

PLURISIGNED HERMITIAN METRICS

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ABSTRACT. Let (X, ω) be a compact hermitian manifold of dimension n . We study the asymptotic behavior of Monge-Ampère volumes $\int_X (\omega + dd^c \varphi)^n$, when $\omega + dd^c \varphi$ varies in the set of hermitian forms that are dd^c -cohomologous to ω . We show that these Monge-Ampère volumes are uniformly bounded if ω is “strongly pluripositive”, and that they are uniformly positive if ω is “strongly plurinegative”. This motivates the study of the existence of such plurisigned hermitian metrics.

We analyze several classes of examples (complex parallelisable manifolds, twistor spaces, Vaisman manifolds) admitting such metrics, showing that they cannot coexist. We take a close look at 6-dimensional nilmanifolds which admit a left-invariant complex structure, showing that each of them admit a plurisigned hermitian metric, while only few of them admit a pluriclosed metric. We also study 6-dimensional solvmanifolds with trivial canonical bundle.

CONTENTS

Introduction	4631
1. Preliminaries	4633
2. Pluripositive hermitian metrics	4636
3. Plurinegative hermitian metrics	4641
4. Homogeneous examples	4648
References	4656

INTRODUCTION

The study of complex Monge-Ampère equations on compact hermitian (non-Kähler) manifolds has gained considerable interest in the last decade. Tosatti-Weinkove [TW10] and then Székelyhidi-Tosatti-Weinkove [STW17] have resolved the Gauduchon-Calabi-Yau conjecture, extending to the hermitian setting Yau’s fundamental result [Yau78]. Associated degenerate complex Monge-Ampère equations have been systematically studied by Dinew, Kołodziej, and Nguyen [DK12, KN15, Din16, KN19], as well as in [LPT21, GL21a, GL21b, GL21c].

By comparison with the setting of Kähler manifolds, a key new difficulty lies in the uniform control of Monge-Ampère volumes. Given X a compact complex

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manifold of complex dimension n equipped with a hermitian metric ω , it is of crucial importance to decide whether

$$v_+(\omega) := \sup \left\{ \int_X (\omega + dd^c\varphi)^n : \varphi \in \mathcal{C}^\infty(X) \text{ and } \omega + dd^c\varphi > 0 \right\}$$

is finite, and whether

$$v_-(\omega) := \inf \left\{ \int_X (\omega + dd^c\varphi)^n : \varphi \in \mathcal{C}^\infty(X) \text{ and } \omega + dd^c\varphi > 0 \right\}$$

is bounded away from zero. Here $d = \partial + \bar{\partial}$ and $d^c = \frac{1}{2i}(\partial - \bar{\partial})$.

It follows from Stokes theorem that $v_-(\omega) = v_+(\omega) = \int_X \omega^n$ when ω is closed or, more generally, when $dd^c\omega = 0$ and $dd^c\omega^2 = 0$. The latter conditions are however rather restrictive and it is an important open problem to decide whether $v_+(\omega)$ (resp. $v_-(\omega)$) is always finite (resp. positive). We refer the reader to [GL21b, Theorem C] for an illustration of how the finiteness of $v_+(\omega)$ is related to a transcendental form of Demailly’s holomorphic Morse inequalities, while [GL21c] strongly motivates the condition $v_-(\omega) > 0$.

It has been shown in [GL21b, Theorem A] that the condition $v_+(\omega) < +\infty$ (resp. $v_-(\omega) > 0$) is independent of the choice of hermitian metric—it only depends on the complex structure—and is a bimeromorphic invariant.

We further study these conditions in this article, testing them on various classes of examples. Our first observation is the following hereditary result.

Theorem A. *Let (X, ω_X) be a compact hermitian manifold and let $Y \subseteq X$ be a closed submanifold equipped with a hermitian form ω_Y . If $v_+(X, \omega_X) < +\infty$ then $v_+(Y, \omega_Y) < +\infty$.*

We then establish the finiteness of $v_+(X)$ (resp. positivity of $v_-(X)$) when X admits special pluripositive (resp. plurinegative) hermitian metrics.

Theorem B. *Let X be a compact complex manifold of dimension n .*

- (1) *If there exists a hermitian metric ω and $\varepsilon > 0$ such that $dd^c\omega \geq 0$ and $dd^c\omega^q \geq \varepsilon\omega \wedge dd^c\omega^{q-1}$, for $2 \leq q \leq n - 2$, then $v_+(\omega) < +\infty$.*
- (2) *If $n = 3$ and X admits a metric ω such that $dd^c\omega \leq 0$, then $v_-(\omega) > 0$.*

In particular if $n = 3$ and ω is pluriclosed then $0 < v_-(\omega) \leq v_+(\omega) < +\infty$.

We also provide a curvature condition to control $v_-(\omega)$ in higher dimension, see Definition 3.1 and Theorem 3.2.

These conditions are always fulfilled when $\dim_{\mathbb{C}} X \leq 2$, so we initiate a systematic study of the 3-dimensional case. Using Hahn-Banach theorem in the spirit of [Mic82, HL83], one can show (Theorems 2.4 and 3.5) that there exists a pluripositive (resp. plurinegative) hermitian metric ω on X if and only if any positive current τ of bidimension $(2, 2)$ such that $dd^c\tau \leq 0$ (resp. $dd^c\tau \geq 0$) satisfies $dd^c\tau = 0$. Thus in dimension 3 such plurisigned hermitian metrics cannot coexist, i.e. the following conditions are mutually exclusive (see Corollary 3.7):

- X admits a hermitian metric ω such that $dd^c\omega \geq 0$ and $dd^c\omega \neq 0$;
- X admits a hermitian metric ω such that $dd^c\omega = 0$;
- X admits a hermitian metric ω such that $dd^c\omega \leq 0$ and $dd^c\omega \neq 0$;
- X does not admit any hermitian metric ω such that $dd^c\omega$ has a sign.

Each case does occur as we show by analyzing several classes of examples:

- Complex parallelisable manifolds and twistor spaces of K3 surfaces admit pluripositive hermitian metrics (see Proposition 4.1 and Theorem 2.7).
- The existence of pluriclosed hermitian metrics has been thoroughly studied in recent years (see [FPS04, FT09, Ver14, COUV16, Ot20, ADOS22]).
- Vaisman manifolds (a special type of locally conformally Kähler manifolds) admit plurinegative hermitian metrics (Proposition 3.10).
- Non-Kähler manifolds from the class \mathcal{C} of Fujiki do not admit any plurisigned hermitian metric (see Proposition 3.8).

This alternative is however no longer valid in higher dimension, see Example 4.5.

We take a closer look at nilmanifolds $X = \Gamma \backslash G$ of (real) dimension 6, where G is a connected and simply connected nilpotent Lie group, and Γ is a discrete co-compact subgroup. There are 34 isomorphism classes of nilpotent Lie algebras in dimension 6, only 18 of which admit a complex structure. Following [ABD11, Uga07, UV14, COUV16] we gather them in four large families (Np), (Ni), (Nii), (Niii) and show the following.

Theorem C. *Consider a six-dimensional nilmanifold $X = \Gamma \backslash G$ endowed with a left-invariant complex structure. There is always a plurisigned hermitian metric. More precisely if X is not a complex torus, then*

- *either X belongs to one of the classes (Np), (Nii), (Niii) and then any left-invariant hermitian metric is pluripositive but not pluriclosed.*
- *or X belongs to the class (Ni) and depending on the complex structure it admits a left-invariant hermitian metric which is either pluriclosed, or pluripositive but not pluriclosed, or else plurinegative but not pluriclosed.*

This analysis largely generalizes the influential work of Fino-Parton-Salamon [FPS04] who characterized the existence of pluriclosed metric in this context. We refer the reader to Section 4.2 for a more precise statement.

Following [FOU15] we also analyze the case of 6-dimensional solvmanifolds, i.e. compact quotients of a connected solvable Lie group by a discrete subgroup, endowed with left-invariant complex structures with holomorphically trivial canonical bundle. These are gathered in ten large families (see Section 4.4), besides the four large families of nilmanifolds that are dealt with by the previous statement.

Theorem D. *Consider a six-dimensional solvmanifold endowed with a left-invariant complex structure with holomorphically-trivial canonical bundle. Then, there is always a plurisigned metric.*

We refer the reader to Theorem 4.7 for a more precise statement, which notably shows that in most cases the plurisigned metric is pluripositive.

1. PRELIMINARIES

In the whole article we let X denote a compact complex manifold of complex dimension $n \geq 1$, and we fix a hermitian form ω_X on X .

1.1. Quasi-plurisubharmonic functions. Let ω be a semi-positive $(1, 1)$ -form.

1.1.1. *Quasi-plurisubharmonic functions.*

Definition 1.1. A function is *quasi-plurisubharmonic* (quasi-psh for short) if it is locally given as the sum of a smooth and a plurisubharmonic function.

Given an open set $U \subset X$, quasi-psh functions $\varphi : U \rightarrow \mathbb{R} \cup \{-\infty\}$ satisfying $\omega_\varphi := \omega + dd^c\varphi \geq 0$ in the weak sense of currents are called ω -psh functions on U . Constant functions are ω -psh functions since ω is semi-positive.

A C^2 -smooth function $u \in C^2(X)$ has bounded Hessian, hence εu is ω -psh on X if $0 < \varepsilon$ is small enough and ω is positive (i.e. hermitian).

Definition 1.2. We let $\text{PSH}(X, \omega)$ denote the set of all ω -plurisubharmonic functions which are not identically $-\infty$.

The set $\text{PSH}(X, \omega)$ is a closed subset of $L^1(X)$, for the L^1 -topology. We refer the reader to [Dem, GZ, Din16] for basic properties of ω -psh functions, and simply recall that:

- $\text{PSH}(X, \omega) \subset \text{PSH}(X, \omega')$ if $\omega \leq \omega'$;
- $\text{PSH}(X, \omega) \subset L^r(X)$ for $r \geq 1$; the induced L^r -topologies are equivalent;
- the subset $\text{PSH}_A(X, \omega) := \{u \in \text{PSH}(X, \omega), -A \leq \sup_X u \leq 0\}$ is compact in $L^r(X)$ for any $r \geq 1$ and any $A > 0$.

Definition 1.3. A quasi-psh function φ has *analytic singularities* if it can be locally written as

$$\varphi(z) = c \log \sum_{j=1}^s |f_j(z)|^2 + \rho(z),$$

where $c > 0$, the f_j 's are holomorphic functions and ρ is a smooth function.

We recall the following fundamental regularization result of Demailly [Dem92].

Theorem 1.4. *Any quasi-psh function is the decreasing limit of smooth quasi-psh functions. Moreover when ω is a hermitian form, any function $\varphi \in \text{PSH}(X, \omega)$ is the decreasing limit of functions $\varphi_j \in \text{PSH}(X, \omega)$ with analytic singularities.*

1.1.2. *Monge-Ampère measure.* The complex Monge-Ampère measure $(\omega + dd^c u)^n$ is well-defined for any ω -psh function u which is *bounded*, as follows from the theory developed by Bedford-Taylor in bounded pseudoconvex domains of \mathbb{C}^n .

If $\beta = dd^c \rho$ is a Kähler form that dominates ω in a local chart U , the function u is β -psh in U hence the positive currents $(\beta + dd^c u)^j$ are well defined for $0 \leq j \leq n$ by [BT82]. This allows one to make the following definition.

Definition 1.5. The *complex Monge-Ampère measure* of u is

$$(\omega + dd^c u)^n := \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} (\beta + dd^c u)^j \wedge (\beta - \omega)^{n-j}.$$

We refer to [DK12] for an adaptation of the main properties of [BT82] to this hermitian and global context.

The mixed Monge-Ampère measures $(\omega + dd^c u)^j \wedge (\omega + dd^c v)^{n-j}$ are also well defined for any $0 \leq j \leq n$, and any bounded ω -psh functions u, v .

A basic property we shall use is the following extension of a fundamental result of Bedford-Taylor:

Lemma 1.6. *Let u, v be bounded ω -psh functions, then $\max(u, v) \in \text{PSH}(X, \omega)$ and*

$$1_{\{u < v\}} (\omega + dd^c \max(u, v))^n = 1_{\{u < v\}} (\omega + dd^c v)^n.$$

The subtle point here is that the set $\{u < v\}$ is not open in the usual sense if v is not continuous, it is merely open for the *plurifine topology*.

