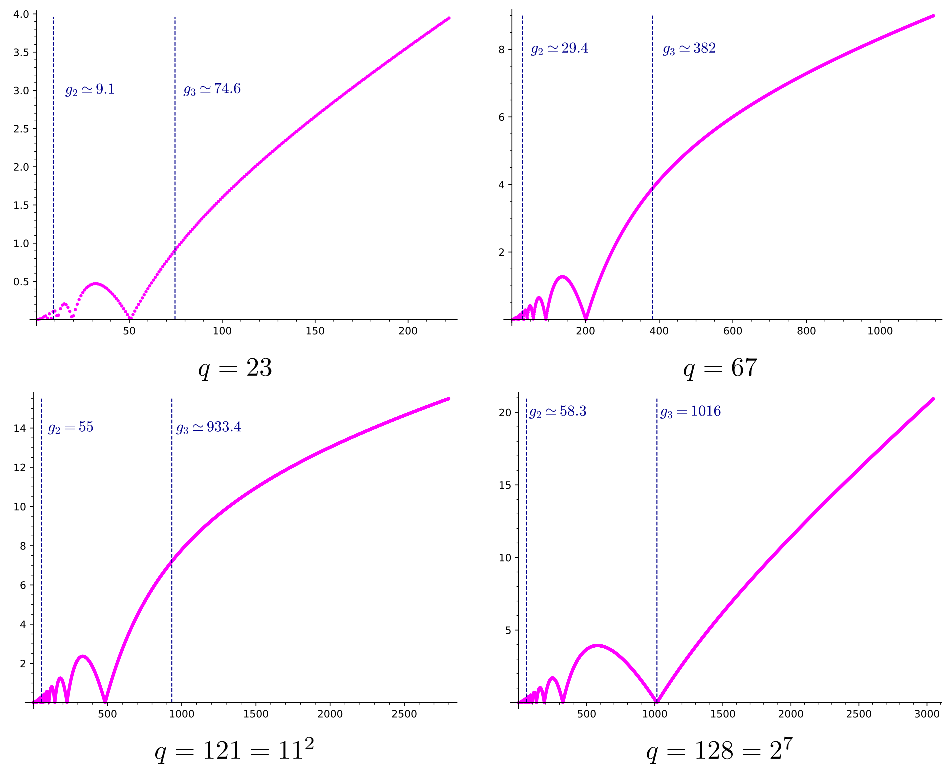

FIGURE 3. The difference (17) for some values of q and $g \leq 3g_3$.

In Figure 3, we plot the gain (17) of our bound compared with Ihara's bound for several values of q, and in each case, values of g up to $3g_3$. One can observe that in comparison with the asymptotic behavior, the ceiling and floor functions in (17) make the the analysis of the gain difficult in the range $[g_2, g_3]$, even though the difference seems to increase with q. Furthermore, note that our bound improves upon Ihara's for several values of g even when q is a square, while Serre's trick does not improve upon Weil's bound in that case. Finally, note that in the last example with $q = 128 = 2^7$, like for every odd power of 2, g_3 and α are integers. This provides an explicit example where our gain is 0. However, it has been shown in [FT98] that for such specific parameters, the Suzuki curves reach Ihara's bound. For a more general study of curves matching Ihara's bound, see the recent work by Aubry, Herbaut, and Monaldi [AHM25].

2.5.3. *A sequence g_q with q increasing.* To prove that we improve upon Ihara's bound for inifinitely many couples (q, g), we study an explicit sequence where the genus g_q is in the Ihara range $[g_2, g_3]$ for q, and q increases. More precisely, let for instance $g_q = 4q$. Then $g_2 \leq g_q \leq g_3$ whenever $q \geq 34$, the parameter α in Theorem 2.5 simplifies to

$$\alpha = \frac{3\sqrt{q} - 1}{2},$$

and (17) becomes

$$\frac{2q}{\lceil \frac{3\sqrt{q}-1}{2} \rceil}(\alpha - \lfloor \alpha \rfloor)(\lceil \alpha \rceil - \alpha). \tag{20}$$

Moreover, since the maximum of the map $x \mapsto (x - \lfloor x \rfloor)(\lceil x \rceil - x)$ is $1/4$ when x is a half-integer, the best gain one can hope for g_q is

$$\frac{q}{2\lceil \frac{3\sqrt{q}-1}{2} \rceil}.$$

In Figure 4 we plot, for the 10000 first prime numbers, the gain achieved with our bound compared with Ihara bound. One can then see that for a high proportion of q, the improvement is close to $\sqrt{q}/3$ when q grows.