1.2. Uniform bounds on Monge-Ampère volumes. Let (X, ω) be a compact hermitian manifold of complex dimension n .

Definition 1.7. We consider

$$v_-(\omega) := \inf \left\{ \int_X (\omega + dd^c u)^n, u \in \text{PSH}(X, \omega) \cap L^\infty(X) \right\}$$

and

$$v_+(\omega) := \sup \left\{ \int_X (\omega + dd^c u)^n, u \in \text{PSH}(X, \omega) \cap L^\infty(X) \right\}.$$

The supremum and infimum in the definition of $v_+(\omega)$ and $v_-(\omega)$ can be taken over $\text{PSH}(X, \omega) \cap C^\infty(X)$ as follows from Theorem 1.4 and Bedford-Taylor’s convergence theorem [BT82]. These quantities have been studied in [GL21b]. A major open problem is the following.

Problem 1.8. *Understand whether $v_+(\omega) < +\infty$ and/or $v_-(\omega) > 0$.*

It follows from Stokes theorem that $0 < v_-(\omega) = v_+(\omega) = \int_X \omega^n < +\infty$ when ω is a Kähler form. The same result holds true as soon as the hermitian form satisfies

$$dd^c \omega = 0 \quad \text{and} \quad d\omega \wedge d^c \omega = 0,$$

a vanishing condition introduced by Guan-Li in [GL10].

This condition actually characterizes the preservation of Monge-Ampère masses, as was observed by Chiose in [Chi16b].

Theorem 1.9 ([Chi16b]). *The following properties are equivalent:*

- (1) $\int_X (\omega + dd^c \varphi)^n = \int_X \omega^n$ for all $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$.
- (2) $dd^c \omega = 0$ and $d\omega \wedge d^c \omega = 0$.

When $n = 2$ the vanishing of $d\omega \wedge d^c \omega$ is automatic for bidegree reasons, hence the preservation of Monge-Ampère volumes is equivalent to ω being a Gauduchon metric. In higher dimension this condition is quite restrictive and unstable.

Chiose further observed that the Guan-Li condition is moreover equivalent to

$$dd^c \omega \geq 0 \quad \text{and} \quad d\omega \wedge d^c \omega \geq 0.$$

Indeed

$$dd^c(\omega^{n-1}) = (n-1)(n-2)d\omega \wedge d^c \omega \wedge \omega^{n-3} + (n-1)dd^c \omega \wedge \omega^{n-2}$$

while Stokes theorem ensures that $\int_X dd^c(\omega^{n-1}) = 0$.

Remark 1.10. Note that X satisfying the $\partial\bar{\partial}$ -Lemma is not enough to ensure the volume preservation property $v_-(\omega) = v_+(\omega) = \int_X \omega^n$. Some splitting-type complex structures arising as deformations of the holomorphically-parallelizable Nakamura solvmanifold satisfy the $\partial\bar{\partial}$ -Lemma [AK17, AOUV17], but they never admit pluriclosed metrics [AOUV17], in particular the Guan-Li conditions are never satisfied (see also Proposition 3.8). It is a folklore conjecture that balanced metrics always exist on compact complex manifolds satisfying the $\partial\bar{\partial}$ -Lemma [Pop17], and it is also expected that balanced and pluriclosed metrics cannot coexist on a compact complex manifold unless it admits Kähler metrics [FV15]. If these two conjectures were true, the volume preservation property would never be satisfied by compact complex non-Kähler manifolds satisfying the $\partial\bar{\partial}$ -Lemma.

In the sequel we analyze several families of compact complex manifolds admitting a hermitian metric ω such that $dd^c\omega \geq 0$ (resp. $dd^c\omega \leq 0$). We can further expect that $d\omega \wedge d^c\omega \leq 0$ (resp. $d\omega \wedge d^c\omega \geq 0$), but we should not expect, in general, that these two forms are both globally positive (resp. negative).

We recall the following properties established in [GL21b].

Theorem 1.11 ([GL21b, Proposition 3.2, Theorem 3.7, Theorem 4.12]). *Let X be a compact complex manifold of dimension n and let ω be a hermitian metric.*

- *The condition $v_+(\omega) < +\infty$ is independent of the choice of the hermitian metric ω , and it is a bimeromorphic invariant.*
- *The condition $v_-(\omega) > 0$ is independent of the choice of the hermitian metric ω , and it is a bimeromorphic invariant.*
- *If $\alpha \in H_{BC}^{1,1}(X, \mathbb{R})$ is a nef class with $\alpha^n > 0$, then α contains a Kähler current (hence X belongs to the Fujiki class \mathcal{C}) if and only if $v_+(\omega) < +\infty$.*

Here $H_{BC}^{1,1}(X, \mathbb{R})$ denotes the first Bott-Chern cohomology group of X . The last item is a partial answer to an important conjecture of Demailly-Paun [DP04].

A main goal of this article is to introduce curvature conditions that allow one to partially answer Problem 1.8, and to try and establish them on various classes of non-Kähler manifolds.

2. PLURIPOSITIVE HERMITIAN METRICS

2.1. The restriction property. We observe in this section that the condition $v_+(\omega) < +\infty$ is stable under restriction.

Theorem 2.1. *Let (X, ω_X) be a compact hermitian manifold and let $Y \subset X$ be a closed submanifold of X equipped with a hermitian form ω_Y . If $v_+(X, \omega_X) < +\infty$ then $v_+(Y, \omega_Y) < +\infty$.*

Proof. Since the finiteness of v_+ is independent of the choice of a hermitian metric, we work here with $\omega_Y = (\omega_X)|_Y$.

It follows from Theorem 1.4 that one can approximate $\varphi \in PSH(Y, \omega_Y)$ by a decreasing sequence of smooth strictly ω_Y -psh functions. Since the complex Monge-Ampère operator is continuous along decreasing sequences [BT82], it suffices to establish a uniform bound from above on $\int_Y (\omega_Y + dd^c\varphi)^k$, where φ is smooth and strictly ω_Y -psh and $k = \dim_{\mathbb{C}} Y$.

It follows from [CGZ13, Proposition 2.1] that there exists a smooth extension $\phi \in PSH(X, \omega_X)$ with $\phi|_Y = \varphi$. It is classical (see e.g. [DP04, Lemma 2.1]) that one can find a function $\psi_Y \in PSH(X, \omega_X)$ which is smooth in $X \setminus Y$, with analytic singularities along Y , and such that

$$(\omega_X + dd^c\psi_Y)^{n-k} \geq \delta_0[Y],$$

for some $\delta_0 > 0$, where $[Y]$ denotes the current of integration along Y . We infer

$$\begin{aligned} \int_Y (\omega_Y + dd^c\varphi)^k &= \int_X (\omega_X + dd^c\phi)^k \wedge [Y] \\ &\leq \delta_0^{-1} \int_X (\omega_X + dd^c\phi)^k \wedge (\omega_X + dd^c\psi_Y)^{n-k} \\ &= \lim_{j \rightarrow +\infty} \delta_0^{-1} \int_X (\omega_X + dd^c\phi)^k \wedge (\omega_X + dd^c\psi_j)^{n-k}, \end{aligned}$$

where $\psi_j = \max(\psi_Y, -j) \in PSH(X, \omega_X) \cap L^\infty(X)$.

Observe now that $u_j = \frac{\phi + \psi_j}{2} \in PSH(X, \omega_X) \cap L^\infty(X)$ with

$$(\omega_X + dd^c \phi)^k \wedge (\omega_X + dd^c \psi_j)^{n-k} \leq 2^n (\omega_X + dd^c u_j)^n,$$

and $\int_X (\omega_X + dd^c u_j)^n \leq v_+(X, \omega_X)$, hence

$$v_+(Y, \omega_Y) = \sup_{\varphi} \int_Y (\omega_Y + dd^c \varphi)^k \leq 2^n \delta_0^{-1} v_+(X, \omega_X) < +\infty,$$

completing the proof. □

2.2. Controlling v_+ . We propose here various curvature conditions on a hermitian metric ω that ensure $v_+(\omega) < +\infty$.

Theorem 2.2. *Let X be a compact complex manifold of dimension n . Assume there exists a hermitian metric ω and $\varepsilon > 0$ such that*

- $dd^c \omega \geq 0$ and
- $dd^c \omega^q \geq \varepsilon \omega \wedge dd^c \omega^{q-1}$ for $2 \leq q \leq n - 2$.

Then $v_+(\omega) < +\infty$.

In complex dimension $n = 3$, the hypothesis boils down to $dd^c \omega \geq 0$. We shall provide several examples of manifolds admitting such pluripositive hermitian metrics in the sequel. Higher dimensional examples satisfying the conditions of Theorem 2.2 will be presented in Example 4.2 and Example 4.6.

Proof. When $n = 3$ this observation is due to Chiose [Chi16b, Question 0.8]. We include a proof as a warm up for the higher dimensional case. It follows from Stokes theorem that $\int_X (dd^c u)^3 = 0$ for any $u \in PSH(X, \omega) \cap C^\infty(X)$, hence

$$\int_X (\omega + dd^c u)^3 = \int_X \omega^3 + 3 \int_X \omega^2 \wedge dd^c u + 3 \int_X \omega \wedge (dd^c u)^2.$$

Integrating by parts we see that

$$\left| \int_X \omega^2 \wedge dd^c u \right| = \left| \int_X u dd^c \omega^2 \right| \leq B \int_X |u| \omega^3 \leq C$$

by compactness (see [GZ, Proposition 8.4]), as we can normalize u by $\sup_X u = 0$. On the other hand

$$\int_X \omega \wedge (dd^c u)^2 = \int_X -dd^c \omega \wedge du \wedge d^c u \leq 0$$

whenever $dd^c \omega \geq 0$. The result follows.

We now treat the case $n \geq 4$. Observe that it suffices to deal with $a\omega$ -psh functions, where $0 < a \leq 1$ is an arbitrarily small fixed constant. Indeed if $u \in PSH(X, \omega) \cap L^\infty(X)$, then $au \in PSH(X, a\omega) \cap L^\infty(X)$ with

$$a^n \omega_u^n \leq ((1 - a)\omega + a\omega_u)^n,$$

hence

$$v_+(\omega) \leq a^{-n} \sup \left\{ \int_X (\omega + add^c u)^n, u \in PSH(X, \omega) \cap L^\infty(X) \right\}.$$

We thus fix $a > 0$ and $u \in PSH(X, \omega)$ with $\sup_X u = 0$. Stokes theorem yields

$$\begin{aligned} & \int_X (\omega + add^c u)^n \\ &= \sum_{0 \leq p \leq n} \binom{n}{p} a^p \int_X \omega^{n-p} \wedge (dd^c u)^p \\ &= \int_X \omega^n + na \int_X \omega^{n-1} \wedge dd^c u + \sum_{2 \leq p \leq n-1} \binom{n}{p} a^p \int_X \omega^{n-p} \wedge (dd^c u)^p \\ &\leq C_1 - \sum_{2 \leq p \leq n-1} \binom{n}{p} a^p \int_X du \wedge d^c u \wedge (\omega_u - \omega)^{p-2} \wedge dd^c(\omega^{n-p}) \\ &= C_1 + S. \end{aligned}$$

We decompose the sum S as follows,

$$\begin{aligned} S &= \sum_{\substack{2 \leq p \leq n-1 \\ 0 \leq k \leq p-2}} B_{k,p} a^p (-1)^{k+1} \int_X du \wedge d^c u \wedge \omega_u^{p-2-k} \wedge \omega^k \wedge dd^c(\omega^{n-p}) \\ &= S_1 + S_2, \end{aligned}$$

where $B_{k,p} = \binom{n}{p} \binom{p-2}{k}$, S_1 is the sum for k even and S_2 is the one for k odd. Using the assumption on $dd^c \omega^{n-p}$ we obtain

$$\begin{aligned} S_1 &= - \sum_{\substack{2 \leq p \leq n-2 \\ 0 \leq k=2l \leq p-2}} B_{k,p} a^p \int_X du \wedge d^c u \wedge \omega_u^{p-2-2l} \wedge \omega^{2l} \wedge dd^c(\omega^{n-p}) \\ &\leq - \sum_{\substack{2 \leq p \leq n-2 \\ 0 \leq k=2l \leq p-2}} B_{k,p} a^p \varepsilon \int_X du \wedge d^c u \wedge \omega_u^{p-2-2l} \wedge \omega^{2l+1} \wedge dd^c(\omega^{n-p-1}), \end{aligned}$$

while

$$\begin{aligned} S_2 &= \sum_{\substack{3 \leq p \leq n-1 \\ 0 \leq k=2l+1 \leq p-2}} B_{k,p} a^p \int_X du \wedge d^c u \wedge \omega_u^{p-3-2l} \wedge \omega^{2l+1} \wedge dd^c(\omega^{n-p}) \\ &= \sum_{\substack{2 \leq p \leq n-2 \\ 0 \leq k=2l+1 \leq p-1}} B_{k,p+1} a^{p+1} \int_X du \wedge d^c u \wedge \omega_u^{p-2-2l} \wedge \omega^{2l+1} \wedge dd^c(\omega^{n-p-1}), \end{aligned}$$

as follows from changing p in $p - 1$. Now $S_1 + S_2 =$

$$= \sum_{\substack{2 \leq p \leq n-2 \\ 0 \leq k=2l+1 \leq p-1}} (-B_{k,p} \varepsilon + B_{k,p+1} a) a^p \int_X du \wedge d^c u \wedge \omega_u^{p-2-2l} \wedge \omega^{2l+1} \wedge dd^c(\omega^{n-p-1}) \leq 0$$

if a is small enough, which yields $S_1 + S_2 \leq 0$ hence $v_+(\omega) \leq a^{-n} C_1$. □

2.3. Pluripositive hermitian metrics.

Definition 2.3. A hermitian metric ω is *pluripositive* if $dd^c \omega \geq 0$, *plurinegative* if $dd^c \omega \leq 0$, and *pluriclosed* if $dd^c \omega = 0$.

Pluriclosed metrics are also often called SKT (strongly Kähler with torsion) in the literature; a manifold X is called SKT if it admits a SKT hermitian metric.

The existence of a pluripositive hermitian metric ω is a condition that is stable under blow-ups with smooth centers. Indeed if $\pi : Y \rightarrow X$ is the blow up of X with smooth center $Z \subset X$, a hermitian metric on Y is obtained by considering

$$\omega_Y = \pi^*\omega_X - \varepsilon\theta_Z,$$

where ω_X is a hermitian metric in $X \setminus Z$ with poles along Z , θ_Z is a closed form cohomologous to the current of integration along the exceptional divisor $\pi^{-1}(Z)$, and $0 < \varepsilon$ is small. Thus $dd^c\omega_Y = \pi^*dd^c\omega_X$ has the same sign as that of $dd^c\omega_X$.

However this condition is not stable under modifications (see Proposition 3.8), and there are obstructions to the existence of pluripositive hermitian metrics:

Theorem 2.4. *The following properties are equivalent.*

- *There exists a hermitian form ω such that $dd^c\omega \geq 0$.*
- *There exists no positive current τ of bidimension $(2, 2)$ such that $S = dd^c\tau \leq 0$ with $S \neq 0$.*

When the complex dimension is $n = 3$, this shows in particular that pluripositive and plurinegative hermitian forms can not coexist (see Corollary 3.7).

Proof. Observe first that the two objects cannot coexist on X . Indeed if ω is a hermitian form such that $dd^c\omega \geq 0$ and τ is a positive current of bidimension $(2, 2)$ such that $dd^c\tau \leq 0$, we obtain

$$0 \leq \int_X dd^c\omega \wedge \tau = \int_X \omega \wedge dd^c\tau \leq 0,$$

which forces $dd^c\tau = 0$ since ω is a hermitian form. Conversely consider

$$\mathcal{C} := \left\{ \text{negative currents } S \text{ of bidimension } (1, 1) \text{ with } \int_X S \wedge \omega = -1 \right\}.$$

This is a compact convex set for the weak topology of currents. We set

$$F := \{ \text{currents } S = dd^c\tau, \text{ where } \tau \geq 0 \text{ has bidimension } (2, 2) \}.$$

This is a closed set for the weak topology. There exists no positive current τ of bidimension $(2, 2)$ such that $S = dd^c\tau \leq 0$ with $S \neq 0$ if and only if the sets \mathcal{C} and F are disjoint. If such is the case, it follows from Hahn-Banach theorem that we can find a continuous functional Φ on the set of bidimension $(1, 1)$ currents that is semi-positive on F and strictly negative on \mathcal{C} . By deRham duality the functional Φ is defined by a form ω of bidegree $(1, 1)$. Now

- $\Phi \geq 0$ on F is equivalent to $dd^c\omega \geq 0$, while
- $\Phi < 0$ on \mathcal{C} is equivalent to ω being hermitian,

as follows from a rescaling argument. □

2.4. Twistor spaces. Twistor spaces provide a large class of examples of compact non-Kähler manifolds which admit a pluripositive hermitian metric.

Dimension 4. Let (M, g) be a compact oriented riemannian 4-manifold. The vector bundle $\Lambda^2 T^*M$ of 2-forms can be decomposed as a direct sum $\Lambda^2 T^*M = \Lambda_+ \oplus \Lambda_-$, where Λ_{\pm} denotes the eigenspace of the Hodge \star -operator corresponding to the ± 1 -eigenvalues of \star (selfdual and anti-selfdual forms).

The riemannian curvature operator $R : \Lambda^2 T^*M \rightarrow \Lambda^2 T^*M$ can be decomposed under the action of the group of special isometries $SO(4)$ as

$$R = \frac{s}{6}Id + W^- + W^+ + \overset{\circ}{r},$$

where s is the scalar curvature, $\overset{\circ}{r}$ is the trace free Ricci curvature, and W^{\pm} are the trace free endomorphisms of Λ^{\pm} . The manifold M is called ASD (anti-selfdual) if $W^+ = 0$; this definition is conformally invariant [AHS78]. A famous result of Taubes [Tau92] provides many examples of such ASD manifolds.

Definition 2.5. The twistor space $X = X(M, [g])$ of $(M, [g])$ is the total space of the sphere bundle of self dual 2-forms.

There is a natural almost complex structure on X which is integrable if and only if M is ASD [AHS78]. In this case X is a compact complex manifold of dimension 3 which is never Kähler unless $M = S^4$ is the sphere (in which case $X = \mathbb{C}\mathbb{P}^3$), or $M = \mathbb{C}\mathbb{P}^2$ (in which case X is the flag space of \mathbb{C}^3), see [Hit81].

Despite being non-Kähler, twistor spaces have a lot of rational curves, in particular all fibers $F \sim \mathbb{P}^1$ of the smooth submersion $\pi : X \rightarrow M$. Note that there also is a holomorphic projection $\pi : X \rightarrow \mathbb{P}^1$.

Dimension 4n. The previous construction generalizes as follows. Let (M, g, D) be a quaternionic Kähler manifold of dimension $4n$, i.e. an oriented complete $4n$ -dimensional riemannian manifold (M, g) whose holonomy group is contained in the product $Sp(1)Sp(n)$ of quaternionic unitary groups. Such a manifold admits a rank 3 subbundle $D \subset \text{End}(TM)$ invariant by the Levi-Civita connection.

Definition 2.6. The twistor space $X = X(M, g, D)$ of (M, g, D) is the bundle of spheres of radius $\sqrt{2}$ of D .

This is a locally trivial bundle with fibre S^2 and structure group $SO(3)$. It can be endowed with a natural metric G and an almost complex structure J that is integrable [Sal82, Theorem 4.1]. Thus (X, J) is a complex manifold of complex dimension $2n + 1$.

When (M, g) is hyperkähler, i.e. when the holonomy group is contained in the quaternionic unitary group $Sp(n)$, it turns out that (M, g) admits three global g -orthogonal integrable Kähler structures I, J, K such that $IJ = -JI = K$. We thus obtain a pencil of complex structures

$$X(M, g, D) \longrightarrow \mathbb{P}^1$$

which is integrable. It is called the Calabi family of (M, g, D) .

Pluripositive hermitian metrics. Twistor spaces admit smooth hermitian $(1, 1)$ -forms ω that are balanced (i.e. $d\omega^{n-1} = 0$, see [Mic82] and [KV98, Proposition 4.5]). There is a natural hermitian form ω whose curvature has been computed by Kaledin-Verbitsky [KV98, Proposition 8.15] and Deschamps-LeDu-Mourougane [DLM17, Corollary 5.6]:

Theorem 2.7. *Let (M, g, D) be a quaternionic Kähler manifold of dimension $4n$ with constant scalar curvature $s \leq 0$. Then the natural hermitian form ω on the twistor space $X(M, g, D)$ is pluripositive $dd^c\omega \geq 0$.*

3. PLURINEGATIVE HERMITIAN METRICS

3.1. Monge-Ampère lower bounds. We show that the condition $v_-(\omega) > 0$ is satisfied if ω satisfies a special plurinegative condition.

Definition 3.1. We say that a hermitian form ω satisfies the condition *Plurineg*(n) if it is a Gauduchon metric and either $n = 2$ or $n \geq 3$ and

$$\sum_{k=1}^{n-\ell-2} (-1)^{n-k-\ell} \binom{n}{k} \binom{n-k-2}{\ell} dd^c\omega^k \wedge \omega^{n-k-2-\ell} \leq 0$$

for any $\ell \in \{0, \dots, n-3\}$.

The condition *Plurineg*(n) requires the Gauduchon condition $dd^c\omega^{n-1} = 0$. It reduces to the latter when $n = 2$, while it moreover asks that

- $dd^c\omega \leq 0$ when $n = 3$;
- $dd^c\omega \leq 0$ and $3dd^c\omega^2 - 2\omega \wedge dd^c\omega \leq 0$ when $n = 4$;
- $dd^c\omega \leq 0$, $dd^c\omega^2 - dd^c\omega \wedge \omega \leq 0$, and $2dd^c\omega^3 - 2\omega \wedge dd^c\omega^2 + \omega^2 \wedge dd^c\omega \leq 0$ when $n = 5$.

Theorem 3.2. *Let X be a compact complex manifold of dimension n . If X admits a hermitian metric ω that satisfies *Plurineg*(n) then $v_-(\omega) = \int_X \omega^n > 0$.*

Proof. Fix u a smooth ω -psh function. We first treat the case $n = 3$ to set the scene. The Gauduchon condition yields $\int_X \omega^2 \wedge dd^c u = 0$ while Stokes theorem ensures that $\int_X (dd^c u)^3 = 0$, thus

$$\begin{aligned} \int_X (\omega + dd^c u)^3 &= \int_X \omega^3 + 3 \int_X \omega^2 \wedge dd^c u + 3 \int_X \omega \wedge (dd^c u)^2 + \int_X (dd^c u)^3 \\ &= \int_X \omega^3 + 3 \int_X -dd^c\omega \wedge du \wedge d^c u \\ &\geq \int_X \omega^3 \end{aligned}$$

if $dd^c\omega \leq 0$, so that $v_-(\omega) \geq \int_X \omega^3 > 0$.

To make the arguments clearer, we also explicitly look at the case $n = 4$. Using the binomial expansion, the Gauduchon condition and Stokes theorem, we obtain

$$\begin{aligned} \int_X \omega_u^4 - \int_X \omega^4 &= 6 \int_X -dd^c\omega^2 \wedge du \wedge d^c u + 4 \int_X -dd^c\omega \wedge dd^c u \wedge du \wedge d^c u \\ &= 2 \int_X [-3dd^c\omega^2 + 2dd^c\omega \wedge \omega] \wedge du \wedge d^c u + 4 \int_X -dd^c\omega \wedge \omega_u \wedge du \wedge d^c u \\ &\geq 0 \end{aligned}$$

if *Plurineg*(4) is satisfied.