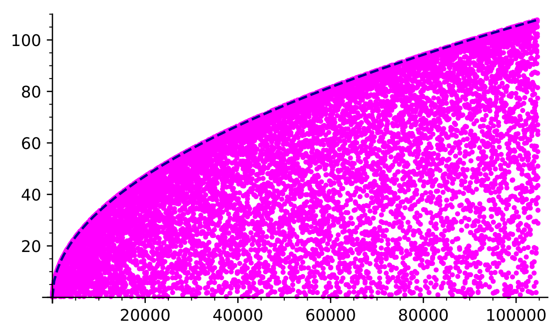

FIGURE 4. The difference (20) for q prime, and $g = 4q$. For comparison, the dashed curve represents the function $\sqrt{q}/3$.

By taking a specific subsequence of q, we show in particular that our bound improves upon Ihara's for infinitely many couples (q, g) where g is in the Ihara range:

Proposition 2.7. *Let $q = 2^{2k}$ for k a large enough integer. Then the gain (17) for $g_q = 4q$ is optimal, namely $\sqrt{q}/3$.*

Proof. Indeed, in this case,

$$\alpha = \frac{3 \cdot 2^k - 1}{2} = 3 \cdot 2^{k-1} - \frac{1}{2}$$

is a half-integer, and $(\alpha - \lfloor \alpha \rfloor)(\lceil \alpha \rceil - \alpha) = 1/4$. □

2.5.4. *Small q: comparison with manYPoints.* One can see in Figure 3 that for small values of q, the improvement in the Ihara range can be small. However, since a lower bound on t_1 gives an upper bound on the number of points N_1 of a curve X over $\mathbb{F}_q$, if the bound obtained by Ihara is close to its floor, our bound can sometimes go below this floor and therefore provide an improvement by 1 in the upper bound. In Table 1 we give the pairs (g, q) for which our bound improves upon Ihara's bound in this sense, among the values of q and g displayed in manYPoints [vdGHLR09]. However, for such small values of g and q, other more specific methods were developed [Sav03, SV86, Rig10, KÖ3, LJP01, How12, HL03, HL12], and give sometimes better bounds than Ihara's, even in the Ihara range. If our bound often meets these improvements with our generic method, in some cases our bound is worse. Also, for g close to g_2, Serre's improvement on Weil's bound can be much stronger than Ihara's bound, and also beat our bound. These cases where our bound does not meet the current records are displayed in parentheses. In a similar way, our bound can be better than the Weil-Oesterlé bound of order 3 when g is slightly above g_3, such cases are marked with a * in Table 1. Nevertheless, our result provides several new records: for the numbers displayed in bold in Table 1, our upper bound on N_1 improves upon the current records by 1.

From a slightly different perspective, we picture in Table 2 the landscape for the methods used to provide the best bounds known in [vdGHLR09] in the Ihara range. In that picture, the gray cells correspond to the couples (q, g) where Ihara's bound still provides the current record. Orange cells correspond to the cases where the Weil-Serre bound is better than Ihara's, therefore in the beginning of the Ihara range. The cells colored in indigo correspond to all the improvements for specific cases or by techniques to exclude Weil polynomials, contained in the numerous works [Sav03, SV86, Rig10, KÖ3, LJP01, How12, HL03, HL12]. When we recover these improvements with our generic closed formula, the cells are two-colored in indigo and magenta, while cells that are fully colored in magenta correspond to our new records. Because the table does not go beyond $g = 50$, we only improve the bounds by 1, but with larger g, one could see better improvements: for example, for $q = 27$ and $g = 72$ we improve Ihara's bound by 2, and by 3 for $q = 47, g = 214$.

3. Extension to higher order Weil-Oesterlé bounds

In principle, our approach can be generalized to Weil-Oesterlé bounds of higher order. However, it becomes harder to obtain a statement similar to Theorem 2.5, where we can explicitely optimise over the matrix A and give a closed formula for the gain compared to the corresponding Weil-Oesterlé bound. Another, more experimental, approach consists, for a fixed couple (q, g), in trying several matrices A and search for the one that provides the best bound.