We now consider the general case.

$$\begin{aligned} \int_X \omega_u^n - \int_X \omega^n &= \sum_{k=0}^{n-1} \binom{n}{k} \int_X \omega^k \wedge (dd^c u)^{n-k} \\ &= \int_X (dd^c u)^n + \sum_{k=1}^{n-2} \binom{n}{k} \int_X \omega^k \wedge (dd^c u)^{n-k} + n \int_X u dd^c \omega^{n-1} \\ &= - \sum_{k=1}^{n-2} \binom{n}{k} \int_X dd^c \omega^k \wedge (dd^c u)^{n-k-2} \wedge du \wedge d^c u, \end{aligned}$$

using the Gauduchon condition $dd^c \omega^{n-1} = 0$. Thus

$$\begin{aligned} \int_X \omega_u^n - \int_X \omega^n &= - \sum_{k=1}^{n-2} \binom{n}{k} \int_X dd^c \omega^k \wedge (\omega_u - \omega)^{n-k-2} \wedge du \wedge d^c u \\ &= - \sum_{k=1}^{n-2} \sum_{\ell=0}^{n-k-2} (-1)^{n-k-\ell} \binom{n}{k} \binom{n-k-2}{\ell} \int_X dd^c \omega^k \wedge \omega^{n-k-2-\ell} \wedge \omega_u^\ell \wedge du \wedge d^c u \\ &= - \sum_{\ell=0}^{n-3} \sum_{k=1}^{n-\ell-2} (-1)^{n-k-\ell} \binom{n}{k} \binom{n-k-2}{\ell} \int_X dd^c \omega^k \wedge \omega^{n-k-2-\ell} \wedge \omega_u^\ell \wedge du \wedge d^c u \\ &= - \sum_{\ell=0}^{n-3} \int_X \sum_{k=1}^{n-\ell-2} (-1)^{n-k-\ell} \binom{n}{k} \binom{n-k-2}{\ell} dd^c \omega^k \wedge \omega^{n-k-2-\ell} \wedge \omega_u^\ell \wedge du \wedge d^c u. \end{aligned}$$

Therefore, if

$$\sum_{k=1}^{n-\ell-2} (-1)^{n-k-\ell} \binom{n}{k} \binom{n-k-2}{\ell} dd^c \omega^k \wedge \omega^{n-k-2-\ell} \leq 0$$

for any $\ell \in \{0, \dots, n-3\}$, we get that $v_-(\omega) \geq \int_X \omega^n$. □

3.2. Monge-Ampère bounds in dimension 3. We focus in this section on the 3-dimensional setting and observe that the condition $v_-(X, \omega_X) > 0$ is satisfied if X admits a plurinegative hermitian metric which is not necessarily Gauduchon. Note that the restriction of a plurinegative metric yields a plurinegative metric, but the Gauduchon condition is in general not preserved under restriction.

Theorem 3.3. *Let X be a compact complex manifold of dimension 3. If X admits a hermitian metric ω such that $dd^c \omega \leq 0$, then $v_-(\omega) > 0$.*

In particular if ω is pluriclosed, then $0 < v_-(\omega) \leq v_+(\omega) < +\infty$.

Proof. Assume by contradiction that there exists a sequence (u_j) such that $u_j \in \text{PSH}(X, \omega) \cap C^\infty(X)$, $\sup_X u_j = -1$ and

$$\int_X (\omega + dd^c u_j)^3 \rightarrow 0.$$

For each j , let $\phi_j \in \text{PSH}(X, \omega) \cap C^\infty(X)$ be the unique solution to

$$(\omega + dd^c \phi_j)^3 = c_j u_j^2 \omega^3$$

normalized by $\sup_X \phi_j = 0$. Here $c_j > 0$ is a positive constant. The existence of ϕ_j (and that of c_j) follows from the main result of [TW10]. Observe that $\int_X |u_j|^4 \omega^3$

is uniformly bounded away from 0 and infinity. Thus by [KN15, Lemma 3.13 and Theorem 5.8],

$$C^{-1} \leq c_j \leq C, \phi_j \geq -C,$$

for some uniform constant $C > 0$.

Let B be a positive constant such that $d\omega \wedge d^c\omega \leq B\omega^3$. For $u, v \in PSH(X, \omega) \cap L^\infty(X)$ we set $\omega_u := \omega + dd^c u, \omega_v := \omega + dd^c v$. When $dd^c\omega \leq 0$ one obtains

$$dd^c(\omega_u \wedge \omega_v) = dd^c\omega \wedge \omega_u + dd^c\omega \wedge \omega_v + 2d\omega \wedge d^c\omega \leq 2B\omega^3,$$

hence

$$dd^c(\omega_{u_j}^2 + \omega_{v_j}^2 + \omega_{u_j} \wedge \omega_{v_j}) \leq 6d\omega \wedge d^c\omega \leq 6B\omega^3.$$

Fix $t > 1$ and set $v_j := \max(u_j, \phi_j - t)$. Using Stokes theorem we obtain

$$\begin{aligned} \int_X (\omega_{v_j}^3 - \omega_{u_j}^3) &= \int_X (v_j - u_j) dd^c(\omega_{u_j}^2 + \omega_{v_j}^2 + \omega_{u_j} \wedge \omega_{v_j}) \\ &\leq 6B \int_X (v_j - u_j) \omega^3 = 6B \int_{\{u_j < \phi_j - t\}} (v_j - u_j) \omega^3 \\ &\leq 6B \int_{\{u_j < \phi_j - t\}} |u_j| \omega^3 \\ &\leq \frac{6B}{tc_j} \int_{\{u_j < \phi_j - t\}} c_j |u_j|^2 \omega^3 \\ &\leq \frac{6BC}{t} \int_{\{u_j < \phi_j - t\}} \omega_{\phi_j}^3 \\ &\leq \frac{6BC}{t} \int_X \omega_{v_j}^3. \end{aligned}$$

In the last line we have used the identity

$$\mathbf{1}_{\{u_j < \phi_j - t\}} \omega_{\phi_j}^3 = \mathbf{1}_{\{u_j < \phi_j - t\}} \omega_{v_j}^3 \leq \omega_{v_j}^3.$$

Choosing $t = 12BC$ we obtain $\int_X \omega_{v_j}^3 \leq 2 \int_X \omega_{u_j}^3$. Using [GL21b, Proposition 3.4], we arrive at a contradiction since the functions v_j are uniformly bounded. \square

3.3. Various obstructions.

3.3.1. *Pluriclosed and Plurinegative metrics.* Constructing pluriclosed hermitian metrics on compact complex manifolds is a problem that has attracted a lot of attention in the last decades. We observe here a rigidity property of this condition.

Proposition 3.4. *Let (X, ω) be a compact hermitian manifold such that $dd^c\omega = 0$. If $dd^c\omega^2 \leq 0$ (resp. $dd^c\omega^2 \geq 0$) then $dd^c\omega^j = 0$ for all $j \geq 1$.*

The restrictive condition $dd^c\omega = 0$ & $dd^c\omega^2 = 0$ has been introduced by Guan-Li [GL10], and further studied by Chiose [Chi16b], it is equivalent to the preservation of the Monge-Ampères volumes, $v_+(\omega) = v_-(\omega) = \int_X \omega^n$ (see Theorem 1.9 and Remark 1.10).

Proof. It follows from Stokes theorem that $\int_X dd^c(\omega^{n-1}) = 0$. Now

$$\begin{aligned} \frac{dd^c(\omega^{n-1})}{n-1} &= \{dd^c\omega \wedge \omega + (n-2)d\omega \wedge d^c\omega\} \wedge \omega^{n-3} \\ &= \left\{ \frac{n-2}{2} dd^c\omega^2 - (n-3) dd^c\omega \wedge \omega \right\} \wedge \omega^{n-3}, \end{aligned}$$

hence

$$\frac{n-2}{2} \int_X dd^c \omega^2 \wedge \omega^{n-3} = (n-3) \int_X dd^c \omega \wedge \omega^{n-2}.$$

The conclusion follows. □

An adaptation of the proof of Theorem 2.4 yields the following characterization of the existence of plurinegative metrics [Eg01, Theorem 3.3].

Theorem 3.5. *The following properties are equivalent.*

- *There exists a hermitian form ω such that $dd^c \omega \leq 0$.*
- *There exists no positive current τ of bidimension $(2, 2)$ such that $S = dd^c \tau \geq 0$ with $S \neq 0$.*

A similar obstruction for the existence of pluriclosed hermitian metrics is well-known [Eg01, Theorem 3.3]: there exists a hermitian form ω such that $dd^c \omega = 0$ if and only if the only dd^c -exact positive current S of bidimension $(1, 1)$ is 0.

Remark 3.6. It follows from the work of Ivashkovich that one can extend meromorphic maps with values in compact non-Kähler manifolds endowed with a plurinegative metric (see [Iv04, Theorem 2.2]).

3.3.2. Mutually exclusive conditions in dimension 3. Let X be a compact complex 3-fold. If X admits a pluripositive hermitian metric ω and another plurinegative hermitian metric ω' , then these are actually both pluriclosed. More generally, it follows from Theorems 2.4 and 3.5 that we have the following alternative.

Corollary 3.7. *Let X be a compact complex 3-fold. The following conditions are mutually exclusive:*

- *X admits a hermitian metric ω such that $dd^c \omega \geq 0$ and $dd^c \omega \neq 0$;*
- *X admits a hermitian metric ω such that $dd^c \omega = 0$;*
- *X admits a hermitian metric ω such that $dd^c \omega \leq 0$ and $dd^c \omega \neq 0$;*
- *X does not admit any hermitian metric ω such that $dd^c \omega$ has a sign.*

We refer the reader to the examples to follow for an illustration of each case.

Proof. Assume ω is a hermitian form such that $dd^c \omega \geq 0$, while ω' is a hermitian form such that $dd^c \omega' \leq 0$. It follows from Theorem 2.4 and Theorem 3.5 that $dd^c \omega = 0$ and $dd^c \omega' = 0$. □

Recall that the class \mathcal{C} of Fujiki consists of compact complex manifolds that are bimeromorphic to a Kähler manifold. The non-existence of plurisigned hermitian metric occurs on non-Kähler Fujiki 3-folds, as we now observe.

Proposition 3.8. *Let X be a compact complex 3-fold in the Fujiki class \mathcal{C} . Then X does not admit any plurisigned hermitian metric, unless X is Kähler.*

A celebrated example of a non-Kähler 3-fold bimeromorphic to $\mathbb{C}\mathbb{P}^3$ has been provided by Hironaka [Hir60]. This shows that [Eg01, Theorem 6.5] is incorrect.

Proof. This is [Chi14, Theorem 2.3]. We include the proof for the reader's convenience. Let X be a compact complex 3-fold in the Fujiki class \mathcal{C} . Let ω be a pluripositive hermitian metric. There exists a Kähler current on X , i.e. a positive closed current T of bidegree $(1, 1)$ which dominates a hermitian form. Up to rescaling, we can assume $T \geq \omega$.

Then $0 \leq \omega \wedge dd^c\omega \leq T \wedge dd^c\omega$, hence

$$0 \leq \int_X \omega \wedge dd^c\omega \leq \int_X T \wedge dd^c\omega = 0,$$

as follows from Stokes theorem. Thus ω is pluriclosed and it follows from [Chi14, Theorem 2.2] that X is Kähler. The proof for plurinegative metrics is similar. \square

3.4. Locally conformally Kähler manifolds. We observe in this section that a large class of non-Kähler manifolds admits a plurinegative hermitian metric ω . This ensures that $v_-(\omega) > 0$ when $\dim_{\mathbb{C}} X = 3$. We then have a closer look at the special subfamily of diagonal Hopf 3-folds.

3.4.1. Existence of plurinegative metrics. Recall that a complex manifold (X, ω) is *locally conformally Kähler* (lck) if one can find local smooth conformal factors $f_j > 0$ in an open cover $\{U_j\}$ of X such that $f_j\omega$ is Kähler in U_j . Thus

$$d\omega = -\frac{df_j}{f_j} \wedge \omega = d(-\log f_j) \wedge \omega$$

and these glue together into a globally well defined closed 1-form θ such that $d\omega = \theta \wedge \omega$.

The *Lee form* θ is unique and defines a conformal invariant, in particular it vanishes if and only if (a conformal multiple of) the metric ω is Kähler. We refer the reader to [DO98] for an introduction to lcK geometry, and to [OV11, OV20, Baz18, OV] for a more recent account.