3.1. **Experimental search.** We sketch this approach for the Weil-Oesterlé bound of order 3, and show that it can successfully improve upon Oesterlé. For $n = 3$, the

q	g
9	(12), (17*)
11	8, 19, 23, 24*
13	16, 19, (22), (25)
16	13, 20, 29, 34, 35, 39, 40, 41
17	15, 17, 22, 24, 29, 42, 45
19	(8), 12, 23, 27, 30, 31, 34, 35, 37, 38, 41, 42, 44, 45, 48, 49
23	14, 22, 27, 30, 32, 35, 38, 43, 46
25	20, 31, 34, 37, 39, 40, 42, 43, 45, 47, 48, 50
27	(11), 14, 48, 49, 50
29	18, 30, 34, 35, 38, 39, 43, 44, 48
31	24, 27, 29, 41, 43, 45, 47, 49, 50
32	26, 29, 41, 46, 48, 50
37	28, 31, 34, 41, 45, 46, 49, 50
41	(18), 29, 30, 39, 40, 43, 44, 47, 50
43	(21), 30, 31, 32, 46, 47, 48, 49, 50
47	(23), 32, 38, 40, 42, 45, 47, **50**
49	33, 37, 46, **49**
53	33, 38, **47**, **48**, **49**, **50**
59	32, 33, 34, 40, 43, 44, **47**, **50**
61	(27), **48**, **49**, **50**
64	39, 43, 47
67	36, **44**, **46**, **48**, **50**
71	(32), 41
73	35, 38, 43
79	(38), (39), **49**
81	**50**
83	**43**, **44**
89	(42), (45), **50**
97	(46), (49)

TABLE 1. The couples (q, g) with $q \leq 100$, $g \leq 50$, $g \in [g_2, g_3]$ such that our bound on N_1 is strictly better than Ihara's. For the two numbers marked with a *, $g > g_3$, but our bound improves upon Oesterlé's. In bold, the new records compared with [vdGHLR09]. In parentheses, better bounds can be found, either by Serre's improvement on Weil's bound, or by the techniques from other works. For all the other numbers, our bound meets the current record.

Gram matrix in (3) is

$$\mathrm{Gram}\left(p(\Gamma^0), p(\Gamma^1), p(\Gamma^2), p(\Gamma^3)\right) = \begin{pmatrix} 2g & t_1 & t_2 & t_3 \\ t_1 & 2gq & qt_1 & qt_2 \\ t_2 & qt_1 & 2gq^2 & q^2t_1 \\ t_3 & qt_2 & q^2t_1 & 2gq^3 \end{pmatrix}.$$

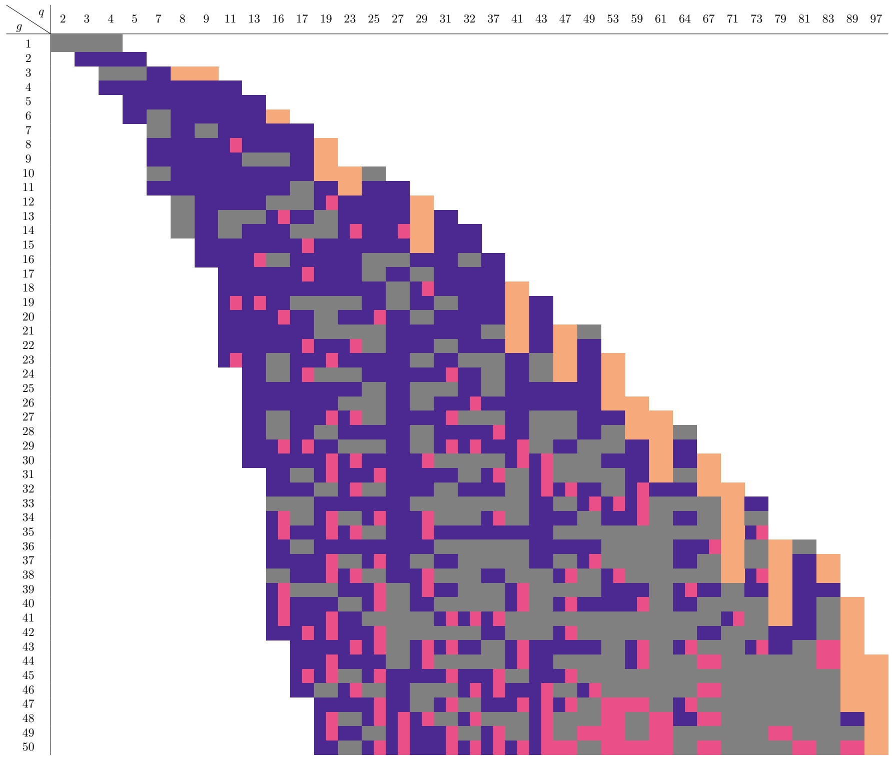

Table 2. The methods used to get the current records in the Ihara range in [vdGHLR09]: the Weil-Serre bound in orange, Ihara's bound in gray, our bound in magenta, and all the other methods in indigo.