Definition 3.9. One says that a compact hermitian manifold (X, ω) is *locally conformally Kähler with potential* if there exists a smooth positive plurisubharmonic function $\varphi : \tilde{X} \rightarrow \mathbb{R}_+^*$ on the universal cover \tilde{X} of X such that $\pi^*\omega = \frac{dd^c\varphi}{\varphi}$ and $\frac{\varphi \circ f}{\varphi} = c_f$ is constant for all deck transformations f .

The manifold (X, ω) is *Vaisman* if $\nabla\theta = 0$, where ∇ denotes the Levi-Civita connection associated to ω . Vaisman manifolds are lck with potential.

The 1-form θ decomposes as $\theta = \alpha + \bar{\alpha}$ where α is a $(1, 0)$ -form which is ∂ -closed and such that $\bar{\partial}\alpha = -\partial\bar{\alpha}$. Thus

$$d\omega \wedge d^c\omega = i\alpha \wedge \bar{\alpha} \wedge \omega^2 \geq 0.$$

When $n = \dim_{\mathbb{C}} X \geq 3$, we observe moreover that this $(3, 3)$ -form is non-zero if $\alpha \neq 0$, which is the case if X is not Kähler.

Since $d\omega \wedge d^c\omega \geq 0$ does not vanish, we have observed after Theorem 1.9 that $dd^c\omega$ cannot be positive. We therefore investigate whether ω is plurinegative.

Proposition 3.10. *Let (X, ω) be a compact hermitian manifold which is lcK with potential. Let $\varphi : \tilde{X} \rightarrow \mathbb{R}_+^*$ be a smooth positive plurisubharmonic function on the universal cover of X such that $\pi^*\omega = \frac{dd^c\varphi}{\varphi}$.*

If (X, ω) is Vaisman then $\psi = \log \varphi$ is plurisubharmonic and for all $k \geq 1$

$$dd^c\omega^k = -k(dd^c\psi)^{k+1} \leq 0.$$

In particular $dd^c\omega \leq 0$ and

- $v_-(X, \omega) > 0$ if $\dim_{\mathbb{C}} X = 3$;
- *there is no pluripositive hermitian metric $\tilde{\omega}$ on X .*

Conversely if $\psi = \log \varphi$ is plurisubharmonic then (X, ω) is Vaisman.

The last statement of this proposition is due to Ornea-Verbitsky [OV20, Corollary 2.4].

Proof. Slightly abusing notation we identify ω with the invariant form $\pi^*\omega$. We set $\psi = \log \varphi$ and observe that

$$dd^c\psi = \frac{dd^c\varphi}{\varphi} - \frac{d\varphi \wedge d^c\varphi}{\varphi^2} \quad \text{and} \quad d\psi \wedge d^c\psi = \frac{d\varphi \wedge d^c\varphi}{\varphi^2}$$

hence $\omega = dd^c\psi + d\psi \wedge d^c\psi$. Observe that $d\psi \wedge d^c\psi$ has rank 1, while ω has rank n , so the rank of $dd^c\psi$ is at least $n - 1$.

Since $d\psi \wedge d^c\psi$ has rank 1, we obtain

$$\omega^k = (dd^c\psi)^k + k(dd^c\psi)^{k-1} \wedge d\psi \wedge d^c\psi$$

and

$$(3.1) \quad dd^c\omega^k = k(dd^c\psi)^{k-1} \wedge dd^c(d\psi \wedge d^c\psi) = -k(dd^c\psi)^{k+1} \leq 0$$

if $\psi = \log \varphi$ is psh. For $k = 1$ we obtain $dd^c\omega \leq 0$, so it follows from Theorem 3.3 that $v_-(\omega) > 0$ when $\dim_{\mathbb{C}} X = 3$.

Assume now that $\tilde{\omega}$ is a hermitian metric on X such that $dd^c\tilde{\omega} \geq 0$. It follows from Stokes theorem that

$$0 \leq \int_X dd^c\tilde{\omega} \wedge \omega^{n-2} = \int_X \tilde{\omega} \wedge dd^c\omega^{n-2} \leq 0,$$

hence $\tilde{\omega}$ is pluriclosed and $-dd^c\omega^{n-2} = (dd^c\psi)^{n-1} \equiv 0$. The latter equality implies that $dd^c\psi$ has rank $\leq n - 2$, a contradiction.

It remains to understand when $\psi = \log \varphi$ is plurisubharmonic. Recall that $d\omega = \theta \wedge \omega$ with $\theta = -d\psi$. Set $\theta^c := \frac{1}{2i}(\alpha - \bar{\alpha}) = -d^c\psi$ so that $d^c\omega = \theta^c \wedge \omega$. Thus

$$\beta := dd^c\psi = -d\theta^c = \omega - \theta \wedge \theta^c$$

is a real $(1, 1)$ -form whose eigenvalues with respect to ω are 1, with multiplicity $(n - 1)$, and $\lambda = 1 - |\theta|_{\omega}^2$, since $\beta^n = \omega^n - n\omega^{n-1} \wedge \theta \wedge \theta^c = [1 - |\theta|_{\omega}^2] \omega^n$. Thus ψ is psh if and only if $|\theta|_{\omega}^2 \leq 1$.

When X is Vaisman then $|\theta|_{\omega}^2$ is constant and there exists a unique conformal choice of ω such that $|\theta|_{\omega}^2 \equiv 1$ (see [Vai82, Remark p232]). In this case ψ is psh hence $\beta \geq 0$ and $\beta^n \equiv 0$. Conversely it follows from (3.1) that

$$\beta^n = (dd^c\psi)^n = -\frac{1}{n-1} dd^c\omega^{n-1}.$$

Stokes theorem ensures that $\int_X \beta^n = 0$. Thus either $|\theta|_{\omega}^2 \equiv 1$, then X is Vaisman and ω is a Gauduchon metric [OV20, Corollary 2.4], or λ changes signs and ψ is not plurisubharmonic. □

Remark 3.11. The existence of plurinegative metrics on Vaisman manifolds can also be deduced from the Ornea-Verbitsky Embedding theorem for Vaisman manifolds [OV10] and Example 3.12 below. More generally, manifolds admitting locally conformally Kähler metrics with potential (this class includes Vaisman manifolds too) can be holomorphically embedded into linear Hopf manifolds [OV10]. Therefore the problem concerning the existence of plurinegative hermitian metrics on locally conformally Kähler manifolds with potential is reduced to study linear Hopf manifolds (which are Vaisman if and only if the generator of the fundamental group is diagonalizable, see e.g. [OV, Theorem 16.3]).

Example 3.12. Let X be a diagonal Hopf manifold $X = \mathbb{C}^n \setminus \{0\} / \sim$, where we identify z and Az in $\mathbb{C}^n \setminus \{0\}$, for some diagonal matrix $A = \text{Diag}(\lambda_1, \dots, \lambda_n)$ with entries $\lambda_j \in \mathbb{D}^*$. We choose $\alpha_i := \frac{\log 2}{2(-\log|\lambda_i|)}$ and set

$$\varphi(z) = \sum_{i=1}^n |z_i|^{2\alpha_i}, \quad \text{so that } \psi = \log \varphi \in PSH(\mathbb{C}^n).$$

Then $\varphi \circ A(z) = \frac{1}{2}\varphi(z)$ hence $\omega(z) := \frac{dd^c\varphi(z)}{\varphi(z)} = dd^c\psi(z) + d\psi \wedge d^c\psi(z)$ defines a hermitian form on X such that $dd^c\omega \leq 0$.

3.4.2. The classical Hopf 3-fold. We consider here $X = \mathbb{C}^3 \setminus \{0\} / \sim$, where we identify z and λz in $\mathbb{C}^3 \setminus \{0\}$, for some $\lambda \in \mathbb{D}^* = \{\zeta \in \mathbb{C}, 0 < |\zeta| < 1\}$. There is a natural holomorphic map $f : X \rightarrow \mathbb{P}^2$ with elliptic fibers.

Set $\eta = \sum_{j=1}^n \bar{z}_j dz_j = \partial|z|^2$. The invariant $(1, 1)$ -form $|z|^{-4}i\eta \wedge \bar{\eta}$ compensates the lack of positivity of $f^*\omega_{FS}$ so that the invariant $(1, 1)$ form

$$\omega = f^*\omega_{FS} + |z|^{-2}i\eta \wedge \bar{\eta} = \frac{\sum_{j=1}^n idz_j \wedge d\bar{z}_j}{|z|^2} = \frac{dd^c\varphi}{\varphi}$$

induces a hermitian form on X that we still denote by ω . Since $\log \varphi = \log |z| \in PSH(\mathbb{C}^3 \setminus \{0\})$, it follows from Proposition 3.10 that $dd^c\omega \leq 0$.

We would like to test the finiteness of $v_+(\omega)$. It follows from the proof of Theorem 3.3.1 that there exists $C > 0$ such that for any $u \in PSH(X, \omega)$ normalized by $\sup_X u = 0$, we have

$$3 \int_X (-dd^c\omega) \wedge du \wedge d^c u - C \leq \int_X (\omega + dd^c u)^3 \leq 3 \int_X (-dd^c\omega) \wedge du \wedge d^c u + C.$$

It is thus tempting to think that one can reach $v_+(\omega) = +\infty$ by constructing ω -psh functions whose gradient does not belong to L^2 . There are several functions $v \in PSH(\mathbb{P}^2, \omega_{FS})$ whose gradient does not belong to L^2 ; they induce functions

$$u = v \circ f \in PSH(X, f^*\omega_{FS}) \subset PSH(X, \omega)$$

with gradient $\nabla u \notin L^2(X)$. However $dv \circ f$ is proportional to η , hence

$$\int_X (-dd^c\omega) \wedge d(v \circ f) \wedge d^c(v \circ f) = 0.$$

One can construct singular ω -psh functions that do not come from \mathbb{P}^2 as follows: the function $\rho = \log \text{dist}_\omega(\cdot, p)$ is psh near $p \in X$ and smooth off p . We multiply it by a cut-off function χ which is identically equal to 1 near p . For $\varepsilon > 0$ small, the function $\varphi = \varepsilon\chi\rho$ belong to $PSH(X, \omega)$ and it has a logarithmic singularity at p . However $\nabla\varphi \in L^2(X)$ as the singularity of φ is isolated. One could consider a convergent series of such functions, this would produce examples with a discrete set of logarithmic singularities (and possibly an uncountable polar set ($\varphi = -\infty$)), but their global gradient would still belong to L^2 .

Question 3.13. *Does one have $v_+(\omega) < +\infty$?*

We tend to expect a positive answer to this problem, but a negative one would be quite interesting as well!

4. HOMOGENEOUS EXAMPLES

We finally study several classes of compact (locally) homogeneous manifolds.

4.1. Complex parallelizable manifolds. Let X be a compact complex manifolds of dimension n that is *complex parallelizable*, i.e. such that the holomorphic tangent bundle $T^{1,0}X$ is holomorphically trivial. It has been shown by Wang [Wan54] that $X = \Gamma \backslash G$ is the quotient of a connected and simply connected complex Lie group G by a discrete subgroup Γ , and that complex tori (for which $G = \mathbb{C}^n$) are the only ones that are Kähler. On the other side, they always admits balanced metrics by [AG86, Proposition 3.1] and they cannot admit pluriclosed metrics, see [FGV19, page 7110].

Proposition 4.1. *A compact complex parallelizable manifold X admits a hermitian metric ω such that $dd^c\omega^k \geq 0$ for all $k \geq 1$.*

In particular, $v_+(X, \omega) < +\infty$ when $\dim_{\mathbb{C}} X = 3$. Moreover X does not admit any plurinegative hermitian metric unless it is a complex torus.

Complex parallelizable manifolds that arise as quotients of complex semisimple Lie groups (e.g. compact quotients of $SL(2; \mathbb{C})$ by a lattice) have been studied in [Yac98]. For a classification of compact complex parallelizable solvmanifolds in complex dimension 3 as well as a partial classification in dimension 4 and 5, we refer the reader to [Nak75, Theorem 1 and Section 6].