As for Ihara, after a convenient change of basis, the Gram matrix

$$\mathrm{Gram}\left(\frac{q^{\frac{3}{2}}p(\Gamma^0)+p(\Gamma^3)}{\sqrt{2}q^{\frac{3}{4}}},\frac{q^{\frac{1}{2}}p(\Gamma^1)+p(\Gamma^2)}{\sqrt{2}q^{\frac{3}{4}}},\frac{q^{\frac{1}{2}}p(\Gamma^1)-p(\Gamma^2)}{\sqrt{2}q^{\frac{3}{4}}},\frac{q^{\frac{3}{2}}p(\Gamma^0)-p(\Gamma^3)}{\sqrt{2}q^{\frac{3}{4}}}\right)$$

$$=\begin{pmatrix} 2gq^{\frac{3}{2}}+t_3 & \sqrt{q}t_1+t_2 & 0 & 0 \\ \sqrt{q}t_1+t_2 & 2g\sqrt{q}+t_1 & 0 & 0 \\ 0 & 0 & 2g\sqrt{q}-t_1 & \sqrt{q}t_1+t_2 \\ 0 & 0 & \sqrt{q}t_1+t_2 & 2gq^{\frac{3}{2}}-t_3 \end{pmatrix}$$

is of course a positive semidefinite matrix, and makes the computations easier. Let

$$M=\begin{pmatrix} 2gq^{\frac{3}{2}}+t_3 & \sqrt{q}t_1+t_2 \\ \sqrt{q}t_1+t_2 & 2g\sqrt{q}+t_1 \end{pmatrix}$$

be the first block of this matrix. Then, the Weil-Oesterlé bound of order 3, optimal for g in the range $[g_3, g_4]$, is given by the intersection with minimal t_1 between the hypersurface given by $\det(M)=0$ and the line given by the equations $t_2=t_1+q^2-q$ and $t_3=t_1+q^3-q$, see [HP19].

We follow the strategy of Section 2.3, and write $M = \sum_{j=1}^{g} M(\omega_j)$, with

$$M(\omega) = \begin{pmatrix} 2q^{\frac{3}{2}} + \tau_3(\omega) & \sqrt{q}\tau_1(\omega) + \tau_2(\omega) \\ \sqrt{q}\tau_1(\omega) + \tau_2(\omega) & 2\sqrt{q} + \tau_1(\omega) \end{pmatrix}.$$

Again, this matrix is a PSD matrix of rank 1, because in addition to $\tau_2(\omega) = \tau_1(\omega)^2 - 2q$ we have $\tau_3(\omega) = \tau_1(\omega)^3 - 3q\tau_1(\omega)$. In this context, we can adapt Lemma 2.3 and Theorem 2.4 to get the following statement.

Theorem 3.1. *Let* $A = \begin{pmatrix} d & a \\ a & b \end{pmatrix}$ *be a positive semi-definite matrix. Assume that* $d, 2a,$ *and* $2a\sqrt{q} + b$ *are natural integers. Then*

$$t_1 \geq \frac{-g\left\lfloor 2q^{3/2}d + 2b\sqrt{q}\right\rfloor - d(q^3 - q) - 2a(q^2 - q)}{d + 2a + 2a\sqrt{q} + b}.$$

Proof. Because $A = \begin{pmatrix} d & a \\ a & b \end{pmatrix}$ and $M(\omega)$ are both PSD, we get for every ω with $|\omega| = \sqrt{q}$ the inequality

$$d\tau_3(\omega) + 2a\tau_2(\omega) + (2a\sqrt{q} + b)\tau_1(\omega) + 2q^{3/2}d + 2b\sqrt{q} \geq 0,$$

which implies the strict inequality

$$d\tau_3(\omega) + 2a\tau_2(\omega) + (2a\sqrt{q} + b)\tau_1(\omega) + \left\lfloor 2q^{3/2}d + 2b\sqrt{q}\right\rfloor + 1 > 0.$$