Proof. By definition there exist $\varphi_1, \dots, \varphi_n \in H^0(X, \Omega_X^1)$ holomorphic 1-forms that are linearly independent at each point. We set

$$\omega := \sum_{j=1}^n i\varphi_j \wedge \overline{\varphi_j}.$$

This is a hermitian $(1, 1)$ -form such that

$$dd^c\omega = i\partial\bar{\partial}\omega = \sum_{j=1}^n i^2\partial\bar{\partial}(\varphi_j \wedge \overline{\varphi_j}).$$

Since $\bar{\partial}\varphi_j = 0$ and $\partial\overline{\varphi_j} = 0$ we obtain, setting $\eta_j = \partial\varphi_j$,

$$dd^c\omega = \sum_{j=1}^n -i^2\eta_j \wedge \overline{\eta_j} \geq 0.$$

Recall indeed that if η is a $(2, 0)$ -form then $-i^2\eta \wedge \overline{\eta} = i^4\eta \wedge \overline{\eta}$ is a weakly positive $(2, 2)$ -form (that is strongly positive if $\eta = \alpha_1 \wedge \alpha_2$ is decomposable).

For $k \geq 1$, we decompose

$$\omega^k = \sum_{j_1, \dots, j_k=1}^n i\varphi_{j_1} \wedge \overline{\varphi_{j_1}} \wedge \dots \wedge i\varphi_{j_k} \wedge \overline{\varphi_{j_k}} = \sum_{J=(j_1, \dots, j_k)} i^{k^2}\alpha_J \wedge \overline{\alpha_J},$$

where $\alpha_J = \varphi_{j_1} \wedge \dots \wedge \varphi_{j_k}$ is a holomorphic k -form. Thus

$$i\partial\bar{\partial}\omega^k = \sum_{J=(j_1, \dots, j_k)} i^{1+k^2}(-1)^k\beta_J \wedge \overline{\beta_J},$$

with $\beta_J = \partial\alpha_J$. Since $i^{1+k^2}(-1)^k = i^{(k+1)^2}$, we conclude that $dd^c\omega^k \geq 0$.

Observe that $dd^c\omega \neq 0$ unless ω is Kähler, in which case X is a complex torus. It follows therefore from Theorem 3.3 and Theorem 3.5 that $v_+(X, \omega) < +\infty$ when

$\dim_{\mathbb{C}} X = 3$, and X does not admit a plurinegative hermitian metric unless it is a torus. \square

In dimension higher than 3, the conditions needed to control v_+ as in Theorem 2.2 are not trivially satisfied. For example, take the complex parallelizable manifold of complex dimension $n = 4$ characterized by a coframe of holomorphic 1-forms $(\varphi^j)_{j \in \{1,2,3,4\}}$ with structure equations

$$d\varphi^1 = d\varphi^2 = d\varphi^3 = 0, \quad d\varphi^4 = -\varphi^2 \wedge \varphi^3,$$

see [Nak75, Type IV.2, page 108]. The metric $\omega = \sum_{j=1}^4 \sqrt{-1}\varphi^j \wedge \bar{\varphi}^j$ satisfies:

$$\begin{aligned} dd^c\omega &= \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \geq 0, \\ dd^c\omega^2 &= 2\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \geq 0, \end{aligned}$$

but

$$\begin{aligned} dd^c\omega^2 - \varepsilon dd^c\omega \wedge \omega &= (2 - \varepsilon)\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \\ &\quad - \varepsilon\sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \wedge \sqrt{-1}\varphi^4 \wedge \bar{\varphi}^4 \end{aligned}$$

is never semi-positive for $\varepsilon > 0$.

The following is an example of a complex parallelizable manifold admitting metrics with the property of Theorem 2.2: if a lattice can be provided for the Lie group, then we would get a compact manifold with $v_+ < \infty$.

Example 4.2. Consider the complex parallelizable solvmanifold of complex dimension $n = 4$ of type IV.5 in [Nak75, page 108], which is characterized by a coframe of holomorphic 1-forms with structure equations

$$\begin{aligned} d\varphi_1 &= 0, \quad d\varphi^2 = \varphi^1 \wedge \varphi^2, \\ d\varphi^3 &= \alpha\varphi^1 \wedge \varphi^3, \quad d\varphi^4 = -(1 + \alpha)\varphi^1 \wedge \varphi^4, \end{aligned}$$

depending on a parameter $\alpha \in \mathbb{R} \setminus \{-1, 0\}$. It is not known whether it admits lattices, see [Nak75, page 110]. Consider the left-invariant metric

$$\omega = a_1\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 + a_2\sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 + a_3\sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 + a_4\sqrt{-1}\varphi^4 \wedge \bar{\varphi}^4,$$

where $a_1, a_2, a_3, a_4 > 0$. A straightforward computation yields

$$\begin{aligned} dd^c\omega &= a_2\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 + a_3|\alpha|^2\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \\ &\quad + a_4|\alpha + 1|^2\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^4 \wedge \bar{\varphi}^4 \geq 0, \\ dd^c\omega^2 &= 2a_2a_3|\alpha + 1|^2\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \\ &\quad + 2a_2a_4|\alpha|^2\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^4 \wedge \bar{\varphi}^4 \\ &\quad + 2a_3a_4\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \wedge \sqrt{-1}\varphi^4 \wedge \bar{\varphi}^4 \geq 0. \end{aligned}$$

Therefore

$$\begin{aligned} dd^c\omega^2 - \varepsilon\omega \wedge dd^c\omega &= (-|\alpha|^2\varepsilon + 2|\alpha + 1|^2 - \varepsilon)a_2a_3\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \\ &\quad + (-|\alpha + 1|^2\varepsilon + 2|\alpha|^2 - \varepsilon)a_2a_4\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^4 \wedge \bar{\varphi}^4 \\ &\quad + (-|\alpha + 1|^2\varepsilon - |\alpha|^2\varepsilon + 2)a_3a_4\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \wedge \sqrt{-1}\varphi^4 \wedge \bar{\varphi}^4. \end{aligned}$$

Therefore, if we take

$$0 < \varepsilon \leq \min \left\{ \frac{2|\alpha + 1|^2}{1 + |\alpha|^2}, \frac{2|\alpha|^2}{1 + |\alpha + 1|^2}, \frac{2}{|\alpha|^2 + |1 + \alpha|^2} \right\},$$

we obtain $dd^c\omega^2 - \varepsilon\omega \wedge dd^c\omega \geq 0$.

4.2. Six-dimensional nilmanifolds. In this section, we consider *nilmanifolds*, namely, compact quotients $\Gamma \backslash G$ of connected simply-connected nilpotent Lie groups G by co-compact discrete subgroups Γ . We recall that a Lie group G is called *nilpotent* if its associated Lie algebra \mathfrak{g} satisfies that the lower central series $\{\mathfrak{g}_j := [\mathfrak{g}, \mathfrak{g}_{j-1}]\}_{j \in \mathbb{N}}$, with $\mathfrak{g}_0 := \mathfrak{g}$, eventually vanishes. In dimension 6, according to [Mor58, Mag86], there are only 34 isomorphism classes of nilpotent Lie algebras over \mathbb{R} , and 10 of them are reducible.

We consider *left-invariant complex structures* on $\Gamma \backslash G$, i.e. complex structures that are induced by complex structures on G being invariant under left-translations. Equivalently, left-invariant complex structures correspond to linear complex structures on the corresponding Lie algebra satisfying an integrability condition. According to [Sal01], only 18 of the above 34 nilpotent Lie algebras admit left-invariant complex structures. We notice that the existence of lattices in nilpotent Lie groups is well-understood thanks to [Mal45], more precisely it corresponds to having rational constant structures, a condition that is satisfied in all the above considered cases.

Among the latter, only four may admit a hermitian metric ω which is pluriclosed, as shown by Fino-Parton-Salamon in [FPS04]. We show in this section that the remaining 14 classes all admit a pluripositive hermitian metric, and we also analyze whether there exists a balanced metric, i.e. a hermitian metric ω such that $d\omega^2 = 0$.

Left-invariant complex structures on six-dimensional nilmanifolds are gathered in four families in [ABD11, Uga07, UV14, COUV16], some of them depending on continuous parameters, up to *linear* equivalence. These families are described by a coframe of left-invariant $(1, 0)$ -forms $(\varphi^1, \varphi^2, \varphi^3)$ with structure equations as follows (we use the short-hands $\varphi^{1\bar{2}} := \varphi^1 \wedge \bar{\varphi}^2$, etc.).

(Np): $d\varphi^1 = d\varphi^2 = 0, d\varphi^3 = \rho\varphi^{1\bar{2}}$ where $\rho \in \{0, 1\}$; these are the *complex parallelizable* structures [Wan54, Nak75]. The manifold is a compact quotient of a *complex* Lie group by a discrete subgroup; $\rho = 0$ corresponds to the complex torus, while $\rho = 1$ corresponds to the Iwasawa manifold [FG86].

(Ni): $d\varphi^1 = d\varphi^2 = 0, d\varphi^3 = \rho\varphi^{1\bar{2}} + \varphi^{1\bar{1}} + \lambda\varphi^{1\bar{2}} + D\varphi^{2\bar{2}}$ where $\rho \in \{0, 1\}, \lambda \in \mathbb{R}^{\geq 0}, D \in \mathbb{C}$ with $\Im D \geq 0$; this class (and the following) contains *nilpotent* complex structures: the ascending series $\{\mathfrak{a}_j(J) := \{X \in \mathfrak{g} : [X, \mathfrak{g}] + [JX, \mathfrak{g}] \subseteq \mathfrak{a}_{j-1}\}\}_{j \in \mathbb{N}}$, with $\mathfrak{a}_0(J) := 0$, eventually equals \mathfrak{g} . The case $\rho = 0$ corresponds to *Abelian* complex structures, namely, the subalgebra of left-invariant $(1, 0)$ -vector fields is Abelian.

(Nii): $d\varphi^1 = 0, d\varphi^2 = \varphi^{1\bar{1}}, d\varphi^3 = \rho\varphi^{1\bar{2}} + B\varphi^{1\bar{2}} + c\varphi^{2\bar{1}}$ where $\rho \in \{0, 1\}, B \in \mathbb{C}, c \in \mathbb{R}^{\geq 0}$ with $(\rho, B, c) \neq (0, 0, 0)$.

(Niii): $d\varphi^1 = 0, d\varphi^2 = \varphi^{1\bar{3}} + \varphi^{1\bar{3}}, d\varphi^3 = \sqrt{-1}\rho\varphi^{1\bar{1}} \pm \sqrt{-1}(\varphi^{1\bar{2}} - \varphi^{2\bar{1}})$ where $\rho \in \{0, 1\}$; this class contains non-nilpotent complex structures.

We refer to [ABD11, Uga07, UV14, COUV16] and [AOUV, Table 1] for more details on the underlying Lie algebras.

We can restrict to study *left-invariant* hermitian structures on $\Gamma \backslash G$, namely, hermitian structures on G being invariant by the whole action of G by left-translations. Such structures correspond to *linear* hermitian structures on the associated Lie algebra. Indeed Belgun Symmetrization Trick [Bel00, Theorem 7] (see also [FG04, Theorem 2.1]) provides the following.

Lemma 4.3 ([Uga07, Proposition 3.6]). *Let $\Gamma \backslash G$ be a compact quotient of a Lie group G endowed with a left-invariant complex structure. If it admits a plurisigned hermitian metric, then it admits a left-invariant plurisigned hermitian metric.*

Proof. The case of pluriclosed metrics is [Uga07, Proposition 3.6]. The same argument applies here; we include a proof for the readers' convenience.

Since G admits a compact quotient, it is *unimodular* [Mil76, Lemma 6.2], namely its left-invariant Haar measure μ is also right-invariant. We consider the symmetrization map

$$\mu: \wedge^\bullet(\Gamma \backslash G) \rightarrow \wedge^\bullet \mathfrak{g}^*,$$

$$\mu(\alpha)(X_1, \dots, X_k) := \int_{\Gamma \backslash G} \alpha(m)(X_1(m), \dots, X_k(m)) \mu(m),$$

where $\wedge^\bullet \mathfrak{g}^*$ is identified with the subspace of left-invariant forms. It is clear that $\mu|_{\wedge^\bullet \mathfrak{g}^*} = \text{id}$ and that $d \circ \mu = \mu \circ d$. Since J is left-invariant, we have $\mu \circ J = J \circ \mu$. Therefore, there hold also $\partial \circ \mu = \mu \circ \partial$ and $\bar{\partial} \circ \mu = \mu \circ \bar{\partial}$.