The assumptions ensuring that all the coefficients are integers, we can apply Lemma 2.1, which yields

$$dt_3 + 2at_2 + (2a\sqrt{q} + b)t_1 \geq -g\left\lfloor 2q^{3/2}d + 2b\sqrt{q}\right\rfloor.$$

As usual, since the coefficients are assumed to be nonnegative we conclude by adding the constraints $t_2 \leq t_1 + q^2 - q$ and $t_3 \leq t_1 + q^3 - q$. □

Using a computer algebra system, one can then compute the bound given by Theorem 3.1 for a large number of A, and check whether it is better than Oesterlé's bound. We can find such matrices A's, and sometimes this leads, like in Section 2.5.4, to a better bound on the number of points N_1 of a curve of genus g over $\mathbb{F}_q$. This even provides new bounds with respect to [vdGHLR09].

Theorem 3.2. *The number* $N_q(g)$ *satisfies*

$$N_q(g) \leq \begin{cases} 53 & \text{if } q = 5, g = 19, \\ 76 & \text{if } q = 7, g = 21, \\ 129 & \text{if } q = 8, g = 36, \\ 163 & \text{if } q = 11, g = 35. \end{cases}$$

Proof. We apply Theorem 3.1 with the matrices A given in the following table.

(q,g)	A	lower bound on t_1	upper bound on N_1
$(5,19)$	$\begin{pmatrix} 1 & 7/2 \\ 7/2 & -7\sqrt{5}+28 \end{pmatrix}$	$-\frac{1723}{36}$	$\frac{1939}{36} < 54$
$(7,21)$	$\begin{pmatrix} 3 & 29/2 \\ 29/2 & -29\sqrt{7}+147 \end{pmatrix}$	$-\frac{12348}{179}$	$\frac{13780}{179} < 77$
$(8,36)$	$\begin{pmatrix} 3 & 15 \\ 15 & -60\sqrt{2}+160 \end{pmatrix}$	$-\frac{23352}{193}$	$\frac{25089}{193} < 130$
$(11,35)$	$\begin{pmatrix} 2 & 14 \\ 14 & -28\sqrt{11}+191 \end{pmatrix}$	$-\frac{33580}{221}$	$\frac{36232}{221} < 164$

□

3.2. **Obstacles and limits.** There are several reasons that make the task of finding a closed formula as in Theorem 2.5 hard for $n \geq 3$.

On the positive side, we have an intuition on how to reduce the domain in which we search for a good matrix A. Indeed, if we denote by $t_W = (t_1, t_2, t_3)$ the solution of the initial Weil-Oesterlé problem, then the matrix M has rank 1 at t_W. If v is a kernel vector of M, then the PSD matrix $A_0 = v^t v$ satisfies $\langle M, A_0 \rangle = 0$. Geometrically, this equation defines the tangent space of the hypersurface defined by $\det(M) = 0$ at t_W. One can then look for a matrix A close to A_0 which satisfies the conditions of Lemma 3.1, this is how we obtained the matrices in Theorem 3.2.

However the vector v, and therefore the matrix A_0, is unique only up to rescaling, and while fixing the top left coefficient of A to 1 appears to be optimal for $n = 2$, this does not seem to be the case for $n = 3$ (see the matrices in the proof of Theorem 3.2). Furthermore, the way to approximate A_0, even after fixing the top left coefficient of A, seems to depend on the arithmetic of the numbers involved when taking floors. This prevents us to find a canonical candidate for the optimal A as we were able to do for $n = 2$. More generally, when n grows, the number of coefficients in A increases, the relations between these coefficients become more complicated, and the initial optimal point t_W is defined by a polynomial equation of increasing degree. All these reasons make the generalisation of Theorem 2.5 a problem that seems difficult.

On the other hand, the numerical experiments we made, even for larger q and g, suggest that the improvement compared to the corresponding standard Weil-Oesterlé bound is weaker when $n = 3$ compared to $n = 2$, which is itself weaker than the improvement by Serre upon Weil's bound. This might be explained by the fact that when n grows, the spectrahedron defined by the Gram matrix of order n is described by equations of higher degree in more variables, and the truncation of this spectrahedron that we obtain with our affine inequalities becomes less powerful.

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