In particular, if ω is a hermitian structure, then $\mu(\omega)$ is a left-invariant $(1, 1)$ -form. It is straightforward to check that $\mu(\omega) > 0$ is still a hermitian structure. We also have: $\partial \bar{\partial}(\mu(\omega)) = \mu(\partial \bar{\partial} \omega)$. In particular if ω is pluripositive (resp. plurinegative), then $\mu(\omega)$ is pluripositive (resp. plurinegative). Indeed, assume that ω is pluripositive. Then, for any $(1, 0)$ -form η

$$dd^c \omega \wedge \sqrt{-1} \eta \wedge \bar{\eta} = c \sqrt{-1} \varphi^{1\bar{1}} \wedge \sqrt{-1} \varphi^{2\bar{2}} \wedge \sqrt{-1} \varphi^{3\bar{3}}$$

for $c \geq 0$. We can restrict to left-invariant $(1, 0)$ -forms η , since positivity is a pointwise notion. Then, thanks to [AK17, Lemma 2.5], $\mu(\omega) \wedge \eta = \mu(\omega \wedge \eta) = \mu(c) \sqrt{-1} \varphi^{1\bar{1}} \wedge \sqrt{-1} \varphi^{2\bar{2}} \wedge \sqrt{-1} \varphi^{3\bar{3}}$ where $\mu(c) \geq 0$. \square

We therefore only consider left-invariant hermitian structures. With respect to a chosen coframe, it is straightforward to check that they are of the form

$$(4.1) \quad 2\omega = \sqrt{-1} r^2 \varphi^{1\bar{1}} + \sqrt{-1} s^2 \varphi^{2\bar{2}} + \sqrt{-1} t^2 \varphi^{3\bar{3}} \\ + u \varphi^{1\bar{2}} - \bar{u} \varphi^{2\bar{1}} + v \varphi^{2\bar{3}} - \bar{v} \varphi^{3\bar{2}} + z \varphi^{1\bar{3}} - \bar{z} \varphi^{3\bar{1}}$$

where $r, s, t \in \mathbb{R}$, $u, v, z \in \mathbb{C}$ satisfy

$$r^2 > 0, \quad s^2 > 0, \quad t^2 > 0, \\ r^2 s^2 > |u|^2, \quad r^2 t^2 > |z|^2, \quad s^2 t^2 > |v|^2, \\ r^2 s^2 t^2 + 2\Re(\sqrt{-1} \bar{u} z \bar{v}) > t^2 |u|^2 + r^2 |v|^2 + s^2 |z|^2,$$

in order for the metric to be positive-definite.

We compute $\sqrt{-1} \partial \bar{\partial} \omega$ for the above families. Note that positivity and strong positivity are equivalent for $(2, 2)$ -forms in dimension 3 (see [Mic82, p. 279–280]), and that it is enough to test positivity pairing with left-invariant forms, since it is a pointwise notion. Recall that, by [BG88, Has89], Kähler metrics do not exist

on non-tori nilmanifolds (even with complex structures that are possibly non-left-invariant).

(Np): For complex paralellizable type, ω is always balanced and

$$\sqrt{-1}\partial\bar{\partial}\omega = \frac{1}{2}\rho^2 t^2 \sqrt{-1}\varphi^{1\bar{1}} \wedge \sqrt{-1}\varphi^{2\bar{2}}$$

is always strongly-positive. More precisely, any left-invariant hermitian metric on the torus is pluriclosed (in fact, Kähler), and any left-invariant hermitian metric on the Iwasawa manifold is pluripositive, balanced, non-pluriclosed. The underlying Lie algebras are respectively \mathfrak{h}_1 and \mathfrak{h}_5 , in the notation of [Sal01], to which we refer for more details.

(Ni): For nilpotent type in Family I, the balanced condition is equivalent to

$$s^2 t^2 + (r^2 t^2 - |z|^2)D + (-\sqrt{-1} t^2 \bar{u} + v\bar{z})\lambda - |v|^2 = 0$$

(it is satisfied on $\mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6$ for metrics such that $r^2 = 1, v = z = 0$, and $s^2 + D = \sqrt{-1} \bar{u} \lambda$) and

$$\sqrt{-1}\partial\bar{\partial}\omega = \frac{t^2}{2} (\lambda^2 + \rho^2 - 2\Re D) \sqrt{-1}\varphi^{1\bar{1}} \wedge \sqrt{-1}\varphi^{2\bar{2}}.$$

Any hermitian metric is either

- pluripositive non-pluriclosed (when $2\Re D < \lambda^2 + \rho^2$, which happens on Lie algebras $\mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6$),
- or pluriclosed (when $2\Re D = \lambda^2 + \rho^2$, which happens on Lie algebras $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_8$),
- or plurinegative non-pluriclosed (when $2\Re D > \lambda^2 + \rho^2$, which happens on Lie algebras $\mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5$).

(Nii): for the complex structures of nilpotent type in Family II, the balanced condition is never satisfied, and

$$\sqrt{-1}\partial\bar{\partial}\omega = \frac{t^2}{2} (c^2 + \rho^2 + |B|^2) \sqrt{-1}\varphi^{1\bar{1}} \wedge \sqrt{-1}\varphi^{2\bar{2}}$$

is always strongly-positive, and never zero. The underlying Lie algebras are $\mathfrak{h}_7, \mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}, \mathfrak{h}_{14}, \mathfrak{h}_{15}, \mathfrak{h}_{16}$.

(Niii): For the complex structures of nilpotent type in Family II, the balanced condition is equivalent to

$$\begin{cases} t^2 \Re u + \Im(z\bar{v}) = 0 \\ \sqrt{-1} s^2 \bar{z} + \bar{u}v = 0 \\ \rho = 0 \end{cases}$$

(it is satisfied on \mathfrak{h}_{19}^- for metrics such that $u = z = 0$) and

$$\sqrt{-1}\partial\bar{\partial}\omega = t^2 \sqrt{-1}\varphi^{1\bar{1}} \wedge \sqrt{-1}\varphi^{2\bar{2}} + s^2 \sqrt{-1}\varphi^{1\bar{1}} \wedge \sqrt{-1}\varphi^{3\bar{3}}$$

is always strongly-positive, and never zero. The underlying Lie algebras are $\mathfrak{h}_{19}^-, \mathfrak{h}_{26}^+$.

Compare the above computations with the results in [LUV17, Proposition 2.7], where the authors study the sign of the Fu-Wang-Wu constant [FWW13] for six-dimensional nilmanifolds with left-invariant Hermitian structures: the sign of $dd^c\omega$ determines the sign of the Fu-Wang-Wu constant.

The following statement summarizes the previous discussion, it can be seen as an extension of [FPS04] to plurisigned metrics.

Theorem 4.4. *Consider a six-dimensional nilmanifold X endowed with a left-invariant complex structure and assume X is not a complex torus. Then, there is always a plurisigned metric. More precisely:*

- *either X belongs to one of the families (Np) , (Nii) and $(Niii)$ then any left-invariant hermitian metric is pluripositive but not pluriclosed.*
- *or X belongs to (Ni) and depending on the complex structure, every left-invariant hermitian metric is either pluriclosed, or pluripositive but not pluriclosed, or else plurinegative but not pluriclosed.*

By Corollary 3.7 the latter three conditions are mutually exclusive.

4.3. Higher dimensional nilmanifolds. As noticed in [FPS04], the pluriclosed condition for left-invariant metrics on six-dimensional nilmanifolds depends only on the complex structure, and we noticed the same behavior for the plurisigned condition in Theorem 4.4.

This is no longer true in higher dimension for the pluriclosed condition [EFV12, Remark 4.1]. More precisely, 8-dimensional nilmanifolds with left-invariant complex structures admitting pluriclosed metrics are classified into two families in [EFV12, Section 4]. For the second family, the pluriclosed condition only depends on the complex structure, while for the first family it also involves the parameters of the metric. We make a similar observation for the plurisigned condition.

Example 4.5. We consider here the eight-dimensional nilmanifold with left-invariant complex structure characterized by the structure equations

$$d\varphi^1 = 0, \quad d\varphi^2 = 0, \quad d\varphi^3 = \varphi^1 \wedge \bar{\varphi}^1 + \frac{1}{2}\varphi^2 \wedge \bar{\varphi}^2, \quad d\varphi^4 = -\varphi^1 \wedge \bar{\varphi}^2,$$

with respect to a left-invariant coframe $\{\varphi^1, \varphi^2, \varphi^3, \varphi^4\}$ of $(1, 0)$ -forms. It belongs to the first family in the above mentioned classification, more precisely, it corresponds to parameters $B_4 = 1$, $C_4 = \frac{1}{2}$, $F_5 = -1$, the others zero, in the notation of [EFV12]. We consider a Hermitian metric of the diagonal form

$$\omega = \sqrt{-1}a_1\varphi^1 \wedge \bar{\varphi}^1 + \sqrt{-1}a_2\varphi^2 \wedge \bar{\varphi}^2 + \sqrt{-1}a_3\varphi^3 \wedge \bar{\varphi}^3 + \sqrt{-1}a_4\varphi^4 \wedge \bar{\varphi}^4,$$

where $a_1, a_2, a_3, a_4 > 0$. Observe that

$$dd^c\omega = (-a_3 + a_4)\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2,$$

showing that they can be either pluripositive, or pluriclosed, or plurinegative, depending on the value of a_3/a_4 .

Note that this example does not satisfy the condition of Theorem 2.2. Indeed,

$$\begin{aligned} dd^c\omega^2 - \varepsilon dd^c\omega \wedge \omega &= (-\varepsilon a_3(a_4 - a_3) + 2a_3a_4)\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^3 \wedge \bar{\varphi}^3 \\ &\quad + (-\varepsilon a_4(a_4 - a_3) - 2a_3a_4)\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1}\varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1}\varphi^4 \wedge \bar{\varphi}^4 \end{aligned}$$

is not positive.

We now provide an 8-dimensional example which satisfies the curvature conditions of Theorem 2.2.

Example 4.6. We consider again an example among the eight-dimensional nilmanifolds in the first family of [EFV12]. More precisely, take parameters $B_1 = 1$, $G_3 = 1$, the others zero, that is consider the structure equations

$$d\varphi^1 = 0, \quad d\varphi^2 = 0, \quad d\varphi^3 = \varphi^1 \wedge \varphi^2, \quad d\varphi^4 = \varphi^2 \wedge \bar{\varphi}^1.$$

Take the diagonal metric

$$\omega = \sqrt{-1} a_1 \varphi^1 \wedge \bar{\varphi}^1 + \sqrt{-1} a_2 \varphi^2 \wedge \bar{\varphi}^2 + \sqrt{-1} a_3 \varphi^3 \wedge \bar{\varphi}^3 + \sqrt{-1} a_4 \varphi^4 \wedge \bar{\varphi}^4,$$

where $a_1, a_2, a_3, a_4 > 0$. We compute

$$\begin{aligned} dd^c \omega &= (a_3 + a_4) \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1} \varphi^2 \wedge \bar{\varphi}^2 \geq 0, \\ dd^c \omega^2 &= 2 a_3 a_4 \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1} \varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1} \varphi^3 \wedge \bar{\varphi}^3 \\ &\quad + 2 a_3 a_4 \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1} \varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1} \varphi^4 \wedge \bar{\varphi}^4, \end{aligned}$$

hence

$$\begin{aligned} dd^c \omega^2 - \varepsilon dd^c \omega \wedge \omega &= a_3 (-\varepsilon a_3 + (2 - \varepsilon) a_4) \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1} \varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1} \varphi^3 \wedge \bar{\varphi}^3 \\ &\quad + a_4 ((2 - \varepsilon) a_3 - \varepsilon a_4) \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1 \wedge \sqrt{-1} \varphi^2 \wedge \bar{\varphi}^2 \wedge \sqrt{-1} \varphi^4 \wedge \bar{\varphi}^4, \end{aligned}$$

which is non-negative for suitable choices of a_3, a_4, ε (take e.g. $a_3 = a_4 = 1, \varepsilon \leq \frac{1}{2}$). In particular, $v_+(\omega) < +\infty$.

4.4. Six-dimensional solvmanifolds with trivial canonical bundle. We now consider *solvmanifolds* i.e. compact quotients of a connected solvable Lie group by a closed subgroup of real dimension 6 admitting a left-invariant complex structure with holomorphically trivial canonical bundle.

According to [FOU15] such complex structures are either nilmanifolds as above (see [Sal01, Theorem 1.3] and [BDV09, Theorem 2.7]) or belong to one of the classes below. We fix a coframe $\{\varphi^1, \varphi^2, \varphi^3\}$ of left-invariant $(1, 0)$ -forms and use the same notations as in the previous section.

- (Si): $d\varphi^1 = A\varphi^{13} + A\varphi^{1\bar{3}}, d\varphi^2 = -A\varphi^{23} - A\varphi^{2\bar{3}}, d\varphi^3 = 0$, where $A = \cos \theta + \sqrt{-1} \sin \theta, \theta \in [0, \pi)$. The complex structures in this family are of splitting type in the sense of [Kas13, Assumption 1], see [AOUV17].
- (Sii): $d\varphi^1 = 0, d\varphi^2 = -\frac{1}{2}\varphi^{13} - (\frac{1}{2} + \sqrt{-1}x)\varphi^{1\bar{3}} + \sqrt{-1}x\varphi^{3\bar{1}}$, and $d\varphi^3 = \frac{1}{2}\varphi^{12} + (\frac{1}{2} - \frac{\sqrt{-1}}{4x})\varphi^{1\bar{2}} + \frac{\sqrt{-1}}{4x}\varphi^{2\bar{1}}$, where $x \in \mathbb{R}^{>0}$.
- (Siii1): $d\varphi^1 = \sqrt{-1}(\varphi^{13} + \varphi^{1\bar{3}}), d\varphi^2 = -\sqrt{-1}(\varphi^{23} + \varphi^{2\bar{3}}), d\varphi^3 = \pm\varphi^{1\bar{1}}$.
- (Siii2): $d\varphi^1 = \varphi^{13} + \varphi^{1\bar{3}}, d\varphi^2 = -\varphi^{23} - \varphi^{2\bar{3}}, d\varphi^3 = \varphi^{1\bar{2}} + \varphi^{2\bar{1}}$.
- (Siii3): $d\varphi^1 = \sqrt{-1}(\varphi^{13} + \varphi^{1\bar{3}}), d\varphi^2 = -\sqrt{-1}(\varphi^{23} + \varphi^{2\bar{3}}), d\varphi^3 = \varphi^{1\bar{1}} + \varphi^{2\bar{2}}$.
- (Siii4): $d\varphi^1 = \sqrt{-1}(\varphi^{13} + \varphi^{1\bar{3}}), d\varphi^2 = -\sqrt{-1}(\varphi^{23} + \varphi^{2\bar{3}}), d\varphi^3 = \pm(\varphi^{1\bar{1}} - \varphi^{2\bar{2}})$.
- (Siv1): $d\varphi^1 = -\varphi^{13}, d\varphi^2 = \varphi^{23}, d\varphi^3 = 0$. This case corresponds to the holomorphically-parallelizable *Nakamura manifold* [Nak75]. This and the next two following cases are complex structures of splitting type in the sense of [Kas13, Assumption 1], see [AOUV17].
- (Siv2): $d\varphi^1 = 2\sqrt{-1}\varphi^{13} + \varphi^{3\bar{3}}, d\varphi^2 = -2\sqrt{-1}\varphi^{23} + x\varphi^{3\bar{3}}, d\varphi^3 = 0$, where $x \in \{0, 1\}$.
- (Siv3): $d\varphi^1 = A\varphi^{13} - \varphi^{1\bar{3}}, d\varphi^2 = -A\varphi^{23} + \varphi^{2\bar{3}}, d\varphi^3 = 0$, where $A \in \mathbb{C}$ with $|A| \neq 1$.
- (Sv): $d\varphi^1 = -\varphi^{3\bar{3}}, d\varphi^2 = \frac{\sqrt{-1}}{2}\varphi^{12} + \frac{1}{2}\varphi^{13} - \frac{\sqrt{-1}}{2}\varphi^{2\bar{1}}, d\varphi^3 = \frac{\sqrt{-1}}{2}(-\varphi^{13} + \varphi^{3\bar{1}})$.

We refer the reader to [Ota14, FOU15] and [AOUV, Table 2] for more details.

Thanks to Lemma 4.3, we can focus our attention to left-invariant hermitian structures, which are of the form (4.1). We compute $\sqrt{-1}\partial\bar{\partial}\omega$ case by case (Compare [FOU15, Theorem 4.3] which studies the sign of the Fu-Wang-Wu function).

- Si:** $\sqrt{-1}\partial\bar{\partial}\omega = 2(-1)((\Re A)^2 r^2 \varphi^{1\bar{1}} + \sqrt{-1}(\Im A)^2 u \varphi^{1\bar{2}} - \sqrt{-1}(\Im A)^2 \bar{u} \varphi^{2\bar{1}} + (\Re A)^2 s^2 \varphi^{2\bar{2}}) \wedge \varphi^{3\bar{3}}$. When $(\cos \theta)^4 \geq (\sin \theta)^4$, any metric is pluripositive; otherwise, there are pluripositive metrics (taking $|u|^2 \leq \frac{\cos \theta}{\sin \theta} r^2 s^2$). When $\cos \theta = 0$, there exist pluriclosed metrics (for $u = 0$) and Kähler metrics (for $u = v = z = 0$, see [Ota14, Theorem 5.1.3]). Plurinegative metrics never exists, while balanced metrics always exist (when $v = z = 0$).
- Sii:** $\sqrt{-1}\partial\bar{\partial}\omega = \left(\frac{4t^2x^2+t^2}{16x^2}\right) (-1)\varphi^{1\bar{1}} \wedge \varphi^{2\bar{2}} + (s^2x^2 + \frac{1}{4}s^2) (-1)\varphi^{1\bar{1}} \wedge \varphi^{3\bar{3}}$. Any metric is pluripositive. There is neither pluriclosed metrics, nor plurinegative metrics, nor Kähler metrics. Balanced metrics always exist (take $u = v = z = 0$).
- Siii1:** $\sqrt{-1}\partial\bar{\partial}\omega = 2(\sqrt{-1}u\sqrt{-1}\varphi^{1\bar{2}} - \sqrt{-1}\bar{u}\sqrt{-1}\varphi^{2\bar{1}}) \wedge \sqrt{-1}\varphi^{3\bar{3}}$. There exist pluriclosed metrics (take $u = 0$), but neither Kähler metrics nor balanced metrics.
- Siii2:** $\sqrt{-1}\partial\bar{\partial}\omega = t^2(-1)\varphi^{1\bar{1}} \wedge \varphi^{2\bar{2}} + 2r^2(-1)\varphi^{1\bar{1}} \wedge \varphi^{3\bar{3}} + 2s^2(-1)\varphi^{2\bar{2}} \wedge \varphi^{3\bar{3}}$. Every metric is pluripositive. There is neither pluriclosed, nor plurinegative, nor Kähler metrics. Balanced metrics exist (take $u \in \mathbb{R}$ and $v = z = 0$).
- Siii3:** $\sqrt{-1}\partial\bar{\partial}\omega = -t^2(-1)\varphi^{1\bar{1}} \wedge \varphi^{2\bar{2}} + 2(-1)\sqrt{-1}(u\varphi^{1\bar{2}} - \bar{u}\varphi^{2\bar{1}}) \wedge \varphi^{3\bar{3}}$. There are neither pluripositive, nor pluriclosed metrics. There exists plurinegative metrics (take $u = 0$), but neither Kähler nor balanced metrics.
- Siii4:** $\sqrt{-1}\partial\bar{\partial}\omega = t^2(-1)\varphi^{1\bar{1}} \wedge \varphi^{2\bar{2}} + 2(-1)(\sqrt{-1}u\varphi^{1\bar{2}} - \sqrt{-1}\bar{u}\varphi^{2\bar{1}}) \wedge \varphi^{3\bar{3}}$. Pluriclosed metrics and plurinegative metrics never exist. There are pluripositive metrics (take $u = 0$), but no Kähler metrics. There are balanced metrics (characterized by parameters $r^2 = s^2, v = z = 0$).
- Siv1:** $\sqrt{-1}\partial\bar{\partial}\omega = \frac{(-1)}{2}(r^2\varphi^{1\bar{1}} + \sqrt{-1}u\varphi^{1\bar{2}} - \sqrt{-1}\bar{u}\varphi^{2\bar{1}} + s^2\varphi^{2\bar{2}}) \wedge \varphi^{3\bar{3}}$. Every metric is pluripositive, there are neither pluriclosed nor plurinegative metrics. There is no Kähler metric, and every metric is balanced.
- Siv2:** $\sqrt{-1}\partial\bar{\partial}\omega = 2(-1)(r^2\varphi^{1\bar{1}} + \sqrt{-1}u\varphi^{1\bar{2}} - \sqrt{-1}\bar{u}\varphi^{2\bar{1}} + s^2\varphi^{2\bar{2}}) \wedge \varphi^{3\bar{3}}$. Every metric is pluripositive, there are neither pluriclosed nor plurinegative metrics. There are no Kähler metrics and no balanced metrics.
- Siv3:** $\sqrt{-1}\partial\bar{\partial}\omega = \frac{(-1)}{2}(|A-1|^2r^2\varphi^{1\bar{1}} + \sqrt{-1}|A+1|^2u\varphi^{1\bar{2}} - \sqrt{-1}|A+1|^2\bar{u}\varphi^{2\bar{1}} + |A-1|^2s^2\varphi^{2\bar{2}}) \wedge \varphi^{3\bar{3}}$. There are pluripositive metrics: when $\Re A \leq 0$, any metric is pluripositive; when $\Re A > 0$, pluripositive metrics are characterized by $|u|^2 \leq \left|\frac{A-1}{A+1}\right|^4 r^2 s^2$. There are neither pluriclosed nor plurinegative, nor Kähler metrics. Balanced metrics always exist (take $v = z = 0$).
- Sv:** $\sqrt{-1}\partial\bar{\partial}\omega = \frac{(-1)}{2}\varphi^{1\bar{1}} \wedge (\sqrt{-1}v\varphi^{2\bar{3}} - \sqrt{-1}\bar{v}\varphi^{3\bar{2}} + \frac{1}{4}s^2\varphi^{3\bar{3}})$. There are pluripositive metrics (take $v = 0$). There are neither pluriclosed, nor plurinegative, nor Kähler, nor balanced metrics.

Notice that it is no longer true that the plurisigned property of left-invariant metrics is completely determined by the complex structure.

We summarize the previous dicussion in the following:

Theorem 4.7. *Let X be a non-Kähler six-dimensional solvmanifold endowed with a left-invariant complex structure with holomorphically-trivial canonical bundle. Then X admits a plurisigned hermitian metric. More precisely:*

- either X belongs to the class (Siii1) and there are pluriclosed metrics;
- or X belongs to (Siii3) and there are plurinegative non-pluriclosed metrics;

- or else X belongs to any of the remaining classes, and it admits a left-invariant pluripositive hermitian metric that is not pluriclosed.

Recall that these three conditions are mutually exclusive by Corollary 3.7. The only Kähler example in the above list corresponds to a special Lie algebra $\mathcal{G} = \mathcal{A}_{5,17}^{0,0,1} \oplus \mathbb{R}$ of splitting type (see [FOU15, Theorem 2.18]).

Remark 4.8. If a hermitian metric is both balanced and pluriclosed, then it is Kähler [AI01, Remark 1]. It is conjectured [FV15, Problem 3] that a compact complex manifold admitting both pluriclosed metrics and balanced metrics also admits Kähler metrics. The conjecture is confirmed for the above nilmanifolds and solvmanifolds by [FV15, Theorem 6.3 and Theorem 6.4]. We notice that these facts fail when replacing pluriclosed with plurisigned: the Iwasawa manifold does not admit any Kähler metric [FG86, BG88, Has89], but every left-invariant hermitian metric on it is both balanced and pluripositive.

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